SU(3) symmetry in the triaxially deformed harmonic oscillator

Kazuko Sugawara-Tanabe,[1] Kosai Tanabe,[2,3] Akito Arima,[4] and Bruno Gruber[5]
[1]*Otsuma Women's University, Tama, Tokyo 206-8540, Japan*
[2]*Department of Physics, Saitama University, Sakura-Ku, Saitama 338-8570, Japan*
[3]*Theoretical Nuclear Physics Laboratory, RIKEN Nishina Center, Wako, Saitama 351-0198, Japan*
[4]*Science Museum, Japan Science Foundation, Tokyo 102-0091, Japan*
[5]*College of Science, Southern Illinois University, Carbondale, Illinois 62901, USA*
(Received 27 June 2009; published 9 October 2009)

An anisotropic harmonic oscillator Hamiltonian can be brought into invariant form under SU(3) transformations by applying nonlinear transformations to the oscillator bosons. The classification of the single-particle levels based on this covering group predicts magic numbers for the triaxial oscillator. It is shown that when the deformation $|\delta|$ is not too large, the physical operators are approximated by the group operators. Estimation is carried out for the alignment of orbital angular momentum in a triaxial field.

DOI: 10.1103/PhysRevC.80.044307 PACS number(s): 21.10.Pc, 21.60.Fw

I. INTRODUCTION

The eigenstates of a many-body Hamiltonian are obtained by projecting the total angular momentum, total nucleon number, parity, etc. from the Slater determinant, which consists of the products of the intrinsic single-particle wave functions. Usually, such an intrinsic single-particle state is determined self-consistently within the approximation which is due to a suitable truncation of the single-particle basis. We adopt the triaxially deformed harmonic oscillator model to approximate the intrinsic single-particle Hamiltonian instead of such self-consistent solutions. Our main aim is to find a general classification scheme for the intrinsic single-particle wave functions for the case of triaxial deformation, in order that we can distinguish which single-particle level is lower than the others, and which kind of energy degeneracy in single-particle levels exists.

We have extended Elliott's SU(3) model [1] to axially symmetric deformed shapes, namely, the superdeformed bands, the hyperdeformed bands, and the highly deformed bands [2], where the original harmonic oscillator boson is replaced by a product of new bosons according to the rational ratio which approximates a given oscillator strength. The boson transformation introduced in Ref. [2] is inappropriate for the triaxial case, since all the harmonic oscillator states are not exhausted. We encounter the difficulty that the new boson number becomes fractional, and that there appears a degeneracy of the vacuum. In the present paper, we propose a new type of nonlinear transformation in order to avoid the degeneracy of the vacuum and in order to map all the oscillator states onto an SU(3) representation. However, the projection operator has to limit the eigenvalues of the new boson numbers to physically allowed integral values. We apply our method not only to the typical case $\gamma = 30^\circ$ but also to $17^\circ \sim 20^\circ$, which has recently been investigated in Lu isotopes [3].

In Sec. II, the deformation parameters δ and γ are defined from the triaxial deformed harmonic oscillator potential. In Sec. III, new bosons for SU(3) are introduced in a general manner. In Sec. IV, the special case of $\gamma = 30^\circ$ is discussed. In Sec. V, the relationship to recent topics regarding the case $\gamma \sim 19^\circ$ is discussed. In Sec. VI, the paper is concluded.

II. THE DEFORMATION PARAMETERS

The harmonic oscillator Hamiltonian with three frequencies ω_x, ω_y, and ω_z, where x, y, and z are the principal axes in the body-fixed frame, is given by

$$H = \frac{1}{2M}\boldsymbol{p}^2 + \frac{M}{2}\left(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2\right). \tag{1}$$

The deformation parameters δ and γ are related to the three frequencies by

$$\begin{aligned}
\omega_x^2 &= \omega_0^2\left[1 + \frac{2\delta}{3}(\sqrt{3}\sin\gamma + \cos\gamma)\right],\\
\omega_y^2 &= \omega_0^2\left[1 + \frac{2\delta}{3}(-\sqrt{3}\sin\gamma + \cos\gamma)\right],\\
\omega_z^2 &= \omega_0^2\left(1 - \frac{4\delta}{3}\cos\gamma\right),
\end{aligned} \tag{2}$$

where ω_0 is the oscillator strength in the spherical limit with $\hbar\omega_0 = 41A^{-1/3}$ MeV, and A denotes the mass number. Here, we choose the Lund convention for γ. Within the region $0^\circ < \gamma < 60^\circ$ and for positive δ, the inequalities $\omega_x > \omega_y > \omega_z$ correspond to $R_x < R_y < R_z$, where R_k describes the nuclear radius in the k direction ($k = x, y, z$). On the other hand, for negative δ, the inequalities $\omega_x < \omega_y < \omega_z$ hold corresponding to $R_x > R_y > R_z$. If we assume δ to be small and take only its first order in the Nilsson potential, Eq. (2) reduces to

$$\begin{aligned}
\omega_x &= \omega_0\left[1 + \frac{\delta}{3}(\sqrt{3}\sin\gamma + \cos\gamma)\right],\\
\omega_y &= \omega_0\left[1 + \frac{\delta}{3}(-\sqrt{3}\sin\gamma + \cos\gamma)\right],\\
\omega_z &= \omega_0\left(1 - \frac{2\delta}{3}\cos\gamma\right).
\end{aligned} \tag{3}$$

The volume conservation condition $\omega_x\omega_y\omega_z = \omega_0^3$ is satisfied by Eq. (3) up to the first order in δ, and its square, i.e., ω_0^6, is satisfied by Eq. (2), also up to the first order in δ. In the present paper, we adopt the parametrization given by Eq. (3) rather than Eq. (2) as a model of triaxial deformation. Here, we remark that δ is the same as $\delta_{\rm osc}$ in Bohr-Mottelson's textbook [4].

The harmonic oscillator boson operators $c_k^\dagger$ and $c_k, k = x, y, z$, in a triaxially deformed field are defined as

$$x = -i\sqrt{\frac{\hbar}{2M\omega_x}}(c_x^\dagger - c_x), \quad p_x = \sqrt{\frac{\hbar M\omega_x}{2}}(c_x^\dagger + c_x),$$
$$y = \sqrt{\frac{\hbar}{2M\omega_y}}(c_y^\dagger + c_y), \quad p_y = i\sqrt{\frac{\hbar M\omega_y}{2}}(c_y^\dagger - c_y), \qquad (4)$$
$$z = -i\sqrt{\frac{\hbar}{2M\omega_z}}(c_z^\dagger - c_z), \quad p_z = \sqrt{\frac{\hbar M\omega_z}{2}}(c_z^\dagger + c_z).$$

Suppose that the three frequencies ω_x, ω_y, and ω_z have an irreducible integral ratio $a : b : c$, i.e.,

$$\omega_x = a\omega_{\rm sh}, \quad \omega_y = b\omega_{\rm sh}, \quad \omega_z = c\omega_{\rm sh}. \qquad (5)$$

We express the eigenstate of $\hat{n}_k = c_k^\dagger c_k$ as

$$|n_k\rangle\rangle = \frac{(c_k^\dagger)^{n_k}}{\sqrt{n_k!}}|0\rangle. \qquad (6)$$

Then the eigenvalues and eigenfunctions of H, for fixed ratio of $a : b : c$, are described by a set of quantum numbers (n_x, n_y, n_z):

$$H|n_x, n_y, n_z\rangle\rangle = \hbar\omega_{\rm sh}\left(N_{\rm sh} + \frac{a+b+c}{2}\right)|n_x, n_y, n_z\rangle\rangle$$
$$= E_{\rm sh}|n_x, n_y, n_z\rangle\rangle, \qquad (7)$$

where the n_k are the eigenvalues of the number operator $\hat{n}_k = c_k^\dagger c_k$, and n_k integers $\geqslant 0$. The shell quantum number $N_{\rm sh}$ is given by

$$N_{\rm sh} = an_x + bn_y + cn_z. \qquad (8)$$

These degeneracies are already known and have been discussed in the literature by many authors [2,5]. However, there has been no explicit discussion regarding the triaxial case in connection with SU(3) symmetry.

From Eq. (3) follow the relations

$$\omega_x + \omega_y + \omega_z = 3\omega_0, \qquad (9a)$$
$$\tan\gamma = \sqrt{3}\frac{\omega_x - \omega_y}{\omega_x + \omega_y - 2\omega_z}, \qquad (9b)$$
$$\frac{2\delta}{3\sqrt{3}}\sin\left(\gamma + \frac{\pi}{3}\right) = \frac{\omega_x - \omega_z}{\omega_x + \omega_y + \omega_z}. \qquad (9c)$$

Equation (9a) is used to determine $\omega_{\rm sh}$ as

$$\omega_{\rm sh} = \frac{3}{a+b+c}\omega_0. \qquad (10)$$

Equations (9b) and (9c) express γ and δ in terms of a, b, and c. For the case of an axially symmetric deformation where $\omega_x = \omega_y$, Eq. (9c) determines δ uniquely corresponding to the ratio $\omega_x : \omega_z$. In the triaxially deformed case, different ratios $\omega_x : \omega_y : \omega_z$ can yield the same value for the parameter γ. Thus, for a fixed value of the parameter γ, Eq. (9b) yields a relation among a, b, and c. Once a set of a, b, and c is given consistent with this relation, Eq. (9c) will give a distinct value for the parameter δ.

III. NEW BOSONS FOR SU(3) AND THE EIGHT GENERATORS

Now, we consider the general case of an integral ratio of $a : b : c$. To construct an SU(3)-invariant expression, we express the harmonic oscillator boson $c_k, k = x, y, z$, Eq. (4), in terms of an m-fold product of new bosons s_m, for positive integer m, by requiring

$$s_m^\dagger s_m = m c_k^\dagger c_k \quad \text{and} \quad [s_m, s_m^\dagger] = 1. \qquad (11)$$

Here, we remark that m represents an integer a, b, or c introduced in Eq. (5). First, we consider the case $m = 2$. Using a procedure analogous to the Holstein-Primakoff transformation for spin operators, we introduce new boson operators s_2 and $s_2^\dagger$ through the relation $c_k = (q_1 + q_2\hat{n}_2)^{-1/2}s_2s_2$, where $\hat{n}_2 = s_2^\dagger s_2$. The coefficients q_1 and q_2 are determined from Eq. (11). Then, it follows

$$c_k = \frac{1}{\sqrt{2(1+\hat{n}_2)}}(s_2)^2, \quad c_k^\dagger = (s_2^\dagger)^2\frac{1}{\sqrt{2(1+\hat{n}_2)}}. \qquad (12)$$

Similarly, for the case of $m = 3$, we find new boson operators s_3 and $s_3^\dagger$ from Eq. (11) as a triproduct,

$$c_k = \frac{1}{\sqrt{3(\hat{n}_3+2)(\hat{n}_3+1)}}(s_3)^3. \qquad (13)$$

From Eq. (13) it follows that $\hat{n}_3 = s_3^\dagger s_3 = 3c_k^\dagger c_k$. Thus, the general form of the new bosons s_m, for any positive integer m, is given by

$$c_k = \left[m\prod_{r=1}^{m-1}(\hat{n}_m + r)\right]^{-1/2}(s_m)^m$$
$$= \left[\frac{\Gamma(\hat{n}_m+1)}{m\Gamma(\hat{n}_m+m)}\right]^{1/2}(s_m)^m, \qquad (14)$$

where $\hat{n}_m = s_m^\dagger s_m$. Equation (14) satisfies the conditions given by Eq. (11), and Eqs. (12) and (13) correspond to the cases $m = 2$ and $m = 3$ in Eq. (14), respectively.

The transformation defined by Eq. (14) differs from the transformation introduced in our previous paper [2], where the new bosons are constructed from the product of the original bosons. In this paper, the original bosons are constructed instead from the product of new bosons as seen from Eq. (14). Thus, there exists a unique vacuum, which is annihilated by s_m as well as c_k. However, since the harmonic boson operators c_k are replaced by the mth power of new boson operators s_m, the many boson Fock space, which is generated by operating $s_m^\dagger$ on a common vacuum, will include unphysical states other than the eigenstates of the original Hamiltonian. In dealing with these unwanted many boson states, the projection operator

$$P_m = \frac{1}{m}\sum_{k=0}^{m-1}e^{i\frac{2\pi k}{m}\hat{n}_m} \quad (m = a, b, c) \qquad (15)$$

is necessary to project out the physical states. It is straightforward to show $P_m^2 = P_m$. The normalized physical states $|n_m\rangle$

are projected out from the Fock space basis $|m)$:

$$|n_m\rangle = \frac{P_m|m)}{\sqrt{(m|P_m|m)}}, \quad (16)$$

with

$$|m) = \frac{(s_m^\dagger)^{n_m}}{\sqrt{n_m!}}|0\rangle, \quad (17)$$

where $|0\rangle$ is the vacuum common to both Eqs. (6) and (17). Setting $m = a, b$, or c in Eq. (14), we determine s_a, s_b, and s_c for a given ratio $a : b : c$, and obtain for $N_{\rm sh} = n_a + n_b + n_c$ in Eq. (7).

The nine operators that commute with the Hamiltonian H are given by

$$(G) \equiv \begin{pmatrix} s_a^\dagger s_a & s_b^\dagger s_a & s_c^\dagger s_a \\ s_a^\dagger s_b & s_b^\dagger s_b & s_c^\dagger s_b \\ s_a^\dagger s_c & s_b^\dagger s_c & s_c^\dagger s_c \end{pmatrix}. \quad (18)$$

These nine operators form a basis for the algebra u(3) as generators of the group U(3). Removing the singlet operator $s_a^\dagger s_a + s_b^\dagger s_b + s_c^\dagger s_c$, the remaining eight operators form an algebra su(3), which generates the group SU(3). From the matrix element $G_{\mu\nu}(\mu, \nu = 1, 2, 3)$ given by Eq. (18), we find the Casimir operator for the single-particle states. From the definition of $A_{\mu\nu}$,

$$A_{\mu\nu} = \sqrt{3}(G_{\mu\nu} - \delta_{\mu\nu}\mathbf{I}/3), \quad \mathbf{I} = \hat{n}_a + \hat{n}_b + \hat{n}_c, \quad (19)$$

the second-order Casimir operator is obtained as

$$\begin{aligned} C &= {\rm Tr}(AA) \\ &= 2(\hat{n}_a + \hat{n}_b + \hat{n}_c)(\hat{n}_a + \hat{n}_b + \hat{n}_c + 3). \end{aligned} \quad (20)$$

With the help of the notation (λ, μ), which labels SU(3) irreducible representation, the expectation value of C becomes $2\lambda(\lambda + 3)$ with $\lambda = n_a + n_b + n_c$ and $\mu = 0$.

The SU(3) group reduces to the subgroup SU(2) × U(1). There are three alternative sets of generators corresponding to subalgebra su(2) [6]. For example, one such subalgebra su(2) is given by

$$I_1 = \frac{A_{12} + A_{21}}{2\sqrt{3}}, \quad I_2 = \frac{i(A_{21} - A_{12})}{2\sqrt{3}}, \quad I_3 = \frac{A_{22} - A_{11}}{2\sqrt{3}}, \quad (21)$$

and U(1) is generated by

$$Y_I = \frac{A_{33}}{\sqrt{3}}. \quad (22)$$

With the use of Eq. (18), Eqs. (21) and (22) are explicitly expressed as

$$\begin{aligned} I_1 &= \tfrac{1}{2}(s_a^\dagger s_b + s_b^\dagger s_a), \quad I_2 = \tfrac{i}{2}(s_a^\dagger s_b - s_b^\dagger s_a), \\ I_3 &= \tfrac{1}{2}(s_b^\dagger s_b - s_a^\dagger s_a), \quad Y_I = \tfrac{1}{3}(2s_c^\dagger s_c - s_a^\dagger s_a - s_b^\dagger s_b). \end{aligned} \quad (23)$$

Elliott [1] has found that the quadrupole operator commutes with the spherical Hamiltonian, which is defined by

$$\hbar Q_q = \sqrt{\frac{4\pi}{5}}\left[r^2 Y_{2q}(\theta,\phi) + {b_0}^4 p^2 Y_{2q}(\theta_p, \phi_p)\right]/{b_0}^2, \quad (24)$$

where ${b_0}^4 = 1/(M^2\omega_0^2)$, and (p, θ_p, ϕ_p) in momentum space corresponds to (r, θ, ϕ) in configuration space. This quadrupole operator does not connect $\pm 2\hbar\omega_0$ excitations, i.e., it is equivalent to an exact quadrupole operator within an N shell. We will make the following substitution for Eq. (24):

$$\begin{aligned} &x \to \tilde{x}, \quad y \to \tilde{y}, \quad z \to \tilde{z}, \\ &p_x \to \tilde{p}_x, \quad p_y \to \tilde{p}_y, \quad p_z \to \tilde{p}_z, \end{aligned} \quad (25)$$

with

$$\begin{aligned} \tilde{x} &= -i\sqrt{\frac{\hbar b_0^2}{2}}(s_a^\dagger - s_a), \quad \tilde{p}_x = \sqrt{\frac{\hbar}{2b_0^2}}(s_a^\dagger + s_a), \\ \tilde{y} &= \sqrt{\frac{\hbar b_0^2}{2}}(s_b^\dagger + s_b), \quad \tilde{p}_y = i\sqrt{\frac{\hbar}{2b_0^2}}(s_b^\dagger - s_b), \\ \tilde{z} &= -i\sqrt{\frac{\hbar b_0^2}{2}}(s_c^\dagger - s_c), \quad \tilde{p}_z = \sqrt{\frac{\hbar}{2b_0^2}}(s_c^\dagger + s_c). \end{aligned} \quad (26)$$

We also adopt the usual definition for the orbital angular momentum operator ℓ_k $(k = x, y, z)$,

$$\hbar\ell_z = xp_y - yp_x, \quad \text{etc.} \quad (27)$$

After the substitution of Eqs. (25) and (26) into Eqs. (24) and (27), we obtain a new set of group operators $\tilde{Q}_q$ for $q = 0, \pm 1$ and ± 2, and $\tilde{\ell}_k$ for $k = a, b$, and c:

$$\begin{aligned} \tilde{Q}_0 &= 2s_c^\dagger s_c - s_a^\dagger s_a - s_b^\dagger s_b, \\ \tilde{Q}_{\pm 1} &= \mp\sqrt{\frac{3}{2}}[s_c^\dagger(s_a \pm s_b) + s_c(s_a^\dagger \mp s_b^\dagger)], \\ \tilde{Q}_{\pm 2} &= \sqrt{\frac{3}{2}}[s_a^\dagger s_a - s_b^\dagger s_b \pm (s_a^\dagger s_b - s_b^\dagger s_a)], \\ \tilde{\ell}_\pm &= \frac{1}{\sqrt{2}}[s_c^\dagger(s_b \pm s_a) + s_c(s_b^\dagger \mp s_a^\dagger)], \\ \tilde{\ell}_c &= -(s_b^\dagger s_a + s_a^\dagger s_b). \end{aligned} \quad (28)$$

Here, we used the definition of $\tilde{\ell}_\pm = (\tilde{\ell}_a \pm i\tilde{\ell}_b)/\sqrt{2}$. In contrast to Elliott's case where $\pm 2\hbar\omega_0$ excitations are excluded, the operators given by Eq. (28) exclude $\pm 2\hbar\omega_{\rm sh}$ excitations. Again the commutation relations among the eight operators defined by Eq. (28) are closed, and they commute with H. The oscillator Hamiltonian H is invariant with respect to the group SU(3) which is generated by these eight operators $\tilde{Q}_q$ and $\tilde{\ell}_q$. The following relations hold:

$$\begin{aligned} \tilde{Q}_0 &= 2\hat{n}_c - \hat{n}_a - \hat{n}_b, \\ \tilde{Q}_2 + \tilde{Q}_{-2} &= \sqrt{6}(\hat{n}_a - \hat{n}_b). \end{aligned} \quad (29)$$

Here, we remark that $\tilde{Q}_0 = 3Y_I$, and $\tilde{Q}_2 + \tilde{Q}_{-2} = -2\sqrt{6}I_3$ [see Eq. (23)].

The Casimir operator in Eq. (20) is also expressed in terms of the generators in Eq. (28) as

$$C = \tfrac{1}{2}(\tilde{Q}\cdot\tilde{Q} + 3\tilde{\ell}\cdot\tilde{\ell}). \quad (30)$$

There are three kinds of bilinear combinations of generators given by Eq. (28), which are expressed in terms of the number operators:

$$\frac{1}{4}\tilde{\ell}_z^2 + \frac{1}{12}(\tilde{Q}_2\tilde{Q}_{-2} + \tilde{Q}_{-2}\tilde{Q}_2) = \left(\frac{\hat{n}_a + \hat{n}_b}{2}\right)\left(\frac{\hat{n}_a + \hat{n}_b}{2} + 1\right), \tag{31a}$$

$$\tilde{Q}_0^2 + (\tilde{Q}_2 + \tilde{Q}_{-2})^2 = (2\hat{n}_c - \hat{n}_a - \hat{n}_b)^2 + 6\,(\hat{n}_a - \hat{n}_b)^2\,, \tag{31b}$$

$$3(\tilde{\ell}_+\tilde{\ell}_- + \tilde{\ell}_-\tilde{\ell}_+) - (\tilde{Q}_1\tilde{Q}_{-1} + \tilde{Q}_{-1}\tilde{Q}_1) = 3\,[4\hat{n}_c\,(\hat{n}_a + \hat{n}_b + 1) + 2\,(\hat{n}_a + \hat{n}_b)]\,. \tag{31c}$$

Equation (31a) is also derived from Eq. (23) as $I_1^2 + I_2^2 + I_3^2$ and is related to the reduction of the SU(3) group to its subgroup SU(2) × U(1). The group U(1) is generated by $\tilde{Q}_0$, while the representations of SU(2) are labeled by integral or half-integral numbers $(\hat{n}_a + \hat{n}_b)/2$, as seen from Eq. (31a). Equation (31b) results from our choice of the principal axis in Eq. (29). Equation (31c) is related to operators $\tilde{Q}_0$ and $(\tilde{Q}_2 + \tilde{Q}_{-2})/\sqrt{6}$. The linear combinations of $\tilde{Q}_{\pm 1}$ and $\tilde{\ell}_{\pm 1}$, i.e.,

$$\xi_\pm = \tilde{Q}_{\pm 1} + \sqrt{3}\tilde{\ell}_\pm \quad (\eta_\pm = \tilde{Q}_{\pm 1} - \sqrt{3}\tilde{\ell}_\pm), \tag{32}$$

become lowering (raising) operators for the value of $\langle\tilde{Q}_0\rangle$ to $\langle\tilde{Q}_0\rangle \mp 3$. The operator $\tilde{Q}_0$ describes the excess of quanta in the c direction (z direction) compared with those in the a-b plane (x-y plane). The operators $\xi_+ + \xi_-$ and $\eta_+ + \eta_-$ shift an oscillator quantum from the c direction into the a-b plane by three units and vice versa, because of the commutation relations

$$[\tilde{Q}_0, \xi_\pm] = -3\xi_\pm, \quad [\tilde{Q}_0, \eta_\pm] = 3\eta_\pm. \tag{33}$$

These operators are also lowering and raising operators for $(\tilde{Q}_2 + \tilde{Q}_{-2})/\sqrt{6}$ by one unit, namely,

$$\left[\frac{\tilde{Q}_2 + \tilde{Q}_{-2}}{\sqrt{6}}, \xi_+ + \xi_-\right] = -(\xi_+ + \xi_-), \qquad \left[\frac{\tilde{Q}_2 + \tilde{Q}_{-2}}{\sqrt{6}}, \eta_+ + \eta_-\right] = \eta_+ + \eta_-. \tag{34}$$

Since $(\tilde{Q}_2 + \tilde{Q}_{-2})/\sqrt{6}$ equals $\hat{n}_a - \hat{n}_b$, the operators $\xi_+ + \xi_-$ and $\eta_+ + \eta_-$ shift an oscillator quantum from the a direction (x direction) to the b direction (y direction) by one unit, and vice versa.

IV. TRIAXIAL DEFORMATION OF $\gamma = 30°$

In this section, we consider the special case of $\gamma = 30°$, where $\tan\gamma = 1/\sqrt{3}$. For this case, Eq. (9b) reduces to

$$a + c = 2b, \tag{35}$$

and Eq. (9c) gives

$$\delta = \frac{3\sqrt{3}(a - c)}{2(a + b + c)}. \tag{36}$$

From Eq. (35), it follows that typical examples for the ratio $a : b : c$ are 3 : 2 : 1, 4 : 3 : 2, and 5 : 4 : 3. Then, Eq. (36) yields the deformation $\delta \sim 0.866$ for 3 : 2 : 1, while $\delta \sim 0.577$ for 4 : 3 : 2 and $\delta \sim 0.433$ for 5 : 4 : 3. If more complex ratios are chosen, the least common multiplier (L.C.M.) for a, b, and c becomes larger, while the corresponding δ becomes smaller. The ratio $a : b : c = 1 : 2 : 3$ corresponds to an oblate nuclear shape ($\omega_x < \omega_y < \omega_z$), while the values of the L.C.M. and the absolute value of δ are the same as for the prolate case with $a : b : c = 3 : 2 : 1$. The value of $|\delta| \sim 0.866$ seems to be very large but is within the region where ω_x^2 and ω_z^2 in Eq. (2) are non-negative.

Choosing the simplest case, namely, $a : b : c = 3 : 2 : 1$ for $\gamma = 30°$, the shell energy $E_{\rm sh}$ in Eq. (7) and shell number $N_{\rm sh}$ in Eq. (8) become

$$E_{\rm sh} = \hbar\omega_{\rm sh}(N_{\rm sh} + 3), \quad N_{\rm sh} = 3n_x + 2n_y + n_z. \tag{37}$$

The relation $\omega_{\rm sh} = \omega_0/2$ is deduced from Eq. (10).

With the help of Eq. (14), we construct new s bosons, namely, s_3 defined by means of a triproduct for the c_x boson, and s_2 defined by means of a biproduct for the c_y boson [both s_2 and s_3 bosons were previously determined in Eqs. (12) and (13)]. The vacuum for the s_3 boson is the same as for the c_x boson, and the vacuum for the s_2 boson is the same as for the c_y boson. However, since the original harmonic oscillator bosons c_k, $k = x, y$, are expressed in the form of triproducts and biproducts of the new bosons s_m, $m = 3, 2$, a projection operator is needed that projects out the states corresponding to the eigenvalues of the s_m bosons and their boson number operators, namely, the multiples of 3 for the case of s_3 and the multiples of 2 for the case of s_2. These projection operators are obtained from the general expression, Eq. (15), as

$$P_3 = \frac{1 + e^{i\frac{2\pi}{3}\hat{n}_3} + e^{i\frac{4\pi}{3}\hat{n}_3}}{3}, \qquad P_2 = \frac{1 + e^{i\pi\hat{n}_2}}{2}. \tag{38}$$

The eight generators in Eq. (28) can be rewritten as

$$\begin{aligned}
\tilde{Q}_0 &= 2c_z^\dagger c_z - s_3^\dagger s_3 - s_2^\dagger s_2,\\
\tilde{Q}_{\pm 1} &= \mp\sqrt{\frac{3}{2}}[c_z^\dagger(s_3 \pm s_2) + c_z(s_3^\dagger \mp s_2^\dagger)],\\
\tilde{Q}_{\pm 2} &= \sqrt{\frac{3}{2}}[s_3^\dagger s_3 - s_2^\dagger s_2 \pm (s_3^\dagger s_2 - s_2^\dagger s_3)],\\
\tilde{\ell}_\pm &= \frac{1}{\sqrt{2}}[c_z^\dagger(s_2 \pm s_3) + c_z(s_2^\dagger \mp s_3^\dagger)],\\
\tilde{\ell}_z &= -(s_2^\dagger s_3 + s_3^\dagger s_2).
\end{aligned} \tag{39}$$

The diagonal number operators given by Eq. (29) become $\tilde{Q}_0 = 2\hat{n}_z - \hat{n}_3 - \hat{n}_2$ and $\tilde{Q}_2 + \tilde{Q}_{-2} = \sqrt{6}(\hat{n}_3 - \hat{n}_2)$. The Casimir operator in Eq. (20) becomes $2(\hat{n}_3 + \hat{n}_2 + \hat{n}_z)(\hat{n}_3 + \hat{n}_2 + \hat{n}_z + 3)$.

In Table I, we summarize the classification of the single-particle states for the case of a triaxial shape for $\gamma = 30°$ and $a : b : c = 3 : 2 : 1$ for the shells $N_{\rm sh} \leqslant 8$. Also listed in Table I are the values for the operators $\langle\tilde{Q}_0\rangle$, $\langle\tilde{Q}_2 + \tilde{Q}_{-2}\rangle/\sqrt{6}$, and $\langle C\rangle$. The states $|\,\rangle$ are the states defined by Eq. (16) in terms of the quantum numbers n_3, n_2, and n_z. The total number of levels for $N_{\rm sh} = 6m$, $6m + 1$, $6m + 2$, $6m + 3$, $6m + 4$, and $6m + 5$ with integer m

TABLE I. Single-particle eigenfunctions of $\tilde{Q}_0$ and $(\tilde{Q}_2+\tilde{Q}_{-2})/\sqrt{6}$ for $\gamma = 30^\circ$ and $a:b:c = 3:2:1$ with $0 \leqslant N_{\rm sh} \leqslant 8$.

$N_{\rm sh}$	n_3	n_2	n_z	$\langle\tilde{Q}_0\rangle$	$\langle\tilde{Q}_2+\tilde{Q}_{-2}\rangle/\sqrt{6}$	$\langle C\rangle$
0	0	0	0	0	0	0
1	0	0	1	2	0	8
2	0	2	0	−2	−2	20
	0	0	2	4	0	
3	3	0	0	−3	3	36
	0	2	1	0	−2	
	0	0	3	6	0	
4	3	0	1	−1	3	56
	0	4	0	−4	−4	
	0	2	2	2	−2	
	0	0	4	8	0	
5	3	2	0	−5	1	80
	3	0	2	1	3	
	0	4	1	−2	−4	
	0	2	3	4	−2	
	0	0	5	10	0	
6	6	0	0	−6	6	108
	3	2	1	−3	1	
	3	0	3	3	3	
	0	6	0	−6	−6	
	0	4	2	0	−4	
	0	2	4	6	−2	
	0	0	6	12	0	
7	6	0	1	−4	6	140
	3	4	0	−7	−1	
	3	2	2	−1	1	
	3	0	4	5	3	
	0	6	1	−4	−6	
	0	4	3	2	−4	
	0	2	5	8	−2	
	0	0	7	14	0	
8	6	2	0	−8	4	176
	6	0	2	−2	6	
	3	4	1	−5	−1	
	3	2	3	1	1	
	3	0	5	7	3	
	0	8	0	−8	−8	
	0	6	2	−2	−6	
	0	4	4	−4	−4	
	0	2	6	10	−2	
	0	0	8	16	0	

is given by $3m^2+3m+1$, $(3m+1)(m+1)$, $(3m+2)(m+1)$, $3(m+1)^2$, $(m+1)(3m+4)$, and $(m+1)(3m+5)$, respectively. The relations $\langle\tilde{Q}_0\rangle = -N_{\rm sh}+3n_z$ and $\langle\tilde{Q}_0\rangle = 2N_{\rm sh}-3(n_3+n_2)$ require that $\langle\tilde{Q}_0\rangle$ starts from $-N_{\rm sh}$ and increases in steps by 3 up to $2N_{\rm sh}$, but $2N_{\rm sh}-3$ does not contribute since n_3+n_2 is always unequal to unity. This differs from the case of the spherical harmonics. From Table I we derive, by taking into account the spin quantum number, a new sequence of magic numbers 2, 4, 8, 14, 22, 32, 46, 62, and 82. These numbers are still meaningful even when the $\vec{\ell}\cdot\vec{s}$ term exists, since the single-particle expectation value of $\vec{\ell}\cdot\vec{s}$ vanishes for triaxial field and does not contribute in the first-order perturbation treatment.

To introduce the realistic operators, we substitute Eq. (4) directly into Eqs. (24) and (27). First we substitute Eq. (4) into Eq. (27) and obtain the following expressions:

$$\begin{aligned}\ell_x &= \frac{1}{2}\Bigg[\left(\sqrt{\frac{c}{b}}-\sqrt{\frac{b}{c}}\right)(c_y^\dagger c_z^\dagger + c_y c_z) \\ &\quad + \left(\sqrt{\frac{c}{b}}+\sqrt{\frac{b}{c}}\right)(c_y^\dagger c_z + c_y c_z^\dagger)\Bigg],\\ \ell_y &= \frac{i}{2}\Bigg[\left(\sqrt{\frac{a}{c}}-\sqrt{\frac{c}{a}}\right)(c_x c_z - c_z^\dagger c_x^\dagger) \\ &\quad + \left(\sqrt{\frac{a}{c}}+\sqrt{\frac{c}{a}}\right)(c_x^\dagger c_z - c_z^\dagger c_x)\Bigg],\\ \ell_z &= \frac{1}{2}\Bigg[\left(\sqrt{\frac{b}{a}}-\sqrt{\frac{a}{b}}\right)(c_x^\dagger c_y^\dagger + c_y c_x) \\ &\quad - \left(\sqrt{\frac{b}{a}}+\sqrt{\frac{a}{b}}\right)(c_x c_y^\dagger + c_y c_x^\dagger)\Bigg].\end{aligned} \tag{40}$$

These operators satisfy the same commutation relations as the group operators given in Eq. (28). If we set $a=b=c=1$ in Eq. (40) and replace c_x and c_y by s_3 and s_2, the operators ℓ go over into the operators $\tilde{\ell}$ in Eq. (39). However, in the general case, Eq. (40) does not commute with H and connects with levels outside the $N_{\rm sh}$ shell. As seen from Eq. (40), the first term in $\ell_k(k=x,y,z)$ disappears for the spherical case ($a=b=c=1$), where we can make one component of $\vec{\ell}$ diagonal, for example ℓ_z. On the contrary, in the triaxial case, no ℓ_k commutes with H, and its expectation value by the state $|n_x,n_y,n_z\rangle\rangle$ defined in Eq. (6) gives $\langle\langle n_x,n_y,n_z|\ell_k|n_x,n_y,n_z\rangle\rangle = 0$, which demonstrates the quenching of orbital angular momentum [7,8].

Now, we compare the diagonal matrix elements of $\tilde{\ell}\cdot\tilde{\ell}$ and $\ell\cdot\ell$. From Eqs. (28) and (40), we get

$$\begin{aligned}&\langle n_a,n_b,n_c|\tilde{\ell}\cdot\tilde{\ell}|n_a,n_b,n_c\rangle \\ &\quad = 2(n_a+n_b+n_c+n_an_b+n_bn_c+n_cn_a),\end{aligned} \tag{41}$$

where $|n_a,n_b,n_c\rangle$ is defined by Eq. (16), and $\langle\langle n_x,n_y,n_z|\ell\cdot\ell|n_x,n_y,n_z\rangle\rangle = \langle\langle\ell_x^2\rangle\rangle + \langle\langle\ell_y^2\rangle\rangle + \langle\langle\ell_z^2\rangle\rangle$with

$$\begin{aligned}\langle\langle\ell_x^2\rangle\rangle &= \left(\frac{b}{c}+\frac{c}{b}\right)\left(n_y+\frac{1}{2}\right)\left(n_z+\frac{1}{2}\right)-\frac{1}{2},\\ \langle\langle\ell_y^2\rangle\rangle &= \left(\frac{c}{a}+\frac{a}{c}\right)\left(n_z+\frac{1}{2}\right)\left(n_x+\frac{1}{2}\right)-\frac{1}{2},\\ \langle\langle\ell_z^2\rangle\rangle &= \left(\frac{a}{b}+\frac{b}{a}\right)\left(n_x+\frac{1}{2}\right)\left(n_y+\frac{1}{2}\right)-\frac{1}{2}.\end{aligned} \tag{42}$$

For the purpose of an explicit application of our dynamical SU(3) model to many-body systems with triaxial deformations along the line of Elliott's SU(3) model, we need to investigate whether the expectation values of $\langle\tilde{\ell}\cdot\tilde{\ell}\rangle$ and $\langle\tilde{Q}\cdot\tilde{Q}\rangle$ approximate the expectation values of the exact operators $\langle\langle\ell\cdot\ell\rangle\rangle$

and $\langle\langle Q \cdot Q \rangle\rangle$. Comparing Eq. (41) with $\langle\langle \ell \cdot \ell \rangle\rangle$, we see that for given values a, b, c, the three coefficients of the boson numbers must add up to the common factor 2 in Eq. (41). We find that both expressions coincide with each other only when $b/a = 1$ and $c/a = 1$, which corresponds to the spherical case, i.e., $\delta = 0$. Therefore, the deviation of $b/a + a/b, a/c + c/a$, and $c/b + b/c$ from the factor 2 indicates the quantitative measure of the approximation. For example, for the case of $a : b : c = 3 : 2 : 1$, Eq. (42) becomes

$$\langle\langle n_x, n_y, n_z | \ell \cdot \ell | n_x, n_y, n_z \rangle\rangle = \left(2 + \frac{1}{6}\right) n_x n_y + \left(2 + \frac{1}{2}\right) n_y n_z + \left(2 + \frac{4}{3}\right) n_z n_x + \left(2 + \frac{3}{4}\right) n_x + \left(2 + \frac{1}{3}\right) n_y + \left(2 + \frac{11}{12}\right) n_z + \frac{1}{2}, \tag{43}$$

whose coefficients of $n_z n_x$ and n_z show large deviation from the factor 2.

On the other hand, for $\gamma = 30^\circ$ and $a : b : c = 4 : 3 : 2 (\delta = 0.577)$, we introduce a new boson s_4 as a tetraproduct for c_x,

$$c_x = \frac{1}{\sqrt{4(\hat{n}_4 + 3)(\hat{n}_4 + 2)(\hat{n}_4 + 1)}} (s_4)^4, \tag{44}$$

where $\hat{n}_4 = s_4^\dagger s_4$. As for the other two axes, the c_y boson is replaced by an s_3 boson, and the c_z boson by an s_2 boson. In this case, $N_{\rm sh} = n_4 + n_3 + n_2$, $\omega_{\rm sh} = \omega_0/3$, $\langle \tilde{Q}_0 \rangle = 2n_2 - n_4 - n_3$, and $\langle \tilde{Q}_2 + \tilde{Q}_{-2} \rangle / \sqrt{6} = n_4 - n_3$. Equation (41) becomes

$$\langle n_4, n_3, n_2 | \tilde{\ell} \cdot \tilde{\ell} | n_4, n_3, n_2 \rangle = 2(n_4 + n_3 + n_2 + n_4 n_3 + n_3 n_2 + n_2 n_4), \tag{45}$$

while Eq. (42) becomes

$$\langle\langle n_x, n_y, n_z | \ell \cdot \ell | n_x, n_y, n_z \rangle\rangle = \left(2 + \frac{1}{12}\right) n_x n_y + \left(2 + \frac{1}{6}\right) n_y n_z + \left(2 + \frac{1}{2}\right) n_z n_x + \left(2 + \frac{7}{24}\right) n_x + \left(2 + \frac{1}{8}\right) n_y + \left(2 + \frac{1}{3}\right) n_z + \frac{3}{16}. \tag{46}$$

Compared with Eq. (43), all the coefficients in Eq. (46) are close to 2, and we may assume that the realistic operators $\ell \cdot \ell$ can be simulated by $\tilde{\ell} \cdot \tilde{\ell}$.

In Table II, we show the classification of the single-particle states for $\gamma = 30^\circ$ and $a : b : c = 4 : 3 : 2$, together with $\langle \tilde{Q}_0 \rangle$, $\langle \tilde{Q}_2 + \tilde{Q}_2 \rangle / \sqrt{6}$, and $\langle C \rangle$. Here, we note that $N_{\rm sh} = 1$ $(= n_4 + n_3 + n_2)$ is not allowed in this scheme, as n_4, n_3, and n_2 are multiples of 4, 3, and 2, respectively. Again, though $\langle \tilde{Q}_0 \rangle$ changes in steps by 3, some levels that exist in the spherical harmonics case do not exist here. We see that the energy degeneracy in Table II (L.C.M. = 12) becomes smaller than that in Table I (L.C.M. = 6). For $N_{\rm sh} = 6$, for example, seven levels are degenerate in Table I, while only three levels are degenerate in Table II. This is because of the impossibility of partitioning certain of the shell numbers $N_{\rm sh}$ into three large integers. In this case, the magic numbers are predicted to be 2, 4, 6, 10, 12, 18, 20, 30, 36, 46, 54, 68, and 78.

TABLE II. Single-particle eigenfunction of $\tilde{Q}_0$ and $(\tilde{Q}_2 + \tilde{Q}_{-2})/\sqrt{6}$ for $\gamma = 30^\circ$ and $a : b : c = 4 : 3 : 2$ with $0 \leqslant N_{\rm sh} \leqslant 12$.

$N_{\rm sh}$	n_4	n_3	n_2	$\langle \tilde{Q}_0 \rangle$	$\langle \tilde{Q}_2 + \tilde{Q}_{-2} \rangle / \sqrt{6}$	$\langle C \rangle$
0	0	0	0	0	0	0
2	0	0	2	4	0	20
3	0	3	0	−3	−3	36
4	4	0	0	−4	4	56
	0	0	4	8	0	
5	0	3	2	1	−3	80
6	4	0	2	0	4	108
	0	6	0	−6	−6	
	0	0	6	12	0	
7	4	3	0	−7	1	140
	0	3	4	5	−3	
8	8	0	0	−8	8	176
	4	0	4	4	4	
	0	6	2	−2	−6	
	0	0	8	16	0	
9	4	3	2	−3	1	216
	0	9	0	−9	−9	
	0	3	6	9	−3	
10	8	0	2	−4	8	260
	4	6	0	−10	−2	
	4	0	6	8	4	
	0	6	4	2	−6	
	0	0	10	20	0	
11	8	3	0	−11	5	308
	4	3	4	1	1	
	0	9	2	−5	−9	
	0	3	8	13	−3	
12	12	0	0	−12	12	360
	8	0	4	0	8	
	4	6	2	−6	−2	
	4	0	8	12	4	
	0	12	0	−12	−12	
	0	6	6	6	−6	
	0	0	12	24	0	

From $\langle\langle n_x, n_y, n_z | \ell_k^2 | n_x, n_y, n_z \rangle\rangle (k = x, y, z)$, we can estimate the direction of alignment in the state $|n_x, n_y, n_z\rangle\rangle$. As seen from Eq. (42), $\langle\langle \ell_k^2 \rangle\rangle$ becomes maximum in the k direction with minimum n_k. When two or three n_k take a common minimum value, the direction of the maximum alignment depends on the magnitudes of $b/a + a/b$, $b/c + c/b$, and $c/a + a/c$. As the relation of $a > b > c (a < b < c)$ holds because of Eq. (5), $c/a + a/c$ becomes maximum, resulting in the maximum alignment along the y axis. For example, for $a : b : c = 4 : 3 : 2$ and $N_{\rm sh} = 6 (= 4n_x + 3n_y + 2n_z)$, the maximum alignment is along the y axis for the $|n_x = 1, n_y = 0, n_z = 1\rangle\rangle$ state, and the next to the largest is along the x axis. For the $|n_x = 0, n_y = 2, n_z = 0\rangle\rangle$ state, the maximum alignment is along the x axis, and the next to the largest along the z axis. For the $|n_x = 0, n_y = 0, n_z = 3\rangle\rangle$ state, the maximum is along the y axis and the next to the largest along the x axis. In the oblate case $(a : b : c = 2 : 3 : 4)$, the alignment is obtained by exchanging the x axis and the z axis. Thus, the maximum alignment is now along the z axis for the $|n_x = 0, n_y = 2, n_z = 0\rangle\rangle$ state, while there is no change in the direction of maximum alignment for the other states in the oblate case.

We can also determine the direction for the maximum value of $\tilde{\ell}_m^2 (m = 4, 3, 2)$ for the $|n_4, n_3, n_2\rangle$ state. Similar to $\langle\langle \ell_k^2 \rangle\rangle$, $\langle \tilde{\ell}_m^2 \rangle$ becomes maximum in the m direction with minimum n_m. When two n_m take a common minimum value, a common maximum value occurs in both directions. For $a : b : c = 4 : 3 : 2$ and $N_{\rm sh} = 6 (= n_4 + n_3 + n_2)$, the direction with the maximum $\langle \tilde{\ell}_m^2 \rangle$ is along the s_3 axis (y axis) for $|n_4 = 4, n_3 = 0, n_2 = 2\rangle$ (corresponding to $|n_x = 2, n_y = 0, n_z = 1\rangle\rangle$), along the s_4 axis (x axis) and along the s_2 axis (z axis) for $|n_4 = 0, n_3 = 6, n_2 = 0\rangle$ (corresponding to $|n_x = 0, n_y = 2, n_z = 0\rangle\rangle$), and along the s_3 axis (y axis) and along the s_4 axis (x axis) for $|n_4 = 0, n_3 = 0, n_2 = 6\rangle$ (corresponding to $|n_x = 0, n_y = 0, n_z = 3\rangle\rangle$). As for $\tilde{\ell}_m^2 (m = 4, 3, 2)$, the direction of maximum value is the same for both oblate (n_2, n_3, n_4) and prolate (n_4, n_2, n_2) cases, as seen from Eq. (45). For the $|n_4 = 0, n_3 = 6, n_2 = 0\rangle$ state, both alignments, along the s_4 axis (x axis) and the s_2 axis (z axis), are the same and do not differ from the result of the realistic oblate state $|n_x = 0, n_y = 2, n_z = 0\rangle\rangle$.

Similar to the case of orbital angular momentum $\boldsymbol{\ell}$, we get the realistic quadrupole operators Q_q for $q = 0, \pm 1, \pm 2$.

$$
\begin{aligned}
Q_0 = \frac{1}{2}\Bigg[& \left(c' - \frac{1}{c'}\right)\left(c_z^{\dagger 2} + c_z^2\right) + \left(c' + \frac{1}{c'}\right)(2\hat{n}_z + 1) \\
& - \frac{1}{2}\left(a' - \frac{1}{a'}\right)\left(c_x^{\dagger 2} + c_x^2\right) - \frac{1}{2}\left(a' + \frac{1}{a'}\right)(2\hat{n}_x + 1) \\
& + \frac{1}{2}\left(b' - \frac{1}{b'}\right)\left(c_y^{\dagger 2} + c_y^2\right) - \frac{1}{2}\left(b' + \frac{1}{b'}\right)(2\hat{n}_y + 1)\Bigg], \\
Q_{\pm 1} = \mp\frac{1}{2}\sqrt{\frac{3}{2}}\Bigg[& \left(\sqrt{a'c'} - \frac{1}{\sqrt{a'c'}}\right)\left(c_z^\dagger c_x^\dagger + c_x c_z\right) \\
& + \left(\sqrt{a'c'} + \frac{1}{\sqrt{a'c'}}\right)\left(c_z^\dagger c_x + c_x^\dagger c_z\right) \\
& \mp \left(\sqrt{b'c'} - \frac{1}{\sqrt{b'c'}}\right)\left(c_z^\dagger c_y^\dagger - c_y c_z\right) \\
& \mp \left(\sqrt{b'c'} + \frac{1}{\sqrt{b'c'}}\right)\left(c_y^\dagger c_z - c_y c_z^\dagger\right)\Bigg], \\
Q_{\pm 2} = \frac{1}{4}\sqrt{\frac{3}{2}}\Bigg[& \left(a' - \frac{1}{a'}\right)\left(c_x^{\dagger 2} + c_x^2\right) + \left(a' + \frac{1}{a'}\right)(2\hat{n}_x + 1) \\
& + \left(b' - \frac{1}{b'}\right)\left(c_y^{\dagger 2} + c_y^2\right) - \left(b' + \frac{1}{b'}\right)(2\hat{n}_y + 1) \\
& \mp 2\left(\sqrt{a'b'} - \frac{1}{\sqrt{a'b'}}\right)\left(c_x^\dagger c_y^\dagger + c_y c_x\right) \\
& \pm 2\left(\sqrt{a'b'} + \frac{1}{\sqrt{a'b'}}\right)\left(c_x^\dagger c_y - c_x c_y^\dagger\right)\Bigg],
\end{aligned}
\tag{47}
$$

where $a' = 3a/(a+b+c)$, $b' = 3b/(a+b+c)$, and $c' = 3c/(a+b+c)$ are deduced from Eq. (10).

We compare the diagonal part of $\tilde{Q} \cdot \tilde{Q}$ and $Q \cdot Q$ for the case of $a : b : c = 4 : 3 : 2$,

$$
\begin{aligned}
\langle n_4, n_3, n_2 | \tilde{Q} \cdot \tilde{Q} | n_4, n_3, n_2 \rangle
&= 4\left(n_4^2 + n_3^2 + n_2^2\right) + 6(n_4 + n_3 + n_2) \\
&\quad + 2(n_4 n_3 + n_3 n_2 + n_2 n_4),
\end{aligned}
\tag{48}
$$

and

$$
\begin{aligned}
&\langle\langle n_x, n_y, n_z | Q \cdot Q | n_x, n_y, n_z \rangle\rangle \\
&= \left(4 + \frac{49}{96}\right) n_x^2 + 4 n_y^2 + \left(4 + \frac{25}{24}\right) n_z^2 \\
&\quad + \left(6 + \frac{91}{288}\right) n_x + \left(6 + \frac{1}{8}\right) n_y + \left(6 + \frac{8}{9}\right) n_z \\
&\quad + \left(2 + \frac{1}{12}\right) n_x n_y + \left(2 + \frac{1}{6}\right) n_y n_z \\
&\quad + \left(2 - \frac{17}{36}\right) n_z n_x + \frac{415}{576}.
\end{aligned}
\tag{49}
$$

Almost all the coefficients in Eqs. (48) and (49) are close to each other. Thus, by comparison between Eqs. (45) and (46) and between Eqs. (48) and (49), we infer that the operators Q and ℓ are approximated by the SU(3) group operators $\tilde{Q}$ and $\tilde{\ell}$.

To extend our treatment to the many-fermion problem, we need the help of the projection operator, $P = P_a P_b P_c$. For example, the many-body quadrupole operator is defined by

$$
\boldsymbol{Q}_\mu = \sum_{i,j} \langle i | P \tilde{Q}_\mu P | j \rangle \alpha_i^\dagger \alpha_j,
\tag{50}
$$

where the suffixes i and j represent the boson state which is specified by (n_a, n_b, n_c), and $\alpha_i^\dagger (\alpha_j)$ denotes a fermion creation (annihilation) operator for the state i (j). It is obvious that if we replace P by 1, we have $\tilde{Q} \cdot \tilde{Q} = C - 3\tilde{\ell} \cdot \tilde{\ell}$, and the expectation value of the many-body Casimir operator C is $2[\lambda^2 + \lambda\mu + \mu^2 + 3(\lambda + \mu)]$ for the SU(3) irreducible representation labeled by (λ, μ) [1]. Such an ideal SU(3) limit realizes an aspect similar to the interacting boson model (IBM) in the axially symmetric limit [9].

V. THE OTHER TRIAXIALITY

While our discussion is based on the parametrization of Eq. (3) as was mentioned below Eq. (3), ω_z^2 becomes negative when δ is larger than a positive critical value δ_c^p, and ω_x^2 becomes negative when δ is smaller than a negative critical value δ_c^n. These critical values are derived from Eq. (2) depending on the value of γ.

Recently, the triaxial strongly deformed bands were observed in Lu isotopes, where γ is estimated to be around $18^\circ \sim 20^\circ$ [3]. When $\tan\gamma = \sqrt{3}/5$, which corresponds to $\gamma \sim 19^\circ$, Eq. (9c) gives $2a + c = 3b$. The simplest ratio for a, b, and c is then 4:3:1, and $\delta \sim 0.99$, which is larger than the critical value $\delta_c^p \sim 0.79$. In the prolate case, the next candidate is $a : b : c = 5 : 4 : 2$ with $\delta \sim 0.72$ and $L.C.M. = 20$. In the oblate case, we obtain the ratio $a : b : c = 1 : 2 : 4$, where L.C.M. is 4 and $\delta \sim -1.13$, whose absolute value seems to be quite large but is still within the limit of $\delta_c^n \sim -1.32$. The other candidate in the oblate case is $a:b:c = 3:4:6$ with $\delta \sim -0.61$ and L.C.M. equal to 12. When $\tan\gamma = \sqrt{3}/4$, which corresponds to $\gamma \sim 23^\circ$, Eq. (9c) gives $3a + 2c = 5b$. As long as L.C.M. is less than 30, we cannot find any ratio within the range $\delta_c^n < \delta < \delta_c^p$ for both the prolate and the oblate shapes. When $\tan\gamma = \sqrt{3}/6$, which corresponds to $\gamma \sim 16^\circ$, Eq. (9c) gives $5a + 2c = 7b$. Again, as long as L.C.M. is assumed to be less than 30, we cannot find any ratio within the range of $\delta_c^n < \delta < \delta_c^p$. The energy degeneracy between the prolate and oblate shapes disappears in all the cases discussed above.

We discuss now the case of $\gamma \sim 19^\circ$. Although the absolute value of δ ($\delta \sim -1.13$) is large, we consider the oblate case of $a:b:c = 1:2:4$, as the L.C.M.(= 4) is small. In this case, c_y is replaced by an s_2 boson and c_z by an s_4 boson. Then, $N_{\rm sh} = n_x + n_2 + n_4$, $\omega_{\rm sh} = \omega_0/2$, $\langle\tilde{Q}_0\rangle = 2n_4 - n_x - n_2$, and $\langle\tilde{Q}_2 + \tilde{Q}_{-2}\rangle/\sqrt{6} = n_x - n_2$. Equation (42) gives

$$\begin{aligned}\langle\langle n_x, n_y, n_z|\ell\cdot\ell|n_x, n_y, n_z\rangle\rangle &= \left(2+\frac{1}{2}\right)n_xn_y + \left(2+\frac{1}{2}\right)n_yn_z + \left(2+\frac{9}{4}\right)n_zn_x \\ &\quad + \left(2+\frac{11}{8}\right)n_x + \left(2+\frac{1}{2}\right)n_y + \left(2+\frac{11}{8}\right)n_z + \frac{13}{16}. \end{aligned} \tag{51}$$

We see that the coefficients of the boson numbers n_x and n_z and their product n_zn_x in Eq. (51) differ significantly from 2. The magic numbers in this case are 2, 4, 8, 12, 20, 28, 40, 52, 70, and 88.

Next we consider the oblate case of $a:b:c = 3:4:6$ ($\delta \sim -0.61$). In this case, a new boson s_6 is introduced for c_z in the form of the sixfold product in Eq. (14), i.e.,

$$c_z = \frac{1}{\sqrt{6(\hat{n}_6+1)(\hat{n}_6+2)(\hat{n}_6+3)(\hat{n}_6+4)(\hat{n}_6+5)}}(s_6)^6. \tag{52}$$

Here, $\hat{n}_6 = s_6^\dagger s_6$, and $N_{\rm sh} = n_3 + n_4 + n_6$, $\omega_{\rm sh} = 3\omega_0/13$, $\langle\tilde{Q}_0\rangle = 2n_6 - n_3 - n_4$, and $\langle\tilde{Q}_2 + \tilde{Q}_{-2}\rangle/\sqrt{6} = n_3 - n_4$. Equation (42) becomes

$$\begin{aligned}\langle\langle n_x, n_y, n_z|\ell\cdot\ell|n_x, n_y, n_z\rangle\rangle &= \left(2+\frac{1}{12}\right)n_xn_y + \left(2+\frac{1}{6}\right)n_yn_z + \left(2+\frac{1}{2}\right)n_zn_x \\ &\quad + \left(2+\frac{7}{24}\right)n_x + \left(2+\frac{1}{8}\right)n_y + \left(2+\frac{1}{3}\right)n_z + \frac{3}{16}. \end{aligned} \tag{53}$$

All the coefficients in Eq. (53) are close to 2. In Table III, we summarize the classification of the single-particle energy level for this case. As seen in Table III, the states for $N_{\rm sh} = 1, 2$, and 5 do not exist, since n_3 is a multiple of 3, n_4 is a multiple of 4, and n_6 is a multiple of 6. Although the L.C.M. is 12, as it is for the case of $\gamma = 30^\circ$ with $a:b:c = 4:3:2$, the level degeneracy in $\gamma \sim 19^\circ$ is less than the level degeneracy for $\gamma = 30^\circ$. For example, only two levels belong to $N_{\rm sh} = 6$ (three levels in Table II), and four levels to $N_{\rm sh} = 12$ (seven levels in Table II). The sequence of magic numbers in Table III

TABLE III. Level degeneracy and the single-particle eigenfunction for $0 \leqslant N_{\rm sh} \leqslant 12$ at $\tan\gamma = \sqrt{3}/5$ with $a:b:c = 3:4:6$ (oblate case).

$N_{\rm sh}$	n_3	n_4	n_6	$\langle\tilde{Q}_0\rangle$	$\langle\tilde{Q}_2 + \tilde{Q}_{-2}\rangle/\sqrt{6}$	$\langle C\rangle$
0	0	0	0	0	0	0
3	3	0	0	−3	3	36
4	0	4	0	−4	−4	56
6	0	0	6	12	0	108
	6	0	0	−6	6	
7	3	4	0	−7	−1	140
8	0	8	0	−8	−8	176
9	3	0	6	9	3	216
	9	0	0	−9	9	
10	0	4	6	8	−4	260
	6	4	0	−10	2	
11	3	8	0	−11	−5	308
12	0	0	12	24	0	360
	6	0	6	6	6	
	0	12	0	−12	−12	
	12	0	0	−12	12	

is 2, 4, 6, 10, 12, 14, 18, 22, 24, 32, 36, 40, 48, 56, 60, 72, and 80.

We can infer the direction of the largest alignment from the calculation of $\langle\langle\ell_k^2\rangle\rangle$. For the level $n_x = n_y = 0$ and $n_z = 1$ in the $N_{\rm sh} = 6 (= 3n_x + 4n_y + 6n_z)$ shell, the largest alignment is found to be along the y axis, and the next to the largest along the x axis. For the level $n_x = 2$ and $n_y = n_z = 0$, the largest alignment is along the y axis and the next to the largest is along the x axis. Similar estimates can be carried out also for $\tilde{\ell}_k^2 (k = 3, 4, 6)$ as SU(3) operators.

For the prolate case of $\gamma \sim 19^\circ$ with $a:b:c = 5:4:2$ ($\delta \sim 0.72$), a new boson s_5 is introduced for c_x in the form of a fivefold product in Eq. (14),

$$c_x = \frac{1}{\sqrt{5(\hat{n}_5+1)(\hat{n}_5+2)(\hat{n}_5+3)(\hat{n}_5+4)}}(s_5)^5, \tag{54}$$

where, $\hat{n}_5 = s_5^\dagger s_5$. Here, c_y is replaced by an s_4 boson, and c_z by an s_2 boson. Subsequently, $N_{\rm sh} = n_5 + n_4 + n_2$, $\omega_{\rm sh} = 3\omega_0/11$, $\langle\tilde{Q}_0\rangle = 2n_2 - n_5 - n_4$, and $\langle\tilde{Q}_2 + \tilde{Q}_{-2}\rangle/\sqrt{6} = n_5 - n_4$. Equation (42) becomes

$$\begin{aligned}\langle\langle n_x, n_y, n_z|\ell\cdot\ell|n_x, n_y, n_z\rangle\rangle &= \left(2+\frac{1}{20}\right)n_xn_y + \left(2+\frac{1}{2}\right)n_yn_z + \left(2+\frac{9}{10}\right)n_zn_x \\ &\quad + \left(2+\frac{19}{40}\right)n_x + \left(2+\frac{11}{40}\right)n_y + \left(2+\frac{7}{10}\right)n_z + \frac{29}{80}. \end{aligned} \tag{55}$$

The coefficients in Eq. (55) are better than those in Eq. (51), but worse than those in Eq. (53) because of a larger δ. In

TABLE IV. Level degeneracy and the single-particle eigenfunction for $0 \leqslant N_{sh} \leqslant 11$ at $\tan\gamma = \sqrt{3}/5$ with $a:b:c = 5:4:2$ (prolate case).

N_{sh}	n_5	n_4	n_2	$\langle\tilde{Q}_0\rangle$	$\langle\tilde{Q}_2+\tilde{Q}_{-2}\rangle/\sqrt{6}$	$\langle C\rangle$
0	0	0	0	0	0	0
2	0	0	2	4	0	20
4	0	4	0	−4	−4	56
	0	0	4	8	0	
5	5	0	0	−5	5	80
6	0	4	2	0	−4	108
	0	0	6	12	0	
7	5	0	2	−1	5	140
8	0	8	0	−8	−8	176
	0	4	4	4	−4	
	0	0	8	16	0	
9	5	4	0	−9	1	216
	5	0	4	3	5	
10	10	0	0	−10	10	260
	0	8	2	−4	−8	
	0	4	6	8	−4	
	0	0	10	20	0	
11	5	4	2	−5	1	308
	5	0	6	7	5	

Table IV, the classification of the single-particle energy levels is summarized. As seen in Table IV, the states $N_{sh} = 1$ and 3 do not exist, since n_5 is a multiple of 5, n_4 a multiple of 4, and n_2 a multiple of 2. We can infer the direction of the maximum alignment from the expectation value of ℓ_k^2. For the level with $n_x = 0, n_y = 1$, and $n_z = 2$ in $N_{sh} = 6 (= 5n_x + 4n_y + 2n_z)$, the largest alignment is found to be along the x axis, and the next to the largest along the y axis. For the level with $n_x = n_y = 0$ and $n_z = 3$, the largest alignment is along the y axis and the next to the largest along the x axis. This is also inferred from the SU(3) group operator space. The shell energy is a little smaller than for the oblate case of 3:4:6. The sequence of magic numbers in the prolate case (Table IV) 2, 4, 8, 10, 14, 16, 22, 26, 34, 38, 48, 54, 66, and 74 differs from that in the oblate case (Table III).

VI. CONCLUSION

We have introduced new bosons corresponding to the integral ratio of three frequencies for a harmonic oscillator potential, by means of a nonlinear transformation that realizes the SU(3) group as a dynamical symmetry group and leaves the anisotropic harmonic oscillator Hamiltonian invariant. In other words, we have constructed a set of operators of SU(3) as a covering group, and all oscillator states are embedded in the SU(3) representation bases. The vacuum is the same as for the original boson, but the new boson numbers in the physical states are restricted to multiples of integral coefficients.

Since various combinations of integral coefficients are allowed for a fixed γ, we have tested several cases with different L.C.M. for the integral coefficients. According to increasing L.C.M., $|\delta|$ becomes smaller and the level degeneracy decreases. We have shown that the physical operators Q and ℓ are approximated by the group operators $\tilde{Q}$ and $\tilde{\ell}$, as long as the absolute value of δ is not large. As examples, we have considered two cases: $\tan\gamma = 1/\sqrt{3}(\gamma = 30^\circ)$ and $\tan\gamma = \sqrt{3}/5(\gamma \sim 19^\circ)$. For the former case, both prolate and oblate nuclear shapes have a common $|\delta|$ and L.C.M.(= 6 or 12), while for the latter case the oblate nuclear shape has a smaller L.C.M. (= 4 or 12) than the prolate nuclear shape (L.C.M. = 20).

Shell closures are predicted by the sequence of magic numbers depending on the deformation parameters δ and γ. For the realistic operator ℓ, we can estimate the direction of orbital angular momentum alignment.

[1] J. P. Elliott, Proc. Roy. Soc. (London) A **245**, 128 (1958); **245**, 562 (1958); J. P. Elliott and M. Harvey, *ibid.* **272**, 557 (1963).

[2] K. Sugawara-Tanabe and A. Arima, Nucl. Phys. **A619**, 88 (1997).

[3] S. W. Ødegård *et al.*, Phys. Rev. Lett. **86**, 5866 (2001); D. R. Jensen *et al.*, *ibid.* **89**, 142503 (2002); G. Schönwaßer *et al.*, Phys. Lett. **B552**, 9 (2003); H. Amro *et al.*, Phys. Rev. C **71**, 011302(R) (2005).

[4] A. Bohr and B. R. Mottelson, *Nuclear Structure* (Benjamin, Reading, MA, 1975), Vol. II, Chap. 5.

[5] F. Duimio and G. Zambotti, Nuovo Cimento A **43**, 1203 (1966); I. Vendramin, *ibid.* **54**, 190 (1968); R. A. Brandt and O. W. Greenberg, J. Math. Phys. **10**, 1168 (1969); A. Cisneros and H. V. McIntosh, *ibid.* **11**, 870 (1970); W. Nazarewicz and J. Dobaczewski, Phys. Rev. Lett. **68**, 154 (1992).

[6] K. Tanabe and H. Sugawara, Prog. Theor. Phys. **32**, 911 (1964).

[7] C. H. Townes and A. L. Shawlow, *Microwave Spectroscopy* (McGraw-Hill, New York, 1955), Chap. 7.

[8] K. Sugawara-Tanabe and K. Tanabe, Nucl. Phys. **A208**, 317 (1973).

[9] F. Iachello and A. Arima, *The Interacting Boson Model* (Cambridge University Press, Cambridge, 1987), Chap. 2.