Two-pion exchange electromagnetic current in chiral effective field theory using the method of unitary transformation

S. Kölling,[1,2,*] E. Epelbaum,[1,2,†] H. Krebs,[2,‡] and U.-G. Meißner[2,1,3,§]

[1]*Forschungszentrum Jülich, Institut für Kernphysik (IKP-3) and Jülich Center for Hadron Physics, D-52425 Jülich, Germany*

[2]*Helmholtz-Institut für Strahlen- und Kernphysik (Theorie) and Bethe Center for Theoretical Physics, Universität Bonn, D-53115 Bonn, Germany*

[3]*Forschungszentrum Jülich, Institut for Advanced Simulation, D-52425 Jülich, Germany*

(Received 4 August 2009; published 29 October 2009)

We derive the leading two-pion-exchange contributions to the two-nucleon electromagnetic current operator in the framework of chiral effective field theory using the method of unitary transformation. Explicit results for the current and charge densities are given in momentum and coordinate space.

DOI: 10.1103/PhysRevC.80.045502 PACS number(s): 13.75.Cs, 21.30.−x

I. INTRODUCTION

Chiral effective field theory (EFT) provides a systematic and model-independent framework to analyze low-energy hadronic processes in harmony with the spontaneously broken approximate chiral symmetry of QCD. This approach has been successfully applied to the derivation of the nuclear forces and, more recently, also hyperon-nucleon and hyperon-hyperon interactions; see Refs. [1–3] for review articles. Exchange vector and axial currents in nuclei have also been studied in the framework of chiral EFT. In their pioneering work, Park, Min, and Rho applied heavy-baryon chiral perturbation theory to derive exchange axial [4] and vector [5] currents at the one-loop level for small values of the photon momentum, focusing, in particular, on the axial-charge and magnetic moment operators, respectively. These calculations were carried out employing time-ordered perturbation theory to extract noniterative contributions to the amplitude. The resulting exchange vector currents were applied within a hybrid approach to analyze magnetic moments and radiative capture cross sections of thermal neutrons on light nuclei [5–8] as well as some polarization observables in radiative neutron capture on the proton [9]. For applications to various electroweak few-nucleon reactions of astrophysical interest see Refs. [10–14].

Deuteron electromagnetic properties [15–19], Compton scattering on the deuteron [20,21] and, more recently, on ^{3}He [22], as well as pion electro- and photoproduction and the corresponding capture reactions [23–28] have also been addressed in the framework of chiral EFT. However, to the best of our knowledge, no applications have so far been performed to electron and photon inelastic few-nucleon reactions with the momentum transfer of the order M_π where a lot of experimental data are available; see Ref. [29] for a recent review article on the theoretical achievements in this field based on conventional framework. Recent progress in the accurate description of the two- [30,31] and more-nucleon systems [32] within the chiral EFT (see also Ref. [3] and references therein) gives a strong motivation to apply this framework to the above-mentioned processes. This requires the knowledge of the consistent electromagnetic exchange current operator for nonvanishing values of the photon momentum. While the leading two-nucleon contributions to the exchange current arise from one-pion exchange and are well known, the corrections at the one-loop level have not yet been completely worked out. An important step in this direction was done recently by Pastore *et al.* [33,34], who considered the electromagnetic two-body current density at the leading one-loop order based on time-ordered perturbation theory. In the present work, we calculate the leading two-pion-exchange two-nucleon four-current operator in chiral EFT based on the method of unitary transformation that we used to derive nuclear forces in Refs. [35–39]. Our work provides an important check of the results presented in Refs. [33,34] but also differs from these works in several important respects. First, as already pointed out, we use a completely different method to compute the current operator. Second, we also give results for the exchange charge density that, to the best of our knowledge, have not yet been published before. Finally, we evaluate analytically all loop integrals to obtain a representation in momentum space in terms of the standard loop functions and the three-point functions. The latter are reduced to a form that can be easily treated numerically. We also succeeded to analytically carry out the Fourier transformation for all contributions leading to an extremely compact representation of the current and charge densities in coordinate space. Notice that contrary to Refs. [33,34], we do not treat the $\Delta(1232)$ isobars as explicit degrees of freedom in this work.

Our manuscript is organized as follows. In Sec. II we provide a short summary of the method of unitary transformation, explain the adopted power counting scheme, list all relevant terms in the effective chiral Lagrangian, and present our results for the exchange current and charge densities in momentum space. The expressions in coordinate space are given in Sec. III. The results of our work are summarized in Sec. IV. The formal operator structure of the effective electromagnetic current can be found in Appendix A, while Appendix B contains the complete momentum-space representation of the

*s.koelling@fz-juelich.de
†e.epelbaum@fz-juelich.de
‡hkrebs@itkp.uni-bonn.de
§meissner@itkp.uni-bonn.de

current. Evaluation of the three-point functions entering this representation is detailed in Appendix C. In Appendix D we verify that the obtained expressions for the current fulfill the continuity equation. Finally, Appendix E collects the formulas needed to obtain the coordinate-space representation given in Sec. III.

II. NUCLEAR CURRENTS USING THE METHOD OF UNITARY TRANSFORMATION

We begin with a brief reminder about the method of unitary transformation. Consider the time-independent Schrödinger equation for interacting pions and nucleons in the absence of electromagnetic sources

$$(H_0 + H_I)|\Psi\rangle = E|\Psi\rangle, \tag{2.1}$$

where $|\Psi\rangle$ denotes an eigenstate of the Hamiltonian H with the eigenvalue E. Let η (λ) be projection operators onto the purely nucleonic (the remaining) part of the Fock space satisfying $\eta^2 = \eta$, $\lambda^2 = \lambda$, $\eta\lambda = \lambda\eta = 0$, and $\lambda + \eta = \mathbf{1}$. To describe the dynamics of few- and many-nucleon systems below the pion production threshold it is advantageous to project Eq. (2.1) onto the η subspace of the full Fock space. The resulting effective equation can then be solved using the standard methods of few- or many-body physics. The decoupling of the η and λ subspaces can be achieved via an appropriately chosen unitary transformation [40,41]

$$\tilde{H} \equiv U^\dagger H U = \begin{pmatrix} \eta\tilde{H}\eta & 0 \\ 0 & \lambda\tilde{H}\lambda \end{pmatrix}; \tag{2.2}$$

see Ref. [42] for an alternative approach. Following Okubo [41], the unitary operator U can be parametrized as

$$U = \begin{bmatrix} \eta(1 + A^\dagger A)^{-1/2} & -A^\dagger(1 + AA^\dagger)^{-1/2} \\ A(1 + A^\dagger A)^{-1/2} & \lambda(1 + AA^\dagger)^{-1/2} \end{bmatrix}, \tag{2.3}$$

in terms of the operator $A = \lambda A \eta$ that has to satisfy the decoupling equation

$$\lambda(H - [A,\ H] - AHA)\eta = 0 \tag{2.4}$$

for the transformed Hamiltonian $\tilde{H}$ to be of block-diagonal form. The effective η-space potential V can be expressed in terms of the operator A via:

$$V = \eta(\tilde{H} - H_0) = \eta\big[(1 + A^\dagger A)^{-1/2}(H + A^\dagger H + HA + A^\dagger HA)(1 + A^\dagger A)^{-1/2} - H_0\big]\eta. \tag{2.5}$$

The unitary transformation U and the effective potential V can be calculated perturbatively based on the most general effective chiral Lagrangian utilizing the chiral power counting. In Ref. [39], a convenient formulation of the power counting has been presented. The low-momentum dimension ν of the effective potential, V, $V \sim \mathcal{O}(Q/\Lambda)^\nu$ with Q and Λ referring to the soft and hard scales of the order of the pion and ρ-meson masses, respectively, is given (modulo the normalization constant -2) by the overall inverse mass dimension of the coupling constants entering the expression for V:

$$\nu = -2 + \sum V_i \kappa_i, \qquad \kappa_i = d_i + \frac{3}{2}n_i + p_i - 4. \tag{2.6}$$

Here, V_i is the number of vertices of type i while d_i, n_i and p_i refer to the number of derivatives or M_π insertions, nucleon field, and pion field operators, respectively. Further, κ_i is simply the canonical field dimension of a vertex of type i (up to the additional constant -4). Writing the effective chiral Hamiltonian H as

$$H = \sum_{\kappa=1}^{\infty} H^{(\kappa)}, \tag{2.7}$$

the operator A can be calculated recursively,

$$A = \sum_{\alpha=1}^{\infty} A^{(\alpha)},$$

$$A^{(\alpha)} = \frac{1}{E_\eta - E_\lambda}\lambda\Bigg[H^{(\alpha)} + \sum_{i=1}^{\alpha-1} H^{(i)}A^{(\alpha-i)} - \sum_{i=1}^{\alpha-1} A^{(\alpha-i)}H^{(i)} - \sum_{i=1}^{\alpha-2}\sum_{j=1}^{\alpha-j-1} A^{(i)}H^{(j)}A^{(\alpha-i-j)}\Bigg]\eta. \tag{2.8}$$

Here, E_η (E_λ) refers to the free energy of nucleons (nucleons and pions) in the state η (λ). The expressions for the unitary operator and the effective potential then follow immediately by substituting Eqs. (2.7) and (2.8) into Eq. (2.5). It is important to emphasize that Eq. (2.3) does not provide the most general parametrization of the unitary operator. Moreover, as found in Ref. [39], the subleading contributions to the three-nucleon force obtained using the parametrization in Eq. (2.3) cannot be renormalized. To restore renormalizability at the level of the Hamilton operator additional unitary transformation U' in the η subspace of the Fock space had to be employed, $\eta U'\eta U'^\dagger\eta = \eta U'^\dagger\eta U'\eta = \eta$, whose explicit form at lowest nontrivial order is given in that work.

It is, in principle, straightforward to extend this formalism to low-energy electromagnetic reactions such as, e.g., electron scattering off light nuclei; see Refs. [43–45] for some early applications of the method of unitary transformation to the derivation of the exchange currents. Here and in what follows, we restrict ourselves to the one-photon-exchange approximation to the scattering amplitude. The effective nuclear current operator $\eta J^\mu_{\rm eff}(x)\eta$ acting in the η space is then defined according to

$$\langle\Psi_f|J^\mu(x)|\Psi_i\rangle = \langle\phi_f|\eta U'^\dagger\eta U^\dagger J^\mu(x) U\eta U'\eta|\phi_i\rangle \equiv \langle\phi_f|\eta J^\mu_{\rm eff}(x)\eta|\phi_i\rangle, \tag{2.9}$$

where $\eta|\phi_{i,f}\rangle = \eta U'^\dagger\eta U^\dagger|\Psi_{i,f}\rangle$ denote the transformed states and we have omitted the components $\lambda|\phi_{i,f}\rangle$, which is justified as long as one stays below the pion production threshold. In the above expression, $J^\mu(x)$ denotes the hadronic current density that enters the effective Lagrangian $\mathcal{L}_{\pi N\gamma}$ describing the interaction of pions and nucleons with an external electromagnetic field $\mathcal{A}^\mu$ and is given by

$$J^\mu(x) = \partial_\nu \frac{\partial\mathcal{L}_{\pi N\gamma}}{\partial(\partial_\nu\mathcal{A}_\mu)} - \frac{\partial\mathcal{L}_{\pi N\gamma}}{\partial\mathcal{A}_\mu}. \tag{2.10}$$

Note that contrary to the Hamilton operator, the unitarily transformed current does, in general, not have the block-diagonal

form, i.e., $\eta U^\dagger J^\mu(x) U\lambda \neq 0$. Again, it is important to realize that the above definition of $\eta J^\mu_{\rm eff}(x)\eta$ does not fully incorporate the freedom in the choice of unitary transformations. Thus, one might expect that this formulation yields the effective current operator that is not renormalizable by a redefinition of the low-energy constants (LECs) entering the underlying Lagrangian. Indeed, renormalizability of the effective current operator implies highly nontrivial constraints in the case of one-pion-exchange contributions at the one-loop level because all β functions of the LECs l_i from $\mathcal{L}_\pi$ [46,47] and d_i from $\mathcal{L}_{\pi N}$ [48,49] are fixed. We have verified that the ultraviolet divergences entering the expressions for the one-pion-exchange contributions using the formulation based on the $\mathcal{A}^\mu$-independent unitary transformation as described above can indeed not be completely removed by the redefinition of the corresponding LECs. Thus, a more general parametrization of the unitary transformation is required to restore renormalizability of the nuclear current. This can be achieved if one allows for the unitary operator to depend explicitly on the external electromagnetic field, $U(\mathcal{A}^\mu)$. The operator $U(\mathcal{A}^\mu)$ then has to be chosen in such a way that the transformed Hamiltonian $U^\dagger(\mathcal{A}^\mu) H_{\pi N\gamma} U(\mathcal{A}^\mu)$ is block-diagonal (with respect to the η and λ spaces) and coincides with the one given in Ref. [39] when the external electromagnetic field is switched off. The effective nuclear current operator $\eta J^\mu_{\rm eff}(x)\eta$ in this more general formulation receives additional contributions that are not included in Eq. (2.9) and result from $\mathcal{A}^\mu$-dependent pieces of $U(\mathcal{A}^\mu)$ in the expression $U^\dagger(\mathcal{A}^\mu) H_{\pi N\gamma} U(\mathcal{A}^\mu)$, whose form is determined by renormalizability of the resulting nuclear current operator. These additional terms in $\eta J^\mu_{\rm eff}(x)\eta$ are found to have no effect on the two-pion-exchange current and will be discussed in detail in a separate publication [50] devoted to the one-pion-exchange contributions. Finally, we emphasize that the power counting employed in the present work implies the following restrictions on the photon momentum k in the two-nucleon rest frame

$$|\vec{k}| \sim \mathcal{O}(M_\pi), \qquad k^0 \sim \mathcal{O}\left(\frac{M_\pi^2}{m}\right) \ll M_\pi, \tag{2.11}$$

where M_π and m refer to the pion and nucleon masses, respectively. In particular, the second constraint implies that for processes with real photons (such as, e.g., bremsstrahlung or radiative capture reactions) our results are applicable only at very low photon three-momenta. For the kinematics with $k^0 \sim \mathcal{O}(M_\pi)$, one will have to systematically keep track of the new soft momentum scale $\sqrt{M_\pi m}$ similar to the case of, e.g., pion production in nucleon-nucleon collisions; see Ref. [51] for a recent review article. In the present work we, however, restrict ourselves to the kinematical conditions specified in Eq. (2.11).

For the calculation of the leading two-pion-exchange two-nuclear current operator in the present work we only need the leading pion and pion-nucleon terms in the effective Lagrangian

$$\begin{aligned} \mathcal{L}^{(2)}_{\pi\pi} &= \frac{F_\pi^2}{4}{\rm tr}\left[D_\mu U D^\mu U^\dagger + M_\pi^2 (U + U^\dagger)\right], \\ \mathcal{L}^{(1)}_{\pi N} &= N^\dagger (iv\cdot D + g_A S\cdot u) N, \end{aligned} \tag{2.12}$$

where the superscript i in $\mathcal{L}^{(i)}$ denotes the number of derivatives and/or pion mass insertions. Here, F_π (g_A) is the pion decay constant (the nucleon axial-vector coupling), N represents a nucleon field in the heavy-baryon formulation, and $S_\mu = \frac{1}{2}\gamma_5\sigma_{\mu\nu}v^\nu$ is the Pauli-Lubanski spin vector that reduces to $S^\mu = (0, \frac{1}{2}\vec{\sigma})$ for $v_\mu = (1, 0, 0, 0)$. At the order we are working and for the contributions to the current operator considered in the present work, all LECs entering Eq. (2.13) should be taken at their physical values. Further, the SU(2) matrix $U = u^2$ collects the pion fields and various covariant derivatives are defined according to

$$\begin{aligned} D_\mu U &= \partial_\mu U - i r_\mu U + i U l_\mu, \\ u_\mu &= i[u^\dagger(\partial_\mu - i r_\mu) u - u(\partial_\mu - i l_\mu) u^\dagger], \\ D_\mu N &= \left[\partial_\mu + \Gamma_\mu - i v^{(s)}_\mu\right] N \quad \text{with} \\ \Gamma_\mu &= \tfrac{1}{2}[u^\dagger(\partial_\mu - i r_\mu) u + u(\partial_\mu - i l_\mu) u^\dagger]. \end{aligned} \tag{2.13}$$

To describe the coupling to an external electromagnetic field, the left- and right-handed currents r_μ and l_μ and the isoscalar current $v^{(s)}_\mu$ have to be chosen as

$$r_\mu = l_\mu = \frac{e}{2}\mathcal{A}_\mu \tau_3, \qquad v^{(s)}_\mu = \frac{e}{2}\mathcal{A}_\mu, \tag{2.14}$$

where e denotes the elementary charge. Expanding the various terms in the effective Lagrangian in powers of the pion field and using the canonical formalism along the lines of Ref. [52], we end up with the following interaction terms in the Hamilton density

$$\begin{aligned} \mathcal{H}^{(1)}_{21} &= \frac{g_A}{2F_\pi} N^\dagger (\vec{\sigma}\,\vec{\tau}\cdot\cdot\vec{\nabla}\vec{\pi}) N, \\ \mathcal{H}^{(2)}_{22} &= \frac{1}{4F_\pi^2} N^\dagger [\vec{\pi}\times\dot{\vec{\pi}}]\cdot\vec{\tau} N, \\ \mathcal{H}^{(4)}_{42} &= \frac{1}{32F_\pi^4} (N^\dagger[\vec{\tau}\times\vec{\pi}]N)\cdot(N^\dagger[\vec{\tau}\times\vec{\pi}]N), \end{aligned} \tag{2.15}$$

and the electromagnetic current density is of the form

$$\begin{aligned} J^{0\,(-1)}_{20} &= \frac{e}{2} N^\dagger(\mathbb{1} + \tau_3) N, \\ J^{0\,(-1)}_{02} &= e[\vec{\pi}\times\dot{\vec{\pi}}]_3, \\ \vec{J}^{\,(-1)}_{02} &= -e[\vec{\pi}\times\vec{\nabla}\vec{\pi}]_3, \\ \vec{J}^{\,(0)}_{21} &= e\,\frac{g_A}{2F_\pi} N^\dagger\vec{\sigma}[\vec{\tau}\times\vec{\pi}]_3 N. \end{aligned} \tag{2.16}$$

In the above expressions we adopt the notation of Ref. [39]. In particular, the subscripts a and b in $\mathcal{H}^{(\kappa)}_{ab}$ and $J^{\mu\,(\kappa)}_{ab}$ refer to the number of the nucleon and pion fields, respectively, while the superscript κ gives the dimension of the operator as defined in Eq. (2.6). Further, the symbol $\cdot\cdot$ in Eq. (2.15) denotes a scalar product in the spin and isospin spaces.

The formal operator structure of the leading two-pion-exchange two-nucleon current at order $\mathcal{O}(eQ)$ is given in Appendix A. For the sake of convenience, we distinguish among seven classes of contributions according to the power of the LEC g_A (i.e., proportional to g_A^0, g_A^2, and g_A^4) and the type of the hadronic current J^μ_{20}, J^μ_{21} or J^μ_{02} as shown in Fig. 1. Note that there are no contributions proportional to g_A^0 and involving J^μ_{20} and J^μ_{21}. We also emphasize that the

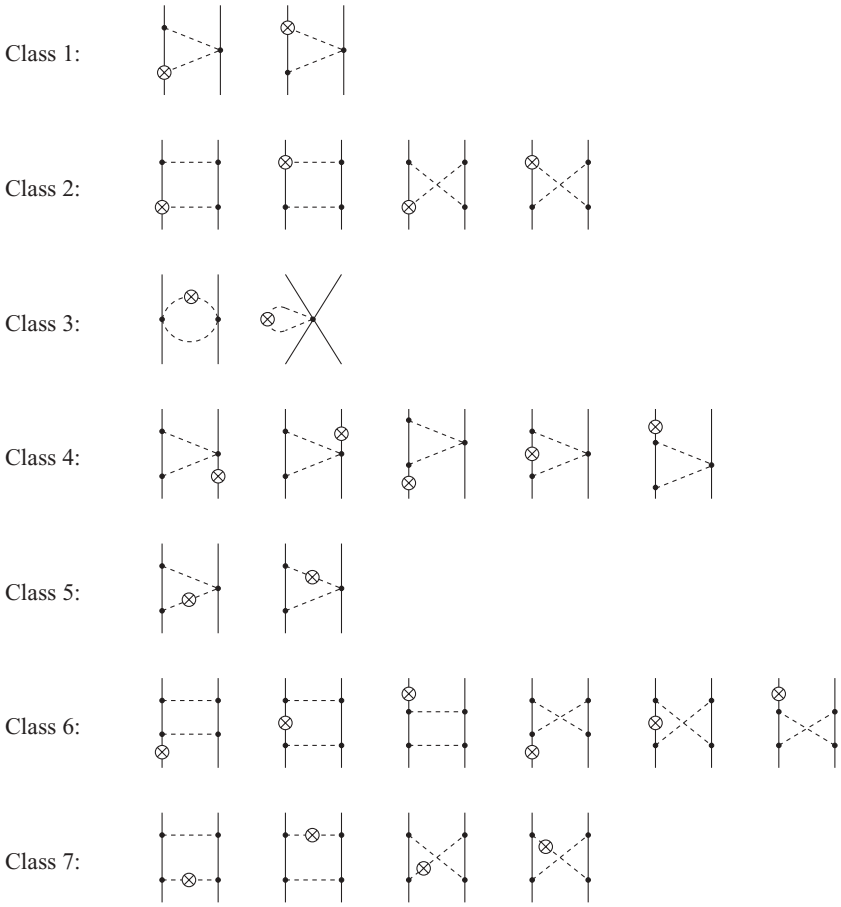

FIG. 1. Diagrams showing contributions to the leading two-pion-exchange currents. Solid and dashed lines refer to nucleons and pions, respectively. Solid dots are the lowest-order vertices from the effective Lagrangian while the crosses represent insertions of the electromagnetic vertices as explained in the text. Diagrams resulting from interchanging the nucleon lines are not shown.

second diagram in the class 3 generates no contribution. It results from the term in the Hamilton density that is absent in the Lagrangian and arises through the application of the canonical formalism. Finally, it should be understood that the meaning of diagrams in the method of unitary transformation is different from the one arising in the context of covariant and/or time-ordered perturbation theory. The diagrams shown in Fig. 1 serve merely to visualize the topology corresponding to a given sequence of operators H and J appearing in the formal expressions given in Appendix A.

It is a fairly straightforward albeit tedious exercise to evaluate the contributions to the nuclear current corresponding to the operators given in Appendix A. Below, we give explicit results for the current and charge densities, $J^\mu = (\rho, \vec{J})$, resulting from the individual classes using the notation

$$\langle \vec{p}_1{}' \, \vec{p}_2{}' | J^\mu | \vec{p}_1 \vec{p}_2 \rangle = \delta(\vec{p}_1{}' + \vec{p}_2{}' - \vec{p}_1 - \vec{p}_2 - \vec{k}) \times \left[\sum_{X=\mathrm{c1}}^{\mathrm{c7}} J_X^\mu + (1 \leftrightarrow 2) \right], \qquad (2.17)$$

where $\vec{p}_i$ ($\vec{p}_i{}'$) denotes the incoming (outgoing) momentum of nucleon i and $\vec{k}$ is the photon momentum. Further, $(1 \leftrightarrow 2)$ refers to the contribution resulting from the interchange of the nucleon labels. We find the following results for the current density from the individual classes of the diagrams shown in Fig. 1:

$$\begin{aligned}
\vec{J}_{\mathrm{c1}}(\vec{q}_1, \vec{q}_2) &= e \frac{g_A^2 i}{16 F_\pi^4} \big[\vec{q}_1 [\vec{\tau}_1 \times \vec{\tau}_2]^3 + 2 [\vec{q}_1 \times \vec{\sigma}_2] \tau_1^3 \big] \\
&\quad \times \int \frac{d^3 l}{(2\pi)^3} \frac{1}{\omega_+ \omega_- (\omega_+ + \omega_-)} \\
&\to -e \frac{g_A^2 i}{64 \pi^2 F_\pi^4} \big[\vec{q}_1 [\vec{\tau}_1 \times \vec{\tau}_2]^3 \\
&\quad + 2 [\vec{q}_1 \times \vec{\sigma}_2] \tau_1^3 \big] L(q_1), \\
\vec{J}_{\mathrm{c2}}(\vec{q}_1, \vec{q}_2) &= e \frac{g_A^4 i}{16 F_\pi^4} \int \frac{d^3 l}{(2\pi)^3} \frac{\omega_+^2 + \omega_+ \omega_- + \omega_-^2}{\omega_+^3 \omega_-^3 (\omega_+ + \omega_-)} \\
&\quad \times \big[4 \tau_2^3 \vec{l} (\vec{l} \cdot [\vec{q}_1 \times \vec{\sigma}_1]) - \big(l^2 - q_1^2 \big) \\
&\quad \times \big(\vec{q}_1 [\vec{\tau}_1 \times \vec{\tau}_2]^3 + 2 [\vec{q}_1 \times \vec{\sigma}_2] \tau_1^3 \big) \big] \\
&\to e \frac{g_A^4 i}{64 \pi^2 F_\pi^4 \big(q_1^2 + 4 M_\pi^2 \big)} \big[\vec{q}_1 [\vec{\tau}_1 \times \vec{\tau}_2]^3 \\
&\quad \times \big(8 M_\pi^2 + 3 q_1^2 \big) + 2 [\vec{q}_1 \times \vec{\sigma}_2] \tau_1^3 \big(8 M_\pi^2 + 3 q_1^2 \big) \\
&\quad - 4 \big(q_1^2 + 4 M_\pi^2 \big) [\vec{q}_1 \times \vec{\sigma}_1] \tau_2^3 \big] L(q_1), \\
\vec{J}_{\mathrm{c3}}(\vec{q}_1, \vec{q}_2) &= e \frac{i}{128 F_\pi^4} [\vec{\tau}_1 \times \vec{\tau}_2] \int \frac{d^3 l}{(2\pi)^3} \frac{1}{\omega_1 \omega_2 \omega_3} \\
&\quad \times \left(\frac{(\omega_1 - \omega_2)(\omega_1 - \omega_3)}{(\omega_1 + \omega_2)(\omega_1 + \omega_3)} + \frac{(\omega_1 + \omega_2)(\omega_1 - \omega_3)}{(\omega_1 + \omega_3)(\omega_2 + \omega_3)} \right. \\
&\quad \left. + \frac{(\omega_1 - \omega_2)(\omega_1 + \omega_3)}{(\omega_1 + \omega_2)(\omega_2 + \omega_3)} \right) \times (\vec{k}_2 + \vec{k}_3),
\end{aligned}$$

$$
\begin{aligned}
\vec{J}_{\rm c4}(\vec{q}_1,\vec{q}_2) &= \vec{J}_{\rm c6}(\vec{q}_1,\vec{q}_2) = 0,\\
\vec{J}_{\rm c5}(\vec{q}_1,\vec{q}_2) &= -e\,\frac{g_A^2 i}{8F_\pi^4}\int\frac{d^3l}{(2\pi)^3}\frac{1}{\omega_1\omega_2\omega_3}\\
&\quad\times\frac{\omega_1+\omega_2+\omega_3}{(\omega_1+\omega_2)(\omega_1+\omega_3)(\omega_2+\omega_3)}(\vec{k}_2+\vec{k}_3)\\
&\quad\times\big[[\vec{\tau}_1\times\vec{\tau}_2]^3\vec{k}_1\cdot\vec{k}_3-2\tau_1^3\vec{\sigma}_2\cdot[\vec{k}_1\times\vec{k}_3]\big],\\
\vec{J}_{\rm c7}(\vec{q}_1,\vec{q}_2) &= e\,\frac{g_A^4 i}{16F_\pi^4}\int\frac{d^3l}{(2\pi)^3}\frac{1}{\omega_1\omega_2\omega_3}\left(\frac{1}{\omega_1\omega_2\omega_3^2}\right.\\
&\quad-\frac{\omega_2}{\omega_1^2(\omega_1^2-\omega_2^2)(\omega_1+\omega_3)}\\
&\quad\left.+\frac{\omega_1}{\omega_2^2(\omega_1^2-\omega_2^2)(\omega_2+\omega_3)}\right)\\
&\quad\times(\vec{k}_2+\vec{k}_3)\big[[\vec{\tau}_1\times\vec{\tau}_2]^3\vec{k}_1\cdot\vec{k}_2\vec{k}_1\cdot\vec{k}_3\\
&\quad+4\tau_2^3\vec{\sigma}_1\cdot[\vec{k}_1\times\vec{k}_2]\vec{k}_1\cdot\vec{k}_3\big],
\end{aligned}
\tag{2.18}
$$

while the contributions to the exchange charge density read

$$
\begin{aligned}
\rho_{\rm c1}(\vec{q}_1,\vec{q}_2) &= \rho_{\rm c2}(\vec{q}_1,\vec{q}_2) = \rho_{\rm c3}(\vec{q}_1,\vec{q}_2) = 0,\\
\rho_{\rm c4}(\vec{q}_1,\vec{q}_2) &= e\,\frac{g_A^2}{16F_\pi^4}\tau_1^3\int\frac{d^3l}{(2\pi)^3}\frac{1}{\omega_+^2\omega_-^2}\big[l^2-q_1^2\big]\\
&\to -e\,\frac{g_A^2}{32\pi F_\pi^4}\tau_1^3\big[M_\pi+\big(2M_\pi^2+q_1^2\big)A(q_1)\big],\\
\rho_{\rm c5}(\vec{q}_1,\vec{q}_2) &= -e\,\frac{g_A^2}{16F_\pi^4}\tau_2^3\int\frac{d^3l}{(2\pi)^3}\frac{1}{\omega_+^2\omega_-^2}(l^2-q_1^2)\\
&\to e\,\frac{g_A^2}{32\pi F_\pi^4}\tau_2^3\big[M_\pi+\big(2M_\pi^2+q_1^2\big)A(q_1)\big],\\
\rho_{\rm c6}(\vec{q}_1,\vec{q}_2) &= -e\,\frac{g_A^4}{8F_\pi^4}\int\frac{d^3l}{(2\pi)^3}\frac{1}{\omega_+^2\omega_-^4}\\
&\quad\times\big[4\tau_2^3\vec{l}\cdot[\vec{q}_1\times\vec{\sigma}_1]\vec{l}\cdot[\vec{q}_1\times\vec{\sigma}_2]+\big(l^2-q_1^2\big)^2\tau_1^3\big]\\
&\to e\,\frac{g_A^2}{32\pi F_\pi^4}\left[\big(\big(4M_\pi^2+2q_1^2\big)\tau_1^3\right.\\
&\quad+\tau_2^3\big[\vec{q}_1\cdot\vec{\sigma}_1\vec{q}_1\cdot\vec{\sigma}_2-q_1^2\vec{\sigma}_1\cdot\vec{\sigma}_2\big]\big)A(q_1)\\
&\quad\left.+M_\pi\frac{\big(11M_\pi^2+3q_1^2\big)}{4M_\pi^2+q_1^2}\tau_1^3\right],\\
\rho_{\rm c7}(\vec{q}_1,\vec{q}_2) &= e\,\frac{g_A^4}{8F_\pi^4}\int\frac{d^3l}{(2\pi)^3}\frac{1}{\omega_1^2\omega_2^2\omega_3^2}\\
&\quad\times\big[\big(\tau_1^3+\tau_2^3\big)(\vec{k}_1\cdot\vec{k}_2\vec{k}_1\cdot\vec{k}_3\\
&\quad+\vec{k}_1\cdot[\vec{k}_2\times\vec{\sigma}_1]\vec{k}_1\cdot[\vec{k}_3\times\vec{\sigma}_2])\\
&\quad+[\vec{\tau}_1\times\vec{\tau}_2]^3\vec{k}_1\cdot\vec{k}_2\vec{k}_1\cdot[\vec{k}_3\times\vec{\sigma}_2]\big],
\end{aligned}
\tag{2.19}
$$

where $\vec{q}_i\equiv\vec{p}_i'-\vec{p}_i$ is the momentum transfer of nucleon i. Clearly, $\vec{\sigma}_i$ and $\vec{\tau}_i$ refer to the spin and isospin Pauli matrices of the nucleon i. Here and in what follows, we label the Cartesian components of various vectors in isospin space by the superscripts rather than subscripts to avoid a possible confusion with the nucleon labels. Further, $\omega_\pm=\sqrt{(\vec{l}\pm\vec{q}_1)^2+4M_\pi^2}$ and $\omega_i=\sqrt{\vec{k}_i^2+M_\pi^2}$ with $\vec{k}_1=\vec{l},\vec{k}_2=\vec{l}-\vec{q}_1$ and $\vec{k}_3=\vec{l}+\vec{q}_2$. We also use here the short-hand notation with $q_i\equiv|\vec{q}_i|$. Finally, it should be understood that the above expressions refer to matrix elements with respect to momenta and operators in spin and isospin spaces, see Eq. (2.17). The expressions for $\vec{J}_{\rm c1,c2}(\vec{q}_1,\vec{q}_2)$ and $\rho_{\rm c4,c5,c6}(\vec{q}_1,\vec{q}_2)$ involve only two-point functions so that the corresponding integrals can be evaluated in terms of the loop functions $L(q)$ and $A(q)$ defined as

$$
\begin{aligned}
L(q) &= \frac{1}{2}\frac{s}{q}\ln\left(\frac{s+q}{s-q}\right),\quad\text{with}\quad s=\sqrt{q^2+4M_\pi^2},\\
A(q) &= \frac{1}{2q}\arctan\left(\frac{q}{2M_\pi}\right).
\end{aligned}
\tag{2.20}
$$

Note that the loop integrals are ultraviolet divergent and thus need to be renormalized by an appropriate redefinition of the LECs accompanying the NN contact interactions. Here we show only the nonpolynomial parts of the resulting expressions that give rise to long-range contributions and are uniquely defined once the regulator (i.e., cutoff) is removed. Short-range contributions involving NN contact interactions will be considered elsewhere. The symbol $\to$ in the above equations signifies that the original ill-defined expression is evaluated using dimensional regularization[1] and only nonpolynomial in momenta and M_π^2 contributions are shown explicitly. The exchange current/charge density contributions $\vec{J}_{\rm c3,c5,c7}(\vec{q}_1,\vec{q}_2)$ and $\rho_{\rm c7}(\vec{q}_1,\vec{q}_2)$ result from loop diagrams where a photon couples to pions in flight. The corresponding loop integrals depend explicitly on two external momenta $\vec{q}_1$ and $\vec{q}_2$ and can be written in terms of the three-point functions. The complete expressions in momentum space are given in Appendix B in terms of a number of scalar integrals, which are easily calculable numerically as detailed in Appendix C. We also emphasize that we have explicitly verified in Appendix D that the derived exchange currents fulfill the continuity equation. Note further that all loop integrals can be carried out analytically in configuration space as discussed in the next section. The results for the current density presented here agree with the ones of [34].[2]

III. TWO-PION-EXCHANGE CURRENT IN CONFIGURATION SPACE

The obtained expressions in momentum space depend only on the momentum transfers of the individual nucleons leading to a local form of the current operator in configuration space that is defined according to

$$
J^\mu(\vec{r}_{10},\vec{r}_{20})\equiv\int\frac{d^3q_1}{(2\pi)^3}\frac{d^3q_2}{(2\pi)^3}e^{i\vec{q}_1\cdot\vec{r}_{10}}\,e^{i\vec{q}_2\cdot\vec{r}_{20}}J^\mu(\vec{q}_1,\vec{q}_2),
\tag{3.1}
$$

where $\vec{r}_{10}=\vec{r}_1-\vec{r}_0,\vec{r}_{20}=\vec{r}_2-\vec{r}_0$ and $\vec{r}_1,\vec{r}_2$ and $\vec{r}_0$ denote the positions of the nucleons 1 and 2 and the photon coupling,

[1] Using, e.g., a cutoff regularization will lead to the same result after taking the limit $\Lambda\to\infty$.

[2] For contributions involving the three-point function we could only check the intermediate results for the integrands involving different pion energies.

respectively. Using the formulas collected in Appendix E and expressions for the current operator in momentum space in terms of three-dimensional loop integrals given in the previous section, we obtain the following surprisingly compact expressions for the current density

$$\begin{aligned}
\vec{J}_{\rm c1}(\vec{r}_{10},\vec{r}_{20}) &= e\,\frac{g_A^2 M_\pi^7}{128\pi^3 F_\pi^4}\big[\vec{\nabla}_{10}[\vec{\tau}_1\times\vec{\tau}_2]^3 \\
&\quad + 2[\vec{\nabla}_{10}\times\vec{\sigma}_2]\tau_1^3\big]\,\delta(\vec{x}_{20})\frac{K_1(2x_{10})}{x_{10}^2}, \\
\vec{J}_{\rm c2}(\vec{r}_{10},\vec{r}_{20}) &= -e\,\frac{g_A^4 M_\pi^7}{256\pi^3 F_\pi^4}\big(3\nabla_{10}^2-8\big)\big[\vec{\nabla}_{10}[\vec{\tau}_1\times\vec{\tau}_2]^3 \\
&\quad + 2[\vec{\nabla}_{10}\times\vec{\sigma}_2]\tau_1^3\big]\,\delta(\vec{x}_{20})\frac{K_0(2x_{10})}{x_{10}} \\
&\quad + e\,\frac{g_A^4 M_\pi^7}{32\pi^3 F_\pi^4}[\vec{\nabla}_{10}\times\vec{\sigma}_1]\tau_2^3\,\delta(\vec{x}_{20})\frac{K_1(2x_{10})}{x_{10}^2}, \\
\vec{J}_{\rm c3}(\vec{r}_{10},\vec{r}_{20}) &= -e\,\frac{M_\pi^7}{512\pi^4 F_\pi^4}[\vec{\tau}_1\times\vec{\tau}_2]^3(\vec{\nabla}_{10}-\vec{\nabla}_{20}) \\
&\quad \times\frac{K_2(x_{10}+x_{20}+x_{12})}{(x_{10}x_{20}x_{12})(x_{10}+x_{20}+x_{12})}, \\
\vec{J}_{\rm c5}(\vec{r}_{10},\vec{r}_{20}) &= -e\,\frac{g_A^2 M_\pi^7}{256\pi^4 F_\pi^4}(\vec{\nabla}_{10}-\vec{\nabla}_{20}) \\
&\quad \times\big[[\vec{\tau}_1\times\vec{\tau}_2]^3\vec{\nabla}_{12}\cdot\vec{\nabla}_{20}-2\tau_1^3\vec{\sigma}_2\cdot[\vec{\nabla}_{12}\times\vec{\nabla}_{20}]\big] \\
&\quad \times\frac{K_1(x_{10}+x_{20}+x_{12})}{(x_{10}x_{20}x_{12})}, \\
\vec{J}_{\rm c7}(\vec{r}_{10},\vec{r}_{20}) &= e\,\frac{g_A^4 M_\pi^7}{512\pi^4 F_\pi^4}(\vec{\nabla}_{10}-\vec{\nabla}_{20}) \\
&\quad \times\big[[\vec{\tau}_1\times\vec{\tau}_2]^3\vec{\nabla}_{12}\cdot\vec{\nabla}_{10}\vec{\nabla}_{12}\cdot\vec{\nabla}_{20} \\
&\quad + 4\tau_2^3\vec{\sigma}_1\cdot[\vec{\nabla}_{12}\times\vec{\nabla}_{10}]\vec{\nabla}_{12}\cdot\vec{\nabla}_{20}\big] \\
&\quad \times\frac{x_{10}+x_{20}+x_{12}}{x_{10}x_{20}x_{12}}K_0(x_{10}+x_{20}+x_{12}), \qquad (3.2)
\end{aligned}$$

and the charge density

$$\begin{aligned}
\rho_{\rm c1}(\vec{r}_{10},\vec{r}_{20}) &= \rho_{\rm c2}(\vec{r}_{10},\vec{r}_{20}) = \rho_{\rm c3}(\vec{r}_{10},\vec{r}_{20}) = 0, \\
\rho_{\rm c4}(\vec{r}_{10},\vec{r}_{20}) &= e\,\frac{g_A^2 M_\pi^7}{256\pi^2 F_\pi^4}\tau_1^3\delta(\vec{x}_{20})\big(\nabla_{10}^2-2\big)\frac{e^{-2x_{10}}}{x_{10}^2}, \\
\rho_{\rm c5}(\vec{r}_{10},\vec{r}_{20}) &= -e\,\frac{g_A^2 M_\pi^7}{256\pi^2 F_\pi^4}\tau_2^3\delta(\vec{x}_{20})\big(\nabla_{10}^2-2\big)\frac{e^{-2x_{10}}}{x_{10}^2}, \\
\rho_{\rm c6}(\vec{r}_{10},\vec{r}_{20}) &= -e\,\frac{g_A^4 M_\pi^7}{256\pi^2 F_\pi^4}\delta(\vec{x}_{20})\big[\tau_1^3\big(2\nabla_{10}^2-4\big) \\
&\quad + \tau_2^3\vec{\sigma}_1\cdot\vec{\nabla}_{10}\vec{\sigma}_2\cdot\vec{\nabla}_{10}-\tau_2^3\vec{\sigma}_1\cdot\vec{\sigma}_2\big]\frac{e^{-2x_{10}}}{x_{10}^2} \\
&\quad - e\,\frac{g_A^4 M_\pi^7}{128\pi^2 F_\pi^4}\delta(\vec{x}_{20})\tau_1^3\big(3\nabla_{10}^2-11\big)\frac{e^{-2x_{10}}}{x_{10}}, \\
\rho_{\rm c7}(\vec{r}_{10},\vec{r}_{20}) &= -e\,\frac{g_A^4 M_\pi^7}{512\pi^3 F_\pi^4}\big[\big(\tau_1^3+\tau_2^3\big)(\vec{\nabla}_{12}\cdot\vec{\nabla}_{10}\vec{\nabla}_{12}\cdot\vec{\nabla}_{20} \\
&\quad + \vec{\nabla}_{12}\cdot[\vec{\nabla}_{10}\times\vec{\sigma}_1]\vec{\nabla}_{12}\cdot[\vec{\nabla}_{20}\times\vec{\sigma}_2]) \\
&\quad + [\vec{\tau}_1\times\vec{\tau}_2]^3\vec{\nabla}_{12}\cdot\vec{\nabla}_{10}\vec{\nabla}_{12}\cdot[\vec{\nabla}_{20}\times\vec{\sigma}_2]\big] \\
&\quad \times\frac{e^{-x_{10}}}{x_{10}}\,\frac{e^{-x_{20}}}{x_{20}}\,\frac{e^{-x_{12}}}{x_{12}}. \qquad (3.3)
\end{aligned}$$

In the above expressions, $K_n(x)$ denote the modified Bessel functions of the second kind and we have introduced dimensionless variables $\vec{x}_{10}=M_\pi\vec{r}_{10}$, $\vec{x}_{20}=M_\pi\vec{r}_{20}$, and $\vec{x}_{12}=M_\pi\vec{r}_{12}=M_\pi(\vec{r}_1-\vec{r}_2)$. Further, $x_{ij}\equiv|\vec{x}_{ij}|$ and all derivatives with respect to $\vec{x}_{10}$, $\vec{x}_{20}$, and $\vec{x}_{12}$ are to be evaluated as if these variables were independent of each other. We also emphasize that the above expressions are valid for $x_{10}+x_{20}>0$. Finally, it should be understood that the behavior of the current and charge densities at short distances will be affected if one uses a regularization with a finite value of the cutoff.

IV. SUMMARY AND OUTLOOK

In this article, we applied the method of unitary transformation to derive the leading two-pion-exchange two-nucleon charge and current densities based on chiral effective field theory. The resulting nuclear current is given both in momentum and configuration space. The results in momentum space involve the standard loop functions $L(q)$ and $A(q)$ and, for certain classes of diagrams, also the three-point functions. In the latter case we expressed all tensor integrals in terms of a set of scalar integrals in different dimensions similar to the method of Ref. [53]; see Appendix B. For the remaining scalar integrals we derive in Appendix C a simple representation using Feynman parameters that is well suited for numerical calculations. In configuration space, we were able to evaluate all loop integrals analytically leading to very compact expressions in terms of the modified Bessel functions of the second kind. We have also explicitly verified that the derived exchange currents fulfill the continuity equation; see Appendix D.

In addition to the two-pion-exchange contributions, there are also one-pion-exchange and short-range terms at order $\mathcal{O}(eQ)$, see Refs. [33,34] for a recent work based on time-ordered perturbation theory. As will be demonstrated in a subsequent publication [50], renormalization of the one-pion-exchange contributions at the one-loop level strongly restricts the ambiguity in the definition of the current and provides a highly nontrivial consistency check of the calculation. In particular, one needs to ensure that *all* appearing ultraviolet divergences are absorbed into redefinition of the LECs d_i and l_i from $\mathcal{L}_{\pi N}^{(3)}$ and $\mathcal{L}_\pi^{(4)}$, respectively, with already known β functions, see, e.g., Refs. [46–49]. This work is in progress [50].

Finally, in the future, one also needs to test the convergence of the chiral expansion for the one- and two-pion-exchange currents by calculating the corrections at order $\mathcal{O}(eQ^2)$. Given the large numerical values of the LECs $c_{3,4}$ from $\mathcal{L}_{\pi N}^{(2)}$, one might expect sizable corrections that, indeed, is well known to be the case for the two-pion-exchange potential [54]. In this context, it might be advantageous to include the $\Delta(1232)$ isobar as an explicit degree of freedom in effective field theory utilizing the small scale expansion [55]; see Refs. [56–61] for recent work along these lines in the purely strong few-nucleon sector.

ACKNOWLEDGMENTS

We thank Daniel Phillips for useful comments on the manuscript and Walter Glöckle, Jacek Golak, Dagmara Rozpedzik, and Henryk Witała for many stimulating

discussions on this topic. S.K. also thanks Jacek Golak and Henryk Witała for their hospitality during his stay in Krakow, where a part of this work was done. This work was supported by funds provided by the Helmholtz Association to the young investigator group "Few-Nucleon Systems in Chiral Effective Field Theory" (grant VH-NG-222) and to the virtual institute "Spin and strong QCD" (VH-VI-231), by the DFG (SFB/TR 16 "Subnuclear Structure of Matter") and by the EU Hadron-Physics2 project "Study of Strongly Interacting Matter."

APPENDIX A: FORMAL STRUCTURE OF THE LEADING TWO-PION-EXCHANGE CURRENT OPERATOR

In this appendix we give the formal structure of the two-pion-exchange current operator that results after applying the unitary transformation. All vertices entering the expressions below should be understood as second-quantized normally ordered operators. Here and in what follows, we adopt the notation of Ref. [39] with a few minor modifications.

- **Class 1** contributions involving $J_{21}^{(0)}$ and proportional to g_A^2

$$J_{\rm c1} = \eta \left[H_{22}^{(2)} \frac{\lambda^2}{E_\pi} J_{21}^{(0)} \frac{\lambda^1}{E_\pi} H_{21}^{(1)} + H_{22}^{(2)} \frac{\lambda^2}{E_\pi} H_{21}^{(1)} \frac{\lambda^1}{E_\pi} J_{21}^{(0)} + J_{21}^{(0)} \frac{\lambda^1}{E_\pi} H_{22}^{(2)} \frac{\lambda^1}{E_\pi} H_{21}^{(1)} \right] \eta + \text{h.c.} \tag{A1}$$

- **Class 2** contributions involving $J_{21}^{(0)}$ and proportional to g_A^4

$$\begin{aligned} J_{\rm c2} = \eta \Bigg[& J_{21}^{(0)} \frac{\lambda^1}{E_\pi^2} H_{21}^{(1)} \eta H_{21}^{(1)} \frac{\lambda^1}{E_\pi} H_{21}^{(1)} \\ & + \frac{1}{2} J_{21}^{(0)} \frac{\lambda^1}{E_\pi} H_{21}^{(1)} \eta H_{21}^{(1)} \frac{\lambda^1}{E_\pi^2} H_{21}^{(1)} \\ & - J_{21}^{(0)} \frac{\lambda^1}{E_\pi} H_{21}^{(1)} \frac{\lambda^2}{E_\pi} H_{21}^{(1)} \frac{\lambda^1}{E_\pi} H_{21}^{(1)} \\ & + \frac{1}{2} H_{21}^{(1)} \frac{\lambda^1}{E_\pi^2} H_{21}^{(1)} \eta J_{21}^{(0)} \frac{\lambda^1}{E_\pi} H_{21}^{(1)} \\ & - H_{21}^{(1)} \frac{\lambda^1}{E_\pi} J_{21}^{(0)} \frac{\lambda^2}{E_\pi} H_{21}^{(1)} \frac{\lambda^1}{E_\pi} H_{21}^{(1)} \Bigg] \eta + \text{h.c.} \end{aligned} \tag{A2}$$

- **Class 3** contributions involving $J_{02}^{(-1)}$ and proportional to g_A^0

$$J_{\rm c3} = \eta \left[H_{22}^{(2)} \frac{\lambda^2}{E_\pi} H_{22}^{(2)} \frac{\lambda^2}{E_\pi} J_{02}^{(-1)} + \frac{1}{2} H_{22}^{(2)} \frac{\lambda^2}{E_\pi} J_{02}^{(-1)} \frac{\lambda^2}{E_\pi} H_{22}^{(2)} - J_{02}^{(-1)} \frac{\lambda^2}{E_\pi} H_{42}^{4} \right] \eta + \text{h.c.} \tag{A3}$$

- **Class 4** contributions involving $J_{20}^{(-1)}$ and proportional to g_A^2

$$\begin{aligned} J_{\rm c4} = \eta \Bigg[& - H_{22}^{(2)} \frac{\lambda^2}{E_\pi} J_{20}^{(-1)} \frac{\lambda^2}{E_\pi} H_{21}^{(1)} \frac{\lambda^1}{E_\pi} H_{21}^{(1)} \\ & + \frac{1}{2} H_{22}^{(2)} \frac{\lambda^2}{E_\pi} H_{21}^{(1)} \frac{\lambda^1}{E_\pi^2} H_{21}^{(1)} \eta J_{20}^{(-1)} \\ & - H_{22}^{(2)} \frac{\lambda^2}{E_\pi} H_{21}^{(1)} \frac{\lambda^1}{E_\pi} J_{20}^{(-1)} \frac{\lambda^1}{E_\pi} H_{21}^{(1)} \\ & + J_{20}^{(-1)} \eta H_{22}^{(2)} \frac{\lambda^2}{E_\pi^2} H_{21}^{(1)} \frac{\lambda^1}{E_\pi} H_{21}^{(1)} \\ & + \frac{1}{2} J_{20}^{(-1)} \eta H_{22}^{(2)} \frac{\lambda^2}{E_\pi} H_{21}^{(1)} \frac{\lambda^1}{E_\pi^2} H_{21}^{(1)} \\ & + \frac{1}{2} J_{20}^{(-1)} \eta H_{21}^{(1)} \frac{\lambda^1}{E_\pi^2} H_{22}^{(2)} \frac{\lambda^1}{E_\pi} H_{21}^{(1)} \\ & + \frac{1}{2} J_{20}^{(-1)} \eta H_{21}^{(1)} \frac{\lambda^1}{E_\pi} H_{22}^{(2)} \frac{\lambda^1}{E_\pi^2} H_{21}^{(1)} \\ & - H_{21}^{(1)} \frac{\lambda^1}{E_\pi} H_{22}^{(2)} \frac{\lambda^1}{E_\pi} J_{20}^{(-1)} \frac{\lambda^1}{E_\pi} H_{21}^{(1)} \Bigg] \eta + \text{h.c.} \end{aligned} \tag{A4}$$

- **Class 5** contributions involving $J_{02}^{(-1)}$ and proportional to g_A^2

$$\begin{aligned} J_{\rm c5} = \eta \Bigg[& - H_{22}^{(2)} \frac{\lambda^2}{E_\pi} J_{02}^{(-1)} \frac{\lambda^2}{E_\pi} H_{21}^{(1)} \frac{\lambda^1}{E_\pi} H_{21}^{(1)} \\ & - J_{02}^{(-1)} \frac{\lambda^2}{E_\pi} H_{22}^{(2)} \frac{\lambda^2}{E_\pi} H_{21}^{(1)} \frac{\lambda^1}{E_\pi} H_{21}^{(1)} \\ & - H_{22}^{(2)} \frac{\lambda^2}{E_\pi} H_{21}^{(1)} \frac{\lambda^1}{E_\pi} H_{21}^{(1)} \frac{\lambda^2}{E_\pi} J_{02}^{(-1)} \\ & - H_{21}^{(1)} \frac{\lambda^1}{E_\pi} H_{22}^{(2)} \frac{\lambda^1}{E_\pi} J_{02}^{(-1)} \frac{\lambda^1}{E_\pi} H_{21}^{(1)} \\ & - H_{22}^{(2)} \frac{\lambda^2}{E_\pi} H_{21}^{(1)} \frac{\lambda^3}{E_\pi} H_{21}^{(1)} \frac{\lambda^2}{E_\pi} J_{02}^{(-1)} \\ & - H_{21}^{(1)} \frac{\lambda^1}{E_\pi} H_{22}^{(2)} \frac{\lambda^3}{E_\pi} J_{02}^{(-1)} \frac{\lambda^1}{E_\pi} H_{21}^{(1)} \\ & - J_{02}^{(-1)} \frac{\lambda^2}{E_\pi} H_{21}^{(1)} \frac{\lambda^1}{E_\pi} H_{22}^{(2)} \frac{\lambda^1}{E_\pi} H_{21}^{(1)} \\ & - H_{22}^{(2)} \frac{\lambda^2}{E_\pi} H_{21}^{(1)} \frac{\lambda^1}{E_\pi} J_{02}^{(-1)} \frac{\lambda^1}{E_\pi} H_{21}^{(1)} \\ & - J_{02}^{(-1)} \frac{\lambda^2}{E_\pi} H_{21}^{(1)} \frac{\lambda^3}{E_\pi} H_{22}^{(2)} \frac{\lambda^1}{E_\pi} H_{21}^{(1)} \\ & - H_{22}^{(2)} \frac{\lambda^2}{E_\pi} H_{21}^{(1)} \frac{\lambda^3}{E_\pi} J_{02}^{(-1)} \frac{\lambda^1}{E_\pi} H_{21}^{(1)} \\ & + \frac{1}{2} H_{22}^{(2)} \frac{\lambda^2}{E_\pi} J_{02}^{(-1)} \eta H_{21}^{(1)} \frac{\lambda^1}{E_\pi^2} H_{21}^{(1)} \\ & + \frac{1}{2} J_{02}^{(-1)} \frac{\lambda^2}{E_\pi} H_{22}^{(2)} \eta H_{21}^{(1)} \frac{\lambda^1}{E_\pi^2} H_{21}^{(1)} \\ & + J_{02}^{(-1)} \frac{\lambda^2}{E_\pi^2} H_{22}^{(2)} \eta H_{21}^{(1)} \frac{\lambda^1}{E_\pi} H_{21}^{(1)} \Bigg] \eta + \text{h.c.} \end{aligned} \tag{A5}$$

- **Class 6** contributions involving $J_{20}^{(-1)}$ and proportional to g_A^4

$$\begin{aligned} J_{\rm c6} = \eta \Bigg[& J_{20}^{(-1)} \eta H_{21}^{(1)} \frac{\lambda^1}{E_\pi^3} H_{21}^{(1)} \eta H_{21}^{(1)} \frac{\lambda^1}{E_\pi} H_{21}^{(1)} \\ & + \frac{3}{8} J_{20}^{(-1)} \eta H_{21}^{(1)} \frac{\lambda^1}{E_\pi^2} H_{21}^{(1)} \eta H_{21}^{(1)} \frac{\lambda^1}{E_\pi^2} H_{21}^{(1)} \\ & - \frac{3}{4} J_{20}^{(-1)} \eta H_{21}^{(1)} \frac{\lambda^1}{E_\pi^2} H_{21}^{(1)} \frac{\lambda^2}{E_\pi} H_{21}^{(1)} \frac{\lambda^1}{E_\pi} H_{21}^{(1)} \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}J_{20}^{(-1)}\eta H_{21}^{(1)}\frac{\lambda^1}{E_\pi}H_{21}^{(1)}\frac{\lambda^2}{E_\pi^2}H_{21}^{(1)}\frac{\lambda^1}{E_\pi}H_{21}^{(1)}\\
&-\frac{1}{4}J_{20}^{(-1)}\eta H_{21}^{(1)}\frac{\lambda^1}{E_\pi}H_{21}^{(1)}\frac{\lambda^2}{E_\pi}H_{21}^{(1)}\frac{\lambda^1}{E_\pi^2}H_{21}^{(1)}\\
&+\frac{1}{8}H_{21}^{(1)}\frac{\lambda^1}{E_\pi^2}H_{21}^{(1)}\eta J_{20}^{(-1)}\eta H_{21}^{(1)}\frac{\lambda^1}{E_\pi^2}H_{21}^{(1)}\\
&-\frac{1}{2}H_{21}^{(1)}\frac{\lambda^1}{E_\pi^2}H_{21}^{(1)}\eta H_{21}^{(1)}\frac{\lambda^1}{E_\pi}J_{20}^{(-1)}\frac{\lambda^1}{E_\pi}H_{21}^{(1)}\\
&-H_{21}^{(1)}\frac{\lambda^1}{E_\pi}J_{20}^{(-1)}\frac{\lambda^1}{E_\pi^2}H_{21}^{(1)}\eta H_{21}^{(1)}\frac{\lambda^1}{E_\pi}H_{21}^{(1)}\\
&+H_{21}^{(1)}\frac{\lambda^1}{E_\pi}J_{20}^{(-1)}\frac{\lambda^1}{E_\pi}H_{21}^{(1)}\frac{\lambda^2}{E_\pi}H_{21}^{(1)}\frac{\lambda^1}{E_\pi}H_{21}^{(1)}\\
&+\frac{1}{2}H_{21}^{(1)}\frac{\lambda^1}{E_\pi}H_{21}^{(1)}\frac{\lambda^2}{E_\pi}J_{20}^{(-1)}\frac{\lambda^2}{E_\pi}H_{21}^{(1)}\frac{\lambda^1}{E_\pi}H_{21}^{(1)}\Bigg]\eta+\text{h.c.}
\end{aligned}
\tag{A6}
$$

- **Class 7** contributions involving $J_{02}^{(-1)}$ and proportional to g_A^4

$$
\begin{aligned}
J_{c7}=\eta\Bigg[&-J_{02}^{(-1)}\frac{\lambda^2}{E_\pi^2}H_{21}^{(1)}\frac{\lambda^1}{E_\pi}H_{21}^{(1)}\eta H_{21}^{(1)}\frac{\lambda^1}{E_\pi}H_{21}^{(1)}\\
&-J_{02}^{(-1)}\frac{\lambda^2}{E_\pi}H_{21}^{(1)}\frac{\lambda^1}{E_\pi^2}H_{21}^{(1)}\eta H_{21}^{(1)}\frac{\lambda^1}{E_\pi}H_{21}^{(1)}\\
&-\frac{1}{2}J_{02}^{(-1)}\frac{\lambda^2}{E_\pi}H_{21}^{(1)}\frac{\lambda^1}{E_\pi}H_{21}^{(1)}\eta H_{21}^{(1)}\frac{\lambda^1}{E_\pi^2}H_{21}^{(1)}\\
&+J_{02}^{(-1)}\frac{\lambda^2}{E_\pi}H_{21}^{(1)}\frac{\lambda^1}{E_\pi}H_{21}^{(1)}\frac{\lambda^2}{E_\pi}H_{21}^{(1)}\frac{\lambda^1}{E_\pi}H_{21}^{(1)}\\
&+H_{21}^{(1)}\frac{\lambda^1}{E_\pi}J_{02}^{(-1)}\frac{\lambda^1}{E_\pi}H_{21}^{(1)}\frac{\lambda^2}{E_\pi}H_{21}^{(1)}\frac{\lambda^1}{E_\pi}H_{21}^{(1)}\\
&+J_{02}^{(-1)}\frac{\lambda^2}{E_\pi}H_{21}^{(1)}\frac{\lambda^3}{E_\pi}H_{21}^{(1)}\frac{\lambda^2}{E_\pi}H_{21}^{(1)}\frac{\lambda^1}{E_\pi}H_{21}^{(1)}\\
&+H_{21}^{(1)}\frac{\lambda^1}{E_\pi}J_{02}^{(-1)}\frac{\lambda^3}{E_\pi}H_{21}^{(1)}\frac{\lambda^2}{E_\pi}H_{21}^{(1)}\frac{\lambda^1}{E_\pi}H_{21}^{(1)}\\
&-\frac{1}{2}H_{21}^{(1)}\frac{\lambda^1}{E_\pi^2}H_{21}^{(1)}\eta J_{02}^{(-1)}\frac{\lambda^2}{E_\pi}H_{21}^{(1)}\frac{\lambda^1}{E_\pi}H_{21}^{(1)}\\
&-\frac{1}{2}H_{21}^{(1)}\frac{\lambda^1}{E_\pi^2}H_{21}^{(1)}\eta H_{21}^{(1)}\frac{\lambda^1}{E_\pi}J_{02}^{(-1)}\frac{\lambda^1}{E_\pi}H_{21}^{(1)}\\
&-H_{21}^{(1)}\frac{\lambda^1}{E_\pi}J_{02}^{(-1)}\frac{\lambda^1}{E_\pi^2}H_{21}^{(1)}\eta H_{21}^{(1)}\frac{\lambda^1}{E_\pi}H_{21}^{(1)}\\
&+\frac{1}{2}H_{21}^{(1)}\frac{\lambda^1}{E_\pi}H_{\pi N}\frac{\lambda^2}{E_\pi}J_{02}^{(-1)}\frac{\lambda^2}{E_\pi}H_{21}^{(1)}\frac{\lambda^1}{E_\pi}H_{21}^{(1)}\Bigg]\eta+\text{h.c.}
\end{aligned}
\tag{A7}
$$

APPENDIX B: LEADING TWO-PION-EXCHANGE CURRENT IN MOMENTUM SPACE

In this appendix we give the expressions for the two-pion-exchange current operator in momentum space. Following Ref. [62], the most general expression for the current and the charge density can be written as

$$
\begin{aligned}
\vec{J}&=\sum_{i=1}^{5}\sum_{j=1}^{24}f_i^j(\vec{q}_1,\vec{q}_2)T_i\vec{O}_j,\\
J^0&=\sum_{i=1}^{5}\sum_{j=1}^{8}f_i^{jS}(\vec{q}_1,\vec{q}_2)T_iO_j^S,
\end{aligned}
\tag{B1}
$$

where $f_i^j\equiv f_i^j(\vec{q}_1,\vec{q}_2)$ are scalar functions and the spin-momentum operators $\vec{O}_i$ and O_i^S are given by

$$
\begin{aligned}
\vec{O}_1&=\vec{q}_1+\vec{q}_2,\\
\vec{O}_2&=\vec{q}_1-\vec{q}_2,\\
\vec{O}_3&=[\vec{q}_1\times\vec{\sigma}_2]+[\vec{q}_2\times\vec{\sigma}_1],\\
\vec{O}_4&=[\vec{q}_1\times\vec{\sigma}_2]-[\vec{q}_2\times\vec{\sigma}_1],\\
\vec{O}_5&=[\vec{q}_1\times\vec{\sigma}_1]+[\vec{q}_2\times\vec{\sigma}_2],\\
\vec{O}_6&=[\vec{q}_1\times\vec{\sigma}_1]-[\vec{q}_2\times\vec{\sigma}_2],\\
\vec{O}_7&=\vec{q}_1(\vec{q}_1\cdot[\vec{q}_2\times\vec{\sigma}_2])+\vec{q}_2(\vec{q}_1\cdot[\vec{q}_2\times\vec{\sigma}_1]),\\
\vec{O}_8&=\vec{q}_1(\vec{q}_1\cdot[\vec{q}_2\times\vec{\sigma}_2])-\vec{q}_2(\vec{q}_1\cdot[\vec{q}_2\times\vec{\sigma}_1]),\\
\vec{O}_9&=\vec{q}_2(\vec{q}_1\cdot[\vec{q}_2\times\vec{\sigma}_2])+\vec{q}_1(\vec{q}_1\cdot[\vec{q}_2\times\vec{\sigma}_1]),\\
\vec{O}_{10}&=\vec{q}_2(\vec{q}_1\cdot[\vec{q}_2\times\vec{\sigma}_2])-\vec{q}_1(\vec{q}_1\cdot[\vec{q}_2\times\vec{\sigma}_1]),\\
\vec{O}_{11}&=(\vec{q}_1+\vec{q}_2)(\vec{\sigma}_1\cdot\vec{\sigma}_2),\\
\vec{O}_{12}&=(\vec{q}_1-\vec{q}_2)(\vec{\sigma}_1\cdot\vec{\sigma}_2),\\
\vec{O}_{13}&=\vec{q}_1(\vec{q}_1\cdot\vec{\sigma}_1)(\vec{q}_1\cdot\vec{\sigma}_2)+\vec{q}_2(\vec{q}_2\cdot\vec{\sigma}_1)(\vec{q}_2\cdot\vec{\sigma}_2),\\
\vec{O}_{14}&=\vec{q}_1(\vec{q}_1\cdot\vec{\sigma}_1)(\vec{q}_1\cdot\vec{\sigma}_2)-\vec{q}_2(\vec{q}_2\cdot\vec{\sigma}_1)(\vec{q}_2\cdot\vec{\sigma}_2),\\
\vec{O}_{15}&=(\vec{q}_1+\vec{q}_2)(\vec{q}_2\cdot\vec{\sigma}_1)(\vec{q}_1\cdot\vec{\sigma}_2),\\
\vec{O}_{16}&=(\vec{q}_1-\vec{q}_2)(\vec{q}_2\cdot\vec{\sigma}_1)(\vec{q}_1\cdot\vec{\sigma}_2),\\
\vec{O}_{17}&=(\vec{q}_1+\vec{q}_2)(\vec{q}_1\cdot\vec{\sigma}_1)(\vec{q}_2\cdot\vec{\sigma}_2),\\
\vec{O}_{18}&=(\vec{q}_1-\vec{q}_2)(\vec{q}_1\cdot\vec{\sigma}_1)(\vec{q}_2\cdot\vec{\sigma}_2),\\
\vec{O}_{19}&=\vec{\sigma}_1(\vec{q}_1\cdot\vec{\sigma}_2)+\vec{\sigma}_2(\vec{q}_2\cdot\vec{\sigma}_1),\\
\vec{O}_{20}&=\vec{\sigma}_1(\vec{q}_1\cdot\vec{\sigma}_2)-\vec{\sigma}_2(\vec{q}_2\cdot\vec{\sigma}_1),\\
\vec{O}_{21}&=\vec{\sigma}_1(\vec{q}_2\cdot\vec{\sigma}_2)+\vec{\sigma}_2(\vec{q}_1\cdot\vec{\sigma}_1),\\
\vec{O}_{22}&=\vec{\sigma}_1(\vec{q}_2\cdot\vec{\sigma}_2)-\vec{\sigma}_2(\vec{q}_1\cdot\vec{\sigma}_1),\\
\vec{O}_{23}&=\vec{q}_1(\vec{q}_2\cdot\vec{\sigma}_1)(\vec{q}_2\cdot\vec{\sigma}_2)+\vec{q}_2(\vec{q}_1\cdot\vec{\sigma}_1)(\vec{q}_1\cdot\vec{\sigma}_2),\\
\vec{O}_{24}&=\vec{q}_1(\vec{q}_2\cdot\vec{\sigma}_1)(\vec{q}_2\cdot\vec{\sigma}_2)-\vec{q}_2(\vec{q}_1\cdot\vec{\sigma}_1)(\vec{q}_1\cdot\vec{\sigma}_2),
\end{aligned}
\tag{B2}
$$

and

$$
\begin{aligned}
O_1^S&=\mathbb{1},\\
O_2^S&=\vec{q}_1\cdot[\vec{q}_2\times\vec{\sigma}_2]+\vec{q}_1\cdot[\vec{q}_2\times\vec{\sigma}_1],\\
O_3^S&=\vec{q}_1\cdot[\vec{q}_2\times\vec{\sigma}_2]-\vec{q}_1\cdot[\vec{q}_2\times\vec{\sigma}_1],\\
O_4^S&=\vec{\sigma}_1\cdot\vec{\sigma}_2,\\
O_5^S&=(\vec{q}_1\cdot\vec{\sigma}_2)(\vec{q}_2\cdot\vec{\sigma}_1),\\
O_6^S&=(\vec{q}_1\cdot\vec{\sigma}_1)(\vec{q}_2\cdot\vec{\sigma}_2),\\
O_7^S&=(\vec{q}_2\cdot\vec{\sigma}_1)(\vec{q}_2\cdot\vec{\sigma}_2)+(\vec{q}_1\cdot\vec{\sigma}_1)(\vec{q}_1\cdot\vec{\sigma}_2),\\
O_8^S&=(\vec{q}_2\cdot\vec{\sigma}_1)(\vec{q}_2\cdot\vec{\sigma}_2)-(\vec{q}_1\cdot\vec{\sigma}_1)(\vec{q}_1\cdot\vec{\sigma}_2).
\end{aligned}
\tag{B3}
$$

As a basis for the isospin operators we choose

$$
\begin{aligned}
T_1&=\tau_1^3+\tau_2^3,\\
T_2&=\tau_1^3-\tau_2^3,\\
T_3&=[\vec{\tau}_1\times\vec{\tau}_2]^3,\\
T_4&=\vec{\tau}_1\cdot\vec{\tau}_2,\\
T_5&=\mathbb{1}.
\end{aligned}
\tag{B4}
$$

The nonvanishing long-range contributions to the scalar functions $f_i^j \equiv f_i^j(\vec{q}_1, \vec{q}_2)$ due to two-pion exchange calculated using dimensional regularization are given by

$$
\begin{aligned}
f_3^1 &= \frac{ieg_A^2 L(q_1)}{128\pi^2 F_\pi^4}\left[\frac{g_A^2\left(8M_\pi^2+3q_1^2\right)}{4M_\pi^2+q_1^2}-1\right] \\
&+ \frac{e\pi}{F_\pi^4}\big[g_A^4 M_\pi^4 I^{(d+2)}_{(2,1,2)} + 4\pi g_A^4 M_\pi^2 q_1^2 I^{(d+4)}_{(2,2,2)} \\
&- 8\pi g_A^4 M_\pi^2 q_1 I^{(d+4)}_{(3,1,2)}(q_1 - q_2 z) - 96\pi^2 g_A^4 q_1^3 q_2 z I^{(d+6)}_{(4,1,2)} \\
&+ 32\pi^2 g_A^4 q_1^2 q_2 I^{(d+6)}_{(3,2,2)}(q_1 z + q_2 z^2 + q_2) \\
&- 2\pi g_A^4 q_1 I^{(d+4)}_{(2,1,2)}(q_1 + 2q_2 z) - 2(g_A^2-1) g_A^2 M_\pi^2 \\
&\times I^{(d+2)}_{(2,1,0)} - 4\pi\left(g_A^2-1\right) g_A^2 q_1^2 I^{(d+4)}_{(2,2,0)} \\
&+ 8\pi\left(g_A^2-1\right) g_A^2 q_1 I^{(d+4)}_{(3,1,0)}(q_1 - q_2 z) \\
&- 2\pi\left(g_A^2-1\right)^2 I^{(d+4)}_{\mathrm{red}\ (2,1,0)}\big] - (1 \leftrightarrow 2), \\
f_3^2 &= \frac{ieg_A^2\left(g_A^2+1\right)}{256\pi^2 F_\pi^4} + \frac{ieg_A^2 L(q_1)}{128\pi^2 F_\pi^4}\left[\frac{g_A^2\left(8M_\pi^2+3q_1^2\right)}{4M_\pi^2+q_1^2}-1\right] \\
&+ \frac{e}{8F_\pi^4}\big[-64\pi^2 g_A^4 q_1 I^{(d+4)}_{(3,1,2)}\big(M_\pi^2(q_1 - q_2 z) + q_1^2 q_2 z\big) \\
&+ 16\pi^2 g_A^4 q_1 I^{(d+4)}_{(2,2,2)}\big(q_1 q_2^2(z^2+1) \\
&- 2M_\pi^2(q_1 - q_2 z)\big) - 2\pi g_A^4 I^{(d+2)}_{(1,1,2)}\big(2M_\pi^2 + q_1 q_2 z\big) \\
&+ 8\pi g_A^4 M_\pi^2 I^{(d+2)}_{(2,1,2)}\big(M_\pi^2 + q_1(q_2 z - q_1)\big) \\
&- 768\pi^3 g_A^4 q_1^3 q_2 z I^{(d+6)}_{(4,1,2)} - 256\pi^3 g_A^4 q_1^2 q_2 I^{(d+6)}_{(3,2,2)} \\
&\times (q_1 z - q_2(z^2+1)) + 16\pi^2 g_A^4 q_1 I^{(d+4)}_{(2,1,2)}(q_1 - 2q_2 z) \\
&+ 2\pi\left(g_A^4-1\right) I^{(d+2)}_{\mathrm{red}\ (1,1,0)} - 8\pi\left(g_A^2-1\right) g_A^2 I^{(d+2)}_{(2,1,0)} \\
&\times \big(2M_\pi^2 + q_1(q_2 z - q_1)\big) + 32\pi^2\left(g_A^2-1\right) g_A^2 q_1 I^{(d+4)}_{(2,2,0)} \\
&\times (q_1 - q_2 z) + 64\pi^2\left(g_A^2-1\right) g_A^2 q_1 I^{(d+4)}_{(3,1,0)}(q_1 - q_2 z) \\
&- 16\pi^2\left(g_A^2-1\right)^2 I^{(d+4)}_{\mathrm{red}\ (2,1,0)} + g_A^4 M_\pi^4 I^{(4)}_{(1,1,2)} \\
&- 2\left(g_A^2-1\right) g_A^2 M_\pi^2 I^{(4)}_{(1,1,0)}\big] + (1 \leftrightarrow 2), \\
f_1^3 &= \frac{ieg_A^2\left(g_A^2+1\right)}{256\pi^2 F_\pi^4} + \frac{ieg_A^2 L(q_1)}{128\pi^2 F_\pi^4} \\
&\times \left[\frac{g_A^2\left(8M_\pi^2+3q_1^2\right)}{4M_\pi^2+q_1^2}-1\right] + (1 \leftrightarrow 2), \\
f_1^4 &= \frac{ieg_A^2 L(q_1)}{128\pi^2 F_\pi^4}\left[\frac{g_A^2\left(8M_\pi^2+3q_1^2\right)}{4M_\pi^2+q_1^2}-1\right] - (1 \leftrightarrow 2), \\
f_1^5 &= -\frac{ieg_A^4}{128\pi^2 F_\pi^4} + \frac{ieg_A^4 L(k)}{32\pi^2 F_\pi^4} - \frac{ieg_A^4 L(q_1)}{64\pi^2 F_\pi^4} + \frac{eg_A^2\pi}{2F_\pi^4} \\
&\times \big[g_A^2\left(-M_\pi^2\right) I^{(d+2)}_{(1,1,2)} + 4\pi g_A^2 q_1 I^{(d+4)}_{(2,1,2)}(q_1 - q_2 z) \\
&+ \left(g_A^2-1\right) I^{(d+2)}_{\mathrm{red}\ (1,1,0)}\big] + (1 \leftrightarrow 2), \\
f_1^6 &= -\frac{ieg_A^4 L(q_1)}{64\pi^2 F_\pi^4} - \frac{2eg_A^4\pi^2}{F_\pi^4} I^{(d+4)}_{(2,1,2)} q_1(q_1 + q_2 z) \\
&- (1 \leftrightarrow 2), \\
f_1^7 &= \frac{eg_A^2\pi}{2F_\pi^4}\big[8\pi g_A^2\left(q_1^2 - 2M_\pi^2\right) I^{(d+4)}_{(3,1,2)} - g_A^2 M_\pi^2 I^{(d+2)}_{(2,1,2)} \\
&+ 192\pi^2 g_A^2 q_1^2 I^{(d+6)}_{(4,1,2)} - 64\pi^2 g_A^2 q_1 q_2 z I^{(d+6)}_{(3,2,2)} \\
&+ 8\pi g_A^2 I^{(d+4)}_{(2,1,2)} + \left(g_A^2-1\right) I^{(d+2)}_{(2,1,0)} \\
&+ 16\pi\left(g_A^2-1\right) I^{(d+4)}_{(3,1,0)}\big] - (1 \leftrightarrow 2), \\
f_1^8 &= \frac{ieg_A^4}{32\pi^2 F_\pi^4 k^2} - \frac{ieg_A^4 L(k)}{32\pi^2 F_\pi^4 k^2} + \frac{eg_A^2\pi}{4F_\pi^4} \\
&\times \big[16\pi g_A^2\left(q_1^2 - 2M_\pi^2\right) I^{(d+4)}_{(3,1,2)} - 2g_A^2 M_\pi^2 I^{(d+2)}_{(2,1,2)} \\
&+ 384\pi^2 g_A^2 q_1^2 I^{(d+6)}_{(4,1,2)} - 8\pi g_A^2 q_1 q_2 z I^{(d+4)}_{(2,2,2)} \\
&- 128\pi^2 g_A^2 q_1 q_2 z I^{(d+6)}_{(3,2,2)} + g_A^2 I^{(d+2)}_{(1,1,2)} + 16\pi g_A^2 I^{(d+4)}_{(2,1,2)} \\
&+ 2(g_A^2-1) I^{(d+2)}_{(2,1,0)} + 32\pi\left(g_A^2-1\right) I^{(d+4)}_{(3,1,0)}\big] + (1 \leftrightarrow 2), \\
f_1^9 &= \frac{eg_A^2\pi}{2F_\pi^4}\big[g_A^2 M_\pi^2 I^{(d+2)}_{(2,1,2)} - 8\pi g_A^2 q_1^2 I^{(d+4)}_{(3,1,2)} \\
&- 64\pi^2 g_A^2 q_1 I^{(d+6)}_{(3,2,2)}(q_1 + q_2 z) + 4\pi g_A^2 I^{(d+4)}_{(2,1,2)} \\
&+ \left(1 - g_A^2\right) I^{(d+2)}_{(2,1,0)}\big] - (1 \leftrightarrow 2), \\
f_1^{10} &= \frac{ieg_A^4}{32\pi^2 F_\pi^4 k^2} - \frac{ieg_A^4 L(k)}{32\pi^2 F_\pi^4 k^2} - \frac{eg_A^2\pi}{4F_\pi^4} \\
&\times \big[-8\pi g_A^2 I^{(d+4)}_{(2,2,2)}\big(2M_\pi^2 + q_1 q_2 z\big) - 2g_A^2 M_\pi^2 I^{(d+2)}_{(2,1,2)} \\
&+ 16\pi g_A^2 q_1^2 I^{(d+4)}_{(3,1,2)} + 128\pi^2 g_A^2 q_1 I^{(d+6)}_{(3,2,2)}(q_1 - q_2 z) \\
&+ g_A^2 I^{(d+2)}_{(1,1,2)} + 8\pi g_A^2 I^{(d+4)}_{(2,1,2)} + 2\left(g_A^2-1\right) I^{(d+2)}_{(2,1,0)} \\
&+ 16\pi\left(g_A^2-1\right) I^{(d+4)}_{(2,2,0)}\big] + (1 \leftrightarrow 2).
\end{aligned}
\tag{B5}
$$

In addition, there are nonvanishing functions f_2^j given by

$$
\begin{aligned}
&f_2^3 = f_1^4, \quad f_2^4 = f_1^3, \quad f_2^5 = -f_1^6, \quad f_2^6 = -f_1^5, \\
&f_2^7 = f_1^8, \quad f_2^8 = f_1^7, \quad f_2^9 = f_1^{10}, \quad f_2^{10} = f_1^9.
\end{aligned}
\tag{B6}
$$

In the above equations, $z \equiv \hat{q}_1 \cdot \hat{q}_2$, $q_i \equiv |\vec{q}_i|$ and the loop functions $L(q)$ and $A(q)$ are defined in Eq. (2.20). Further, the functions I correspond to the three-point functions via

$$
I^{(d)}_{(\nu_1,\nu_2,\nu_3)} \equiv I(d; 0, 1; q_1, \nu_1; -q_2, \nu_2; 0, \nu_3) \ \text{ with } \ q_i = (0, \vec{q}_i)
\tag{B7}
$$

and

$$
I(d; p_1, \nu_1; p_2, \nu_2; p_3, \nu_3; p_4, \nu_4) = \mu^{4-d}\int \frac{d^d\ell}{(2\pi)^d}\frac{1}{\left[(\ell+p_1)^2 - M_\pi^2\right]^{\nu_1}\left[(\ell+p_2)^2 - M_\pi^2\right]^{\nu_2}\left[(\ell+p_3)^2 - M_\pi^2\right]^{\nu_3}[v\cdot(\ell+p_4)]^{\nu_4}}.
\tag{B8}
$$

Here, all propagators are understood to have an infinitesimal positive imaginary part. Note that here and in what follows, we will need only the functions I for four-momenta with vanishing zeroth component. We further emphasize that all functions $I^{(d+n)}_{(\nu_1,\nu_2,\nu_3)}$ that enter the above equations except $I^{(d+2)}_{(1,1,0)}$, $I^{(d+4)}_{(2,1,0)}$, and $I^{(d+4)}_{(1,2,0)}$ are finite in dimensional regularization in the limit $d \to 4$. For these functions, we define reduced functions by subtracting the poles in four dimensions

$$
\begin{aligned}
I^{(d+2)}_{\text{red }(1,1,0)} &= I^{(d+2)}_{(1,1,0)} - \frac{i}{4\pi} L(\mu) - \frac{i}{128\pi^3} \ln\left(\frac{M_\pi^2}{\mu^2}\right), \\
I^{(d+4)}_{\text{red }(2,1,0)} &= I^{(d+4)}_{(2,1,0)} + \frac{i}{48\pi^2} L(\mu) + \frac{i}{1536\pi^4} \ln\left(\frac{M_\pi^2}{\mu^2}\right), \\
I^{(d+4)}_{\text{red }(1,2,0)} &= I^{(d+4)}_{(1,2,0)} + \frac{i}{48\pi^2} L(\mu) + \frac{i}{1536\pi^4} \ln\left(\frac{M_\pi^2}{\mu^2}\right),
\end{aligned}
\tag{B9}
$$

where

$$
L(\mu) = \frac{\mu^{d-4}}{16\pi^2}\left\{\frac{1}{d-4} + \frac{1}{2}\left[\gamma_E - 1 - \ln(4\pi)\right]\right\}. \tag{B10}
$$

Here, μ is the scale introduced in dimensional regularization and $\gamma_E = -\Gamma'(1) \simeq 0.577$. Finally, for scalar functions contributing to the charge density we obtain the following expressions:

$$
\begin{aligned}
f_3^{2S} &= \frac{eg_A^4 A(k)}{64\pi F_\pi^4} + \frac{i\pi^2 eg_A^4}{F_\pi^4}\big[-2M_\pi^2 I^{(d+4)}_{(2,1,3)} + 16\pi q_1^2 I^{(d+6)}_{(3,1,3)} \\
&\quad - 8\pi q_1 q_2 z I^{(d+6)}_{(2,2,3)} + I^{(d+4)}_{(1,1,3)}\big] + (1 \leftrightarrow 2), \\
f_3^{3S} &= \frac{2i\pi^2 eg_A^4}{F_\pi^4}\big[8\pi q_1^2 I^{(d+6)}_{(3,1,3)} - M_\pi^2 I^{(d+4)}_{(2,1,3)}\big] - (1 \leftrightarrow 2), \\
f_1^{1S} &= \frac{eg_A^4 M_\pi\big(12M_\pi^4 + 7M_\pi^2 q_1^2 + q_1^2 q_2^2\big)}{64\pi F_\pi^4\big(4M_\pi^2 + q_1^2\big)\big(4M_\pi^2 + q_2^2\big)} \\
&\quad - \frac{eg_A^4 A(k)\big(2M_\pi^2 + q_1^2\big)}{16\pi F_\pi^4} + \frac{eg_A^4 A(q_1)\big(2M_\pi^2 + q_1^2\big)}{32\pi F_\pi^4} \\
&\quad + \frac{ieg_A^4\pi}{F_\pi^4}\big[M_\pi^4\big(-I^{(d+2)}_{(1,1,3)}\big) + 8\pi M_\pi^2 q_1 I^{(d+4)}_{(2,1,3)}(q_1 - q_2 z) \\
&\quad + 64\pi^2 q_1^3 q_2 z I^{(d+6)}_{(3,1,3)} - 16\pi^2 q_1^2 q_2^2 (z^2+1) I^{(d+6)}_{(2,2,3)} \\
&\quad + 2\pi q_1 q_2 z I^{(d+4)}_{(1,1,3)}\big] + (1 \leftrightarrow 2), \\
f_1^{4S} &= -\frac{eg_A^4 q_1^2 A(q_1)}{64\pi F_\pi^4} - \frac{2ieg_A^4\pi^2}{F_\pi^4}\big[8\pi q_1^2 q_2^2 (z^2-1) I^{(d+6)}_{(2,2,3)} \\
&\quad - q_1 q_2 z I^{(d+4)}_{(1,1,3)}\big] + (1 \leftrightarrow 2), \\
f_1^{5S} &= \frac{2ieg_A^4\pi^2}{F_\pi^4}\big[8\pi q_1 q_2 z I^{(d+6)}_{(2,2,3)} - I^{(d+4)}_{(1,1,3)}\big] + (1 \leftrightarrow 2), \\
f_1^{6S} &= \frac{16i\pi^3 eg_A^4}{F_\pi^4} q_1 q_2 z I^{(d+6)}_{(2,2,3)} + (1 \leftrightarrow 2), \\
f_1^{7S} &= \frac{eg_A^4 A(q_1)}{128\pi F_\pi^4} - \frac{16i\pi^3 eg_A^4}{F_\pi^4} q_1^2 I^{(d+6)}_{(2,2,3)} + (1 \leftrightarrow 2), \\
f_1^{8S} &= -\frac{eg_A^4 A(q_1)}{128\pi F_\pi^4} - \frac{16i\pi^3 eg_A^4}{F_\pi^4} q_1^2 I^{(d+6)}_{(2,2,3)} - (1 \leftrightarrow 2),
\end{aligned}
$$

$$
\begin{aligned}
f_2^{1S} &= \frac{eg_A^4 M_\pi^3 q_1^2}{64\pi F_\pi^4\big(4M_\pi^2 + q_1^2\big)\big(4M_\pi^2 + q_2^2\big)} \\
&\quad + \frac{eg_A^2\big(g_A^2 - 1\big)A(q_1)\big(2M_\pi^2 + q_1^2\big)}{32\pi F_\pi^4} - (1 \leftrightarrow 2), \\
f_2^{7S} &= -\frac{eg_A^4 A(q_1)}{128\pi F_\pi^4} - (1 \leftrightarrow 2), \\
f_2^{8S} &= \frac{eg_A^4 A(q_1)}{128\pi F_\pi^4} + (1 \leftrightarrow 2).
\end{aligned}
\tag{B11}
$$

APPENDIX C: EVALUATION OF THE THREE-POINT FUNCTION

The momentum-space expressions given in the previous appendix involve the three-point function defined in Eq. (B8). Below we show how loop integrals of this kind can be evaluated by introducing the corresponding Feynman parameters. In particular, we consider the following integrals

$$
\begin{aligned}
\mathit{\Pi}_1 &= I(d; 0, \nu_1; p_2, \nu_2; p_3, \nu_3; 0, \nu_4), \\
\mathit{\Pi}_2 &= I(d; 0, \nu_1; p_2, \nu_2; p_3, \nu_3; 0, 0),
\end{aligned}
\tag{C1}
$$

in the following kinematics:

$$
q_i = (0, \vec{q}_i), \quad \hat{q}_1 \cdot \hat{q}_2 = z, \quad v^2 = 1, \quad v \cdot q_1 = v \cdot q_2 = 0. \tag{C2}
$$

Here, we consider the case with $\nu_i = 1, 2, 3, \ldots$ and $|\vec{q}_i| > 0$. Starting from this point, we denote by q_i the length of the corresponding three-momentum in units of the pion mass, i.e., $q_i \equiv |\vec{q}_i|/M_\pi$. Introducing the Feynman parameters and carrying out the integration over the loop momentum, we obtain the following result for the first integral in Eq. (C1):

$$
\begin{aligned}
\mathit{\Pi}_1 &= \mu^{4-d} \frac{2^{\nu_4-1}\Gamma\left(\frac{\nu_4}{2}\right)\Gamma\left(\nu_1+\nu_2+\nu_3+\frac{\nu_4}{2}-\frac{d}{2}\right)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_4)} \\
&\quad \times \frac{(-1)^{\nu_1+\nu_2+\nu_3+\nu_4} i}{(4\pi)^{(d/2)}} \int_0^1 dt \int_0^t dy (t-y)^{\nu_1-1} \\
&\quad \times (1-t)^{\nu_2-1} y^{\nu_3-1} \left[\frac{1}{M_\pi^2 D(y_1 - y_2)}\right]^{\nu_1+\nu_2+\nu_3+\frac{\nu_4}{2}-\frac{d}{2}} \\
&\quad \times \left(\frac{1}{y_1 - y} + \frac{1}{y - y_2}\right)^{\nu_1+\nu_2+\nu_3+\frac{\nu_4}{2}-\frac{d}{2}},
\end{aligned}
\tag{C3}
$$

where

$$
y_1 = \frac{E}{2D} + \sqrt{\frac{E^2 + 4AD}{4D^2}}, \qquad y_2 = \frac{E}{2D} - \sqrt{\frac{E^2 + 4AD}{4D^2}}, \tag{C4}
$$

and we have introduced

$$
\begin{aligned}
A &\equiv 1 + q_1^2(1-t)t \;>\; 0, \\
B &\equiv 2q_1 q_2 z + q_2^2, \\
C &\equiv -2q_1 q_2 z, \\
D &\equiv q_2^2 \;>\; 0, \\
E &\equiv B + tC.
\end{aligned}
\tag{C5}
$$

It can be shown that the following inequalities hold true in the integration region for the considered kinematics:

$$y_1 > t \geqslant y \geqslant 0 > y_2. \tag{C6}$$

Thus, the remaining two-dimensional integral in Eq. (C3) can be easily calculated numerically for all desired values of ν_i.

Similarly, for the second integral in Eq. (C1) we obtain:

$$\begin{aligned} I\!I_2 = {} & \mu^{4-d} \frac{\Gamma\left(\nu_1+\nu_2+\nu_3-\frac{d}{2}\right)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)} \frac{(-1)^{\nu_1+\nu_2+\nu_3} i}{(4\pi)^{(d/2)}} \\ & \times \int_0^1 dt \int_0^t dy (t-y)^{\nu_1-1}(1-t)^{\nu_2-1} y^{\nu_3-1} \\ & \times \left[\frac{1}{M_\pi^2 D(y_1-y_2)}\right]^{\nu_1+\nu_2+\nu_3-\frac{d}{2}} \\ & \times \left(\frac{1}{y_1-y}+\frac{1}{y-y_2}\right)^{\nu_1+\nu_2+\nu_3-\frac{d}{2}}, \end{aligned} \tag{C7}$$

where, again, the inequalities given in Eq. (C6) hold true.

For the reduced functions $I^{(d+2)}_{\text{red }(1,1,0)}$, $I^{(d+4)}_{\text{red }(1,2,0)}$, and $I^{(d+4)}_{\text{red }(2,1,0)}$ we obtain the expressions:

$$\begin{aligned} I^{(d+2)}_{\text{red }(1,1,0)} = {} & \frac{i}{64\pi^3} \int_0^1 dt \\ & \times \int_0^t dy \left\{1 - \ln\left[\frac{1}{D(y_1-y)(y-y_2)}\right]\right\}, \\ I^{(d+4)}_{\text{red }(2,1,0)} = {} & \frac{i}{256\pi^4} \int_0^1 dt \\ & \times \int_0^t dy (t-1) \left\{1 - \ln\left[\frac{1}{D(y_1-y)(y-y_2)}\right]\right\}, \\ I^{(d+4)}_{\text{red }(1,2,0)} = {} & -\frac{i}{256\pi^4} \int_0^1 dt \\ & \times \int_0^t dy\, y \left\{1 - \ln\left[\frac{1}{D(y_1-y)(y-y_2)}\right]\right\}, \end{aligned} \tag{C8}$$

that do not depend on μ.

APPENDIX D: CURRENT CONSERVATION AND THE CONTINUITY EQUATION

Current conservation implies that the electromagnetic current operator $\vec{J}(\vec{x})$ should fulfill the continuity equation

$$\vec{\nabla} \cdot \vec{J}(\vec{r}) = -\frac{\partial \rho}{\partial t} = -i[H, \rho] = -i[H_0 + \bar{V}, \rho], \tag{D1}$$

where $\rho = \rho(\vec{r}) \equiv J^0(\vec{r})$ is the charge density and H, H_0, and $\bar{V}$ refer to the two-nucleon Hamilton operator, kinetic energy term, and the potential, respectively. The continuity equation thus provides a powerful check for the calculation. For the leading two-pion-exchange contributions to the current operator the continuity equation takes the form

$$\begin{aligned} \vec{\nabla} \cdot \vec{J}^{(1)}_{2\pi}(\vec{r}) = {} & e[\vec{\tau}_1 \times \vec{\tau}_2]^3 W^{(2)}_{2\pi}(\vec{r}_1 - \vec{r}_2) \\ & \times [\delta(\vec{r}-\vec{r}_1) - \delta(\vec{r}-\vec{r}_2)], \end{aligned} \tag{D2}$$

where the potential in the isospin limit is written in the form

$$\bar{V} = V + \vec{\tau}_1 \cdot \vec{\tau}_2 W, \tag{D3}$$

$\vec{r}_i$ denotes the position of the nucleon i and the superscripts of J and W refer to the chiral order. Further, we made use of the explicit form of the leading one-body charge density

$$\rho(\vec{r}) = \frac{e}{2}\left[\left(\mathbb{1} + \tau_1^3\right)\delta(\vec{r}-\vec{r}_1) + \left(\mathbb{1} + \tau_2^3\right)\delta(\vec{r}-\vec{r}_2)\right], \tag{D4}$$

and the absence of the leading one-pion-exchange charge density, $\rho^{(-1)}_{1\pi} = 0$. Equation (D2) can be transformed into momentum space

$$i(\vec{q}_1+\vec{q}_2) \cdot \vec{J}^{(1)}_{2\pi}(\vec{q}_1,\vec{q}_2) = e[\vec{\tau}_1 \times \vec{\tau}_2]^3 \left[W^{(2)}_{2\pi}(\vec{q}_1) - W^{(2)}_{2\pi}(\vec{q}_2)\right]. \tag{D5}$$

The left-hand side of the above equation can be expressed in terms of the basis operators O_i^S defined in Appendix B. Using the representation of the current operator in Eq. (B1), the spin-momentum operators appearing on the left-hand side of the above equation can be expressed in terms of the operators O_j^S as follows:

$$\sum_{j=1}^{24} f_i^j(\vec{q}_1,\vec{q}_2)\vec{k} \cdot \vec{O}_j = \sum_{j=1}^{8} g_i^{jS}(\vec{q}_1,\vec{q}_2) O_j^S, \tag{D6}$$

where the scalar functions $g_i^{jS}(\vec{q}_1,\vec{q}_2)$ read

$$\begin{aligned} g_i^{1S} = {} & k^2 f_i^1 + \left(q_1^2 - q_2^2\right) f_i^2, \\ g_i^{2S} = {} & -f_i^4 - f_i^6 + \tfrac{1}{2}k^2 f_i^7 + \tfrac{1}{2}\left(q_1^2-q_2^2\right) f_i^8 \\ & + \tfrac{1}{2}k^2 f_i^9 - \tfrac{1}{2}\left(q_1^2-q_2^2\right) f_i^{10}, \\ g_i^{3S} = {} & -f_i^3 + f_i^5 + \tfrac{1}{2}k^2 f_i^8 + \tfrac{1}{2}\left(q_1^2-q_2^2\right) f_i^7 \\ & + \tfrac{1}{2}k^2 f_i^{10} - \tfrac{1}{2}\left(q_1^2-q_2^2\right) f_i^9, \\ g_i^{4S} = {} & k^2 f_i^{11} + \left(q_1^2-q_2^2\right) f_i^{12}, \\ g_i^{5S} = {} & k^2 f_i^{15} + \left(q_1^2-q_2^2\right) f_i^{16} + 2 f_i^{19}, \\ g_i^{6S} = {} & k^2 f_i^{17} + \left(q_1^2-q_2^2\right) f_i^{18} + 2 f_i^{21}, \\ g_i^{7S} = {} & f_i^{19} + f_i^{21} + \tfrac{1}{2}k^2 f_i^{13} + \tfrac{1}{2}\left(q_1^2-q_2^2\right) f_i^{14} \\ & + \tfrac{1}{2}k^2 f_i^{23} + \tfrac{1}{2}\left(q_1^2-q_2^2\right) f_i^{24}, \\ g_i^{8S} = {} & -f_i^{20} + f_i^{22} - \tfrac{1}{2}k^2 f_i^{14} - \tfrac{1}{2}\left(q_1^2-q_2^2\right) f_i^{13} \\ & + \tfrac{1}{2}k^2 f_i^{24} + \tfrac{1}{2}\left(q_1^2-q_2^2\right) f_i^{23}, \end{aligned} \tag{D7}$$

and $\vec{k} \equiv \vec{q}_1 + \vec{q}_2$. Substituting the expressions for f_i^j for the leading two-pion-exchange current operator from Eq. (B5), one finds that the only nonvanishing function $g_i^{jS(2\pi)}$ is given by

$$\begin{aligned} g_3^{1S(2\pi)} = {} & e\frac{iL(q_1)}{384\pi^2 F_\pi^4}\left[4M_\pi^2\left(5g_A^4 - 4g_A^2 - 1\right)\right. \\ & \left. + q_1^2\left(23g_A^4 - 10g_A^2 - 1\right) + \frac{48 g_A^4 M_\pi^4}{q_1^2 + 4M_\pi^2}\right] \\ & + \alpha - (1 \leftrightarrow 2). \end{aligned} \tag{D8}$$

where α denotes a polynomial in q_1 and q_2 whose form depends on the choice of the subtraction scheme. Using the explicit

form of $W_{2\pi}^{(2)}$,

$$W_{2\pi}^{(2)} = -\frac{L(q)}{384\pi^2 F_\pi^4}\left[4M_\pi^2\left(5g_A^4 - 4g_A^2 - 1\right) + q^2\left(23g_A^4 - 10g_A^2 - 1\right) + \frac{48g_A^4 M_\pi^4}{4M_\pi^2 + q^2}\right] + \alpha', \quad \text{(D9)}$$

where α' again denotes a subtraction-scheme dependent polynomial contribution, it is easy to see that Eq. (D5) is indeed fulfilled.

APPENDIX E: COORDINATE-SPACE REPRESENTATION OF VARIOUS LOOP INTEGRALS

In this appendix we collect the formulas needed to obtain the expressions for various loop integrals in coordinate space. The two-pion-exchange contributions to the current operator resulting in the method of unitary transformation are given in terms of three-dimensional loop integrals with the integrands being rational functions of the pion energies $\omega_i = \sqrt{k_i^2 + M_\pi^2}$, where $k_i \equiv |k_i|$ refers to the pion momentum. To end up with simple expressions in coordinate space, it is convenient to rewrite the integrands as continuous superpositions of the propagators using the following expressions:

$$\begin{aligned}
\frac{1}{\omega_1\omega_2(\omega_1+\omega_2)} &= \frac{2}{\pi}\int_0^\infty d\beta \frac{1}{\omega_1^2+\beta^2}\frac{1}{\omega_2^2+\beta^2}, \\
\frac{\omega_1^2+\omega_1\omega_2+\omega_2^2}{\omega_1^3\omega_2^3(\omega_1+\omega_2)} &= -\frac{1}{2}\frac{\partial}{\partial M_\pi^2}\frac{1}{\omega_1\omega_2(\omega_1+\omega_2)}, \\
&= \frac{1}{\pi}\int_0^\infty d\beta\left[\frac{1}{\left(\omega_1^2+\beta^2\right)^2\left(\omega_2^2+\beta^2\right)} + \frac{1}{\left(\omega_1^2+\beta^2\right)\left(\omega_2^2+\beta^2\right)^2}\right]. \quad \text{(E1)}
\end{aligned}$$

In addition, we also need the following relations which involve three different pion energies:

$$\begin{aligned}
D_1 \equiv &= \frac{1}{\omega_1\omega_2\omega_3}\frac{\omega_1+\omega_2+\omega_3}{(\omega_1+\omega_2)(\omega_1+\omega_3)(\omega_2+\omega_3)} \\
&= \frac{2}{\pi}\int_0^\infty d\beta \frac{1}{\left(\omega_1^2+\beta^2\right)\left(\omega_2^2+\beta^2\right)\left(\omega_3^2+\beta^2\right)}, \\
D_2 \equiv &\frac{1}{\omega_1\omega_2\omega_3}\left[\frac{(\omega_1-\omega_2)(\omega_1-\omega_3)}{(\omega_1+\omega_2)(\omega_1+\omega_3)} + \frac{(\omega_1+\omega_2)(\omega_1-\omega_3)}{(\omega_1+\omega_3)(\omega_2+\omega_3)} + \frac{(\omega_1-\omega_2)(\omega_1+\omega_3)}{(\omega_1+\omega_2)(\omega_2+\omega_3)}\right] \\
&= -\frac{4}{\pi}\int_0^\infty d\beta\left[\frac{4\beta^2}{\left(\omega_1^2+\beta^2\right)\left(\omega_2^2+\beta^2\right)\left(\omega_3^2+\beta^2\right)} - \frac{1}{\left(\omega_2^2+\beta^2\right)\left(\omega_3^2+\beta^2\right)}\right], \\
D_3 \equiv &\frac{1}{\omega_1\omega_2\omega_3}\left[\frac{1}{\omega_1\omega_2\omega_3^2} - \frac{\omega_2}{\omega_1^2\left(\omega_1^2-\omega_2^2\right)(\omega_1+\omega_3)} + \frac{\omega_1}{\omega_2^2\left(\omega_1^2-\omega_2^2\right)(\omega_2+\omega_3)}\right] \\
&= -2\frac{\partial}{\partial M_\pi^2}\frac{1}{\omega_1\omega_2\omega_3}\frac{\omega_1+\omega_2+\omega_3}{(\omega_1+\omega_2)(\omega_1+\omega_3)(\omega_2+\omega_3)} \\
&= \frac{4}{\pi}\int_0^\infty d\beta\left[\frac{1}{\left(\omega_1^2+\beta^2\right)^2\left(\omega_2^2+\beta^2\right)\left(\omega_3^2+\beta^2\right)} + \frac{1}{\left(\omega_1^2+\beta^2\right)\left(\omega_2^2+\beta^2\right)^2\left(\omega_3^2+\beta^2\right)} + \frac{1}{\left(\omega_1^2+\beta^2\right)\left(\omega_2^2+\beta^2\right)\left(\omega_3^2+\beta^2\right)^2}\right]. \quad \text{(E2)}
\end{aligned}$$

Using the above expressions, it is straightforward to carry out the Fourier transformation $\mathcal{F}$,

$$\mathcal{F}(f(\vec{q}_1,\vec{q}_2)) \equiv \int \frac{d^3q_1}{(2\pi)^3}\frac{d^3q_2}{(2\pi)^3} e^{i\vec{q}_1\cdot\vec{r}_{10}} e^{i\vec{q}_2\cdot\vec{r}_{20}} f(\vec{q}_1,\vec{q}_2), \quad \text{(E3)}$$

of the integrals which appear in the calculation. For example,

$$\begin{aligned}
&\mathcal{F}\left(\int \frac{d^3\ell}{(2\pi)^3}\frac{1}{\omega_+\omega_-(\omega_+ + \omega_-)}\right) \\
&= \delta(\vec{r}_{20})\int\frac{d^3q_1}{(2\pi)^3}\int\frac{d^3\ell}{(2\pi)^3} e^{i\vec{q}_1\cdot\vec{r}_{10}}\frac{2}{\pi} \times \int_0^\infty d\beta \frac{1}{(\omega_+^2+\beta^2)(\omega_-^2+\beta^2)} \\
&= \delta(\vec{r}_{20})\frac{16}{\pi}\int_0^\infty d\beta\int\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3} e^{i\vec{k}_1\cdot\vec{r}_{10}} e^{-i\vec{k}_2\cdot\vec{r}_{10}} \times \frac{1}{\left(4\omega_1^2+\beta^2\right)\left(4\omega_2^2+\beta^2\right)} \\
&= \frac{M_\pi^3}{4\pi^3}\delta(\vec{r}_{20})\int_0^\infty d\beta \frac{e^{-\sqrt{1+\beta^2}2x_{10}}}{2x_{10}^2} \\
&= \frac{M_\pi^6}{8\pi^3}\delta(\vec{x}_{20})\frac{K_1(2x_{10})}{x_{10}^2}, \quad \text{(E4)}
\end{aligned}$$

where $\omega_\pm = \sqrt{(\vec{l}\pm\vec{q}_1)^2 + 4M_\pi^2}$ and the K_i refer to the modified Bessel functions. We made use of Eq. (E1) in the first line and carried out the substitutions $\vec{\ell}+\vec{q}_1 \to 2\vec{k}_1$, $\vec{\ell}-\vec{q}_1 \to 2\vec{k}_2$ in the second line. Notice further that the last equality is valid for $x_{10} > 0$. In a similar way one obtains

$$\begin{aligned}
\mathcal{F}\left(\int\frac{d^3\ell}{(2\pi)^3}\frac{l^2}{\omega_+\omega_-(\omega_+ + \omega_-)}\right) &= \frac{M_\pi^8}{12\pi^3}\delta(\vec{x}_{20})\left(\nabla_{10}^2 - 4\right) \times \frac{K_1(2x_{10})}{x_{10}^2}, \\
\mathcal{F}\left(\int\frac{d^3\ell}{(2\pi)^3}\frac{\omega_+^2+\omega_+\omega_-+\omega_-^2}{\omega_+^3\omega_-^3(\omega_+ + \omega_-)}\right) &= \frac{M_\pi^4}{16\pi^3}\delta(\vec{x}_{20})\frac{K_0(2x_{10})}{x_{10}}. \quad \text{(E5)}
\end{aligned}$$

Again, these expressions are only valid for $x_{10} > 0$. Clearly, the above results may also be obtained by first carrying out the momentum-space loop integrals. The resulting non-polynomial expressions are given in terms of the loop functions L and A, see Eq. (2.20), and can be easily Fourier-transformed

using the following relations

$$\int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \frac{L(q)}{s^2} \to \frac{M_\pi}{4\pi}\frac{K_0(2x)}{x},$$
$$\int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} (s^2)^k L(q) \to (-1)^{(k+1)}\left(\nabla_x^2 - 4\right)^k \times \frac{M_\pi^{(3+2k)}}{2\pi}\frac{K_1(2x)}{x^2},$$
$$\int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} (s^2)^k A(q) \to (-1)^k\left(\nabla_x^2 - 4\right)^k \frac{M_\pi^{2+2k}}{8\pi}\frac{e^{-2x}}{x^2}, \tag{E6}$$

where $k = 0, 1, 2, \ldots$.

Consider now the more complicated expressions involving three different pion energies which appear in the calculation:

$$\begin{aligned}
&\mathcal{F}\left(\int \frac{d^3k_1}{(2\pi)^3} d^3k_2 d^3k_3\ \delta(\vec{k}_2 - \vec{k}_1 + \vec{q}_1)\ \delta(\vec{k}_3 - \vec{k}_1 - \vec{q}_2)(\vec{k}_2 + \vec{k}_3) D_2\right) \\
&= -\frac{4}{\pi}\int_0^\infty d\beta \int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\frac{d^3k_3}{(2\pi)^3} e^{-i\vec{k}_2\cdot\vec{r}_{10}} e^{i\vec{k}_3\cdot\vec{r}_{20}} e^{i\vec{k}_1\cdot\vec{r}_{12}} \left(\frac{4\beta^2}{\left(\omega_1^2+\beta^2\right)\left(\omega_2^2+\beta^2\right)\left(\omega_3^2+\beta^2\right)} - \frac{1}{\left(\omega_2^2+\beta^2\right)\left(\omega_3^2+\beta^2\right)}\right)(\vec{k}_2 + \vec{k}_3) \\
&= \frac{iM_\pi^7}{4\pi^4}(\vec{\nabla}_{10} - \vec{\nabla}_{20}) \int_0^\infty d\beta\beta^2 \frac{e^{-\sqrt{1+\beta^2}x_{10}}}{x_{10}} \frac{e^{-\sqrt{1+\beta^2}x_{20}}}{x_{20}} \frac{e^{-\sqrt{1+\beta^2}x_{12}}}{x_{12}} \\
&= \frac{iM_\pi^7}{4\pi^4}(\vec{\nabla}_{10} - \vec{\nabla}_{20}) \frac{K_2(x_{10}+x_{20}+x_{12})}{(x_{10}\,x_{20}\,x_{12})(x_{10}+x_{20}+x_{12})},
\end{aligned} \tag{E7}$$

where in the first line we used Eq. (E2). The last relation is valid for positive values of the argument of the Bessel function. Similarly, we obtain:

$$\begin{aligned}
&\mathcal{F}\left(\int \frac{d^3k_1}{(2\pi)^3} d^3k_2 d^3k_3\ \delta(\vec{k}_2 - \vec{k}_1 + \vec{q}_1)\ \delta(\vec{k}_3 - \vec{k}_1 - \vec{q}_2) k_1^{i_1}\cdots k_1^{i_m} k_2^{j_1}\cdots k_2^{j_n} k_3^{k_1}\cdots k_3^{k_p} D_1\right) \\
&\quad = (-1)^{m+p}\frac{M_\pi^4 (iM_\pi)^{n+m+p}}{32\pi^4}\nabla_{12}^{i_1}\cdots\nabla_{12}^{i_m}\nabla_{10}^{j_1}\cdots\nabla_{10}^{j_n}\nabla_{20}^{k_1}\cdots\nabla_{20}^{k_p}\frac{K_1(x_{10}+x_{20}+x_{12})}{(x_{10}\,x_{20}\,x_{12})}, \\
&\mathcal{F}\left(\int \frac{d^3k_1}{(2\pi)^3} d^3k_2 d^3k_3\ \delta(\vec{k}_2 - \vec{k}_1 + \vec{q}_1)\ \delta(\vec{k}_3 - \vec{k}_1 - \vec{q}_2) k_1^{i_1}\cdots k_1^{i_m} k_2^{j_1}\cdots k_2^{j_n} k_3^{k_1}\cdots k_3^{k_p} D_3\right) \\
&\quad = (-1)^{m+p}\frac{M_\pi^2 (iM_\pi)^{m+n+p}}{32\pi^4}\nabla_{12}^{i_1}\cdots\nabla_{12}^{i_m}\nabla_{10}^{j_1}\cdots\nabla_{10}^{j_n}\nabla_{20}^{k_1}\cdots\nabla_{20}^{k_p}\frac{x_{10}+x_{20}+x_{12}}{x_{10}x_{20}x_{12}}K_0(x_{10}+x_{20}+x_{12}).
\end{aligned} \tag{E8}$$

[1] P. F. Bedaque and U. van Kolck, Annu. Rev. Nucl. Part. Sci. **52**, 339 (2002).
[2] E. Epelbaum, Prog. Part. Nucl. Phys. **57**, 654 (2006).
[3] E. Epelbaum, H. W. Hammer, and U.-G. Meißner, arXiv:0811.1338 [nucl-th] (to appear in Rev. Mod. Phys).
[4] T. S. Park, D. P. Min, and M. Rho, Phys. Rep. **233**, 341 (1993).
[5] T. S. Park, D. P. Min, and M. Rho, Nucl. Phys. **A596**, 515 (1996).
[6] Y. H. Song, R. Lazauskas, T. S. Park, and D. P. Min, Phys. Lett. **B656**, 174 (2007).
[7] Y.-H. Song, R. Lazauskas, and T.-S. Park, Phys. Rev. C **79**, 064002 (2009).
[8] R. Lazauskas, Y. H. Song, and T. S. Park, arXiv:0905.3119 [nucl-th].
[9] T. S. Park, K. Kubodera, D. P. Min, and M. Rho, Phys. Lett. **B472**, 232 (2000).
[10] T. S. Park, K. Kubodera, D. P. Min, and M. Rho, Astrophys. J. **507**, 443 (1998).
[11] T. S. Park *et al.*, arXiv:nucl-th/0107012.
[12] S. Ando, T. S. Park, K. Kubodera, and F. Myhrer, Phys. Lett. **B533**, 25 (2002).
[13] T. S. Park *et al.*, Phys. Rev. C **67**, 055206 (2003).
[14] K. Kubodera and T. S. Park, Annu. Rev. Nucl. Part. Sci. **54**, 19 (2004).
[15] D. R. Phillips and T. D. Cohen, Nucl. Phys. **A668**, 45 (2000).
[16] M. Walzl and U.-G. Meißner, Phys. Lett. **B513**, 37 (2001).
[17] D. R. Phillips, Phys. Lett. **B567**, 12 (2003).

[18] D. R. Phillips, J. Phys. G **34**, 365 (2007).
[19] M. P. Valderrama, A. Nogga, E. Ruiz Arriola, and D. R. Phillips, Eur. Phys. J. A **36**, 315 (2008).
[20] S. R. Beane, M. Malheiro, J. A. McGovern, D. R. Phillips, and U. van Kolck, Nucl. Phys. **A747**, 311 (2005).
[21] D. Choudhury and D. R. Phillips, Phys. Rev. C **71**, 044002 (2005).
[22] D. Shukla, A. Nogga, and D. R. Phillips, Nucl. Phys. **A819**, 98 (2009).
[23] S. R. Beane, V. Bernard, T. S. H. Lee, U.-G. Meißner, and U. van Kolck, Nucl. Phys. **A618**, 381 (1997).
[24] V. Bernard, H. Krebs, and Ulf.-G. Meißner, Phys. Rev. C **61**, 058201 (2000).
[25] H. Krebs, V. Bernard, and U.-G. Meißner, Nucl. Phys. **A713**, 405 (2003).
[26] H. Krebs, V. Bernard, and U.-G. Meißner, Eur. Phys. J. A **22**, 503 (2004).
[27] A. Gardestig and D. R. Phillips, Phys. Rev. C **73**, 014002 (2006).
[28] V. Lensky, V. Baru, J. Haidenbauer, C. Hanhart, A. E. Kudryavtsev, and U.-G. Meißner, Eur. Phys. J. A **26**, 107 (2005).
[29] J. Golak, R. Skibinski, H. Witala, W. Glöckle, A. Nogga, and H. Kamada, Phys. Rep. **415**, 89 (2005).
[30] E. Epelbaum, W. Glöckle, and U.-G. Meißner, Nucl. Phys. **A747**, 362 (2005).
[31] D. R. Entem and R. Machleidt, Phys. Rev. C **68**, 041001(R) (2003).
[32] E. Epelbaum, A. Nogga, W. Glöckle, H. Kamada, Ulf.-G. Meißner, and H. Witala, Phys. Rev. C **66**, 064001 (2002).
[33] S. Pastore, R. Schiavilla, and J. L. Goity, Phys. Rev. C **78**, 064002 (2008).
[34] S. Pastore, L. Girlanda, R. Schiavilla, M. Viviani, and R. B. Wiringa, Phys. Rev. C **80**, 034004 (2009).
[35] E. Epelbaum, W. Glöckle, and U.-G. Meißner, Nucl. Phys. **A637**, 107 (1998).
[36] E. Epelbaum, W. Glöckle, and U.-G. Meißner, Nucl. Phys. **A671**, 295 (2000).
[37] E. Epelbaum, U.-G. Meißner, and W. Glöckle, Nucl. Phys. **A714**, 535 (2003).
[38] E. Epelbaum, Phys. Lett. **B639**, 456 (2006).
[39] E. Epelbaum, Eur. Phys. J. A **34**, 197 (2007).
[40] N. Fukuda, K. Sawada, and M. Taketani, Prog. Theor. Phys. **12**, 156 (1954).
[41] S. Okubo, Prog. Theor. Phys. **12**, 603 (1954).
[42] H. Krebs, V. Bernard, and U.-G. Meißner, Annals Phys. **316**, 160 (2005).
[43] M. Gari and H. Hyuga, Z. Phys. A **277**, 291 (1976).
[44] H. Hyuga and H. Ohtsubo, Nucl. Phys. **A294**, 348 (1978).
[45] J. A. Eden and M. F. Gari, Phys. Rev. C **53**, 1102 (1996).
[46] J. Gasser and H. Leutwyler, Nucl. Phys. **B250**, 517 (1985).
[47] G. Ecker and M. Mojžiš, Phys. Lett. **B365**, 312 (1996).
[48] N. Fettes, U.-G. Meißner, and S. Steininger, Nucl. Phys. **A640**, 199 (1998).
[49] J. Gasser, M. A. Ivanov, E. Lipartia, M. Mojžiš, and A. Rusetsky, Eur. Phys. J. C **26**, 13 (2002).
[50] S. Kölling, E. Epelbaum, H. Krebs, U.-G. Meißner (in preparation).
[51] C. Hanhart, Phys. Rep. **397**, 155 (2004).
[52] I. S. Gerstein, R. Jackiw, S. Weinberg, and B. W. Lee, Phys. Rev. D **3**, 2486 (1971).
[53] A. I. Davydychev, Phys. Lett. **B263**, 107 (1991).
[54] N. Kaiser, R. Brockmann, and W. Weise, Nucl. Phys. **A625**, 758 (1997).
[55] T. R. Hemmert, B. R. Holstein, and J. Kambor, J. Phys. G **24**, 1831 (1998).
[56] N. Kaiser, S. Gerstendorfer, and W. Weise, Nucl. Phys. **A637**, 395 (1998).
[57] C. Ordóñez, L. Ray, and U. van Kolck, Phys. Rev. C **53**, 2086 (1996).
[58] H. Krebs, E. Epelbaum, and U.-G. Meißner, Eur. Phys. J. A **32**, 127 (2007).
[59] E. Epelbaum, H. Krebs, and U.-G. Meißner, Nucl. Phys. **A806**, 65 (2008).
[60] E. Epelbaum, H. Krebs, and U.-G. Meißner, Phys. Rev. C **77**, 034006 (2008).
[61] V. R. Pandharipande, D. R. Phillips, and U. van Kolck, Phys. Rev. C **71**, 064002 (2005).
[62] L. L. Foldy and J. A. Lock, in *Mesons in Nuclei*, edited by M. Rho and D. Wilkinson (North-Holland, Amsterdam, 1979), Vol. II, pp. 465–493.