New exact solutions of the standard pairing model for well-deformed nuclei

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A new step-by-step diagonalization procedure for evaluating exact solutions of the nuclear deformed mean-field plus pairing interaction model is proposed via a simple Bethe ansatz in each step from which the eigenvalues and corresponding eigenstates can be obtained progressively. This new approach draws upon an observation that the original one- plus two-body problem in a k-particle Hilbert subspace can be mapped onto a one-body grand hard-core boson picture that can be solved step by step with a simple Bethe ansatz known from earlier work. Based on this new procedure, it is further shown that the extended pairing model for deformed nuclei [Feng Pan, V. G. Gueorguiev, and J. P. Draayer, Phys. Rev. Lett. **92**, 112503 (2004)] is similar to the standard pairing model with the first step approximation, in which only the lowest energy eigenstate of the standard pure pairing interaction part is taken into consideration. Our analysis shows that the standard pairing model with the first step approximation displays similar pair structures of the first few exact low-lying states of the model, which, therefore, provides a link between the two models.

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I. INTRODUCTION

Pairing is an important residual interaction when a mean-field approach is used as a starting approximation for a description of nuclear structure. In particular, pairing correlations are essential for a description of binding energies, odd-even effects, single-particle occupancies, excitation spectra, electromagnetic transition rates, beta-decay probabilities, transfer reaction amplitudes, low-lying collective modes, level densities, and moments of inertia, etc. [1–3]. Two commonly used methods, the Bardeen-Cooper-Schrieffer (BCS) [4] and Hartree-Fock-Bogolyubov (HFB) [2] technique for finding approximate solutions are well known. The limitations of these methods, when applied in nuclear physics, are well understood [5], which is also the case when using these methods to determine the energy spectra of nanoscale metallic grains [6,7]. Various procedures have been used to correct the approximation deficiencies, such as particle-number projected mean-field treatments [8–10], the use of coherent states [11], stochastic number projection techniques [12], statistical descriptions [13], treatments of the residual parts of the Hamiltonian in the random phase approximation [14,15], and various recursive approaches [16,17]. Typically, however, these procedures have been found to have only limited applicability because the results usually yield insufficient accuracy. Other methods are likewise limited by their own sets of complications.

On the other hand, an exact treatment of the nuclear mean-field plus pairing type Hamiltonian was initiated by Richardson and Gaudin, known as the Richardson-Gaudin method [18–21]. Recently, extensions to the Richardson-Gaudin theory have been made using the Bethe ansatz methodology [22–31], especially, an application of the Richardson-Gaudin solution to Sm isotopes with less than seven pairs of valence nucleons was made [32]. Though these approaches show that the mean-field plus pairing model is exactly solvable, the solutions are generally not simple and normally require extensive numerical work, especially when the number of levels and valence pairs are large in spite of the recent efforts in improving the procedure [33]. In [34], an extended pairing model with many-pair interaction terms was proposed, which can be solved based on a simpler Bethe ansatz, and describes even-odd mass differences in $^{154-171}$Yb isotopes rather well. However, it was not clear why the pairing interaction strength in the model should be drastically reduced when increasing the number of valence nucleon pairs in order to fit experimental data of even-odd mass differences. Moreover, since the standard form of the pairing Hamiltonian is commonly adopted, it should be interesting to see whether the extended pairing model in some aspects covers main features of the standard model.

In the following, a new step-by-step diagonalization procedure for evaluating exact solutions of the nuclear deformed mean-field plus pairing interaction model will be proposed via a simple Bethe ansatz in each step from which the eigenvalues and corresponding eigenstates can be obtained progressively, which is shown in Sec. II. In Sec. III, as a demonstration of the procedure, a system with $p = 10$ levels will be analyzed. In Sec. IV, it will be shown that the extended pairing model for deformed nuclei is similar to the standard pairing model with the first step approximation, in which only the lowest energy eigenstate of the standard pure pairing interaction part is taken into consideration. Our analysis reveals that the standard pairing Hamiltonian with the first step approximation displays similar pair structures of the first few exact low-lying states of the model, which, therefore, provides a link between the two models. A short discussion regarding implications of our results is given in Sec. V.

II. A NEW DIAGONALIZATION PROCEDURE FOR THE MEAN-FIELD PLUS STANDARD PAIRING HAMILTONIAN

The deformed mean-field plus standard pairing model Hamiltonian is given by

$$\hat{H} = \sum_{j=1}^{p} \epsilon_j \hat{n}_j - GS^+S^- = \sum_{j=1}^{p} \epsilon_j \hat{n}_j - G\sum_{ij} S_i^+ S_j^-, \quad (1)$$

where p is the total number of levels considered, $G > 0$ is the overall pairing strength, $\{\epsilon_j\}$ are single-particle energies taken from any deformed mean-field, such as the Nilsson model [2] or the relativistic mean-field theory [35–37], $\hat{n}_j = c^\dagger_{j\uparrow}c_{j\uparrow} + c^\dagger_{j\downarrow}c_{j\downarrow}$ is the fermion number operator for the jth level, and $S^+_i = c^\dagger_{i\uparrow}c^\dagger_{i\downarrow}(S^-_i = (S^+_i)^\dagger = c_{i\downarrow}c_{i\uparrow})$ are pair creation (annihilation) operators. The up and down arrows in these expressions refer to time-reversed states. Since the formalism for even-odd systems is similar, in the following we only focus on the even-even seniority zero case.

Since each deformed level can be occupied by no more than a single pair due to the Pauli principle, the Hamiltonian (1) is also equivalent to a finite-site hard-core Bose-Hubbard model with infinite range hopping and infinite on-site repulsion. Let $B^+_{i_1 i_2 \cdots i_k} = S^+_{i_1}S^+_{i_2}\cdots S^+_{i_k}$ with $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant p$. In the following, we set $\mu = (i_1 i_2 \cdots i_k)$ to be the μth normal order sequence with $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant p$. The operator B^+_μ can be regarded as a grand hard-core boson creation operator [34]. The total number of these operators is $p!/((p-k)!k!)$ which exactly equals the dimension of the Hilbert subspace of k pairs with no double occupancy.

For k-pair excitations, by using the standard second quantization formalism, the Hamiltonian (1) can effectively be reduced to the following 'one-body' Hamiltonian in the grand hard-core boson picture:

$$\hat{H}_k = \sum_{\mu\nu}\langle\mu|\sum_{j=1}^{p}\epsilon_j\hat{n}_j|\nu\rangle B^+_\mu B_\nu - G\sum_{\mu\nu}\langle\mu|S^+S^-|\nu\rangle B^+_\mu B_\nu, \tag{2}$$

where $B_\mu = (B^+_\mu)^\dagger$, because any k-pair eigenstate of Eq. (1) can be expanded in terms of single grand hard-core boson states $\{|\mu\rangle = B^+_\mu|0\rangle\}$, where $|0\rangle$ is the pair vacuum state. In Eq. (2), the mean-field one-body term in the Hilbert subspace spanned by $\{|\mu\rangle = B^+_\mu|0\rangle\}$ is diagonal with the matrix elements

$$\langle\mu|\sum_{j=1}^{p}\epsilon_j\hat{n}_j|\nu\rangle = \delta_{\mu\nu}2\bar{\epsilon}_\mu = \delta_{\mu\nu}2\sum_{t=1}^{k}\epsilon_{i_t}, \tag{3}$$

while the matrix elements of the pairing interaction term are

$$\begin{aligned}\langle\mu|S^+S^-|\nu\rangle = &\sum_{q\rho}\langle\mu|p/2-q\rho k\rangle_{\mathrm{QQ}}\\ &\times\langle p/2-q\rho k|S^+S^-|p/2-q\rho k\rangle_{\mathrm{QQ}}\\ &\times\langle p/2-q\rho k|\nu\rangle,\end{aligned} \tag{4}$$

in which

$$\begin{aligned}&{}_{\mathrm{Q}}\langle p/2-q\rho k|S^+S^-|p/2-q\rho k\rangle_{\mathrm{Q}}\\ &\quad = h^{(q)}_k = (k-q)(p-k-q+1)\end{aligned} \tag{5}$$

is the matrix element of S^+S^- in the Racah quasispin formalism [38] with the total quasispin $S_{\mathrm{Q}} = p/2-q$, where $q = 0, 1, \ldots, \min[k, p-k]$, and ρ is an additional quantum number needed in distinguishing from different states with the same quasispin for a given p and k. For a given p, the total number of different quasispin states with the same $S_{\mathrm{Q}} = p/2-q$ is given by [39]

$$\omega_q = \frac{(p-2q+1)p!}{(p-q+1)(p-q)!q!}. \tag{6}$$

Furthermore, $\alpha^{q\rho}_\mu = \langle\mu|p/2-q\rho k\rangle_{\mathrm{Q}}$ used in Eq. (4) is an overlap of the quasispin state $|p/2-q\rho k\rangle_{\mathrm{Q}}$ with μth single grand hard-core boson state $|\mu\rangle$, which can be chosen as real. It can easily be verified that the total number of different quasispin states equals exactly to the dimension of the Hilbert subspace of k pairs with no double occupancy:

$$d = \sum_{q=0}^{\min[k,p-k]}\omega_q = \frac{p!}{(p-k)!k!}. \tag{7}$$

Thus, Eq. (2) can explicitly be written as

$$\hat{H}_k = \sum_{\mu=1}^{d}2\bar{\epsilon}_\mu B^+_\mu B_\mu - G\sum_{q=0}^{\min[k,p-k]}h^{(q)}_k\sum_{\rho=1}^{\omega_q}\sum_{\mu\nu}\alpha^{q\rho}_\mu\alpha^{q\rho}_\nu B^+_\mu B_\nu. \tag{8}$$

In order to diagonalize the Hamiltonian (8), let us consider a simpler Hamiltonian with only the first term and $q = 0$ part in the second term of Eq. (8):

$$h_0 = \sum_{\mu=1}^{d}2\bar{\epsilon}_\mu B^+_\mu B_\mu - Gh^{(0)}_k\sum_{\mu\nu}\alpha^{01}_\mu\alpha^{01}_\nu B^+_\mu B_\nu. \tag{9}$$

As shown in [34,40], the Hamiltonian (9) can be diagonalized into the following form:

$$h_0 = \sum_{\tau_0}E^{(\tau_0)}D^+(E^{(\tau_0)})D(E^{(\tau_0)}) \tag{10}$$

with

$$D^+(E^{(\tau_0)}) = \sqrt{\frac{1}{\mathcal{N}_{\tau_0}}}\sum_\mu\frac{\alpha^{01}_\mu}{2\bar{\epsilon}_\mu - E^{(\tau_0)}}B^+_\mu, \tag{11}$$

where $E^{(\tau_0)}$ is the τ_0-th root of the following equation:

$$Gh^{(0)}_k\sum_\mu\frac{\left(\alpha^{01}_\mu\right)^2}{2\bar{\epsilon}_\mu - E^{(\tau_0)}} = 1, \tag{12}$$

and $\mathcal{N}_{\tau_0}$ is the normalization constant obtained from

$$\sum_\mu\frac{\left(\alpha^{01}_\mu\right)^2}{(E^{(\tau_0)}-2\bar{\epsilon}_\mu)(E^{(\tau_0')}-2\bar{\epsilon}_\mu)} = \delta_{\tau_0\tau_0'}\mathcal{N}_{\tau_0}. \tag{13}$$

In this case, Eq. (12) will provide exactly d different roots $E^{(\tau_0)}$ as long as all combinations of the single-particle energies $\bar{\epsilon}_\mu = \sum_{t=1}^{k}\epsilon_{i_t}$ are different for all k-pair excitation cases. Fortunately, this is always the case when single-particle energies $\{\epsilon_j\}$ are generated from any deformed mean-field theory. To the contrary, let us consider an extreme situation with all levels degenerate, $\bar{\epsilon}_\mu = \epsilon$ for any μ. In this case, Eq. (12) becomes

$$Gh^{(0)}_k\sum_\mu\frac{\left(\alpha^{01}_\mu\right)^2}{2\epsilon - E^{(\tau_0)}} = 1, \tag{14}$$

which only has one solution with $E^{(\tau_0)} = 2\epsilon - Gh_k^{(0)}$ since $\sum_\mu (\alpha_\mu^{01})^2 = 1$. Thus, other $d-1$ solutions cannot be obtained from such a procedure. A similar situation also occurs in the former Richardson-Gaudin solution to the standard pairing Hamiltonian (1). Though the Richardson-Gaudin method is effective to get exact solutions for both deformed and nondeformed cases, the method can only get complete solutions for cases with nondegenerate single-particle energies.

Since Eqs. (8) and (9) should be diagonalized within the single grand hard-core boson subspace spanned by $\{B_\mu^+|0\rangle\}$, the effective commutation relations needed to prove that Eq. (9) can indeed be expressed in the form shown in Eq. (10) are

$$[B_\nu, B_\mu^+] = \delta_{\mu\nu}, \tag{15}$$

which are only valid when they are applied into the vacuum state. Using Eq. (9) and the ansatz (11), we have

$$\begin{aligned}[h_0, D^+(E^{(\tau_0)})] &= E^{(\tau_0)} D^+(E^{(\tau_0)}) \\ &\quad - \sqrt{\frac{1}{\mathcal{N}_{\tau_0}}} \left(1 - Gh_k^{(0)} \sum_\mu \frac{(\alpha_\mu^{01})^2}{2\bar{\epsilon}_\mu - E^{(\tau_0)}}\right) \\ &\quad \times \sum_\nu \alpha_\nu^{01} B_\nu^+. \end{aligned} \tag{16}$$

Though a direct proof is in demand, it can be checked numerically with any set of parameters, $Gh_k^{(0)}$, $\{\alpha_\mu^{01}\}$, and $\{\bar{\epsilon}_\mu\}$ with $\bar{\epsilon}_1 \neq \bar{\epsilon}_2 \cdots \neq \bar{\epsilon}_d$, that the orthnormal condition (13) is automatically satisfied when $E^{(\tau_0)}$ satisfies Eq. (12). Therefore, Eq. (9) can indeed be expressed as that shown in Eq. (10) as long as Eq. (12) is satisfied.

In order to simplify our expression, in the following the indices (q, ρ) are relabeled by r with $r = (q, \rho)$. Thus, the Hamiltonian (8) can be rewritten as

$$\begin{aligned}\hat{H}_k &= \sum_{\tau_0} E^{(\tau_0)} D^+(E^{(\tau_0)}) D(E^{(\tau_0)}) \\ &\quad - G \sum_{r=1}^{d-1} h_k^{(r)} \sum_{\mu\nu} \alpha_\mu^r \alpha_\nu^r B_\mu^+ B_\nu. \end{aligned} \tag{17}$$

In the next step, we consider

$$h_1 = \sum_{\tau_0} E^{(\tau_0)} D^+(E^{(\tau_0)}) D(E^{(\tau_0)}) - Gh_k^{(1)} \sum_{\mu\nu} \alpha_\mu^1 \alpha_\nu^1 B_\mu^+ B_\nu \tag{18}$$

in Eq. (17). Using Eq. (11), we have

$$B_\mu^+ = \sum_{\tau_0} \sqrt{\frac{1}{\mathcal{N}_{\tau_0}}} \frac{\alpha_\mu^{01}}{2\bar{\epsilon}_\mu - E^{(\tau_0)}} D^+(E^{(\tau_0)}). \tag{19}$$

By using Eq. (19), $\sum_\mu \alpha_\mu^1 B_\mu^+$ in the second term of Eq. (18) can be expressed as

$$\sum_\mu \alpha_\mu^1 B_\mu^+ = \sum_{\tau_0} \sqrt{\frac{1}{\mathcal{N}_{\tau_0}}} \sum_\mu \frac{\alpha_\mu^{01} \alpha_\mu^1}{2\bar{\epsilon}_\mu - E^{(\tau_0)}} D^+(E^{(\tau_0)}). \tag{20}$$

Then, we similarly have

$$\begin{aligned}&\sum_{\tau_0} E^{(\tau_0)} D^+(E^{(\tau_0)}) D(E^{(\tau_0)}) - Gh_k^{(1)} \sum_{\mu\nu} \alpha_\mu^1 \alpha_\nu^1 B_\mu^+ B_\nu \\ &= \sum_{\tau_0} E^{(\tau_0)} D^+(E^{(\tau_0)}) D(E^{(\tau_0)}) \\ &\quad - Gh_k^{(1)} \sum_{\tau_0 \tau_0'} \Lambda_{\tau_0} \Lambda_{\tau_0'} D^+(E^{(\tau_0)}) D(E^{(\tau_0')}) \\ &= \sum_{\tau_1} E^{(\tau_1)} D^+(E^{(\tau_1)}) D(E^{(\tau_1)}), \end{aligned} \tag{21}$$

where

$$\Lambda_{\tau_0} = \sqrt{\frac{1}{\mathcal{N}_{\tau_0}}} \sum_\mu \frac{\alpha_\mu^{01} \alpha_\mu^1}{2\bar{\epsilon}_\mu - E^{(\tau_0)}}, \tag{22}$$

$$D^+(E^{(\tau_1)}) = \sqrt{\frac{1}{\mathcal{N}_{\tau_1}}} \sum_{\tau_0} \frac{\Lambda_{\tau_0}}{E^{(\tau_0)} - E^{(\tau_1)}} D^+(E^{(\tau_0)}), \tag{23}$$

$E^{(\tau_1)}$ is the τ_1-th root of the following equation:

$$Gh_k^{(1)} \sum_{\tau_0} \frac{(\Lambda_{\tau_0})^2}{E^{(\tau_0)} - E^{(\tau_1)}} = 1, \tag{24}$$

and $\mathcal{N}_{\tau_1}$ is the normalization constant obtained from

$$\sum_{\tau_0} \frac{(\Lambda_{\tau_0})^2}{(E^{(\tau_0)} - E^{(\tau_1)})(E^{(\tau_0)} - E^{(\tau_1')})} = \delta_{\tau_1 \tau_1'} \mathcal{N}_{\tau_1}. \tag{25}$$

Thus, using the results shown in Eqs. (21)– (25) and following the above procedure consecutively, we finally have

$$\begin{aligned}\hat{H}_k &= \sum_{\tau_0} E^{(\tau_0)} D^+(E^{(\tau_0)}) D(E^{(\tau_0)}) - G \sum_{r=1}^{d-1} h_k^{(r)} \sum_{\mu\nu} \alpha_\mu^r \alpha_\nu^r B_\mu^+ B_\nu \\ &= \sum_{\tau_{d-1}} E^{(\tau_{d-1})} D^+(E^{(\tau_{d-1})}) D(E^{(\tau_{d-1})}), \end{aligned} \tag{26}$$

where

$$D^+(E^{(\tau_{d-1})}) = \sqrt{\frac{1}{\mathcal{N}_{\tau_{d-1}}}} \sum_{\tau_{d-2}} \frac{\Lambda_{\tau_{d-2}}}{E^{(\tau_{d-2})} - E^{(\tau_{d-1})}} D^+(E^{(\tau_{d-2})}) \tag{27}$$

with

$$\Lambda_{\tau_s} = \sqrt{\frac{1}{\mathcal{N}_{\tau_s}}} \sum_{\tau_0 \tau_1 \cdots \tau_{s-1}} \prod_{\nu=0}^{s-1} \frac{\Lambda_{\tau_\nu}}{E^{(\tau_\nu)} - E^{(\tau_{\nu+1})}} \sum_\mu \frac{\alpha_\mu^{s+1} \alpha_\mu^{01}}{2\bar{\epsilon}_\mu - E^{\tau_0}}, \tag{28}$$

$$D^+(E^{(\tau_{s+1})}) = \sqrt{\frac{1}{\mathcal{N}_{\tau_{s+1}}}} \sum_{\tau_s} \frac{\Lambda_{\tau_s}}{E^{(\tau_s)} - E^{(\tau_{s+1})}} D^+(E^{(\tau_s)}), \tag{29}$$

$$Gh_k^{(s+1)} \sum_{\tau_s} \frac{(\Lambda_{\tau_s})^2}{E^{(\tau_s)} - E^{(\tau_{s+1})}} = 1, \tag{30}$$

and

$$\sum_{\tau_s} \frac{(\Lambda_{\tau_s})^2}{(E^{(\tau_s)} - E^{(\tau_{s+1})})(E^{(\tau_s)} - E^{(\tau_{s+1}')})} = \delta_{\tau_{s+1} \tau_{s+1}'} \mathcal{N}_{\tau_{s+1}}, \tag{31}$$

for $s = 0, 1, 2, \ldots, d-2$. Hence, after d steps, the Hamiltonian (8) is diagonalized as shown in Eq. (26), of which the eigenstate is

$$|k, \tau_{d-1}\rangle = D^+(E^{(\tau_{d-1})})|0\rangle \tag{32}$$

with the corresponding eigenenergy $E^{(\tau_{d-1})}$.

This new step-by-step diagonalization procedure needs at most d steps to get final exact results, but in each step the corresponding Bethe ansatz equation (30) contains only one variable, of which roots can easily be obtained numerically similar to what is required in the extended pairing model proposed previously [34], and in the TDA and RPA approximations with separable potentials [2]. Though this method may be unpractical for large size systems because one needs to get all d roots from Eq. (30) in each step, this procedure can also be used to check contributions from pairing potential in the Racah quasispin formalism for different q of the second term in Eq. (8), and is certainly applicable to relatively small systems. Actually, for k pair excitation, though each term with different q from the second term of Eq. (8) will contribute to the final eigenenergy and correlate with eigenstates, the first few of these terms are key to determining properties of the first few low-lying states of the model as will be shown in the next section.

III. A NUMERICAL EXAMPLE FOR $p = 10$

In this section, we will apply this new step-by-step diagonalization procedure to the deformed mean-field plus standard pairing model for $p = 10$ levels with number of pairs $k = 1, 2, \ldots, 10$, in which the single particle energies are given by $\epsilon_i = i + \chi_i$ for $i = 1, 2, \ldots, 10$, where χ_i are random numbers within the interval (0, 1) to avoid accidental degeneracy, and the pairing strength is set to be $G = 0.5$. Since $h_k^{(0)} > h_k^{(1)} \geqslant \cdots \geqslant h^{d-1}$ is always satisfied, the lowest quasispin term with $q = 0$ from the pairing potential should be most important to the first few eigenstates of the model, which is indeed the case as can be seen from results shown in Table I.

In each step, we need to calculate the overlaps of the quasispin states $|p/2 - q\rho k\rangle_{\mathrm{Q}}$ with μth single grand hard-core boson states $|\mu\rangle$. For $q = 0$, the k-pair state

$$\begin{aligned}|p/2k\rangle_{\mathrm{Q}} &= \sqrt{(p-k)!/(k!p!)}(S^+)^k|0\rangle \\ &= \sqrt{k!(p-k)!/p!} \sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant p} S_{i_1}^+ S_{i_2}^+ \cdots S_{i_k}^+|0\rangle \\ &= \sqrt{1/d}\sum_\mu B_\mu^+|0\rangle \end{aligned} \tag{33}$$

is the eigenstate of the operator S^+S^-. Thus we have

$$\alpha_\mu^{01} = \langle\mu|p/2k\rangle_{\mathrm{Q}} = \sqrt{1/d}. \tag{34}$$

For $q \geqslant 1$, the quasispin states $|p/2 - q\rho k\rangle_{\mathrm{Q}}$ can be obtained by directly diagonalizing S^+S^- as shown in Eq. (5), or by using the representation theory of $SU_{\mathrm{Q}}(2) \times S_p$ summarized in [39]. Then, one can use them to calculate the overlaps $\alpha_\mu^{q\rho} = \langle\mu|p/2 - q\rho k\rangle_{\mathrm{Q}}$.

In Table I, we list the first five eigenenergies and overlaps of the eigenstates with the corresponding exact ones for number of pairs $k = 1, 2, \ldots, 10$ calculated with only the h^0 term involved, which is called the first step approximation. With the first step approximation, it can be seen from Table I that results of $k = 1$ and $k = 10$ cases are exact because $h^{(q)} = 0$ for $q \geqslant 1$. The approximate energy eigenvalues will gradually increase the corresponding exact ones with increasing the number of pairs k since pairing potential terms $h^{(q)}$ with $q = 1, 2, \ldots, \min[p-k, k]$ will contribute more and more to the final eigenenergies. However, the overlaps of the eigenstates with the corresponding exact ones are always greater than 88% for the ground and the first excited states for any number of pairs k. Therefore, the $h^{(0)}$ term of the pairing potential is dominant in determining pairing structure of the first two excitation states in the model though the corresponding energy eigenvalues are different from the exact ones.

Since the largest deviation of the energy eigenvalues from the exact ones occurs at the half-filling case, using the procedure shown in the previous section, we calculated energy eigenvalues step by step for the $k = 5$ case with $q = 0, 1, \ldots, 4$ since $h^{(5)} = 0$, of which the results are shown in Table II. For a given q, there are actually ω_q substeps involved in the diagonalization process according to the procedure shown in the previous section. It can be seen from Table II that the overlaps of the first four eigenstates with the corresponding exact ones will reach 99% after three diagonalization steps though there is still a deviation in eigenenergies, which shows that the first few $h^{(q)}$ terms with $q = 0, 1, 2$ are key to determine the pair structure of the first few low-lying states of the model. While high-lying quasispin states mainly correlate with high excited states of the model and keep the low part of the spectrum less affected.

IV. COMPARISON WITH THE EXTENDED PAIRING MODEL

For k pair excitations, if only the $h^{(0)}$ term from the standard pairing potential is considered for the standard pairing Hamiltonian (8), namely,

$$\hat{H}_k^{(1)} = \sum_{\mu=1}^{d} 2\bar{\epsilon}_\mu B_\mu^+ B_\mu - G(k(p-k+1)/d)\sum_{\mu\nu} B_\mu^+ B_\nu, \tag{35}$$

following Eqs. (9)–(13), the eigenstate of Eq. (35) can be written as

$$|k;\tau) = \sum_{\mu=1}^{d} \frac{1}{2\bar{\epsilon}_\mu - E^{(\tau)}} B_\mu^+|0\rangle, \tag{36}$$

where τ is an additional quantum number used to distinguish different excitation states, and $E^{(\tau)}$ is an unknown variable to be determined in diagonalizing Eq. (35). In solving the following eigenequation:

$$\hat{H}_k^{(1)}|k;\tau) = E_k^{(\tau)}|k;\tau) \tag{37}$$

TABLE I. First five eigenenergies of the standard pairing model with $p = 10$ levels for $k = 0, 1, 2 \ldots, 10$ obtained from the first step approximation (appro.) and compared with the corresponding exact results (exact), and overlap (olp) of the eigenstates obtained from the first step approximation with the corresponding exact ones, in which the single-particle energies $\epsilon_1 = 1.706$, $\epsilon_2 = 2.754$, $\epsilon_3 = 3.440$, $\epsilon_4 = 4.349$, $\epsilon_5 = 5.743$, $\epsilon_6 = 6.604$, $\epsilon_7 = 7.591$, $\epsilon_8 = 8.959$, $\epsilon_9 = 9.335$, $\epsilon_{10} = 10.125$, and pairing strength $G = 0.5$, where the single-particle energies and G are given in arbitrary units.

	$k=1$			$k=2$			$k=3$			$k=4$			$k=5$		
	exact	appro.	olp	exact	appro.	olp	exact	appro.	olp	exact	appro.	olp	exact	appro.	olp
E_1	2.255	2.255	100%	6.662	8.133	98%	12.873	15.423	93%	21.101	24.272	91%	31.856	35.748	88%
E_2	4.797	4.797	100%	8.999	9.808	97%	15.508	17.213	92%	24.293	26.909	90%	34.529	37.447	88%
E_3	6.438	6.438	100%	10.756	11.392	78%	16.961	18.576	89%	26.161	28.312	65%	36.407	39.209	83%
E_4	8.272	8.272	100%	13.128	12.2618	75%	17.973	19.832	82%	26.176	29.058	67%	36.961	40.167	81%
E_5	10.969	10.969	100%	13.375	13.625	85%	18.936	20.598	65%	27.621	29.882	62%	38.828	41.206	55%
	$k=6$			$k=7$			$k=8$			$k=9$			$k=10$		
	exact	appro.	olp	exact	appro.	olp	exact	appro.	olp	exact	appro.	olp	exact	appro.	olp
E_1	44.638	48.868	91%	59.532	63.713	96%	76.780	79.982	96%	95.625	97.091	95%	116.212	116.212	100%
E_2	47.415	50.152	90%	62.610	65.911	93%	78.949	82.641	97%	97.888	101.652	96%			
E_3	49.165	52.261	81%	63.852	67.440	82%	80.277	83.861	81%	99.015	102.961	99%			
E_4	50.908	53.366	76%	64.573	69.232	71%	81.182	85.191	82%	101.540	105.221	98%			
E_5	51.728	54.227	74%	65.263	69.651	55%	83.148	86.404	75%	103.552	107.341	98%			

with $B_\mu B_\nu^+|0\rangle = \delta_{\mu\nu}|0\rangle$, it shows that the variable $E^{(\tau)}$ must satisfy the following equation:

$$G(k(p-k+1)/d)\sum_{\mu=1}^{d}\frac{1}{2\bar{\epsilon}_\mu - E^{(\tau)}} = 1, \tag{38}$$

and the eigenenergy $E_k^{(\tau)} = E^{(\tau)}$. Thus, the additional quantum number τ labels different roots of Eq. (34). This is the so-called first step approximation shown in Sec. II. The solution is complete so long as all combinations of the single-particle energies $\sum_{t=1}^{k}\epsilon_{i_t}$ are different for all k-pair excitation cases. Fortunately, this is always the case when single-particle energies $\{\epsilon_j\}$ are generated from any deformed mean-field theory. Since the single grand particle energies $2\bar{\epsilon}_\mu$ are all different, there are exactly $p!/((p-k)!k!)$ distinct roots in Eq. (38). The resultant eigenstates (33), which are mutually orthogonal but not normalized, satisfy

$$(k;\tau|k;\tau') = \delta_{\tau\tau'}\mathcal{N}_\tau, \tag{39}$$

where

$$\mathcal{N}_\tau = \sum_{\mu=1}^{d}\frac{1}{(2\bar{\epsilon}_\mu - E^{(\tau)})^2}. \tag{40}$$

It follows that the normalized eigenstate can be expressed as $|k;\tau\rangle = \sqrt{1/\mathcal{N}_\tau}|k;\tau)$.

As shown in [34], a Nilsson mean-field plus extended pairing interaction Hamiltonian

$$\hat{H}_{\rm ex} = \sum_{j=1}^{p}\epsilon_j n_j - G_{\rm ex}\sum_{i,j}S_i^+S_j - G_{\rm ex}\left(\sum_{\rho=2}^{\infty}\frac{1}{(\rho!)^2}\times\sum_{i_1\neq i_2\neq\cdots\neq i_{2\rho}}S_{i_1}^+S_{i_2}^+\cdots S_{i_\rho}^+S_{i_{\rho+1}}S_{i_{\rho+2}}\cdots a_{i_{2\rho}}\right), \tag{41}$$

where no pair of indices among $\{i_1, i_2, \cdots, i_{2\rho}\}$ is the same for any ρ, can also be solved exactly by using a simple

TABLE II. First five eigenenergies of the standard pairing model with $p = 10$ levels for $k = 5$ obtained from step-by-step diagonalization procedure, in which the parameters used are the same as those shown in Table I, where only $h^{(q)}$ with $q = 0, 1, 2, \ldots, s$ terms from the pairing potential are involved in the sth step approximation as described in the previous section.

	1st step $q=0,1$		2nd step $q=0,1,2$		3rd step $q=0,1,2,3$		4th step $q=0,1,2,3,4$		exact $q=0$	
	eigenvalue	olp	eigenvalue	olp	eigenvalue	olp	eigenvalue	olp	eigenvalue	olp
E_1	35.748	88%	33.862	98%	32.481	99%	32.176	99%	31.856	100%
E_2	37.447	88%	36.566	97%	35.631	99%	35.102	99%	34.528	100%
E_3	39.209	83%	38.398	90%	37.451	99%	36.847	99%	36.407	100%
E_4	40.167	81%	39.422	89%	37.942	99%	37.637	99%	36.961	100%
E_5	41.206	55%	40.821	57%	39.779	97%	38.819	97%	38.828	100%

Bethe ansatz that is similar to what is proposed in this work. Besides the usual Nilsson mean-field and the standard pairing interaction, this form includes many-pair hopping terms that allow nucleon pairs to simultaneously scatter (hop) between and among different Nilsson levels. Furthermore, the extended pairing interaction Hamiltonian (41) can be used to describe even-odd mass differences rather well as long as the extended pairing interaction strength $G_{\rm ex}$ decreases with an increasing number of pairs k. It follows from this that it is interesting to compare results of the deformed mean-field plus standard pairing Hamiltonian (1) with those from the extended pairing model [34]. And indeed, it is not difficult to show that the expressions for eigenstates of the extended pairing Hamiltonian and those of the standard pairing Hamiltonian (1) in the first step approximation are the same. For k-pair excitations, the eigenenergies $E_k^{(\tau)}({\rm ex})$ of the extended pairing Hamiltonian (41) are given by

$$E_k^{(\tau)}({\rm ex}) = E_{\rm ex}^{(\tau)} - (k-1)G_{\rm ex}, \tag{42}$$

where $E_{\rm ex}^{\tau}$ is the τ-th root of the Bethe ansatz equation,

$$G_{\rm ex}\sum_{\mu=1}^{d}\frac{1}{2\bar{\epsilon}_\mu - E_{\rm ex}^{(\tau)}} = 1. \tag{43}$$

A comparison of Eq. (43) with the Bethe ansatz equation (36) for the standard pairing Hamiltonian in the first step approximation (35) shows that the two Hamiltonians yield exactly the same excitation energies and the corresponding eigenstates so long as the parameter $G_{\rm ex}$ in the extended pairing model and the parameter G in the standard pairing Hamiltonian (1) satisfy the following relation:

$$G_{\rm ex} = ((p-k)!k!(p-k+1)k/p!)\,G. \tag{44}$$

Furthermore, while the ground states of the two Hamiltonians are also the same, the ground-state energies are different. However, once the overall pairing strength G is fixed, and the parameter $G_{\rm ex}$ is chosen according to Eq. (44), it is easy to show that the difference between the ground-state energy of the extended model and that of the standard pairing model in the first step approximation is given by

$$E_k^{(\rm g)}({\rm ex}) - E_k^{(\rm g)} = -(k-1)G_{\rm ex}. \tag{45}$$

This expression shows that the extended pairing interaction contributes a little more attraction among valence pairs than the standard pairing interaction in the first step approximation, but reproduces excitation energies exactly the same as those in the standard pairing model with first step approximation. Since $G_{\rm ex}$ decreases drastically with an increasing of k toward the half-filling, the ground state energy difference of the two Hamiltonians becomes negligible with an increasing number of pairs k with $k \leqslant [p/2]-1$ when p is even, and $k \leqslant [p/2]$ when p is odd, where $[x]$ denotes the integer part of x.

As an example, the ground-state energy difference (45) of the two Hamiltonians in the sixth (82–126) major shell with the standard pairing strength $G = 0.2$ MeV, which is a typical parameter value for describing deformed nuclei in this region, shows that the ground-state energy difference of the two Hamiltonians are rather small in this case. The largest deviation of the ground-state energy of the two Hamiltonians is at $k = 2$ with $E_k^{(\rm g)}({\rm ex}) - E_k^{(\rm g)} = -36.3636$ keV. Notwithstanding, since the only difference between the two Hamiltonians, so long as $G_{\rm ex}$ is taken to be related to G by prescription (44), is in the overall binding energy, and since an analytic expression for this difference in also known in terms of $G_{\rm ex}$ through Eq. (45), for practical purposes the two Hamiltonians yield the same results, even though the Hamiltonians are quite different. This in itself is interesting, since it shows that a many-pair interaction Hamiltonian can have identical solutions to the two-pair interaction with truncations. Obviously, it follows that for such systems the structure of fixed-Z (isotopic) and fixed-N (isotonic) chains follow solely from the structure of the simplest single-pair member of the chain and simple "pair-counting" factors related to the pairing interaction strength and single-particle energies.

Thus, we conclude that, basically, the extended pairing model is different from the standard pairing model. However, if only the first few eigenstates are considered, the pair structure of these states in the two models are similar, especially in the ground state, as can be seen from an analysis of the overlaps in the previous section. It can be expected that the difference of the two models will be negligible when the number of pairs k or pairing interaction strength G is small. In addition, since the extended pairing model can be solved exactly with a single one variable equation (43), which is simpler than the Richardson-Gaudin equations with k variables for the standard pairing Hamiltonian, the extended pairing model can be applied to relatively large systems, especially when one only wants to know the first few low-lying eigenstates and corresponding eigenenergies.

V. CONCLUSION

A new step-by-step diagonalization procedure for evaluating exact solutions of the nuclear deformed mean-field plus pairing interaction model is proposed via a simple Bethe ansatz in each step from which the eigenvalues and corresponding eigenstates can be obtained progressively. This new approach draws upon an observation that the original one- plus two-body problem in a k-particle Hilbert subspace can be mapped into a one-body grand hard-core boson picture that can be solved step by step with a simple Bethe ansatz known from earlier work, in which one only needs to solve a single variable nonlinear equation instead of a set of coupled nonlinear equations with k variables as is required, for example, within the framework of the well-known Richardson-Gaudin method. Though this method may be unpractical for large size systems because one needs to get all d roots from the Bethe ansatz equation in each step, this procedure can be used to check contributions from the pairing potential in the Racah quasispin formalism, and is certainly applicable to relatively small systems.

As is shown in the example with $p = 10$ levels, though each term with different q from the pure pairing interaction will contribute to the final eigenenergy and correlate with eigenstates, the first few of these terms are key to determine the first few low-lying states of the model. While high-lying

quasispin states mainly correlate with high excited states of the model and keep the low part of the spectrum less affected.

Based on this new procedure, it is further shown that the extended pairing model for deformed nuclei [34] is similar to the standard pairing model with the first step approximation, in which only the lowest energy eigenstate of the standard pure pairing interaction part is taken into consideration. Our analysis shows that the standard pairing Hamiltonian with the first step approximation displays similar pair structures of the first few low-lying states of the standard pairing model, which, therefore, provides a link between the two models.

Furthermore, the new method proposed is not limited to the deformed mean-field plus pairing problem only, as it should also prove useful for solving a much larger class of quantum many-body problems in which model Hamiltonians are described by

$$\hat{H} = \hat{H}_0 + \lambda \hat{H}_1, \tag{46}$$

where λ is a real parameter, and $\hat{H}_0$ and $\hat{H}_1$ do not commute, $[\hat{H}_0, \hat{H}_1] \neq 0$. According to our procedure, if the particle number is a conserved quantity, and $\hat{H}_0$ and $\hat{H}_1$ can be diagonalized independently in a k-particle basis, then the Hamiltonian (46) is exactly solvable by using the step-by-step exact diagonalization procedure. Moreover, the method can also be extended to deal with Hamiltonians with more than two noncommutative terms by using a similar procedure consecutively. Research in this direction is in progress.

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