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Pairing theory of the symmetry energy

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A model is investigated that displays a picture of the symmetry energy as an energy of rotation in isospace of a Cooper pair condensate, briefly "superfluid isorotation." The Hamiltonian is isobarically invariant and has a one- and a two-nucleon term, where the two-nucleon interaction is composed of an isovector pairing force and an interaction of isospins. It is analyzed in the Hartree-Bogolyubov plus random-phase approximation (RPA). The Hartree-Bogolyubov energy minus Lagrangian multiplier terms proportional to the number of valence nucleons and the z component of the isospin is shown to be locally minimized by a product of neutron and proton Bardeen-Cooper-Schrieffer states. The equations of the RPA can be reduced to independent equations for two-neutron, two-proton, and neutron-proton quasiparticle pairs. In each of these spaces, they have a Nambu-Goldstone solution due to the global gauge invariance and isobaric invariance of the Hamiltonian. Except for the Nambu-Goldstone solutions, the RPA solutions are independent of the strength of the isospin interaction. If, in one space, the pertinent single-nucleon spectrum has a particle-hole symmetry, the RPA solutions are twofold degenerate except for the Nambu-Goldstone solution and one more solution. In an idealized case of infinitely many equidistant single-nucleon levels, the one-nucleon term in the Hamiltonian and the isospin interaction contribute terms in the symmetry energy quadratic in the isospin T. The pairing force and the two-neutron and two-proton RPA correlation energies do not contribute. The contribution of the neutron-proton correlation energy is dominated by the Nambu-Goldstone solution, which gives a linear term that makes the total symmetry energy proportional to $T(T+1)$. The rest of this contribution is negative and can be written as the difference of two terms of the form $\sqrt{(aT)^2+b^2}-b$. Observations reported from Skyrme force calculations are discussed in the light of these results. Calculations with deformed Woods-Saxon single-nucleon levels give results similar to those of the idealized case. In calculations for the mass numbers $A=56$ and $A=100$ with spherical Woods-Saxon levels, the promotion of nucleons across magic gaps in the single-nucleon spectrum and the onset of superfluidity with the departure from magicity give rise to large linear terms in the symmetry energy. The calculations with Woods-Saxon single-nucleon levels reproduce surprisingly well the empirical symmetry energy. An experimental signature of superfluid isorotation is discussed.

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I. INTRODUCTION

The concept of a symmetry energy originates in Weizsäcker's early attempt [1] to construct a formula for the nuclear binding energy. Guided by Majoranas ideas [2] as to the nature of the internucleon force, Weizsäcker suggests an expression for the binding energy of a doubly even nucleus that, except for an electrostatic term, is symmetric in the numbers N and Z of neutrons and protons. If $B_s(N,Z)$ is the symmetric part of such an expression, the symmetry energy is $B_s(A/2,A/2)-B_s(N,Z)$ with $A=N+Z$. As discussed, for example, by Bohr and Mottelson [3], it carries information on basic aspects of the nuclear structure. Models of the nucleosynthesis in core-collapse supernovae rely on estimates of the masses of nuclei inaccessible to experiment. The accuracy of such estimates depends on a valid understanding of the origin of the symmetry energy, and data on the abundancies of nuclides in the nearby universe may in turn constrain nuclear models, as discussed in recent reviews by Arnould and Goriely [4] and Arnould, Goriely, and Takahashi [5]. In infinite nuclear matter, the symmetry energy per nucleon is a function of the nucleon density. The form of this function has a bearing on the surface structure of finite nuclei, the structure of neutron stars, and the dynamics of heavy-ion reactions. This research was reviewed recently by Li, Chen, and Ko [6].

Because $N-Z$ is twice the eigenvalue M_T of the z component T_z of the isospin $\boldsymbol{T}$, and most nuclear ground states are approximate eigenstates of $\boldsymbol{T}^2$ with the eigenvalue $T(T+1)$ given by $T=|M_T|$, the symmetry energy may be conceived as the T-dependent part of the nuclear ground-state energy in the limit of isobaric invariance. Much theoretical and analytical work, reviewed in Sec. II, aims at describing its dependence on T. The empirical evidence seems, at least, compatible with the conjecture that, in a first approximation, the symmetry energy is proportional to the Casimir invariant of the isospin algebra, $T(T+1)$.

In two previous brief articles [7], I discuss a schematic, micoscopic model that gives this T dependence approximately for low T. It is inspired by the Goswami's [8] observation that the potential of interaction, in a superfluid nucleus, of the individual nucleons with the condensate of Cooper [9] pairs, in the following referred to briefly as the "pair potential," is not only nondiagonal in N and Z but also isobarically noninvariant. Frauendorf and Sheikh [10] point out that the symmetry energy may be conceived accordingly as an energy of rotation of the condensate in isospace. An expression for the symmetry energy proportional to $T(T+1)$ is then analogous

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to the well-known expression for the energy levels of a quantal rotor. It should be stressed that it is, in this picture, the entire symmetry energy that is proportional to $T(T+1)$ and not only the contribution from the two-nucleon term in the Hamiltonian. In this respect the isorotational picture differs basically from several models reviewed in Sec. II. For brevity, I call a rotation in isospace of a Cooper pair condensate *superfluid isorotation*.

The model introduced in Ref. [7] is designed to display this physics. It thus involves valence nucleons obeying a schematic Hamiltonian with a one- and a two-nucleon term that conserves the number $A_{\rm v}$ of valence nucleons and the isospin. To calculate states with arbitrary $A_{\rm v}$ and T by a minimization, Lagrangian multiplier terms proportional to $\hat{A}_{\rm v}$ and T_z are subtracted from the Hamiltonian, where $\hat{A}_{\rm v}$ is the operator with the eigenvalues $A_{\rm v}$. This is equivalent to imposing neutron and proton chemical potentials. The Hamiltonian minus Lagrangian multiplier terms is treated in the Hartree-Bogolyubov plus random-phase approximation (RPA), which is known to lead to a separation of the collective degrees of freedom associated with spontaneously broken symmetries of a many-body system. Although the Hamiltonian is schematic, the principles of its treatment are thus general and may be applied to any energy functional of a Bogolyubov quasinucleon vacuum provided this functional is invariant under global gauge transformations and isobaric transformations.

The two-nucleon interaction of the model has a pairing and a particle-hole part. When the latter is omitted, the theory is for a spherical nucleus equivalent to that of Ginoccio and Wesener [11]. Some of the results derived below are known from their study. However, the present methods are quite different from theirs. The relation of the present work to that of Ginoccio and Wesener is discussed in Sec. III.

In the present article, my model is analyzed to a much greater depth than in Ref. [7]. Its mathematical structure is discussed in detail, its symmetries are explored, and the formulas used in the calculations are given explicitly. Furthermore, the calculations have been considerably extended. In the idealized case of equidistant single-nucleon levels considered in Ref. [7], the number of such levels has been enlarged by a factor more than 40 to rule out any spurious effect of the its finiteness. These calculations have been carried out for parameters appropriate for different mass numbers, and the results are described in a more general form than previously. A deviation from a linear T dependence of the contribution to the symmetry energy from the correlation energy calculated in the RPA has been traced to the correlations of neutron-proton quasiparticle pairs and its origin understood. Finally, to approach a description of actual nuclei, calculations with Woods-Saxon single-nucleon levels have been carried out for several isobaric chains.

To set up a background for the present work, I review in Sec. II previous theories of the symmetry energy dating back to the era of the birth of nuclear physics in the 1930s. Certain aspects of some early models are elucidated in an appendix. The present empirical evidence as to the T dependence of the symmetry energy is also discussed in Sec. II. The formalism is then developed in Sec. III and the calculations are described and discussed in Sec. IV. After some brief remarks in Sec. V on the issue of an experimental signature of superfluid isorotation, the study is summarized in Sec. VI.

II. THEORY AND PHENOMENOLOGY OF THE SYMMETRY ENERGY: A REVIEW

Bethe and Bacher [12] introduce the assumption, which has since then been common in the literature, that the symmetry energy depends quadratically on $N-Z$. This means, in terms of isospin, that it is proportional to T^2. A different T dependence is derived theoretically by Wigner [13]. He assumes that nucleons interact by a two-body force and makes the (now obsolete) assumption that this force is invariant under arbitrary unitary transformations of the nucleonic spin and isospin. This implies, in particular, that any exchange force must be of the Majorana type. Wigner then infers that the two-nucleon force gives a contribution to the symmetry energy equal to $T(T+4)$ times a factor that he supposes depends only weakly on T. If this factor is constant, the contribution of the two-nucleon force to the symmetry energy has a term linear in T in addition to the quadratic one. This is the celebrated "Wigner term." Wigner's derivation is easily redone with the SU(4) group of unitary transformations of the nucleonic spin and isospin replaced with the SU(2) group of isobaric transformations. As shown in the appendix, a factor $T(T+1)$ then replaces $T(T+4)$. A term in the symmetry energy linear in T gives rise to a cusp at $N=Z$ in the curve of masses along an isobaric chain. Such cusps are found, in fact, in an analysis of measured masses by Myers and Swiatecki [14].

Wigner estimates the contribution of the nucleon kinetic energy by the Fermi gas model, which gives a leading term proportional to T^2. In his model, the contributions of the one- and two-nucleon terms in the Hamiltonian thus depend differently on T. This is true for most models with a Hamiltonian composed of a one- and a two-nucleon term.

In the framework of the spherical shell model, and assuming conservation of isospin and seniority, Talmi and Unna [15] show that, for nuclei whose valence nucleons occupy a single j shell, the symmetry energy is proportional to $T(T+1)$. In the spherical shell model, the single-nucleon term in the Hamiltonian has the role played in the Fermi-gas model by the nucleon kinetic energy. When all valence nucleons occupy the same j shell, the sum of their single-nucleon energies is constant for a given mass number, so the entire symmetry energy stems from the residual two-nucleon interaction. While the result of Talmi and Unna thus agrees superficially with that of Wigner's argument applied to the isobaric SU(2), it is shown in the appendix to imply that the basic assumption of the latter, namely that the average interaction energy in two-nucleon states with a definite symmetry in position and spin is independent of T, does not hold for the isobarically invariant isovector pairing force acting in a single j shell.

In isobarically invariant shell-model calculations for $A=24$ and $A=48$, Satuła *et al.* [16] find that omitting the interaction of isoscalar nucleon pairs essentially eliminates the deviation from a quadratic T dependence of the symmetry energy. Because the $A=48$ nuclei whose calculated binding energies are analysed in Ref. [16] belong to the $1f_{7/2}$ shell,

this seems to indicate, in view of the result of Talmi and Unna, that the interaction of isovector pairs used in the calculation has a major component that does not conserve seniority. In line with an earlier study by Brenner *et al.* [17], Satuła *et al.* also consider a certain linear combination of the binding energies of nuclei differing in N and Z by small numbers. These linear combination are constructed so as to filter out a posible enhancement of the binding energy for $N = Z$. Both Brenner *et al.* and Satuła *et al.* find that, for a range of *sd*- and *fp*-shell nuclei, their linear combinations vanish essentially when the interaction of isoscalar pairs is turned off in their shell-model calculations. They hence infer that the observed deviation from a quadratic T dependence of the symmetry energy stems from this part of the two-nucleon interaction.

Myers and Swiatecki [14] fit the measured nuclear masses with a formula where the deviation from a quadratic dependence of the symmetry energy on $N - Z$ decreases exponentially with $|N - Z|$. More recently, Myers [18] suggests that a term in the symmetry energy linear in $|N - Z|$ could arise because two nucleons in identical states of orbital motion are more strongly bound than other nucleon pairs. He assumes that the single-nucleon levels are fourfold degenerate and the states of each level differ only by the directions of their spins and isospins. Thus he neglects the single-nucleon spin-orbit potential. His independent-nucleon model, in fact, has Wigner's SU(4) symmetry. When N neutrons and Z protons occupy the lowest levels in a potential well, and N and Z are even, the number of pairs in identical states of orbital motion is then equal to $3A/2 - |N - Z|$. In the presense of a single-nucleon spin-orbit potential, this is the number of pairs of nucleons in identical or time-reversed states of orbital motion and spin. Jensen, Hansen, and Jonson [19] work out a version of Myers's argument which maintains the SU(4) symmetry of the independent-nucleon model but includes exchange terms in the assumed delta-force interaction of the nucleons. In Ref. [20], Myers and Swiatecki explain the extra binding of nucleon pairs in identical states of orbital motion by the congruence of the nodal surfaces of the single-nucleon wave functions. They call the resulting term in the nuclear binding energy, accordingly, a *congruence* energy. In Ref. [21], these authors consider only pairs of a neutron and a proton and give for the number of particularly strongly bound pairs of this kind the expression $(A - |N - Z|)/2$, which is the number of neutron-proton pairs in identical states of orbital motion and spin. Because a congruence of nodal surfaces requires that the stationary states of a nucleon have unique orbital wave functions, and thus that the single-nucleon spin-orbit potential is neglected, counting only nucleon pairs with parallel spins seems, however, inconsistent with the congruence picture.

A contribution to the symmetry energy proportional to $T(T + 1)$ is pointed out by Bohr and Mottelson [3] to arise from the separable particle-hole interaction that, in the Hartree approximation, generates a term in the single-nucleon potential proportional to $\boldsymbol{T} \cdot \boldsymbol{t}$, where $\boldsymbol{t}$ is the single-nucleon isospin. Satuła and Wyss [22] study the symmetry energy in Hartree-Fock calculations with Skyrme forces where they ensure isobaric invariance by omitting the Coulomb force and assuming equal neutron and proton masses. They find that when the isospin-dependent parts of the Skyrme forces are omitted, the symmetry energy is roughly proportional to T^2 with T taken here equal to M_T. The isospin-dependent parts give additional contributions that are nearly proportional to $T(T + 1)$. They thus act similarly to the separable particle-hole interaction of Bohr and Mottelson. The quadratic T dependence of the remainder of the symmetry energy may be understood to result from a redistribution of the nucleons on their self-consistent energy levels. Figure 4 of Ref. [22] shows that when a Bardeen-Cooper-Schrieffer [23] pairing term is added to the energy functional, the contribution of the isospin-dependent forces to the symmetry energy is no longer proportional to $T(T + 1)$, where T is now taken equal to $\langle T_z \rangle$. A roughly quadratic dependence on T can be inferred from the article's Figs. 2 and 4. This is explained in Sec. IV B below.

In a model with nucleons in a deformed potential and a pairing force acting on both isoscalar and isovector nucleon pairs, Satuła and Wyss [24] find that an approximate particle number projection allows isoscalar and isovector components of the pair potential to coexist. In calculations for doubly even nuclei in the *fp* shell, the isoscalar component turns out to be for certain values of the coupling constants particularly large for $N \approx Z$, which leads to an enhanced binding of such nuclei. The authors therefore suggest that the Wigner term arises from isoscalar pairing. In Ref. [25], they consider an isoscalar pair potential with a different structure. Without particle number projection, Civitarese, Reboiro, and Vogel [26] obtain a similar enhanced binding for $N \approx Z$ with an isovector pairing force that acts with a larger strength on neutron-proton than on neutron and proton pairs and is thus isobarically noninvariant.

The empirical evidence as to the T dependence of the symmetry energy is ambiguous. Thus in the recent analyses by Royer and Gautier [27], Royer [28], Kirson [29,30], and Dieperink and Van Isacker [31], several terms that are plausible parts of a mass formula compete to improve the fit to the empirical masses. It seems safe, though, to conclude that a Wigner term in some form is called for and that a symmetry energy propotional to $T(T + 1)$ is compatible with the data. If a factor $T(T + x)$ is assumed, the tendency is that x is somewhat less than 1. The analyses by Zeldes [32] and Kirson [30] also indicate that the deviation from a quadratic T dependence is not confined to the region of nuclei with approximately equal N and Z, such as implied by the exponential parametrization of Myers and Swiatecki [14] and the theories of Brenner *et al.* [17], Satuła *et al.* [16], Satuła and Wyss [24,25], and Civitarese, Reboiro, and Vogel [26].

III. THEORY

A. Hamiltonian

The Hamiltonian considered may be thought of as that of a spherical or deformed shell model. It is written

$$H = \sum_j \epsilon_j a_j^\dagger a_j + \frac{1}{2} \sum_{jklm} v_{jklm} a_j^\dagger a_k^\dagger a_m a_l, \qquad (1)$$

where a_j annihilates a nucleon in the state $|j\rangle$ and the summations runs over an orthonormal set of single-nucleon states spanning a valence space. The basic single-nucleon

states form quadruples $|j\rangle = |q\sigma\tau\rangle$ with a common energy $\epsilon_j = \epsilon_q$, where $q\sigma$ is q or $\overline{q}$, and τ is n, denoting a neutron state, or p, denoting a proton state. These states are related by $|\overline{q}\tau\rangle = \overline{|q\tau\rangle}$ and $|qp\rangle = t_-|qn\rangle$, where a bar over a ket denotes time reversal and $t_\pm = t_x \pm i t_y$ in terms of the single-nucleon isospin $\boldsymbol{t} = (t_x, t_y, t_z)$. Note that time reversal is defined so as to commute with t_-, so t_x and t_z are even, and t_y odd, under time reversal.

The interaction matrix element v_{jklm} has a pairing and a particle-hole part, which are treated differently below,

$$v_{jklm} = v^{\text{pair}}_{jklm} + v^{\text{ph}}_{jklm}.$$

The pairing part is the matrix element of the isobarically invariant isovector pairing force,

$$v^{\text{pair}}_{jklm} = -G\langle j|\boldsymbol{t}|\tilde{k}\rangle \cdot \langle \tilde{m}|\boldsymbol{t}|l\rangle, \quad |\tilde{j}\rangle = -2it_y\overline{|j\rangle}. \tag{2}$$

The components of

$$\boldsymbol{P} = (P_x, P_y, P_z) = \frac{1}{\sqrt{2}}\sum_{jk}\langle\tilde{k}|\boldsymbol{t}|j\rangle a_k a_j$$

are given by

$$\begin{aligned} P_+ &= P_x + iP_y = \sqrt{2}\sum_q a_{\overline{q}p}a_{qp} = \sqrt{2}P_p, \\ P_- &= P_x - iP_y = -\sqrt{2}\sum_q a_{\overline{q}n}a_{qn} = -\sqrt{2}P_n, \\ P_z &= \frac{1}{\sqrt{2}}\sum_q (a_{\overline{q}p}a_{qn} + a_{\overline{q}n}a_{qp}). \end{aligned} \tag{3}$$

It follows from $\langle j|\boldsymbol{t}|k\rangle = -\langle\tilde{k}|\boldsymbol{t}|\tilde{j}\rangle$, which is easily verified, that $\boldsymbol{P}$ commutes as an isovector with the isospin

$$\boldsymbol{T} = (T_x, T_y, T_z) = \sum_{jk}\langle j|\boldsymbol{t}|k\rangle a^\dagger_j a_k.$$

Note

$$\begin{aligned} T_+ = T_-^\dagger = T_x + iT_y &= \sum_{q\sigma} a^\dagger_{q\sigma n}a_{q\sigma p}, \\ T_z &= \frac{1}{2}\sum_{q\sigma}(a^\dagger_{q\sigma n}a_{q\sigma n} - a^\dagger_{q\sigma p}a_{q\sigma p}). \end{aligned} \tag{4}$$

The particle-hole part of v_{jklm} is the matrix element of the separable interaction of Bohr and Mottelson [3], given by

$$v^{\text{ph}}_{jklm} = \kappa\langle j|\boldsymbol{t}|l\rangle \cdot \langle k|\boldsymbol{t}|m\rangle \tag{5}$$

with a coupling constant κ. I call this interaction the *symmetry force*.

The Hamiltonian (1) commutes with $\boldsymbol{T}$ and

$$\hat{A}_{\text{v}} = \sum_j a^\dagger_j a_j.$$

It therefore has a complete orthogonal set of eigenstates that are also eigenstates of $\hat{A}_{\text{v}}$, $\boldsymbol{T}^2$, and T_z. I denote the eigenvalues of these operators by A_{v}, $T(T+1)$, and M_T. The numbers N_τ of valence neutrons and protons are then given by $N_\tau = A_{\text{v}}/2 + 2m_t M_T$ with $m_t = \pm 1/2$ for $\tau = n$ and p. Because we are concerned with isobaric multiplets where these numbers are even for $M_T = T$, it follows that A_{v} is even, T is an integer, and $A_{\text{v}}/2$ and T have equal parities. The eigenstates of H form degenerate multiplets with $M_T = T, T-1, \ldots, -T$.

If $G = 0$, the lowest eigenvalue of H for given A_{v} and T is

$$E = 2\sum_\tau \sum_{\epsilon_q < \lambda_\tau} \epsilon_q + \frac{1}{2}\kappa\left[T(T+1) - \frac{3}{4}A_{\text{v}}\right], \tag{6}$$

where the Fermi level λ_τ is such that $N_\tau/2$ levels ϵ_q satisfy $\epsilon_q < \lambda_\tau$ with $N_\tau = A_{\text{v}}/2 + 2m_t T$. The eigenvalue of the symmetry force in Eq. (6) reflects $\sum_{a\neq b}\boldsymbol{t}_a \cdot \boldsymbol{t}_b = \boldsymbol{T}^2 - \sum_a \boldsymbol{t}_a^2$ and $\boldsymbol{t}^2 = 3/4$, where $\boldsymbol{t}_a$ is the isospin of the ath nucleon. The energy E given by Eq. (6) is a convex function of A_{v} and T and an increasing function of T for given A_{v}. These properties of the lowest eigenvalue of H are likely to persist for $G \neq 0$. Because the actual ground state energy, reduced for the electrostatic energy, of a doubly even nucleus with $N = Z$ is not a convex function of A, the Hamiltonian (1) therefore does not reproduce the absolute reduced energies, but it may reproduce their differences for a fixed A.

This Hamiltonian is evidently very schematic. Its relevance for a study of superfluid isorotation is due to the fact that it obeys the symmetries that are violated by the pair potential of Cooper pairing, global gauge invariance and isobaric invariance. Another virtue of the Hamiltonian is its simplicity, which makes its behavior transparent. Other symmetries of a more realistic nuclear Hamiltonian than the global gauge invariance and the isobaric invariance are violated by the present one. Like every shell-model Hamiltonian, it is thus not invariant under translational and Galilean transformations. If the single-nucleon energies are derived from a deformed potential, as in the calculations in Sec. IV C below, it is also not invariant under rotations in space. If the Hamiltonian would obey these symmetries, one could have solutions of the Hartree-Bogolyubov problem which would break the symmetries and thus give rise to Nambu-Goldstone solutions of the RPA problem additional to those discussed in Sec. III H below. The translational and Galilean invariances are necessarily broken by the solution of the Hartree-Bogolyubov problem. Imposing, in particular, rotational invariance would be important in a study of rotationally excited states. For the present study, which deals with nuclear ground states, and where excited states are considered only in so far as they are isobaric analogs of ground states, rotational invariance may be assumed to be less important.

B. Routhian

The eigenstates of H with the quantum numbers A_{v}, T, and M_T are also eigenstates of

$$\hat{R} = H - \lambda\hat{A}_{\text{v}} - \mu T_z,$$

where λ and μ are parameters and the lowest eigenvalues E and R of H and $\hat{R}$ for given A_{v}, T, and M_T are related by

$$R = E - \lambda A_{\text{v}} - \mu M_T.$$

If E is a convex function of A_{v} and T, and an increasing function of T for given A_{v}, then for any set of A_{v} and T there exists a set of λ and μ with $\mu = 0$ for $T = 0$ and $\mu > 0$ for $T > 0$, so R is minimal for this set of A_{v} and T and

$M_T = T$. Hence the lowest eigenstates of H for given A_v and T and with $M_T = T$ is also the lowest eigenstate of $\hat{R}$ for some values of λ and $\mu \geqslant 0$. Because the allowed values of A_v and T form a discrete set, λ and μ are not unique functions of these variables. They can be chosen freely within certain limits. Pashkevich and I [33] call a quantity analogous to R, involving the angular momentum, a Routhian, a term borrowed from analytical mechanics.

C. Quasinucleon vacuum

I now set out to calculate R approximately by perturbation theory starting from a vacuum $|\Phi\rangle$ of Bogolyubov [34] quasinucleons. The state $|\Phi\rangle$ is determined up to a phase by

$$\alpha_j|\Phi\rangle = 0 \tag{7}$$

in terms of a complete set of quasinucleon annihilators

$$\alpha_j = \sum_k (u_{jk}a_k + v_{jk}a_k^\dagger)$$

obeying

$$\{\alpha_j, \alpha_k\} = 0, \quad \{\alpha_j, \alpha_k^\dagger\} = \delta_{jk}.$$

It will be assumed to minimize the "Hartree-Bogolyubov" Routhian R_HB given by

$$\begin{aligned} R_\mathrm{HB} &= E_\mathrm{HB} - \lambda\langle\hat{A}_\mathrm{v}\rangle - \mu\langle T_z\rangle, \qquad (8)\\ E_\mathrm{HB} &= \sum_j \epsilon_j\langle a_j^\dagger a_j\rangle \\ &\quad + \frac{1}{2}\sum_{jklm}\left(v_{jklm}^\mathrm{pair}\langle a_j^\dagger a_k^\dagger\rangle\langle a_m a_l\rangle + v_{jklm}^\mathrm{ph}\langle a_j^\dagger a_l\rangle\langle a_k^\dagger a_m\rangle\right) \\ &= \langle H_0\rangle - G|\langle\boldsymbol{P}\rangle|^2 + \frac{1}{2}\kappa\langle\boldsymbol{T}\rangle^2, \qquad (9)\\ H_0 &= \sum_j \epsilon_j a_j^\dagger a_j. \end{aligned}$$

In these expressions, the expectation values are in the state $|\Phi\rangle$. This is my convention from now on unless otherwise stated in the context. If R_HB is minimal, $|\Phi\rangle$ is an eigenstate of

$$\begin{aligned} R_0 &= \sum_{\iota\kappa}\frac{\partial R_\mathrm{HB}}{\partial\langle a_\iota a_\kappa\rangle}a_\iota a_\kappa \\ &= H_0 - G(\langle\boldsymbol{P}\rangle^*\cdot\boldsymbol{P} + \langle\boldsymbol{P}\rangle\cdot\boldsymbol{P}^\dagger) + \kappa\langle\boldsymbol{T}\rangle\cdot\boldsymbol{T} \\ &\quad - \lambda\hat{A}_\mathrm{v} - \mu T_z, \qquad (10) \end{aligned}$$

where the convention has been introduced that a Greek letter subscript takes a value j or j^{-1} with $a_{j^{-1}} = a_j^\dagger$. (A similar notation is used in an early article [35] by Vogel and me.) The partial derivatives in Eq. (10) refer to the expressions (8) and (9) with all $\langle a_\iota a_\kappa\rangle$ considered mutually independent. Because $|\Phi\rangle$ is an eigenstate of R_0, the annihilators α_j can be chosen as eigenvectors of the linear map $\alpha \mapsto [\alpha, R_0]$.

If the isovector $\langle\boldsymbol{T}\rangle$ differs from the zero vector, it can be rotated into an arbitrary direction by an isobaric transformation of $|\Phi\rangle$. This does not change the first two terms in expression (8). For $\mu > 0$, the isovector $\langle\boldsymbol{T}\rangle$ therefore points into the z direction if R_HB is minimal. For $\mu = 0$, the value of R_HB does not depend on the direction of $\langle\boldsymbol{T}\rangle$, which can therefore be just assumed to point into the z direction. Generally, then, $\langle T_x\rangle = \langle T_y\rangle = 0$, which also holds if $\langle\boldsymbol{T}\rangle$ is the zero vector.

The exact ground state of $\hat{R}$ is an eigenstate of $\hat{N}_\tau = \hat{A}_\mathrm{v}/2 + 2m_t T_z$ with even eigenvalues, and therefore an eigenstate of $e^{-i\pi\hat{N}_\tau}$ with the eigenvalues one. If $|\Phi\rangle$ is an eigenstate of $e^{-i\pi\hat{N}_\tau}$, the eigenvalues should therefore be one as well; that is,

$$e^{-i\pi\hat{N}_\tau}|\Phi\rangle = |\Phi\rangle. \tag{11}$$

Suppose R_HB has been minimized with these constraints, and consider an infinitesimal variation $|\delta\Phi\rangle$ that violates the constraints. Any quasinucleon vacuum is an eigenstate of $e^{-i\pi\hat{A}_\mathrm{v}} = e^{-i\pi\hat{N}_n}e^{-i\pi\hat{N}_p}$, which has the eigenvalues ± 1. By continuity, therefore, $e^{-i\pi\hat{N}_n}e^{-i\pi\hat{N}_p}|\delta\Phi\rangle = |\delta\Phi\rangle$. Because $e^{-i\pi\hat{N}_\tau}$ has the eigenvalues ± 1, the variation must then satisfy $e^{-i\pi\hat{N}_\tau}|\delta\Phi\rangle = -|\delta\Phi\rangle$. It is easily verified that R_HB is stationary with respect to such a variation, and it will be seen in Sec. III H that R_HB is locally minimized by a state that satifies the constraints (11). I cannot prove that it is also globally minimized by this state; this issue is discussed a little further in Sec. III H. Anyway, I impose from now on the constraints (11). This entails $\langle P_z\rangle = 0$, so Eq. (10) becomes

$$\begin{aligned} R_0 &= H_0 - \sum_\tau(\Delta_\tau^* P_\tau + \Delta_\tau P_\tau^\dagger) \\ &\quad + (\kappa\langle T_z\rangle - \mu)\cdot T_z - \lambda\hat{A}_\mathrm{v}, \quad \Delta_\tau = G\langle P_\tau\rangle. \end{aligned}$$

A transformation $|\Phi\rangle \mapsto e^{-i\sum_\tau\phi_\tau\hat{N}_\tau}|\Phi\rangle$ with suitable angles ϕ_τ makes $\Delta_\tau \geqslant 0$. This leads to an expression for R_0 in the form of the single-quasinucleon Hamiltonian of the theory of Cooper pairing of Bogolyubov [36] and Valatin [37],

$$\begin{aligned} R_0 &= H_0 - \sum_\tau(\Delta_\tau(P_\tau + P_\tau^\dagger) + \lambda_\tau\hat{N}_\tau), \\ \lambda_\tau &= \lambda + m_t(\mu - \kappa\langle T_z\rangle), \qquad (12) \end{aligned}$$

so that the solution of the eigenproblem

$$[\alpha_j, R_0] = E_j\alpha_j, \quad E_j > 0,$$

is obtained immediately from this theory,

$$\alpha_{q\tau} = u_{q\tau}a_{q\tau} - v_{q\tau}a_{\bar{q}\tau}^\dagger, \quad \alpha_{\bar{q}\tau} = u_{q\tau}a_{\bar{q}\tau} + v_{q\tau}a_{q\tau}^\dagger, \tag{13}$$

$$u_{q\tau} = \sqrt{\frac{1}{2}\left(1 + \frac{\epsilon_q - \lambda_\tau}{E_{q\tau}}\right)}, \quad v_{q\tau} = \sqrt{\frac{1}{2}\left(1 - \frac{\epsilon_q - \lambda_\tau}{E_{q\tau}}\right)}, \tag{14}$$

$$E_{q\tau} = E_{\bar{q}\tau} = \sqrt{(\epsilon_q - \lambda_\tau)^2 + \Delta_\tau^2}. \tag{15}$$

Because $\alpha_j^\dagger$ is an eigenvector of $\alpha \mapsto [\alpha, R_0]$ with the eigenvalue $-E_j$, either α_j or $\alpha_j^\dagger$ could annihilate $|\Phi\rangle$. It is shown, however, in Sec. III H that a quasinucleon vacuum annihilated by an operator $\alpha_j^\dagger$ cannot locally minimize R_HB if it satisfies Eq. (11). The operators (13) therefore annihilate $|\Phi\rangle$, and Eq. (11) holds by the construction of $|\Phi\rangle$.

The relations $\Delta_\tau = G\langle P_\tau\rangle$ are equivalent to

$$\sum_q \frac{1}{E_{q\tau}} = \frac{2}{G} \quad \text{or} \quad \Delta_\tau = 0. \tag{16}$$

Unless otherwise stated in the context, $\Delta_\tau > 0$ will be assumed. It remains to fix λ_τ. The conventional way of the theory of Cooper pairing, which I shall follow, consists in demanding

$$\langle \hat{N}_\tau\rangle = 2\sum_q v_{q\tau}^2 = N_\tau = \frac{A_\mathrm{v}}{2} + 2m_t T. \tag{17}$$

This implies

$$E = E_\mathrm{HB} + (R - R_\mathrm{HB}).$$

In other words, corrections to E_HB can be calculated as corrections to R_HB. From Eqs. (12) and (17) one gets

$$\begin{aligned} \lambda_\tau &= \lambda + m_t(\mu - \kappa T), \\ \lambda &= (\lambda_n + \lambda_p)/2, \quad \mu = \lambda_n - \lambda_p + \kappa T, \end{aligned} \tag{18}$$

and Eq. (9) becomes

$$E_\mathrm{HB} = E_0 + E_\mathrm{pair} + \frac{1}{2}\kappa T^2, \tag{19}$$

$$E_0 = \langle H_0\rangle = 2\sum_{q\tau} v_{q\tau}^2 \epsilon_q, \tag{20}$$

$$E_\mathrm{pair} = -G|\langle \boldsymbol{P}\rangle|^2 = -\frac{\Delta_n^2 + \Delta_p^2}{G}. \tag{21}$$

If and only if $T = 0$, one has $u_{qn} = u_{qp}$, $v_{qn} = v_{qp}$, and $E_{qn} = E_{qp}$. Furthermore, $\lambda_n - \lambda_p$ and, hence, μ are increasing functions of T, so $T = 0$, $\lambda_n = \lambda_p$, and $\mu = 0$ are equivalent. For $T = 0$, also $\Delta_n = \Delta_p$, so in terms of its Cartesian components, $\langle \boldsymbol{P}\rangle$ is purely imaginary and points into the y direction. Other states with the same R_HB are generated in this case by arbitrary global gauge transformations and isobaric transformations of $|\Phi\rangle$. These transformations generally change the complex argument and the direction of $\langle \boldsymbol{P}\rangle$. Rotating, in particular, $\langle \boldsymbol{P}\rangle$ into the z direction by an isobaric transformation yields $\Delta_n = \Delta_p = 0$ and $\Delta_{np} = G\langle P_z\rangle \neq 0$. This solution of the Hartree-Bogolyubov problem for the isobarically invariant isovector pairing force is mentioned by Engel *et al.* [7] as alternative to the former. Both solutions are in fact just two out of an infinity of equivalent solutions related by global gauge transformations and isobaric transformations. In general, these solutions violate the isobarically noninvariant constraints (11).

D. RPA

In my brief articles [7], I derive the RPA part of the theory from a boson expansion. This allows some shortcuts by reference to the literature. Boson expansions are, however, ambiguous due to the noncommutability of boson field operators. Following Thouless [39], I base the following derivation on perturbation theory.

For any Hamiltonian $\mathcal{H}$, and any pair of operators X and Y, I define the propagator

$$\begin{aligned} \mathcal{G}_t(X, Y, t, \mathcal{H}) &= -i\langle T\{X'(t)Y'(0)\}\rangle, \\ X(t) &= e^{i\mathcal{H}t} X e^{-i\mathcal{H}t}, \quad X' = X - \langle X\rangle, \end{aligned} \tag{22}$$

where $T\{\ldots\}$ indicates time ordering and the expectation values are in the ground state of $\mathcal{H}$. It is easily proved by Thouless's [40] method that $\mathcal{G}_t(X, Y, t, \mathcal{H})$ is the sum of all Feynman diagrams, referring to some independent-particle Hamiltonian, where the vertices representing X are linked to the vertices representing Y and with no unlinked part. In particular, the subtraction of $\langle X\rangle$ in the last of the equations (22) cancels diagrams without an unlinked part where the vertices representing X are not linked to the vertices representing Y. If X and Y are linear combinations of products of even numbers of fermion field operators, the Fourier transform of $\mathcal{G}_t(X, Y, t, \mathcal{H})$ is

$$\begin{aligned} \mathcal{G}_\omega(X, Y, \omega, \mathcal{H}) &= \int_{-\infty}^{\infty} e^{i\omega t}\mathcal{G}_t(X, Y, t, \mathcal{H})dt \\ &= \left\langle X' \frac{1}{\omega - (\mathcal{H} - \mathcal{E} - i\eta)} Y' \right. \\ &\quad \left. - Y' \frac{1}{\omega + (\mathcal{H} - \mathcal{E} - i\eta)} X' \right\rangle, \end{aligned}$$

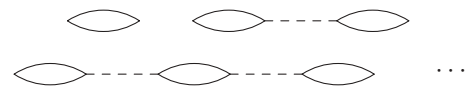
where $\mathcal{E}$ is the lowest eigenvalue of $\mathcal{H}$ and $\eta > 0$ is infinitesimal. I consider propagators

$$\begin{aligned} G(X, Y, \omega) &= \mathcal{G}_\omega(X, Y, \omega, \hat{R}), \\ G_0(X, Y, \omega) &= \mathcal{G}_\omega(X, Y, \omega, R_0), \end{aligned}$$

and Feynman diagrams referring to R_0 as the independent-particle Hamiltonian. In particular,

$$\begin{aligned} G_0(\alpha_k\alpha_j, \alpha_l^\dagger\alpha_m^\dagger, \omega) &= \frac{\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}}{\omega - (E_j + E_k - i\eta)}, \\ G_0(\alpha_l^\dagger\alpha_m^\dagger, \alpha_k\alpha_j, \omega) &= \frac{\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}}{-\omega - (E_j + E_k - i\eta)}, \\ G_0(\alpha_\iota\alpha_\kappa, \alpha_\lambda\alpha_\mu, \omega) &= 0 \quad \text{otherwise}. \end{aligned}$$

The propagator $G(a_\iota a_\kappa, a_\lambda a_\mu, \omega)$ is approximated by the sum of the diagrams



(23)

where a "lens" represents $G_0(a_\iota a_\kappa, a_\lambda a_\mu, \omega)$ and a dashed line represents the interaction matrix element $w_{\iota\kappa,\lambda\mu}$, given by

$$\begin{aligned} w_{j^{-1}k^{-1},ml} &= w_{ml,j^{-1}k^{-1}} = \tfrac{1}{2}v_{jklm}^\mathrm{pair}, \\ w_{j^{-1}l,k^{-1}m} &= v_{jklm}^\mathrm{ph}, \\ w_{\iota\kappa,\lambda\mu} &= 0 \quad \text{otherwise}. \end{aligned} \tag{24}$$

Due to the separable form of these matrix elements according to Eqs. (2) and (5), the diagrams (23) involve free propagators $G_0(X, Y, \omega)$ with many coherent terms over the configurations

of two quasinucleons. They therefore give a large contribution to the exact propagator $G(a_\iota a_\kappa, a_\lambda a_\mu, \omega)$. Summation of these diagrams results in the RPA equation

$$G(a_\iota a_\kappa, a_\lambda a_\mu, \omega) = G_0(a_\iota a_\kappa, a_\lambda a_\mu, \omega) + \sum_{\pi\rho\sigma\tau} G_0(a_\iota a_\kappa, a_\pi a_\rho, \omega) w_{\pi\rho,\sigma\tau} \times G(a_\sigma a_\tau, a_\lambda a_\mu, \omega).$$

In a basis of a complete set of independent operators $\alpha_k\alpha_j$ and their Hermitian conjugates $\alpha_{j^{-1}}\alpha_{k^{-1}}$, this equation takes the matrix form

$$\mathsf{G}(\omega) = \mathsf{G}_0(\omega) + \mathsf{G}_0(\omega)\mathsf{V}\mathsf{G}(\omega) = \left(\mathsf{G}_0(\omega)^{-1} - \mathsf{V}\right)^{-1}, \quad (25)$$

where

$$\mathsf{V}_{\iota\kappa,\lambda\mu} = \sum_{\pi\rho\sigma\tau} (a_\pi a_\rho)_{\iota\kappa} (a_\sigma a_\tau)_{\lambda\mu} w_{\pi\rho,\sigma\tau} \quad (26)$$

with

$$\left(X\right)_{\iota\kappa} = \langle\{\alpha_\kappa^\dagger \alpha_\iota^\dagger, X\}\rangle. \quad (27)$$

Equation (25) implies that $\mathsf{G}(\omega)$ has poles where

$$\mathsf{G}_0(\omega)^{-1} - \mathsf{V}$$

is singular.

It is convenient to introduce also block matrices corresponding to a division of the basis into a first part consisting of the operators $\alpha_k\alpha_j$ and a second part consisting of the operators $\alpha_{j^{-1}}\alpha_{k^{-1}}$ with a common order of the pairs (j,k) in both part. Because

$$\left(\mathsf{G}_0(\omega)^{-1}\right)_{j^{-1}k^{-1},kj} = \omega - (E_j + E_k - i\eta),$$
$$\left(\mathsf{G}_0(\omega)^{-1}\right)_{kj,j^{-1}k^{-1}} = -\omega - (E_j + E_k - i\eta),$$
$$\left(\mathsf{G}_0(\omega)^{-1}\right)_{\iota\kappa,\lambda\mu} = 0 \quad \text{otherwise},$$

the poles of $\mathsf{G}(\omega)$ are then the eigenvalues of

$$\mathsf{R} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (\mathsf{E} + \mathsf{V}), \quad (28)$$

where $\mathsf{0}$ and $\mathsf{1}$ denote the zero and identity matrices, and

$$\mathsf{E}_{j^{-1}k^{-1},kj} = \mathsf{E}_{kj,j^{-1}k^{-1}} = E_j + E_k - i\eta, \quad (29)$$
$$\mathsf{E}_{\iota\kappa,\lambda\mu} = 0 \quad \text{otherwise}.$$

The properties of R and

$$\mathsf{K} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (\mathsf{E} + \mathsf{V}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathsf{R} \quad (30)$$

are discussed by Thouless [39]. I shall follow his discussion partially. Because $\mathsf{E} + \mathsf{V}$ is symmetric,

$$\mathsf{R}^T = (\mathsf{E} + \mathsf{V}) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathsf{R} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Therefore, R has an eigenvalue $-\omega$ for every eigenvalue ω. For $\eta = 0$, K is Hermitian. If x is a right eigenvector of R with the eigenvalue ω, therefore

$$\mathsf{x}^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathsf{R} = \mathsf{x}^\dagger \mathsf{K} = \mathsf{x}^\dagger \mathsf{K}^\dagger = \mathsf{x}^\dagger \mathsf{R}^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \omega^* \mathsf{x}^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (31)$$

so $\mathsf{x}^\dagger \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is a left eigenvector with the eigenvalue ω^*. Two right eigenvectors x_1 and x_2 with the eigenvalues ω_1 and ω_2 satisfy

$$\omega_1^* \mathsf{x}_1^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathsf{x}_2 = \mathsf{x}_1^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathsf{R}\mathsf{x}_2^\dagger = \omega_2 \mathsf{x}_1^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathsf{x}_2,$$

so

$$\mathsf{x}_1^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathsf{x}_2 = 0 \quad \text{if} \quad \omega_1^* \neq \omega_2. \quad (32)$$

If K is also positive definite, the matrix

$$\mathsf{K}^{1/2}\mathsf{R}\mathsf{K}^{-1/2} = \mathsf{K}^{1/2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathsf{K}^{1/2}$$

is Hermitian, so the eigenvalues of R are real, and its right eigenvectors span the space of column matrices of their dimension. (This simple argument is due to Ring and Schuck [41]; Thouless [39] has a more complicated one.) If x is a right eigenvector of R with the eigenvalue ω, one then gets from Eq. (31)

$$\omega \mathsf{x}^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathsf{x} = \mathsf{x}^\dagger \mathsf{K} \mathsf{x} > 0, \quad (33)$$

so ω cannot be zero. When the imaginary term in Eq. (29) is included, R acquires the additional term

$$\delta\mathsf{R} = -i\eta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It follows from Eq. (31) that the resulting change $\delta\omega$ of ω is given by

$$\delta\omega \mathsf{x}^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathsf{x} = \mathsf{x}^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \delta\mathsf{R}\mathsf{x} = -i\eta|\mathsf{x}|^2.$$

A comparison with Eq. (33) then shows that the positive eigenvalues of R move into the lower imaginary half-plane.

It will be seen in Sec. III G that K is, actually, only positive semidefinite for $\eta = 0$. To escape the complications arising from the eigenvalues zero of K in this limit, it is understood in the next section that an infinitesimal positive definite term has been added to K.

E. Correction $R - R_{\rm HB}$

The correction $R - R_{\rm HB}$ can be divided into two parts,

$$R - R_{\rm HB} = (\langle\hat{R}\rangle - R_{\rm HB}) + (R - \langle\hat{R}\rangle).$$

A general result derived by Goldstone [42] implies that the second bracket in this expression is the sum of all linked Feynman diagrams without external lines and with at least two interaction lines. The first bracket is the part of the expectation value of the two-nucleon interaction in the quasinucleon vacuum that is not included in $R_{\rm HB}$. It is given by

$$\begin{aligned}\langle\hat{R}\rangle - R_{\rm HB} &= \frac{1}{2}\sum_{jklm}\big[v^{\rm pair}_{jklm}(\langle a^\dagger_j a^\dagger_k a_m a_l\rangle - \langle a^\dagger_j a^\dagger_k\rangle\langle a_m a_l\rangle)\\ &\quad + v^{\rm ph}_{jklm}(\langle a^\dagger_j a^\dagger_k a_m a_l\rangle - \langle a^\dagger_j a_l\rangle\langle a^\dagger_k a_m\rangle)\big]\\ &= \frac{1}{2}\sum_{jklm}\Big\langle\frac{1}{2}v^{\rm pair}_{jklm}(\{a^\dagger_j a^\dagger_k - \langle a^\dagger_j a^\dagger_k\rangle, a_m a_l - \langle a_m a_l\rangle\}\\ &\quad + [a^\dagger_j a^\dagger_k, a_m a_l])\\ &\quad + v^{\rm ph}_{jklm}((a^\dagger_j a_l - \langle a^\dagger_j a_l\rangle)(a^\dagger_k a_m - \langle a^\dagger_k a_m\rangle)\\ &\quad + a^\dagger_j[a^\dagger_k a_m, a_l])\Big\rangle.\end{aligned}$$

It will be seen that the part

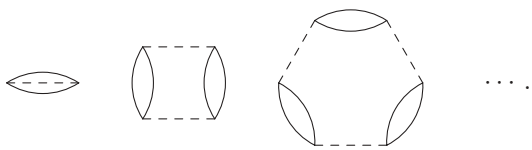
$$\begin{aligned}&\frac{1}{2}\sum_{jklm}\Big\langle\frac{1}{2}v^{\rm pair}_{jklm}\{a^\dagger_j a^\dagger_k - \langle a^\dagger_j a^\dagger_k\rangle, a_m a_l - \langle a_m a_l\rangle\}\\ &\quad + v^{\rm ph}_{jklm}(a^\dagger_j a_l - \langle a^\dagger_j a_l\rangle)(a^\dagger_k a_m - \langle a^\dagger_k a_m\rangle)\Big\rangle \qquad (34)\end{aligned}$$

of these terms can be combined with terms in $R - \langle\hat{R}\rangle$. The remainder c can be calculated from Eqs. (2), (5), and (17),

$$\begin{aligned}c &= \frac{1}{2}\sum_{jklm}\Big\langle\frac{1}{2}v^{\rm pair}_{jklm}[a^\dagger_j a^\dagger_k, a_m a_l] + v^{\rm ph}_{jklm}a^\dagger_j[a^\dagger_k a_m, a_l]\Big\rangle\\ &= \frac{3}{4}\left[G(2d - A_{\rm v}) - \frac{1}{2}\kappa A_{\rm v}\right], \qquad (35)\end{aligned}$$

where $4d$ is the dimension of the valence space. The factor $3/4$ is just $\boldsymbol{t}^2$. It is seen that c does not depend on T and thus does not contribute to the symmetry energy. The pairing-force part of c vanishes when the valence space is halfway filled. The symmetry-force part is $\frac{1}{2}\kappa\big(\sum_{a\neq b}\boldsymbol{t}_a\cdot\boldsymbol{t}_b - \boldsymbol{T}^2\big)$; compare the remark after Eq. (6).

The expression (34) is equal to the expression (37) below with $n = 1$, which is the first diagram in the series



(36)

I approximate $R - R_{\rm HB} - c$ by the sum $E_{\rm RPA}$ of this series. This is expected to be a good approximation for the reason mentioned in connection with the diagrams (23). The nth diagram equals

$$\frac{i}{2n}\int_{-\infty}^{\infty}\mathrm{tr}\,(\mathsf{V}\mathsf{G}_0(\omega))^n\frac{d\omega}{2\pi}, \qquad (37)$$

where the denominator $2n$ appears because the diagram has $2n$ equivalent vertices. Hence

$$E_{\rm RPA} = -\frac{i}{2}\int_{-\infty}^{\infty}\left(-\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{tr}\,(\mathsf{V}\mathsf{G}_0(\omega))^n\right)\frac{d\omega}{2\pi}. \qquad (38)$$

When the positive eigenvalues of R for $\eta = 0$ are denoted by ω_n, and the set of pairs (j,k) of subscripts of the basic operators $\alpha_k\alpha_j$ by $\mathcal{S}$, the integrand in Eq. (38) can be expressed as follows.

$$\begin{aligned}&-\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{tr}(\mathsf{V}\mathsf{G}_0(\omega))^n\\ &= \mathrm{tr}\log(1 - \mathsf{V}\mathsf{G}_0(\omega)) = \log\det(1 - \mathsf{V}\mathsf{G}_0(\omega))\\ &= \log\det((\mathsf{G}_0(\omega)^{-1} - \mathsf{V})\mathsf{G}_0(\omega))\\ &= \log\det\left((\omega - \mathsf{R})\left(\omega - \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\mathsf{E}\right)^{-1}\right)\\ &= \log\left(\prod_n(\omega - (\omega_n - i\eta))(\omega + (\omega_n - i\eta))\right.\\ &\quad\left./\prod_{(j,k)\in\mathcal{S}}(\omega - (E_j + E_k - i\eta))(\omega + (E_j + E_k - i\eta))\right)\\ &= \sum_n(\log(\omega - (\omega_n - i\eta)) + \log(\omega + (\omega_n - i\eta)))\\ &\quad - \sum_{(j,k)\in\mathcal{S}}(\log(\omega - (E_j + E_k - i\eta))\\ &\quad + \log(\omega + (E_j + E_k - i\eta))).\end{aligned}$$

Because this is proportional to ω^{-2} for large ω, the integration path can be extended with an infinite semicircle in the lower imaginary half-plane. When this semicircle is deformed into a linear path running backward below the real axis at the distance 2η, the integral collects contributions only from the discontinuity of $\log z$ for $z < 0$ in the terms in the integrand that have this cut in the lower imaginary half-plane. The result is

$$E_{\rm RPA} = \frac{1}{2}\left[\sum_n\omega_n - \sum_{(j,k)\in\mathcal{S}}(E_j + E_k)\right]. \qquad (39)$$

This expression has a very simple interpretation if ω_n and $E_j + E_k$ are conceived as frequencies of harmonic oscillators. Then $E_{\rm RPA}$ is just the change in total oscillator zero point energy induced by the interaction of the quasinucleons.

Truncating the expansion in Feynman diagrams to the sum of the diagrams where a pair of quasinucleons created together is also annihilated together is equivalent to treating such a pair as a single boson. Therefore, the result in Eq. (39) is the same as obtained in the quasiboson approximation. To my knowledge, the derivation above does not appear in the literature. A somewhat different derivation by contour integration is due to Shimizu *et al.* [43]. In the quasiboson approximation,

Marshalek [44] derives an expression for the RPA correction to a Hartree-Bogolyubov energy that corresponds to the present $c + E_{\mathrm{RPA}}$ when the difference between the Routhians considered is taken into account.

F. Matrices E and V

From now on, η is set to zero without further notice, so E is real. For any operator X, I denote by the corresponding sans-serif symbol X the column matrix of the brackets $(X)_{\iota\kappa}$ defined by Eq. (27). From Eqs. (2), (5), (24), and (26) then follows

$$\begin{aligned}\mathsf{V} &= -G\left[\begin{pmatrix}0&1\\1&0\end{pmatrix}\mathbf{P}^*\cdot\mathbf{P}^T+\mathbf{P}\cdot\mathbf{P}^\dagger\begin{pmatrix}0&1\\1&0\end{pmatrix}\right]\\ &\quad+\kappa\mathbf{T}\cdot\mathbf{T}^T\\ &= -G\left\{\sum_\tau\left[\begin{pmatrix}0&1\\1&0\end{pmatrix}\mathsf{P}_\tau^*\mathsf{P}_\tau^T+\mathsf{P}_\tau\mathsf{P}_\tau^\dagger\begin{pmatrix}0&1\\1&0\end{pmatrix}\right]\right.\\ &\quad\left.+\begin{pmatrix}0&1\\1&0\end{pmatrix}\mathsf{P}_z^*\mathsf{P}_z^T+\mathsf{P}_z\mathsf{P}_z^\dagger\begin{pmatrix}0&1\\1&0\end{pmatrix}\right\}\\ &\quad+\kappa\left(\frac{1}{2}(\mathsf{T}_+\mathsf{T}_-^T+\mathsf{T}_-\mathsf{T}_+^T)+\mathsf{T}_z\mathsf{T}_z^T\right).\end{aligned}\tag{40}$$

It is seen from Eqs. (3), (4), (13), (14), and (27) that P_τ, P_z, $\mathsf{T}_\pm$, and T_z are real. Furthermore, if X is any of these matrices, then, with $j = q\sigma\tau$ and $k = q'\sigma'\tau'$, its elements $(X)_{kj}$ and $(X)_{j^{-1}k^{-1}}$ are nonzero only for $q' = q$ and $\sigma' \neq \sigma$. Therefore, only within the subspace spanned by the corresponding basic operators do the eigenvalues of R differ from those of $\begin{pmatrix}0&1\\-1&0\end{pmatrix}\mathsf{E}$ and thus contribute to the difference in Eq. (39). Within this subspace, one can choose $j = q\tau$ and $k = \bar{q}\tau'$. I call the space spanned by such basic operators with $\tau = \tau'$ the $\tau\tau$ space, that spanned by such basic operators with $\tau \neq \tau'$ the np space, and the direct product of the nn and pp spaces the $nn + pp$ space. Because for any of the aforementioned matrices X, the corresponding operator X is time reversal even, it follows from Eq. (27) that $(X)_{\bar{q}pqn} = (X)_{\bar{q}nqp}$ and $(X)_{(qn)^{-1}(\bar{q}p)^{-1}} = (X)_{(qp)^{-1}(\bar{q}n)^{-1}}$. Therefore, if the basic operators of the np space are replaced with their linear combinations

$$\begin{aligned}[\alpha^\dagger\alpha^\dagger]_{q\bar{q}\pm} &= \tfrac{1}{\sqrt{2}}(\alpha^\dagger_{qn}\alpha^\dagger_{\bar{q}p}\pm\alpha^\dagger_{qp}\alpha^\dagger_{\bar{q}n}),\\ [\alpha\alpha]_{\bar{q}q\pm} &= \tfrac{1}{\sqrt{2}}(\alpha_{\bar{q}p}\alpha_{qn}\pm\alpha_{\bar{q}n}\alpha_{qp}),\end{aligned}\tag{41}$$

only the corresponding brackets

$$\begin{aligned}(X)_{\bar{q}q\pm} &= \tfrac{1}{\sqrt{2}}[(X)_{\bar{q}pqn}\pm(X)_{\bar{q}nqp}],\\ (X)_{q^{-1}\bar{q}^{-1}\pm} &= \tfrac{1}{\sqrt{2}}[(X)_{(qn)^{-1}(\bar{q}p)^{-1}}\pm(X)_{(qp)^{-1}(\bar{q}n)^{-1}}]\end{aligned}$$

with the subscript $+$ differ from zero. The only nonzero elements of E in the basis of the operators (41) are

$$\mathsf{E}_{q^{-1}\bar{q}^{-1}\pm,\bar{q}q\pm}=\mathsf{E}_{\bar{q}q\pm,q^{-1}\bar{q}^{-1}\pm}=E_{qn}+E_{qp}.\tag{42}$$

For the calculation of E_{RPA} one therefore needs to keep only the part of the np space spanned by the operators with the subscript $+$. From now on, I call this subspace the np space.

Equations (3), (4), (13), and (27) give

$$\begin{aligned}(P_\tau)_{\bar{q}\tau q\tau} &= \langle P_\tau\alpha^\dagger_{q\tau}\alpha^\dagger_{\bar{q}\tau}\rangle = u^2_{q\tau},\\ (P_\tau)_{(q\tau)^{-1}(\bar{q}\tau)^{-1}} &= \langle\alpha_{\bar{q}\tau}\alpha_{q\tau}P_\tau\rangle = -v^2_{q\tau},\\ (P_z)_{\bar{q}q+} &= \langle P_z[\alpha^\dagger\alpha^\dagger]_{q\bar{q}+}\rangle = u_{qn}u_{qp},\\ (P_z)_{q^{-1}\bar{q}^{-1}+} &= \langle[\alpha\alpha]_{\bar{q}q+}P_z\rangle = -v_{qn}v_{qp},\\ (T_+)_{\bar{q}q+} &= (T_-)_{q^{-1}\bar{q}^{-1}+} = \langle T_+[\alpha^\dagger\alpha^\dagger]_{q\bar{q}+}\rangle\\ &= \sqrt{2}v_{qn}u_{qp},\\ (T_-)_{\bar{q}q+} &= (T_+)_{q^{-1}\bar{q}^{-1}+} = \langle T_-[\alpha^\dagger\alpha^\dagger]_{q\bar{q}+}\rangle\\ &= \sqrt{2}u_{qn}v_{qp},\\ (T_z)_{\bar{q}\tau q\tau} &= (T_z)_{(q\tau)^{-1}(\bar{q}\tau)^{-1}} = \langle T_z\alpha^\dagger_{q\tau}\alpha^\dagger_{\bar{q}\tau}\rangle\\ &= 2m_t u_{q\tau}v_{q\tau},\\ (\hat{A}_{\mathrm{v}})_{\bar{q}\tau q\tau} &= (\hat{A}_{\mathrm{v}})_{(q\tau)^{-1}(\bar{q}\tau)^{-1}} = \langle\hat{A}_{\mathrm{v}}\alpha^\dagger_{q\tau}\alpha^\dagger_{\bar{q}\tau}\rangle\\ &= 2u_{q\tau}v_{q\tau},\\ (X)_a &= 0\quad\text{otherwise},\end{aligned}\tag{43}$$

where a is $\bar{q}\tau q\tau$, $(q\tau)^{-1}(\bar{q}\tau)^{-1}$, $\bar{q}q+$, or $q^{-1}\bar{q}^{-1}+$. The elements of V connecting the $nn + pp$ and np spaces are seen to vanish. The part of V acting in the $nn + pp$ space is given by the terms in the sum (40) with P_τ, $\mathsf{P}^\dagger_\tau$, and T_z, and the part acting in the np space by the terms with P_z, $\mathsf{P}^\dagger_z$, and $\mathsf{T}_\pm$. The nonzero elements of E in the $nn + pp$ space are

$$\mathsf{E}_{(q\tau)^{-1}(\bar{q}\tau)^{-1},\bar{q}\tau q\tau}=\mathsf{E}_{\bar{q}\tau q\tau,(q\tau)^{-1}(\bar{q}\tau)^{-1}}=2E_{q\tau}.\tag{44}$$

In the np space, they are given by Eq. (42). Note that for $\kappa = 0$, the elements of V connecting the nn and pp spaces vanish. When also $\mu = 0$, the nn, pp, and np parts of E or V are identical.

G. Definiteness of K

Because the matrices E and V are real, the symmetries mentioned in Sec. III D imply

$$\mathsf{E}+\mathsf{V}=\begin{pmatrix}\mathsf{B}&\mathsf{A}\\\mathsf{A}&\mathsf{B}\end{pmatrix},\tag{45}$$

where A and B are symmetric. Hence follow

$$\mathsf{K}=\begin{pmatrix}\mathsf{A}&\mathsf{B}\\\mathsf{B}&\mathsf{A}\end{pmatrix},\quad\mathsf{R}=\begin{pmatrix}\mathsf{A}&\mathsf{B}\\-\mathsf{B}&-\mathsf{A}\end{pmatrix},$$

whence

$$\begin{aligned}\mathsf{K}' &= \frac{1}{2}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\mathsf{K}\begin{pmatrix}1&1\\1&-1\end{pmatrix}=\begin{pmatrix}\mathsf{A}+\mathsf{B}&0\\0&\mathsf{A}-\mathsf{B}\end{pmatrix},\\ \mathsf{R}' &= \frac{1}{2}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\mathsf{R}\begin{pmatrix}1&1\\1&-1\end{pmatrix}=\begin{pmatrix}0&\mathsf{A}-\mathsf{B}\\\mathsf{A}+\mathsf{B}&0\end{pmatrix}.\end{aligned}\tag{46}$$

I shall prove that K is positive semidefinite. From $T_z = T_z^\dagger$ and $T_\pm = T_\mp^\dagger$ follow $\mathsf{T}_z = \begin{pmatrix}0&1\\1&0\end{pmatrix}\mathsf{T}_z$ and $\mathsf{T}_\pm = \begin{pmatrix}0&1\\1&0\end{pmatrix}\mathsf{T}_\mp$. Because κ is positive, the symmetry force is seen from Eqs. (30) and (40) to contribute, then, a positive semidefinite

term to K. It is therefore sufficient to consider the case $\kappa = 0$. From the remark at the end of Sec. III F, it follows that, in this case, the *nn* and *pp* spaces can be treated separately. Ginoccio and Wesener [11] prove that K is positive semidefinite in these spaces for $\kappa = 0$. The present proof differs somewhat from theirs and is extended to the *np* space. It also gives the dimension of K's kernel.

With the $\tau\tau'$ parts of A and B denoted by $\mathsf{a}_{\tau\tau'}$ and $\mathsf{b}_{\tau\tau'}$, it is according to Eq. (46) sufficients to prove that the matrices $\mathsf{a}_{\tau\tau'} \pm \mathsf{b}_{\tau\tau'}$ are positive semidefinite for $\kappa = 0$. Eqs. (40), (42), (43), (44), and (45) give

$$\mathsf{a}_{\tau\tau'} \pm \mathsf{b}_{\tau\tau'} = \mathsf{e}_{\tau\tau'} - G\mathsf{p}_{\tau\tau'\mp}\mathsf{p}^T_{\tau\tau'\mp}, \tag{47}$$

where $\mathsf{e}_{\tau\tau'}$ is the diagonal matrix with the diagonal elements $E_{q\tau} + E_{q\tau'}$ and $\mathsf{p}_{\tau\tau'\pm}$ the column matrix with the elements

$$p_{q\tau\tau'\pm} = u_{q\tau}u_{q\tau'} \pm v_{q\tau}v_{q\tau'}. \tag{48}$$

For any column matrix x, thus

$$\begin{aligned}\mathsf{x}^\dagger(\mathsf{a}_{\tau\tau'} \pm \mathsf{b}_{\tau\tau'})\mathsf{x} &= \mathsf{x}^\dagger\big(\mathsf{e}_{\tau\tau'} - G\mathsf{p}_{\tau\tau'\mp}\mathsf{p}^T_{\tau\tau'\mp}\big)\mathsf{x} \\ = \mathsf{x}^\dagger\mathsf{e}_{\tau\tau'}\mathsf{x} - G\big|\mathsf{p}^T_{\tau\tau'\mp}\mathsf{x}\big|^2 &\geqslant \big(1 - G\mathsf{p}^T_{\tau\tau'\mp}\mathsf{e}^{-1}_{\tau\tau'}\mathsf{p}_{\tau\tau'\mp}\big)\mathsf{x}^\dagger\mathsf{e}_{\tau\tau'}\mathsf{x} \\ &= (1 - Gf_{\tau\tau'\mp})\mathsf{x}^\dagger\mathsf{e}_{\tau\tau'}\mathsf{x}\end{aligned}$$

with

$$f_{\tau\tau'\pm} = \mathsf{p}^T_{\tau\tau'\pm}\mathsf{e}^{-1}_{\tau\tau'}\mathsf{p}_{\tau\tau'\pm} = \sum_q \frac{p^2_{q\tau\tau'\pm}}{E_{q\tau} + E_{q\tau'}},$$

where equality holds if and only if x is proportional to $\mathsf{e}^{-1}_{\tau\tau'}\mathsf{p}_{\tau\tau'\mp}$. Equations (16) and (48) give

$$\begin{aligned}f_{\tau\tau'\pm} &\leqslant \sum_q \frac{1}{E_{q\tau} + E_{q\tau'}} \\ &\leqslant \frac{1}{4}\sum_q\left(\frac{1}{E_{q\tau}} + \frac{1}{E_{q\tau'}}\right) = \frac{1}{G},\end{aligned}$$

where, as it follows from the remark at the end of Sec. III C, both equalities hold if and only if the sign is $+$ and either $\tau = \tau'$ or $\mu = 0$. Hence, the matrices $\mathsf{a}_{\tau\tau'} \pm \mathsf{b}_{\tau\tau'}$ are positive semidefinite. Only a matrix $\mathsf{a}_{\tau\tau'} - \mathsf{b}_{\tau\tau'}$ can have an eigenvalue zero, which it has if either $\tau = \tau'$ or $\mu = 0$. The multiplicity of these eigenvalues is 1. They are discussed from another point of view in Sec. III H.

Let the symmetry force be included again. Because, as it has now been shown, $\mathsf{A} + \mathsf{B}$ is positive definite, one can apply to R one more similarity transformation,

$$\begin{aligned}\mathsf{R}'' &= \begin{pmatrix}(\mathsf{A}+\mathsf{B})^{1/2} & 0 \\ 0 & (\mathsf{A}+\mathsf{B})^{-1/2}\end{pmatrix} \\ &\quad\times \mathsf{R}'\begin{pmatrix}(\mathsf{A}+\mathsf{B})^{-1/2} & 0 \\ 0 & (\mathsf{A}+\mathsf{B})^{1/2}\end{pmatrix} \\ &= \begin{pmatrix}0 & (\mathsf{A}+\mathsf{B})^{1/2}(\mathsf{A}-\mathsf{B})(\mathsf{A}+\mathsf{B})^{1/2} \\ 1 & 0\end{pmatrix} = \begin{pmatrix}0 & \mathsf{M} \\ 1 & 0\end{pmatrix}.\end{aligned}$$

The matrix M is real, symmetric, and positive semidefinite. It therefore has only non-negative eigenvalues ω^2, and the corresponding eigenvectors x span the space of column matrices of their dimension. If $\omega^2 > 0$, then

$$\begin{pmatrix}\omega\mathsf{x} \\ \mathsf{x}\end{pmatrix}$$

are eigenvectors of R'' with the eigenvalues ω. Note that ω can have both signs, so two eigenvalues of R'' with opposite signs correspond to each such eigenvalue of M. If $\omega^2 = 0$, one has

$$\mathsf{R}''\begin{pmatrix}0 \\ \mathsf{x}\end{pmatrix} = 0, \quad \mathsf{R}''\begin{pmatrix}\mathsf{x} \\ 0\end{pmatrix} = \begin{pmatrix}0 \\ \mathsf{x}\end{pmatrix}.$$

The column matrix in the first of these equations is an eigenvector of R'' with the eigenvalue zero, whereas the first one in the second equation is not an eigenvector of R''. Following Thouless [39,45] and Marshalek and Wesener [46], one can interpret the linear combinations of two-quasinucleon annihilators and creators corresponding to these column matrices as proportional to leading terms in expansions of a conserved momentum and its conjugate coordinate. Together, all the pairs of column matrices mentioned above span the space of column matrices of their dimension.

It follows from Eq. (30) that K and R have a common right kernel, whose dimension I denote by d_k. For $\kappa = 0$, the preceding discussion implies $d_\mathrm{k} = 2$ if $\mu > 0$ and $d_\mathrm{k} = 3$ if $\mu = 0$. Because d_k is also the dimension of the right kernel of M and

$$\det(\omega - \mathsf{R}) = \det(\omega - \mathsf{R}'') = \det(\omega^2 - \mathsf{M}),$$

the characteristic root zero of R has the multiplicity $2d_\mathrm{k}$. When K is made positive definite by an infinitesimal perturbation, as it was assumed to have been done for the purpose of the derivation in Sec. III E, the eigenvalue zero of R then splits into d_k pairs of nonzero infinitesimal real eigenvalues with opposite signs. Being infinitesimal, these eigenvalues do not contribute to E_RPA, which can therefore be calculated from the positive eigenvalues of the original singular matrix R. These are the square roots of the positive eigenvalues of M. The eigenvalues of M are calculated most efficiently from

$$\mathsf{M}' = (\mathsf{A}+\mathsf{B})^{1/2}\mathsf{M}(\mathsf{A}+\mathsf{B})^{-1/2} = (\mathsf{A}+\mathsf{B})(\mathsf{A}-\mathsf{B}). \tag{49}$$

That the dimension of the matrix to be diagonalized can be reduced in this way when R is real is pointed out by Ring and Schuck [41] in the case when K is positive definite.

H. Nambu-Goldstone solutions

It follows from a theorem proved by Thouless [45] that a general Bogolyubov quasinucleon vacuum in the vicinity of $|\Phi\rangle$ is obtained from $|\Phi\rangle$ by a transformation of the form

$$|\Phi\rangle \mapsto \mathcal{N}\exp\left(\sum_{(j,k)\in\mathcal{S}} \zeta_{kj}\alpha^\dagger_j\alpha^\dagger_k\right)|\Phi\rangle, \tag{50}$$

where ζ_{kj} are complex parameters and $\mathcal{N}$ is a normalization factor. Because expectation values in the transformed vacuum do not depend on its phase, they are functions of ζ_{kj}. Thouless's results in Ref. [45] imply that $\mathsf{E} + \mathsf{V}$ is the Hessian matrix of R_HB with respect to ζ_{kj} and $\zeta_{j^{-1}k^{-1}} = \zeta^*_{kj}$ for $\zeta_{kj} = 0$ and that, if some symmetry of E_HB is violated by the quasinucleon

vacuum, special solutions of the eigenproblem of R result from considering variations of $|\Phi\rangle$ within the symmetry group. These solutions are analogous to the Nambu-Goldstone [47] boson solutions in field theories with a so-called spontaneous symmetry breaking, that is, a violation of a symmetry by the vacuum.

In the present case, the variations to be considered are

$$|\delta\Phi\rangle = -i\delta\chi X|\Phi\rangle, \tag{51}$$

where X is any one of the Hermitian operators $\hat{N}_\tau$, T_x, and T_y and $\delta\chi$ is infinitesimal and real. Because the first two terms in the expression (8) are invariant under global gauge transformations and isobaric transformations,

$$\frac{\delta R_{\mathrm{HB}}}{\delta\chi} = -\mu\frac{\delta\langle T_z\rangle}{\delta\chi} = i\mu\langle Y\rangle, \quad Y = [T_z, X],$$

whence for $\zeta_{kj} = 0$, by $\partial R_{\mathrm{HB}}/\partial\zeta_{\iota\kappa} = 0$, follows

$$\begin{aligned}
&\sum_{\substack{(\mu,\lambda)\in\mathcal{S}\ \text{or}\\ (\lambda^{-1},\mu^{-1})\in\mathcal{S}}} \frac{\partial^2 R_{\mathrm{HB}}}{\partial\zeta_{\iota\kappa}\partial\zeta_{\lambda\mu}}\frac{\delta\zeta_{\lambda\mu}}{\delta\chi} \\
&\quad= \sum_{\substack{(\mu,\lambda)\in\mathcal{S}\ \text{or}\\ (\lambda^{-1},\mu^{-1})\in\mathcal{S}}} \left[\frac{\partial}{\partial\zeta_{\iota\kappa}}\left(\frac{\partial R_{\mathrm{HB}}}{\partial\zeta_{\lambda\mu}}\frac{\delta\zeta_{\lambda\mu}}{\delta\chi}\right)\right. \\
&\qquad\left. -\left(\frac{\partial}{\partial\zeta_{\iota\kappa}}\frac{\delta\zeta_{\lambda\mu}}{\delta\chi}\right)\frac{\partial R_{\mathrm{HB}}}{\partial\zeta_{\lambda\mu}}\right] \\
&\quad= \sum_{\substack{(\mu,\lambda)\in\mathcal{S}\ \text{or}\\ (\lambda^{-1},\mu^{-1})\in\mathcal{S}}} \frac{\partial}{\partial\zeta_{\iota\kappa}}\left(\frac{\partial R_{\mathrm{HB}}}{\partial\zeta_{\lambda\mu}}\frac{\delta\zeta_{\lambda\mu}}{\delta\chi}\right) \\
&\quad= \frac{\partial}{\partial\zeta_{\iota\kappa}}\frac{\delta R_{\mathrm{HB}}}{\delta\chi} = i\mu\frac{\partial\langle Y\rangle}{\partial\zeta_{\iota\kappa}} = i\mu\big(Y\big)_{\iota\kappa}.
\end{aligned} \tag{52}$$

For $\zeta_{kj} = 0$, Eqs. (50) and (51) give

$$\begin{aligned}
\delta\zeta_{kj} &= \langle\Phi|\alpha_k\alpha_j|\delta\Phi\rangle = -i\delta\chi\langle\alpha_k\alpha_j X\rangle \\
&= -i\delta\chi\big(X\big)_{j^{-1}k^{-1}}, \\
\delta\zeta_{j^{-1}k^{-1}} &= \delta\zeta^*_{kj} = i\delta\chi\langle X\alpha^\dagger_j\alpha^\dagger_k\rangle = i\delta\chi\big(X\big)_{kj}.
\end{aligned}$$

Eq. (52) can therefore be written

$$(\mathsf{E}+\mathsf{V})\begin{pmatrix}0 & -\mathsf{i}\\ \mathsf{i} & 0\end{pmatrix}\mathsf{X} = i\mu\mathsf{Y},$$

or

$$\mathsf{R}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\mathsf{X} = -\mu\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\mathsf{Y}.$$

Explicitly, these relations are

$$\begin{aligned}
&\mathsf{R}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\hat{\mathsf{N}}_\tau = 0, \\
&\mathsf{R}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\mathsf{T}_\mathsf{x} = -i\mu\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\mathsf{T}_\mathsf{y}, \\
&\mathsf{R}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\mathsf{T}_\mathsf{y} = i\mu\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\mathsf{T}_\mathsf{x}.
\end{aligned} \tag{53}$$

The column matrices $\hat{\mathsf{N}}_\tau$ and $\mathsf{T}_\mp$ do not vanish because $|\Phi\rangle$ is not an eigenstate of $\hat{N}_\tau$ or $T_\pm$. Therefore $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\hat{\mathsf{N}}_\tau$ and $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\mathsf{T}_\mp$ are right eigenvectors of R with the eigenvalues zero and $\pm\mu$. They belong to the $\tau\tau$ and np spaces, respectively. For $\mu = 0$, one has $\mathsf{T}_\mathsf{y} = 0$ because $\langle\boldsymbol{P}\rangle$ points into the y direction so that $|\Phi\rangle$ is an eigenvector of T_y. The column matrices $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\mathsf{T}_\mp$ then merge into a single right eigenvector $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\mathsf{T}_\mathsf{x}$ of R with the eigenvalue zero.

For $\kappa = 0$, the column matrices in the common right kernel of R and K found here must be those whose existence was proved in Sec. III G. It is now seen that they persist for $\kappa \neq 0$. It follows from the discussion in Sec. III G that they span the kernel. The derivatives $\partial^2 R_{\mathrm{HB}}/\partial\zeta^*_{\iota\kappa}\partial\zeta_{\lambda\mu}$ are for $\zeta_{kj} = 0$ the elements of K. Because it has now been proved that K is positive semidefinite and its entire kernel stems from symmetries of R_{HB}, it follows that $|\Phi\rangle$ minimizes R_{HB} locally. For a valence space consisting of a single j shell, and an energy functional without the symmetry force and including the complete expectation value of the pairing force, Camiz, Covello, and Jean [48] prove that a state with annihilators of the form (13) minimizes this functional globally on a certain class of Bogolyubov quasinucleon vacua with contant $\langle\hat{N}_\tau\rangle$. Using their method, one can easily prove that $|\Phi\rangle$ minimizes E_{HB} globally on the same class of Bogolyubov quasinucleon vacua.

I promised in Sec. III C to prove that a quasinucleon vacuum annihilated by an operator $\alpha^\dagger_j$ and obeying Eq. (11) cannot minimize R_{HB} locally. Assume, to the contrary, that $|\Phi\rangle$ is annihilated by an operator $\alpha^\dagger_j$. To satisfy Eq. (11), the number of such operators must be even for each kind of nucleon, so for one kind of nucleon there must be at least two such operators, $\alpha^\dagger_{q\tau}$ and $\alpha^\dagger_{q'\tau}$, say. These operators replace $\alpha_{q\tau}$ and $\alpha_{q'\tau}$ in Eq. (50). Matrices E and V can be defined as the Hessian matrices of $\langle R_0\rangle$ and $R_{\mathrm{HB}} - \langle R_0\rangle$ with respect to $\zeta_{\iota\kappa}$ for $\zeta_{kj} = 0$. In particular,

$$\begin{aligned}
\mathsf{E}_{(q\tau)^{-1}(q'\tau)^{-1},q'\tau q\tau} &= \mathsf{E}_{q'\tau q\tau,(q\tau)^{-1}(q'\tau)^{-1}} \\
&= -E_{q\tau} - E_{q'\tau} < 0,
\end{aligned}$$

while V is still given by an expression in the form of Eq. (40) with real P_τ, P_z, $\mathsf{T}_\pm$, and T_z. Now, let a column matrix x be defined by

$$\mathsf{x}_{q'\tau q\tau} = -\mathsf{x}_{(q\tau)^{-1}(q'\tau)^{-1}} = 1, \quad \mathsf{x}_{\iota\kappa} = 0 \quad \text{otherwise}.$$

Because T_z is Hermitian, $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\mathsf{T}_z = \mathsf{T}_z$, so

$$\mathsf{T}^T_\pm\mathsf{x} = \mathsf{T}^T_z\mathsf{x} = 0.$$

The contribution to $\mathsf{x}^T\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\mathsf{V}\mathsf{x}$ from the symmetry force therefore vanishes. The contribution from the pairing force is nonpositive. With K given by Eq. (30), then

$$\mathsf{x}^T\mathsf{K}\mathsf{x} < 0,$$

so K is not positive semidefinite and $|\Phi\rangle$ does not minimize $R_{\rm HB}$ locally.

The eigenvalues zero of R in the $\tau\tau$ spaces can be understood to result from the ground state of $\hat{R}$ being an eigenstate of $\hat{N}_\tau$. The similar applies to the eigenvalues zero in the np space for $\mu = 0$. For $\mu > 0$, the eigenvalues $\pm\mu$ can be understood to originate in the $M_T = T - 1$ isobaric analog of the ground state, which has the excitation energy μ as an eigenstate of $\hat{R}$. This interpretation differs from that of Ginoccio and Wesener [11], who, in the quasiboson picture, find that the vibrational quantum in question increases T by one unit. They infer this from the commutation relation, which they do not prove, written in the line after their Eq. (88b).

In the oscillator interpretation of the expression (39), the term $\frac{1}{2}\mu$ arises from the quantal fluctuations of the variables $T_\pm$, which are in the approximation $[T_+, T_-] = 2T_z \approx 2T$ proportional to conjugate coordinates. In an eigenstate of $\boldsymbol{T}^2$ and T_z, these fluctuations are determined by the isospin algebra. Assume, in accordance with the isorotational picture, that intrinsic classical variables, invariant under isorotation, could be defined so that the Hamiltonian is a function of $|\boldsymbol{T}|$ and these intrinsic variables, and let $E_{\rm col}(|\boldsymbol{T}|)$, where the subscript col stands for collective, be the energy of the intrinsic equilibrium for given $|\boldsymbol{T}|$, where $\boldsymbol{T}$ is a "classical" isospin. The term $\frac{1}{2}\mu$ in Eq. (39) then correponds to the following estimate of the energy due to the quantal fluctuations of $\boldsymbol{T}$.

$$E_{\rm col}[\sqrt{T(T+1)}] - E_{\rm col}(T) \approx \frac{1}{2}\frac{\Delta E_{\rm col}(T)}{\Delta T} \approx \frac{1}{2}\frac{\Delta E}{\Delta T} \approx \frac{1}{2}\mu,$$

where the differences are taken between the discrete allowed values of T and the T dependence of the total zero point energy is neglected in the second approximation. A related interpretation of the a term involving the angular velocity that emerges analogously in models with rotational invariance is discussed by Marshalek [44] in a boson expansion picture.

Because $\mathsf{T_z} = \frac{1}{2}(\hat{\mathsf{N}}_{\rm n} - \hat{\mathsf{N}}_{\rm p})$ and

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\mathsf{T}_\pm = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathsf{T}_\mp = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\mathsf{T}_\mp,$$
$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\mathsf{T}_z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathsf{T}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\mathsf{T}_z,$$

it follows from Eq. (32) that any other right eigenvector of R than those mentioned so far is orthogonal to the components of T. As seen from Eqs. (28) and (40), it then belongs to the kernel of the symmetry-force term in Eq. (40). Therefore, $\pm\mu$ are the only eigenvalues of R that depend on κ, and the right eigenvectors of R are independent of κ. Because, as seen from Eqs. (14), (15), (16), and (17), λ_τ and Δ_τ are independent of κ, it hence follows that it is sufficient to determine the eigenvalues of R for $\kappa = 0$, in which case the nn and pp spaces separate. The symmetry force is then taken into account by adding the term κT in Eq. (18).

This is a remarkable result. Its interpretation in the isorotational picture is that the symmetry force does not influence the intrinsic excitations. It contributes only to the collective energy. This is consistent with the fact, following from the remark after Eq. (6), that the symmetry force differs from $\frac{1}{2}\kappa\boldsymbol{T}^2$ only by a constant.

From Eqs. (18), (19), (35), and (39) follow

$$E_{\rm RPA} = E_{{\rm RPA},\kappa=0} + \frac{1}{2}\kappa T, \tag{54}$$

$$E = E_{\rm HB} + c + E_{\rm RPA} = E_{\kappa=0} + \frac{1}{2}\kappa[T(T+1) - \frac{3}{4}A_{\rm v}]. \tag{55}$$

Using Eq. (45), $\mathsf{T_z} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathsf{T_z}$, and the fact that $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\mathsf{T_z}$ belongs to the kernel of $\mathsf{E} + \mathsf{V}$, one can easily prove that M' defined by Eq. (49) is independent of κ in the $nn + pp$ space. Thus, its elements between the nn and pp spaces vanish.

I. Case $\Delta_\tau = 0$

Most of the preceding discussion applies for $\Delta_\tau = 0$. The Fermi level λ_τ can be placed, in this case, anywhere between such a pair of single-nucleon levels that $N_\tau/2$ levels ϵ_q satisfy $\epsilon_q < \lambda_\tau$. The pairing energy $E_{\rm pair}$ vanishes according to Eq. (21), and one gets from Eqs. (19) and (20)

$$E_{\rm HB} = 2\sum_\tau \sum_{\epsilon_q<\lambda_\tau} \epsilon_q + \frac{1}{2}\kappa T^2. \tag{56}$$

The column matrices N_τ vanish, so R has no eigenvalue zero in the $nn + pp$ space. The brackets $(T_-)_{\bar{q}q+}$ also vanish, and $(T_-)_{q^{-1}\bar{q}^{-1}+}$ is equal to $\sqrt{2}$ if $\lambda_p < \epsilon_q < \lambda_n$ and zero otherwise. For $T = 0$, the column matrix $\mathsf{T_x}$ then vanishes because no level ϵ_q satisfies $\lambda_p < \epsilon_q < \lambda_n$, so R has neither in the np space an eigenvalue zero. For $T > 0$, consider the part of the np space spanned by the operators $[\alpha^\dagger\alpha^\dagger]_{q\bar{q}\pm}$ and $[\alpha\alpha]_{\bar{q}q\pm}$ with $\lambda_p < \epsilon_q < \lambda_n$. In this subspace, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\mathsf{T}_-$ has the elements $\sqrt{2}$ in its upper part and zero in its lower part. From $(P_z)_{\bar{q}q+} = (P_z)_{q^{-1}\bar{q}^{-1}+} = 0$ follows $\mathsf{R} = \begin{pmatrix} \mathsf{E} & 0 \\ 0 & -\mathsf{E} \end{pmatrix}$ for $\kappa = 0$, where the diagonal elements of E are $E_{qn} + E_{qp} = (\lambda_{\rm n} - \epsilon_{\rm q}) + (\epsilon_{\rm q} - \lambda_{\rm p}) = \lambda_n - \lambda_p$. Thus $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\mathsf{T}_-$ is for $\kappa = 0$ an eigenvector of R with the eigenvalue $\lambda_n - \lambda_p = \mu$. This is the only positive eigenvalue of R that depends on κ. For $\kappa > 0$, it becomes $\lambda_n - \lambda_p + \kappa T = \mu$.

For $G = \kappa = 0$ one has $E_{\rm RPA} = 0$. Generally for $G = 0$, Eq. (54) then gives

$$E_{\rm RPA} = \frac{1}{2}\kappa T. \tag{57}$$

Equations (35), (55), (56), and (57) imply Eq. (6), so the RPA is exact in this case. For $G > 0$, the pairing force gives an additional contribution to $E_{\rm RPA}$.

J. Case of particle-hole symmetry

Special phemonema occur when the single-nucleon spectrum is symmetric about some energy and the valence space is halfway filled. For brevity, I call this symmetry of the

single-nucleon spectrum a *particle-hole symmetry*. Without loss of generality, the spectrum can be assumed, in this case, to be centered at zero and labeled so that $\epsilon_q = -\epsilon_{-q} > 0$ for $q > 0$. Then, $\lambda = 0$, $\lambda_p = -\lambda_n$, $\Delta_p = \Delta_n$, $E_{qp} = E_{(-q)n}$, $u_{qp} = v_{(-q)n}$, $v_{qp} = u_{(-q)n}$, and M' defined by Eq. (49) has equal eigenvalues in the nn and pp spaces.

In the np space, the eigenvalues of M' turn out to have the multiplicity 2 with the exception of the eigenvalue μ^2 found in Sec. III H and an eigenvalue $4(\lambda_n^2 + \Delta_n^2)$, both of which are for $\kappa = 0$ less than all $(E_{qn} + E_{qp})^2$ and less than all other eigenvalues of M'. To see how this happens, first consider the case $\kappa = 0$. Let $\mathsf{a}_{\tau\tau'} \pm \mathsf{b}_{\tau\tau'}$ be divided into blocks corresponding to an ordering of the subscripts q by their signs with $q > 0$ first and a common order of $|q|$ for each sign. Equations (47) and (48) then give

$$\begin{aligned}
&\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}(\mathsf{a}_{np} + \mathsf{b}_{np})(\mathsf{a}_{np} - \mathsf{b}_{np})\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\
&\quad = \begin{pmatrix} \mathsf{e} & 0 \\ 0 & \mathsf{e} - 2G\mathsf{p}_-\mathsf{p}_-^{\mathsf{T}} \end{pmatrix}\begin{pmatrix} \mathsf{e} - 2G\mathsf{p}_+\mathsf{p}_+^{\mathsf{T}} & 0 \\ 0 & \mathsf{e} \end{pmatrix} \\
&\quad \sim \begin{pmatrix} \mathsf{e}^2 - 2G\mathsf{e}^{1/2}\mathsf{p}_+\mathsf{p}_+^{\mathsf{T}}\mathsf{e}^{1/2} & 0 \\ 0 & \mathsf{e}^2 - 2G\mathsf{e}^{1/2}\mathsf{p}_-\mathsf{p}_-^{\mathsf{T}}\mathsf{e}^{1/2} \end{pmatrix},
\end{aligned}$$

where the similarity is effected by the matrix $\begin{pmatrix} \mathsf{e}^{-1/2} & 0 \\ 0 & \mathsf{e}^{1/2} \end{pmatrix}$. The matrices e and $\mathsf{p}_\pm$ are the upper left and upper parts of e_{np} and $\mathsf{p}_{np\pm}$. From Eqs. (14), (15), and (48), one gets by some algebra

$$(E_{qn} + E_{qp})p_{qnp\pm}^2 = \frac{1}{4}\left(\frac{1}{E_{qn}} + \frac{1}{E_{qp}}\right)[(E_{qn} + E_{qp})^2 - \Lambda_\pm] \tag{58}$$

with

$$\Lambda_+ = 4\lambda_n^2, \quad \Lambda_- = 4\left(\lambda_n^2 + \Delta_n^2\right). \tag{59}$$

It follows from Eq. (16) that the column matrix s with the dimension of $\mathsf{p}_{np\pm}$ and the element

$$s_q = \sqrt{\frac{1}{2}G\left(\frac{1}{E_{qn}} + \frac{1}{E_{qp}}\right)} \tag{60}$$

is a unit vector. From Eq. (14) one can derive $u_{qn}u_{qp} > v_{qn}v_{qp}$ for $q > 0$. It then follows from Eq. (48) that the elements of $\mathsf{p}_\pm$ are positive. It follows from Eq. (58) that $\Lambda_\pm$ are then less than all $(E_{qn} + E_{qp})^2$, so the diagonal matrices

$$\mathsf{f}_\pm = \mathsf{e}^2 - \Lambda_\pm \tag{61}$$

are positive definite and the positive eigenvalues of

$$\mathsf{g} = (1 - \mathsf{s}\mathsf{s}^T)\mathsf{e}^2(1 - \mathsf{s}\mathsf{s}^T)$$

are greater than $\Lambda_\pm$.

From Eqs. (58), (60), and (61), one gets

$$\begin{aligned}
\mathsf{e}^2 - 2G\mathsf{e}^{1/2}\mathsf{p}_\pm\mathsf{p}_\pm^T\mathsf{e}^{1/2} &= \mathsf{f}_\pm - \mathsf{f}_\pm^{1/2}\mathsf{s}\mathsf{s}^T\mathsf{f}_\pm^{1/2} + \Lambda_\pm \\
&\sim \mathsf{f}_\pm(1 - \mathsf{s}\mathsf{s}^T) + \Lambda_\pm = \mathsf{h}_\pm.
\end{aligned}$$

It is seen that s is an eigenvector of $\mathsf{h}_\pm$ with the eigenvalue $\Lambda_\pm$. Furthermore, if $\omega^2 > 0$ is an eigenvalue of g and x the corresponding eigenvector, then

$$\mathsf{x} - \frac{1}{\omega^2 - \Lambda_\pm}\mathsf{s}\mathsf{s}^T\mathsf{e}^2\mathsf{x}$$

is an eigenvector of $\mathsf{h}_\pm$ with this eigenvalue. The matrices $\mathsf{h}_\pm$ thus have these eigenvalues in common. The eigenvalue $\Lambda_+ = (2\lambda_n)^2 = \mu^2$ of M' is just that discussed in Sec. III H. It is the only eigenvalue of M' that depends on κ. For $\kappa > 0$, it becomes $(2\lambda_n + \kappa T)^2 = \mu^2$. The eigenvalues ω^2 of $\mathsf{h}_\pm$ are seen to satisfy

$$\mathsf{p}_\pm^T\frac{2\mathsf{e}}{\mathsf{e}^2 - \omega^2}\mathsf{p}_\pm = \frac{1}{G}. \tag{62}$$

Thus, M' has in the np space one twofold degenerate eigenvalue in each interval between consecutive squares of $E_{qn} + E_{qp}$, $q > 0$.

For $T = 0$, the eigenvalues of M' in a $\tau\tau$ space and their multiplicities are the same as in the np space. The eigenvalues with the multiplicity one are zero and $4\Delta_n^2$ in this case. It follows that, more generally, if the energy levels of the nucleons with isospin m_t are symmetrically distributed about λ_τ, then the eigenvalues of M' in the $\tau\tau$ space have the multiplicity two with the exception of eigenvalues zero and $4\Delta_\tau^2$, which have the multiplicity one. This was proved previously for $\kappa = 0$ by Högaasen-Feldman [49] and Bès and Broglia [50].

IV. CALCULATIONS

A. Equidistant single-nucleon levels

An insight into the general behavior of the model is obtained by studying the idealized case of infinitely many equidistant single-nucleon levels. This is the topic of the present section. More precisely, I assume that the system has d valence single-nucleon levels ϵ_q, spaced by a constant D and symmetrically distributed about zero, and that $A_v = 2d$. The discussion in Sec. III J then applies, and I denote by Δ the value of Δ_n for $T = 0$. This system is considered in the limit $d \to \infty$ with D, Δ, and κ kept fixed in the limit. Appropriate values of D, Δ, and κ can be derived from empirical formulas in the literature. Thus, $D = 4/l(6a/\pi^2)$ with $a = 0.176A(1 - 1.0A^{-1/3})$ MeV^{-1} according to Kataria, Ramamurthy, and Kapoor [51]. An empirical formula of Bohr and Mottelson [3] for the odd-even mass difference gives $\Delta = 12A^{-1/2}$ MeV. The constant $(D + \kappa)/2$ will be seen to be close to the coefficient of $T(T + 1)$ in a mass formula. According to an empirical mass formula by Duflo and Zucker [52], then $D + \kappa = 2(134.4A^{-1} - 203.6A^{-4/3})$ MeV. Table I shows, for

TABLE I. Parameters and results of the calculations with infinitely many equidistant single-nucleon levels.

A	D (MeV)	Δ/D	κ/D	a	b	x
48	1.1	1.6	2.0	.873	4.08	1.010
100	.5	2.5	2.8	.859	5.74	1.008
240	.2	4.2	3.6	.842	8.71	1.005

some mass numbers A, the values of D, Δ/D, and κ/D given by these expressions.

Because the dependence of E on κ is trivial according to Eq. (55), I assume for now $\kappa = 0$. In units of D, the energy E is then a function of Δ/D and T. Some quantities mentioned in the following are infinite in the limit $d \to \infty$. When such quantities are said to obey some relation, this is meant to be be arbitrarily accurately true when d is sufficiently large. I have checked that these statements hold very accurately in calculations for $d = 500$ and $\Delta/D = 2.5$.

The Hartree-Bogolyubov energy $E_{\rm HB}$ is according to Eq. (19) the sum of E_0 and $E_{\rm pair}$, both of which are infinite in the limit $d \to \infty$. In this limit, the only change in the solution of Eq. (16) and (17) when N_τ changes by two units is a change of λ_τ by the level distance D. Therefore, $\Delta_n = \Delta$ independently of T, so $E_{\rm pair}$ does not contribute to the symmetry energy. However, E_0 does so. This is because the isospin T is produced by the promotion of T nucleons from proton levels with the average energy $\lambda_p/2$ to neutron levels with the average energy $\lambda_n/2$. The result is an increase

$$E_0 - E_{0,T=0} = T\frac{\lambda_n - \lambda_p}{2} = \frac{1}{2}DT^2,$$

where

$$\frac{\lambda_n - \lambda_p}{2} = \lambda_n = -\lambda_p = T\frac{D}{2} = \frac{1}{2}DT \tag{63}$$

has been used.

Equations (39), (59), and (63) give

$$E_{\rm RPA} = E_{nn} + E_{np} + \frac{1}{2}DT + \sqrt{\left(\frac{1}{2}DT\right)^2 + \Delta^2}, \tag{64}$$

$$E_{nn} = \sum_n \omega_{nn,n} - 2\sum_q E_{qn},$$

$$E_{np} = \sum_n \omega_{np,n} - \sum_q E_{qn},$$

where $\omega_{nn,n}$ are the positive eigenvalues of R in the nn or pp space and $\omega_{np,n}$ the twofold degenerate, positive eigenvalues of R in the np space. By adding the third term in the expression (64) to E_0, one gets a contribution to the symmetry energy proportional to $T(T+1)$,

$$E_0 - E_{0,T=0} + \tfrac{1}{2}DT = \tfrac{1}{2}DT(T+1).$$

The energies E_{nn} and E_{np} are infinite in the limit $d \to \infty$. Because the quasinucleon energies E_{qn} and the elements of R in the nn space are just relabeled when N_n is changed by two units, E_{nn} is independent of T. However, E_{np} gives a nonzero contribution to the symmetry energy. This is demonstrated in Fig. 1 by a calculation for $\Delta/D = 2.5$ and $d = 1000$. A calculation for $d = 500$ gives a curve that cannot be distinguished from this one. The limit $d \to \infty$ is thus realized in practice for d of this order.

The decrease of E_{np} with increasing T is understood from the fact noticed in connection with Eq. (62) that the frequencies $\omega_{np,n}$ lie between consecutive two-quasinucleon energies $E_{qn} + E_{qp}$, $q > 0$. The lowest two-quasinucleon energy increases with T, whereas when $E_{qn} + E_{qp}$ is sufficiently large, $E_{qn} + E_{qp} \approx (\epsilon_q - \lambda_n) + (\epsilon_q - \lambda_p) = 2\epsilon_q$, which is

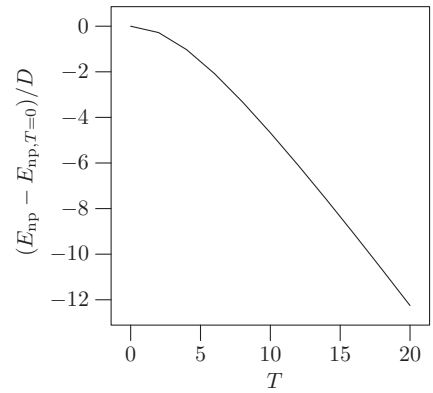


FIG. 1. The contribution of E_{np} to the symmetry energy in units of D for $\Delta/D = 2.5$ and $d = 1000$.

a constant. Therefore the difference between the sum of $\omega_{np,n}$ and the sum of $E_{qn} + E_{qp}$, $q > 0$, that is, $\sum_q E_{qn}$, decreases. The sum $E_{np} + \sqrt{\Lambda_+}$ increases because all the positive roots ω of Eq. (62), including $\sqrt{\Lambda_+}$, and all the two-quasinucleon energies are bounded below by $\sqrt{\Lambda_+}$, which increases. Therefore also $E_{\rm RPA}$, which differs from the average of $E_{np} + \sqrt{\Lambda_+}$ only by the constant E_{nn}, increases.

The calculation of E_{np} for $\Delta/D = 2.5$ and $d = 1000$ was extended to larger T than shown in Fig. 1, up to $T = 100$. The entire set of results is well described by the expression

$$E_{np} - E_{np,T=0} = -D[\sqrt{(aT)^2 + b^2} - b],$$
$$a = .859, \quad b = 5.74.$$

In the view of Fig. 1, this gives a curve which can hardly be distinguished from the calculated one. Similar results are obtained for $\Delta/D = 1.6$ and $\Delta/D = 4.2$ with a and b given in Table I.

Now collecting all the contributions discussed above and adding the last term in Eq. (55), one gets

$$E - E_{T=0} = \frac{1}{2}(D+\kappa)T(T+1) - D\left[\sqrt{(aT)^2 + b^2} - \sqrt{\left(\frac{T}{2}\right)^2 + \left(\frac{\Delta}{D}\right)^2} - b + \frac{\Delta}{D}\right]. \tag{65}$$

For the paramters in Table I, the term subtracted from $\frac{1}{2}(D+\kappa)T(T+1)$ in this expression is positive for $T > 0$. It amounts to 11% of $\frac{1}{2}(D+\kappa)T$ for $A = 48$ and $T = 8$ (^{48}S), 10% for $A = 100$ and $T = 14$ (^{100}Kr), and 9% for $A = 240$ and $T = 28$ (^{240}U). The linear term $\frac{1}{2}(D+\kappa)T$ is thus the dominant correction to the quadratic term $\frac{1}{2}(D+\kappa)T^2$. The sign of the additional term in Eq. (65) is, however, consistent with the experience, mentioned in Sec. II, that in global fits to the empirical masses with formulas that include a term proportional to $T(T+x)$, the constant x tends to be somewhat

less than 1. For $T \approx 0$, Eq. (65) gives

$$E - E_{T=0} \approx \frac{1}{2}(D+\kappa)T(T+1) - D\left(\frac{a^2}{2b} - \frac{D}{8\Delta}\right)T^2$$
$$= \frac{1}{2}\left[D\left(1 - \frac{a^2}{b} + \frac{D}{4\Delta}\right) + \kappa\right]T(T+x),$$
$$x = \left(1 - \frac{\frac{a^2}{b} - \frac{D}{4\Delta}}{1 + \frac{\kappa}{D}}\right)^{-1}. \qquad (66)$$

Table I shows x calculated from Eq. (66) for the parameters in the table. It is seen that $x \approx 1.01$ in these cases.

As discussed in Sec. III I, the RPA gives the exact energy (6) for $G = 0$. The symmetry energy derived from this expression in the present case is

$$E - E_{T=0} = \tfrac{1}{2}[(D+\kappa)T^2 + \kappa T], \qquad (67)$$

without the term $\frac{1}{2}DT$. The spontaneous breaking of the isobaric invariance by the pairing force is thus required for this term to appear.

B. Comparison with Skyrme force models

Nuclear masses are often compared with Hartree-Fock (HF), Hartree-Fock-BCS (HFBCS), or Hartree-Fock-Bogolyubov (HFB) calculations with phenomenological energy functionals based on Skyrme forces. The HFB method is described in detail in a recent article by Chamel, Goriely, and Pearson [53], where also references to earlier work in this line of research are found. Applying the HF, HFBCS or HFB scheme in an approximate way to the present Hamiltonian sheds a light on the origin of certain phenomena observed in the Skyrme force calculations. In the formalism of Ref. [53], a pairing and a particle-hole part of the two-nucleon interaction are treated differently, as in the present theory. For the present pairing interaction, and $\langle P_z \rangle = 0$, there is then no difference between the HFBCS and HFB schemes.

Apart from the different interactions, the only difference between the formalism of Sec. III C and the HFB formalism of Ref. [53] is that the particle-hole matrix element is antisymmetrized in the latter. In the formalism of Sec. III C, this amounts to replacing $R_{\rm HB}$ with

$$R_{\rm HB} - \tfrac{1}{2}\kappa \,\mathrm{tr}\, \boldsymbol{t}\cdot\rho\boldsymbol{t}\rho, \quad \langle j|\rho|k\rangle = \langle a_k^\dagger a_j\rangle. \qquad (68)$$

The resulting Routhian is stationary at the quasinucleon vacuum $|\Phi\rangle$ given by Eqs. (7), (13), (14), (15), (16), and (17) when ϵ_q is replaced with $\epsilon_q - \frac{1}{4}\kappa(v^2_{q\tau} + 2v^2_{q\tau'})$, $\tau' \neq \tau$. I neglect this modification of the self-consistent single-nucleon energy and calculate the second term in Eq. (68), the "Fock term," with $u_{q\tau}$ and $v_{q\tau}$ given by Eqs. (14) and (15). The

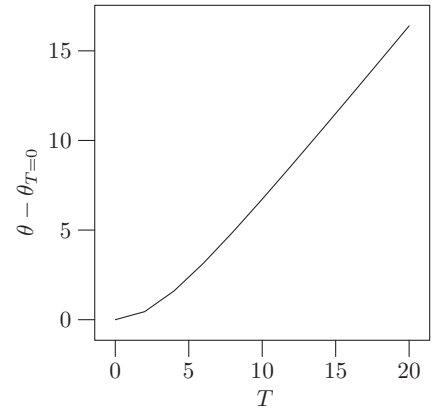


FIG. 2. The quantity $\theta - \theta_{T=0}$ as a function of T for $d = 1000$ and $\Delta/D = 2.5$.

essential factor is

$$\mathrm{tr}\, \boldsymbol{t}\cdot\rho\boldsymbol{t}\rho$$
$$= \sum_q \left[\frac{1}{2}\left(v_{qn}^4 + v_{qp}^4\right) + 2v_{qn}^2 v_{qpE}^2\right]$$
$$= \frac{3}{4}A_{\rm v} - \sum_q \left[\frac{1}{2}\left(u_{qn}^2 v_{qn}^2 + u_{qp}^2 v_{qp}^2\right) + u_{qn}^2 v_{qp}^2 + u_{qp}^2 v_{qn}^2\right]. \qquad (69)$$

The first term in this expression gives the second term in the outer bracket in Eq. (35), and the second term a part of the first of the diagrams (36). In the case of infinitely many equidistant single-nucleon levels, the only part of the expression (69) that depends on T is

$$\theta = \sum_q \left(u_{qn}^2 v_{qp}^2 + u_{qp}^2 v_{qn}^2\right). \qquad (70)$$

The quantity $\theta - \theta_{T=0}$ is shown in Fig. 2 as a function of T for $d = 1000$ and $\Delta/D = 2.5$. It is seen that the asymtotic slope is 1. This is easily understood from Eq. (70). Indeed, for $T \gg 1$ the first term in the bracket is practically zero, while the second term is close to 1 for $\lambda_p < \epsilon_q < \lambda_n$ and close to zero otherwise. Hence, for $T \gg 1$, the contribution of the Fock term to the symmetry energy deviates from the linear term in the expression (67) only by a constant. For $T \approx 0$, it is, however, quadratic in T because θ is by Eqs. (14), (15), (63), and (70) analytic and even as a function of T. Therefore, the HFB scheme does not produce any cusp at $N = Z$ in the curve of masses along an isobaric chain. The study by Satuła and Wyss [22] shows that this is true also in HFBCS calculations with Skyrme forces, and it is true in the HFB calculations discussed in Ref. [53], where, to reproduce the empirical masses, a so-called Wigner correction is added to the HFB energy.

HF calculations with Skyrme forces correpond to the case $G = 0$ of the present theory. I this case, the Fock term is equal to $\frac{1}{2}\kappa(T - \frac{3}{4}A_{\rm v})$, so the exact energy (6) is recovered. With infinitely many equidistant single-nucleon levels, the symmetry energy is given by Eq. (67) and thus includes the linear term $\frac{1}{2}\kappa T$. Satuła and Wyss [22] find that such a

term appears also in HF calculations with Skyrme forces. As in the present theory, there is in these calculations no term corresponding to $\frac{1}{2}DT$. The present schematic model thus seems representative of the Skyrme force models as to the basic mechanisms responsible for the absense or presense of such linear terms of various origins.

C. Deformed Woods-Saxon single-nucleon levels

Moving now from the idealized case of a uniform single-nucleon spectrum to assumptions closer to the reality, I discuss in this section calculations with single-nucleon levels derived from a deformed Woods-Saxon potential. I have chosen for this study isobaric chains whose $T = 0$ members have low-lying 2^+ levels and can thus be supposed to have appreciable quadrupole deformations. For each isobaric chain, the single-nucleon energies are calculated for a deformation pertaining to the $T = 0$ nucleus. The definition of the deformed Woods-Saxon potential follows Dudek *et al.* [54]. The average nucleon mass and the average of the "universal" neutron and proton parameters of Dudek *et al.* [55] are employed, and no Coulomb potential is included. The half-depth surface of the spin-independent part of the potential is assumed to have a pure, prolate quadrupole shape with the deformation β derived by the relations of Raman, Nestor, and Tikkanen [56] from $B(E2, 0^+ \to 2^+)$, when it is known, and otherwise from the 2^+ excitation energy. All bound single-nucleon states are included in the valence space.

The pairing force contant G is determined for each isobaric chain by demanding $\Delta_n = \Delta_p = 12A^{-1/2}$ MeV for $T = 0$. Most other nuclei in the isobaric chain then acquire similar values of Δ_τ, but Δ_τ may vanish if λ_τ reaches a gap in the single-nucleon spectrum. The calculated symmetry energy turns out to be minor sensitive to the precise value of G. The constant κ is chosen so as to best fit the data. A summary of the resulting parameters is given in the first, third, and fourth rows of Table II. A somewhat disturbing feature of these parameters is the slightly irregular A dependence of the optimal κ. This irregularity may be related to the crude treatment of the shape degrees of freedom.

For comparison with the calculated results, I have derived an empirical symmetry energy from the masses compiled by Audi, Wapstra, and Thibault in their 2003 Atomic Mass Evaluation [57]. It is calculated from the binding energy for $M_T \geqslant 0$ minus the electrostatic term extracted by Kirson [29]

TABLE II. Parameters and results of the calculation with Woods-Saxon single-nucleon levels.

A	β	G (MeV)	Interval of Δ_τ (MeV)	κ (MeV)
48	0.337	0.450	1.62–1.85	1.4
56	0	0.4	0–1.94	1.1
68	0.234	0.286	0–1.65	1.5
80	0.342	0.262	1.08–1.69	1.1
100	0	.2	0–1.55	.8

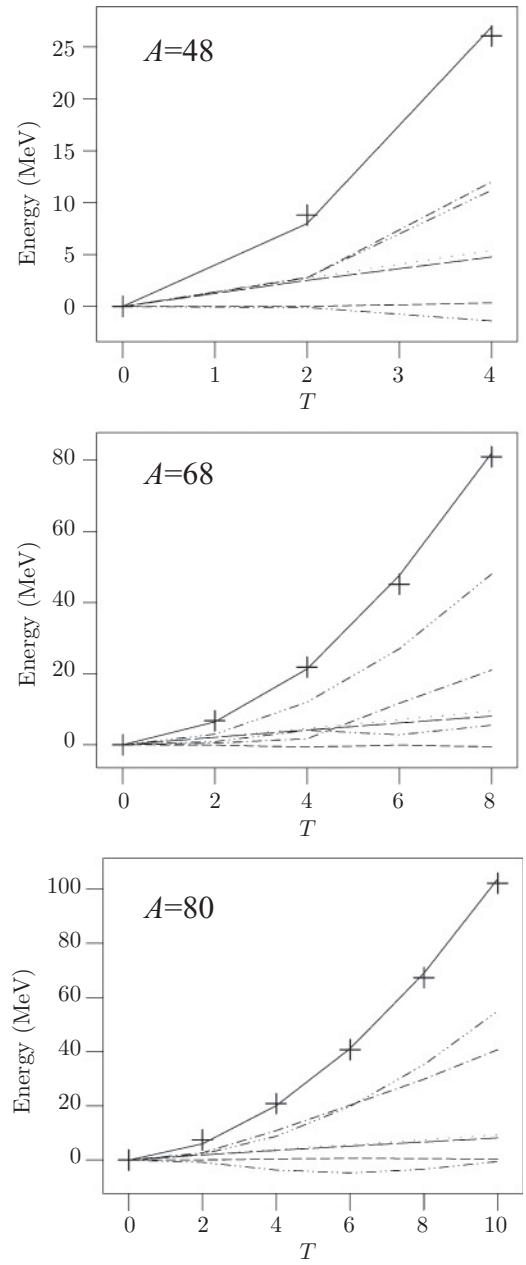


FIG. 3. Calculated (solid line) and empirical (crosses) symmetry energy for some values of the mass number A. Shown with different signatures are also the following components of the calculated symmetry energy. (Dash-dot) $E_0 - E_{0,T=0}$, (dash-dot-dot) $E_{\text{pair}} - E_{\text{pair},T=0}$, (dash-dot-dot-dot) $\frac{1}{2}\kappa T^2$, (short dash) $E_{\text{RPA},nn+pp} - E_{\text{RPA},nn+pp,T=0}$, (long dash) $E_{\text{RPA},np} - E_{\text{RPA},np,T=0}$, where $E_{\text{RPA},nn+pp}$ and $E_{\text{RPA},np}$ are the contributions to the expression (39) from the $nn + pp$ and np spaces. The dotted curve is $\frac{1}{2}\mu$.

from the differences of the binding energies of mirror nuclei,

$$\frac{-0.716Z^2 + 0.993Z^{4/3}}{A^{1/3}} \text{ MeV}.$$

The calculated and empirical symmetry energies are shown in Fig. 3. In view of the schematic character of the model, where, in particular, any variation of shape degrees of freedom is absent, the agreement is suprisingly good. Also shown in the

figure is the composition of the calculated symmetry energy. The term $\frac{1}{2}\kappa T^2$ typically makes up about half of it for large T. The contribution of the pairing energy $E_{\rm pair}$ is plus or minus a few MeV. It arises from the variation of Δ_τ along the isobaric chain. This variation is due, in turn, to the variation with λ_τ of the local single-nucleon level density. The irregularity of the single-nucleon spectrum is also responsible for a fairly irregular behavior of the contribution from E_0 as compared to its quadratic dependence on T in the idealized case.

The part of the contribution from $E_{\rm RPA}$ originating in the $nn + pp$ space is essentially zero, as in the idealized case. The part originating in the np space is dominated by the single eigenvalue $\frac{1}{2}\mu$ of R, which increases essentially linearly with T. Its deviation from $\frac{1}{2}\mu$ appears to be of higher than linear order in T and always negative, just as in the idealized case, and it amounts to about -1.5 MeV for the largest T's for which the binding energy is measured. The ratio $(E_{\rm RPA} - E_{{\rm RPA},T=0})/(E - E_{T=0})$ is close to $T/(T^2 + T) = 1/3$ for $T = 2$ in the three cases. This is understood from the following facts. (i) $E_{\rm HB} - E_{{\rm HB},\langle T_z\rangle=0} \propto \langle T_z\rangle^2$ for small $\langle T_z\rangle$ because $E_{\rm HB}$ is analytic and even as a funtion of $\langle T_z\rangle$. (ii) $E_{\rm RPA} - E_{{\rm RPA},T=0}$ is dominated by $\frac{1}{2}\mu = \frac{1}{2}dE_{\rm HB}/d\langle T_z\rangle$. (iii) $\langle T_z\rangle = T$.

D. Spherical Woods-Saxon single-nucleon levels: yet another mechanism

For $A = 56$ and $A = 100$, the $T = 0$ nucleus is doubly magic. Therefore, I have made calculations for these mass numbers with a spherical Woods-Saxon potential. I have chosen G in this case so that $\Delta_\tau \approx 12A^{-1/2}$ MeV for $T > 0$; then $\Delta_\tau = 0$ for $T = 0$. The adopted parameters are given in the second and fifth rows of Table II, and the results of the calculations are shown in Fig. 4. The valence space consists of the levels below the $N_\tau = 50$ shell closure for $A = 56$ and below the $N_\tau = 82$ shell closure for $A = 100$.

The contributions to the symmetry energy from $\frac{1}{2}\kappa T^2$ and $E_{\rm RPA}$ are similar to those of the deformed case. But, due to the increase of Δ_τ when λ_τ moves into a shell above or below the magic gap, $E_{\rm pair}$ gives in this case a significant negative contribution. Because the largest decrease of $E_{\rm pair}$ takes place for low T, $E_{\rm pair}$ then contributes to the nonlinear part of the symmetry energy. However, E_0 is practically linear. This is because the increase of E_0 results from the promotion of nucleons from proton states below the shell gap to neutron states above the gap. The slope of E_0 is thus roughly equal to the shell gap. The slope of $E_0 + E_{\rm pair}$ at low T is similar to that of $E_{\rm RPA}$ for $A = 56$ and several times that of $E_{\rm RPA}$ for $A = 100$. In both cases, it thus gives a significant contribution to the linear part of the symmetry energy. This could be expected to be true in general in isobaric chains with a doubly magic $T = 0$ nucleus.

V. EXPERIMENTAL SIGNATURE OF SUPERFLUID ISOROTATION

An experimental signature of a nuclear rotation in space is the sequence of large and approximately equal reduced $E2$ transition probabilities between consecutive members of a rotational band produced by the quadrupole deformation of the intrinsic charge distribution. The quadrupole deformation is measured by the mass or charge quadrupole tensor. In the case of superfluid isorotation, the intrinsic deformation is measured by the isovector $\boldsymbol{P}$. Because the consecutive members of an isorotational band are separated by two units of isospin, they are not connected directly by an isovector. However, as pointed out by Bohr and Mottelson [3], the superfluid isorotational bands participate in a larger structure consisting of the ground states of all the doubly even nuclei and their isobaric analog states. The ground states of the doubly even isotopes are connected by P_n, and the ground states of the doubly even isotones by P_p. The isovector component P_z connects such states with $|M_T| = T - 1$ isobaric analog states in doubly odd nuclei. The chains of superfluid isotopes or isotones in fact form the pair rotational bands discussed by Bohr [58]. Yoshida [59] shows that superfluidity enhances the ground-state-to-ground-state cross section of two-neutron or two-proton transfer between doubly even nuclei by a factor about $(2\Delta_\tau/G)^2$, where $\tau = n$ or p. Isovector one-neutron-one-proton transfer between a ground state of a doubly even nucleus and an isobaric analog state is then similarly enhanced.

The experimental signature of superfluid isorotation therefore coincides with that of pair rotation. The picture of a superfluid isorotation implies that the enhancement factors of two-nucleon transfer involving doubly even ground states

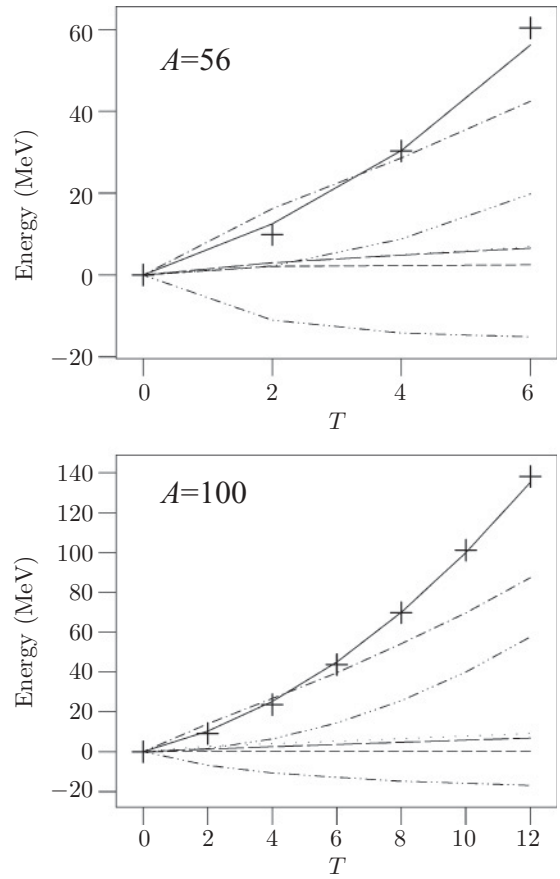


FIG. 4. See the caption to Fig. 3.

or their isobaric analogs remains approximately constant all the way down to $T = 0$. Near closed shells, the pair rotational bands may develop into the pair vibrational bands dicussed by Bohr [58], which, as well, have enhanced two-nucleon transfer cross sections. If, however, doubly even nuclei with $T = 0$ would have a structure radically different from that of doubly even nuclei with $T > 0$, a major deviation from such a smooth behavior would be seen. No such effect seems to be indicated by the enhancement factors compiled by Bès *et al.* [60].

VI. SUMMARY

A Hamiltonian with a single-nucleon term and a two-nucleon interaction was investigated. The single-nucleon Hamiltonian has fourfold degenerate eigenvalues correponding to time-reversed pairs of neutron and proton states. The two-nucleon interaction has a pairing and a particle-hole part. The paring part is the isobarically invariant isovector pairing force, and the particle-hole part an interaction of isospins, which I call the symmetry force. A Routhian was contructed by subtracting from the Hartree-Bogolyubov energy functional Lagrangian multiplier terms propotional to the expectation values of the number of valence nucleons and the z component of the isospin, and it was shown that this Routhian is locally minimized by a product of neutron and proton Bardeen-Cooper-Schrieffer [23] states.

This quasinucleon vacuum and a single-quasinucleon Routhian operator derived from the Hartree-Bogolyubov Routhian was taken as the starting point for a calculation of the ground-state energy as a function of the number of valence particles and the isospin quantum number T in the RPA. The correction to the Hartree-Bogolyubov energy is the sum of a term that does not depend on T and a term E_{RPA} equal to half the sum of the poles of the RPA two-quasinucleon propagator minus the sum of the two-quasinucleon energies. The poles of the two-quasinucleon propagator that differ from two-quasinucleon energies can be determined separately in a two-neutron, a two-proton, and a neutron-proton quasiparticle space. In each of these spaces, there is a Nambu-Goldstone pole due to the global gauge invariance and isobaric invariance of the Hamiltonian. The two-neutron and two-proton Nambu-Goldstone poles have the frequency zero, while the neutron-proton Nambu-Goldstone pole is equal to the Lagrangian multiplier of the z component of the isospin. The term in E_{RPA} resulting from this pole was interpreted in a picture of a collective rotation in isospace to be due to the quantal fluctuation of the isospin. The pole in question is the only one that depends on the strength κ of the symmetry force. The only contribution of the symmetry force to the symmetry energy is therefore a term $\frac{1}{2}\kappa T(T+1)$.

If the single-nucleon spectrum is symmetric about a certain energy and the valence space halfway filled, the neutron-proton poles are twofold degenerate except for the Nambu-Goldstone pole and one more pole, which has an analytic expression. Related results are known from the literature to pertain to the neutron-neutron and proton-proton spaces.

In an idealized case of infinitely many equidistant single-nucleon levels, the neutron and proton systems have a common pair gap Δ that does not depend on T. Therefore, the pairing force does not contribute to the symmetry energy. Neither do the two-neutron and two-proton parts of E_{RPA}. For $\kappa = 0$, the single-nucleon term in the Hamiltonian and the neutron-proton Nambu-Goldstone pole give together a contribution equal to $\frac{1}{2}DT(T+1)$, where D is the single-nucleon level spacing. The second nondegenerate neutron-proton pole gives a contribution equal to $\sqrt{(\frac{1}{2}DT)^2+\Delta^2}-\Delta$. The remainder of the contribution of the neutron-proton part of E_{RPA} was calculated numerically. In a very good approximation, it is $-D[\sqrt{(aT)^2+b^2}-b]$, where a and b are functions of Δ/D. For realistic parameters, the sum of these two terms is negative and amounts to about -10% of the linear term $\frac{1}{2}(D+\kappa)T$ for the largest T's of observed nuclei. For $T \approx 0$, they give a contribution to the symmetry energy quadratic in T that makes the symmetry energy proportional to $T(T+x)$ with $x \approx 1.01$. In the absence of the pairing force, the RPA gives the exact symmetry energy, which is $\frac{1}{2}[(D+\kappa)T^2+\kappa T]$. This expression does not have the linear term $\frac{1}{2}DT$, which thus appears only when the isobaric invariance is spontaneously broken by the pairing force.

If the matrix element of the symmetry force is antisymmetrized and the contribution of the exchange term to the self-consistent single-nucleon energy and the RPA correlations are neglected, the total contribution of the symmetry force to the symmetry energy is asymptotically for large T equal to $\frac{1}{2}\kappa T(T+1)$ plus a constant. For $T \approx 0$, it is quadratic in T. Therefore the curves of masses along isobaric chains get no cusps at $T = 0$. This corresponds to observations reported from Hartree-Fock-Bogolyubov calculations with Skyrme forces. In the absence of the pairing force, the exact symmetry energy without the linear term $\frac{1}{2}DT$ is recovered also in this approximation. This correponds to observations reported from Skyrme force Hartree-Fock calculations.

Calculations with Woods-Saxon single-nucleon levels give results in surprisingly good agreement with the empirical variation of the binding energy of doubly even nuclei along isobaric chains. In the case of a deformed Woods-Saxon potential, the behavior of the individual components of the calculated symmetry energy is similar to their behavor in the idealized case. In calculations for $A = 56$ and $A = 100$ with spherical Woods-Saxon levels, the promotion of nucleons across the $N = Z = 28$ and $N = Z = 50$ gaps in the single-nucleon spectrum gives a large linear contribution. Due to the onset of superfluidity when the neutron and proton Fermi levels move into the shells around these gaps, the pairing force gives another large contibution in this case. Together, these two contributions give a linear term that is comparable to or larger than that of E_{RPA}.

In a picture of a collective rotation in isospace, the intrinsic deformation is measured by the pair annihilation isovector. The isorotation can therefore be characterized as superfluid. The pair annihilation isovector does not connect directly consecutive members of a superfluid isorotational band, which differ by two units of isospin. However, the superfluid isorotational bands participate in a larger structure that includes the pair rotational and pair vibrational bands discussed in the literature. Within these bands, the cross sections for two-nucleon transfer

are enhanced by the superfluid correlations. The picture of a superfluid isorotation implies that the enhancement factors of isovector two-nucleon transfer should develop smoothly down to $T = 0$. This seems to be consistent with the empirical evidence.

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APPENDIX: WIGNER ARGUMENT FOR THE ISOBARIC SU(2)

Consider a system of valence nucleons and a two-nucleon interaction $\sum v$, where the summation is over pairs of nucleons, and let P_s and P_a project to the spaces of two-nucleon states symmetric and antisymmetric in position and spin. In any state of the system,

$$\left\langle \sum v \right\rangle = \left\langle \sum (P_\mathrm{s} + P_\mathrm{a}) v \right\rangle = \frac{\langle \sum P_\mathrm{s} v \rangle}{\langle \sum P_\mathrm{s} \rangle} \left\langle \sum P_\mathrm{s} \right\rangle + \frac{\langle \sum P_\mathrm{a} v \rangle}{\langle \sum P_\mathrm{a} \rangle} \left\langle \sum P_\mathrm{a} \right\rangle. \quad \text{(A1)}$$

The total antisymmetry implies $P_\mathrm{s} = \frac{1}{4} - \boldsymbol{t}_1 \cdot \boldsymbol{t}_2$, where $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ are the isospins of the two nucleons. Hence

$$\sum P_\mathrm{s} = \frac{\hat{A}_\mathrm{v}(\hat{A}_\mathrm{v} + 2)}{8} - \frac{\boldsymbol{T}^2}{2},$$

$$\sum P_\mathrm{a} = \sum (1 - P_\mathrm{s}) = \frac{3\hat{A}_\mathrm{v}(\hat{A}_\mathrm{v} - 2)}{8} + \frac{\boldsymbol{T}^2}{2}.$$

The assumption in the case of the isobaric SU(2) symmetry corresponding to Wigner's in the case of the SU(4) symmetry is that the ground-state values of $\langle \sum P_\mathrm{s} v \rangle / \langle \sum P_\mathrm{s} \rangle$ and $\langle \sum P_\mathrm{a} v \rangle / \langle \sum P_\mathrm{a} \rangle$ are smooth functions of A_v and T. If, in particular, they do not depend on T, the symmetry energy is proportional to $T(T + 1)$.

The assumption of T-independent $\langle \sum P_\mathrm{s} v \rangle / \langle \sum P_\mathrm{s} \rangle$ and $\langle \sum P_\mathrm{a} v \rangle / \langle \sum P_\mathrm{a} \rangle$ is easily seen to be invalid in the case of the isobarically invariant isovector pairing force acting in a single j shell. In this case, $v = -(2j + 1) G P_0$, where P_0 projects to two-nucleon angular momentum zero. From an expression for the eigenvalues of $(2j + 1) \sum P_0$ derived by Edmonds and Flowers [61], it follows that the lowest eigenvalue of $\sum v$ for fixed even A_v and $A_\mathrm{v}/2 + T$ is

$$G \left[T(T + 1) - \frac{A_\mathrm{v}(4j + 8 - A_\mathrm{v})}{4} \right]. \quad \text{(A2)}$$

Because a state of two nucleons from the same j shell with angular momentum zero is antisymmetric in position and spin, and v is negative semidefinite, $\langle \sum P_\mathrm{s} v \rangle = 0$ and $\langle \sum P_\mathrm{a} v \rangle < 0$ unless $\langle \sum v \rangle = 0$. Taking $\langle \sum P_\mathrm{a} v \rangle / \langle \sum P_\mathrm{a} \rangle$ to be a negative constant, one gets from Eq. (A1)

$$\left\langle \sum v \right\rangle \propto \left\langle \sum P_\mathrm{a} \right\rangle = \frac{3A_\mathrm{v}(A_\mathrm{v} - 2)}{8} + \frac{T(T + 1)}{2}$$

with a negative constant of proportionality. This expression obviously conflicts with expression (A2). It even differs from the latter by the sign of its contribution to the symmetry energy.

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