Analytical relativistic ideal hydrodynamical solutions in $(1+3)$D with longitudinal and transverse flows

Jinfeng Liao* and Volker Koch†

Nuclear Science Division, Lawrence Berkeley National Laboratory, MS70R0319, 1 Cyclotron Road, Berkeley, California 94720, USA

(Received 1 June 2009; published 15 September 2009)

A new method for solving relativistic ideal hydrodynamics in $(1+3)$D is developed. Longitudinal and transverse radial flows are explicitly embedded into the ansatz for the velocity field and the hydrodynamic equations are reduced to a single equation for the transverse velocity field only, which is analytically more tractable as compared to the full hydrodynamic equations. As an application we use the method to find analytically all possible solutions whose transverse velocity fields have power dependence on the proper time and transverse radius. The possible applications to relativistic heavy ion collisions and possible generalizations of the method are discussed.

DOI: 10.1103/PhysRevC.80.034904 PACS number(s): 47.75.+f, 25.75.−q, 24.10.Nz

I. INTRODUCTION

Relativistic hydrodynamics has wide applications in a variety of physical phenomena, ranging from the largest scales such as in cosmology and astrophysics [1] to the smallest scales such as in relativistic nuclear collisions [2,3]. For an introduction to the general formalism, see, for example, Refs. [4–7].

Recently there has been a remarkably successful application of relativistic ideal hydrodynamics (RIHD) to the description of the space-time evolution of the hot dense QCD matter created in the BNL Relativistic Heavy Ion Collider (RHIC) experiments. In the collisions of two relativistically moving heavy nuclei, a lot of energy is deposited in a small volume, which soon creates an equilibrated system of high energy density with special initial geometry: extremely thin in the beam direction $\hat{z}$, whereas in the transverse plane $\hat{x}$-$\hat{y}$ it is of the size of the nuclei. The space-time evolution at the RHIC is characterized by fast longitudinal expansion (longitudinal flow) and strong transverse expansion (radial and elliptic flow). In noncentral collisions the created matter on the transverse plane $\hat{x}$-$\hat{y}$ is initially anisotropic: such initial spatial anisotropy leads to different pressure gradients and thus different accelerations of the flow along different azimuthal directions. The resulting anisotropic transverse flow velocity eventually translates into the anisotropic azimuthal distribution of the final particle yield, which is represented by the experimental observable called elliptic flow v_2—one of the milestone measurements at the RHIC [8]. RIHD model calculations [9–11] (and more recently its extension to include viscous corrections [12]), performed with realistic initial conditions and Equation of State (E.o.S) for RHIC, are able to reproduce the elliptic flow data at low-to-intermediate transverse momenta for almost all particle species and for various centralities, beam energies, and colliding nuclei. These achievements of RIHD have been the basis for the RHIC discovery that the matter being created is a strongly coupled, nearly perfect fluid [13–16] with an extremely short mean free path. It has been suggested [17–21] that the microscopic origin could be due to the strong scattering via the Lorentz force between the electric and magnetic degrees of freedom coexisting in the created matter, with the magnetic ones ultimately connected with the mechanism of QCD deconfinement transition.

The great success of RIHD at the RHIC has also generated considerable interest in the formal aspects of relativistic hydrodynamics, particularly in analytical solutions of the RIHD equations with an emphasis on possible application to RHIC (see, e.g., Refs. [22–29]. The idea to use exact simple RIHD solutions to describe the multiparticle production in high energy collisions dated back to Landau and Khalatnikov [30]. An important solution came from the work of Hwa [31] and Bjorken [32], that is, the rapidity boost invariant $(1+1)$D solution that is widely used to describe the longitudinal expansion at the RHIC. Many of the above-mentioned recent works [23–26] concentrate on finding $(1+1)$D solutions that give an alternative description of the longitudinal expansion and a more realistic (non-boost-invariant) multiplicity distribution over rapidity.

Despite the progress in solving RIHD equations in $(1+1)$D, it is quite difficult to solve them in higher dimensions. To develop methods and find solutions in a realistic $(1+3)$D setting with potential application for RHIC remains an attractive but demanding task. In this work, we develop a new method to find solutions in $(1+3)$D with both longitudinal and transverse flows. In Sec. III, we show how the method can reduce the hydrodynamics equations to a single constraint equation for the transverse velocity field only. In Sec. IV, using the derived equation, we find all solutions with power-law dependence on the proper time and transverse radius. The physical relevance of our results to RHIC and possible generalizations are discussed in Sec. V. We also include a brief introduction to RIHD in Sec. II and an illustration of the method in $(1+1)$D in Appendices B and C.

*jliao@lbl.gov
†vkoch@lbl.gov

II. REVIEW OF RELATIVISTIC IDEAL HYDRODYNAMICS IN (1 + 3)D

The hydrodynamics equations in general are simply the conservation laws for energy and momenta, that is,

$$T^{mn}{}_{;n} = 0, \tag{1}$$

with m, n running over $3+1$ space-time indices. Following usual convention (in, e.g., Refs. [4,5]), the subscript "$;n$" denotes the covariant derivative D_n whereas the subscript "$,n$" is for the ordinary derivative ∂_n. Below we introduce curved coordinates to simplify the hydrodynamics equations. Therefore, we use the general form for the hydrodynamics equations that involves covariant derivatives (see, e.g., Refs. [4,5]). Throughout this article we discuss only hydrodynamics without any conserved charge, leaving the situation with conserved currents for further investigation.

For relativistic ideal hydrodynamics, the stress tensor is given by

$$T^{mn} = (\epsilon + p)u^m u^n - p g^{mn}, \tag{2}$$

with ϵ and p being the energy density and pressure defined in the flowing matter's local rest frame (L.R.F), which by definition are Lorentz scalars. The flow field $u^m(x)$ is constrained by $u^m \cdot u_m = 1$. In the usual $(t,\vec{x})$ coordinates one can express $u^m(x)$ as $\gamma(1,\vec{v})$ with $\gamma = 1/\sqrt{1-\vec{v}^2}$ and $\vec{v} = d\vec{x}/dt$.

We further need to specify an E.o.S relating the energy density ϵ and the pressure p of the underlying fluid. Here we employ a simple, linear E.o.S, which is typically used in analytic studies of RIHD [23–26,31,32]:

$$p = \nu(\epsilon + p). \tag{3}$$

The above means $\epsilon = \frac{1-\nu}{\nu} p$, implying a speed of sound $c_s = \sqrt{\frac{\partial p}{\partial \epsilon}} = \sqrt{\frac{\nu}{1-\nu}}$, and, to assure $c_s \leqslant 1$, we require $0 < \nu \leqslant 1/2$. We note that a solution obtained with the above E.o.S (3) remains valid in the presence of a nonzero bag constant B: in such case, the right-hand side of Eq. (3), that is, $\epsilon + p$, remains unchanged with $p \to p - B$ and $\epsilon \to \epsilon + B$, and thus Eq. (3) is modified only by adding a constant to its left-hand side. The hydrodynamics equations together with the E.o.S thus form a complete set of five equations for the five field variables: $\epsilon(x), p(x)$, and the three independent components of $u^m(x)$.

A. Hydro equations in curved coordinates

When formulating hydrodynamics for application to, for example, the relativistic heavy ion collisions, it is often useful to use alternative coordinate systems that are curved. For our purpose of studying the $(1+3)$D solutions with longitudinal and transverse flow, we use a coordinate system of (τ,η,ρ,ϕ): that is, the proper time, the spatial (longitudinal) rapidity, the transverse radius, and the azimuthal angle. They are related to the usual (t,x,y,z) in the following way:

$$\begin{aligned} \tau &= \sqrt{t^2 - z^2}, \quad \eta = \frac{1}{2}\ln\frac{t+z}{t-z}, \\ \rho &= \sqrt{x^2+y^2}, \quad \phi = \frac{1}{2i}\ln\frac{x+y\cdot i}{x-y\cdot i}, \end{aligned} \tag{4}$$

and inversely

$$\begin{aligned} t &= \tau\cosh\eta, \quad z = \tau\sinh\eta, \\ x &= \rho\cos\phi, \quad y = \rho\sin\phi. \end{aligned} \tag{5}$$

The velocity field u^m in these coordinates is related to $u^\mu = \gamma(1,\vec{v})$ in flat coordinates $(t,\vec{x})$ via

$$\begin{aligned} u^\tau &= \gamma(\cosh\eta - v_z\sinh\eta), \quad u^\eta = \frac{\gamma}{\tau}(v_z\cosh\eta - \sinh\eta), \\ u^\rho &= \gamma(v_x\cos\phi + v_y\sin\phi), \quad u^\phi = \frac{\gamma}{\rho}(v_y\cos\phi - v_x\sin\phi). \end{aligned} \tag{6}$$

The metric tensor associated with the (τ,η,ρ,ϕ) coordinates is

$$\begin{aligned} g_{mn} &= \mathrm{Diag}(1, -\tau^2, -1, -\rho^2), \\ g^{mn} &= \mathrm{Diag}\left(1, -\frac{1}{\tau^2}, -1, -\frac{1}{\rho^2}\right). \end{aligned} \tag{7}$$

For the covariant derivatives we need the Affine connections $\Gamma^j_{mn} = g^{jk}\Gamma_{kmn} = g^{jk}\frac{1}{2}(g_{km,n} + g_{kn,m} - g_{mn,k})$. In our case, the nonvanishing connections are

$$\begin{aligned} \Gamma^\tau_{\eta\eta} &= \tau, \quad \Gamma^\eta_{\eta\tau} = \Gamma^\eta_{\tau\eta} = \frac{1}{\tau}, \\ \Gamma^\rho_{\phi\phi} &= -\rho, \quad \Gamma^\phi_{\rho\phi} = \Gamma^\phi_{\phi\rho} = \frac{1}{\rho}. \end{aligned} \tag{8}$$

We also give the explicit form of covariant derivatives in the present coordinates for an arbitrary contra-variant-vector A^k (i.e., with upper indices $k = \tau,\eta,\rho,\phi$):

$$\begin{aligned} A^k{}_{;\tau} &= A^k{}_{,\tau} + \Gamma^k_{\tau i}A^i = A^k{}_{,\tau} + \frac{1}{\tau}\delta^k_\eta A^\eta, \\ A^k{}_{;\eta} &= A^k{}_{,\eta} + \Gamma^k_{\eta i}A^i = A^k{}_{,\eta} + \tau\delta^k_\tau A^\eta + \frac{1}{\tau}\delta^k_\eta A^\tau, \\ A^k{}_{;\rho} &= A^k{}_{,\rho} + \Gamma^k_{\rho i}A^i = A^k{}_{,\rho} + \frac{1}{\rho}\delta^k_\phi A^\phi, \\ A^k{}_{;\phi} &= A^k{}_{,\phi} + \Gamma^k_{\phi i}A^i = A^k{}_{,\phi} - \rho\delta^k_\rho A^\phi + \frac{1}{\rho}\delta^k_\phi A^\rho. \end{aligned} \tag{9}$$

Inserting Eqs. (2), (6), and (7) into the general hydrodynamics Eqs. (1) and making use of Eqs. (8) and (9), one obtains the hydrodynamics equations explicitly in the curved coordinates (τ,η,ρ,ϕ).

B. Some known simple exact solutions

We now recall some known simple exact solutions that are pertinent for our approach.

One of the most famous examples is the so-called Hwa-Bjorken solution [31,32] which is essentially the Hubble expansion in $(1+1)$D. The pressure and velocity fields of this solution are given by

$$p_{Bj.} = \frac{\text{constant}}{\tau^{1/(1-\nu)}}, \quad u_{Bj.} = (1, 0, 0, 0). \tag{10}$$

It is more transparent to look at the components of $\vec{v}$ in flat coordinates, which are simply $v_z = \tanh\eta = \frac{z}{t}$ and $v_x = v_y = 0$.

A generalization of the Hwa-Bjorken solution to radial Hubble flow in $(1+3)$D is straightforward. The pressure and velocity fields are

$$p_{\text{Hu.}} = \frac{\text{constant}}{(\tau^2-\rho^2)^{\frac{3}{2(1-\nu)}}}, \quad u_{\text{Hu.}} = \gamma\left(\frac{1}{\cosh\eta}, 0, \frac{\rho}{\tau}, 0\right), \quad \gamma = \frac{\cosh\eta}{\sqrt{1-(\rho/\tau)^2\cosh^2\eta}}, \tag{11}$$

with the domain of validity being $\rho < \tau$. In flat coordinates the velocity fields are given in simple form: $\vec{v} = \vec{x}/t = (x/t, y/t, z/t)$.

Further generalization of the spherically symmetric Hubble flows to accelerating, non-Hubble flows in arbitrary $(1+\text{d})$D has been done in Ref. [23]. An ellipsoidally expanding hydrodynamical solution has also been discussed (see the first article in Ref. [22]). There have also been studies of adding radial flow to a longitudinal Bjorken profile (see, e.g., Ref. [33]).

III. THE NEW REDUCTION METHOD

In this section, we use a new reduction method to find solutions for $(1+3)$D RIHD equations. The general idea is to first embed known solutions in lower dimensions that automatically solve two out of the total of four-component hydrodynamics equations and then reduce the remaining two equations into a single equation for the velocity field only. As usual, one starts with a certain ansatz for the flow velocity field: in our case we use an ansatz with built-in longitudinal and transverse radial flow, aiming at possible application for RHIC. It would be even more interesting to include *transverse elliptic flow*, which requires suitable curved coordinates (like certain hyperbolic coordinates) other than the ones used here. However generally in those cases, more Affine connections are nonvanishing, which makes the reduction method discussed below much more involved: we leave this for future investigation.

A. Including longitudinal and transverse flow

We first embed the boost-invariant longitudinal flow as many numerical hydrodynamics calculations do, which is a suitable approach for RHIC related phenomenology. To do that, we simply set $v_z = z/t = \tanh\eta$, that is, $u^\eta = 0$.

Next we include the transverse radial flow, which is isotropic in the transverse plane. Radial flow is substantial and important at the RHIC. To do so, we introduce the radial flow field v_ρ and set the flat-coordinate transverse flow fields to be $v_x = v_\rho\cos\phi$ and $v_y = v_\rho\sin\phi$, which implies for the curved coordinates $u^\rho = \gamma v_\rho$ and $u^\phi = 0$, with the latter meaning an axially symmetric velocity field. We note that this ansatz goes beyond a simple change to cylindrical coordinates, because we require that $u^\phi = 0$, which considerably simplifies the hydrodynamics equations.

To summarize, to describe a situation with both longitudinal flow and transverse radial flow we have made the following ansatz for the flow fields u^m in the coordinates (τ, η, ρ, ϕ):

$$u^m = \bar{\gamma}(1, 0, \bar{v}_\rho, 0), \quad \bar{v}_\rho \equiv v_\rho\cosh\eta, \quad \bar{\gamma} \equiv 1/\sqrt{1-\bar{v}_\rho^2}. \tag{12}$$

Note that we need to require $\bar{v}_\rho \leqslant 1$.

B. The equation for transverse velocity

With the flow fields given in Eq. (12), we can now explicitly express the stress tensor components. The nonvanishing ones are given below:

$$T^{\tau\tau} = \bar{\gamma}^2(\epsilon+p) - p = \left(\frac{\bar{\gamma}^2}{\nu} - 1\right)p, \tag{13}$$

$$T^{\rho\rho} = \bar{\gamma}^2\bar{v}_\rho^2(\epsilon+p) + p = \left(\frac{\bar{\gamma}^2\bar{v}_\rho^2}{\nu} + 1\right)p, \tag{14}$$

$$T^{\tau\rho} = \bar{\gamma}^2\bar{v}_\rho(\epsilon+p) = \frac{\bar{\gamma}^2\bar{v}_\rho}{\nu}p, \tag{15}$$

$$T^{\eta\eta} = \frac{p}{\tau^2}, \quad T^{\phi\phi} = \frac{p}{\rho^2}. \tag{16}$$

For the second equalities in each of the first three lines we have used the E.o.S (3) to substitute $\epsilon + p$ by p/ν.

With the above expressions and using Eqs. (8) and (9), the hydrodynamics Eqs. (1) then become

$$T^{\tau\lambda}{}_{;\lambda} = T^{\tau\tau}{}_{,\tau} + \frac{T^{\tau\tau}}{\tau} + \frac{p}{\tau} + T^{\tau\rho}{}_{,\rho} + \frac{T^{\tau\rho}}{\rho} = 0, \tag{17}$$

$$T^{\eta\lambda}{}_{;\lambda} = \frac{1}{\tau^2}p_{,\eta} = 0, \tag{18}$$

$$T^{\rho\lambda}{}_{;\lambda} = T^{\rho\rho}{}_{,\rho} + \frac{T^{\rho\rho}}{\rho} - \frac{p}{\rho} + T^{\tau\rho}{}_{,\tau} + \frac{T^{\tau\rho}}{\tau} = 0, \tag{19}$$

$$T^{\phi\lambda}{}_{;\lambda} = \frac{1}{\rho^2}p_{,\phi} = 0. \tag{20}$$

The two equations involving derivatives over η and ϕ are trivially solved by setting $p(x) = p(\tau, \rho)$ (and the same for energy density $\epsilon(\tau, \rho)$ due to the E.o.S) and accordingly $\bar{v}_\rho(x) = \bar{v}_\rho(\tau, \rho)$. We note that the simple forms of Eqs. (18) and (20) are a direct consequence of the vanishing components $u^\eta = u^\phi = 0$ in the flow field ansatz, Eq. (12).

Finally we introduce a combined field variable $\mathcal{K}$ defined as

$$\mathcal{K} \equiv \frac{T^{\tau\tau}+p}{(\rho\tau)} = \frac{\bar{\gamma}^2 p}{\nu\rho\tau} \rightarrow p = \frac{\nu\rho\tau}{\bar{\gamma}^2}\mathcal{K}. \tag{21}$$

We then substitute the pressure p in Eqs. (17) and (19) and obtain two equations for the fields $\mathcal{K}$ and $\bar{v}_\rho$, which can be expressed as

$$D_a \cdot \mathcal{K}_{,\tau} + D_b \cdot \mathcal{K}_{,\rho} = D_1 \cdot \mathcal{K}, \tag{22}$$
$$D_b \cdot \mathcal{K}_{,\tau} + D_c \cdot \mathcal{K}_{,\rho} = D_2 \cdot \mathcal{K}. \tag{23}$$

The coefficients D_a, D_b, D_c, D_1, and D_2 are given by

$$\begin{aligned} D_a &= (1-\nu) + \nu \bar{v}_\rho^2, \\ D_b &= \bar{v}_\rho, \\ D_c &= \nu + (1-\nu)\bar{v}_\rho^2, \\ D_1 &= -2\nu\bar{v}_\rho\bar{v}_{\rho,\tau} - \nu\big(1-\bar{v}_\rho^2\big)/\tau - \bar{v}_{\rho,\rho}, \\ D_2 &= -2(1-\nu)\bar{v}_\rho\bar{v}_{\rho,\rho} + \nu\big(1-\bar{v}_\rho^2\big)/\rho - \bar{v}_{\rho,\tau}. \end{aligned} \tag{24}$$

From Eqs. (22) and (23) we obtain

$$\begin{aligned} \frac{\mathcal{K}_{,\tau}}{\mathcal{K}} &= (\ln \mathcal{K})_{,\tau} = \frac{D_c D_1 - D_b D_2}{D_a D_c - D_b^2} \equiv \mathcal{F}[\tau,\rho], \\ \frac{\mathcal{K}_{,\rho}}{\mathcal{K}} &= (\ln \mathcal{K})_{,\rho} = \frac{D_a D_2 - D_b D_1}{D_a D_c - D_b^2} \equiv \mathcal{G}[\tau,\rho], \end{aligned} \tag{25}$$

with the functions $\mathcal{F}$ and $\mathcal{G}$ given by

$$\begin{aligned} \mathcal{F}[\bar{v}_\rho(\tau,\rho)] = \frac{1}{\nu(1-\nu)\big(1-\bar{v}_\rho^2\big)^2} \Big\{ &\big[(1-\nu)\bar{v}_\rho^2 - \nu\big]\bar{v}_{\rho,\rho} \\ &+ \big[(1-2\nu^2) + 2\nu(\nu-1)\bar{v}_\rho^2\big]\bar{v}_\rho\bar{v}_{\rho,\tau} \\ &- \nu\big[1-\bar{v}_\rho^2\big]\big[\nu+(1-\nu)\bar{v}_\rho^2\big]/\tau \\ &- \nu\bar{v}_\rho\big[1-\bar{v}_\rho^2\big]/\rho\Big\}, \end{aligned} \tag{26}$$

$$\begin{aligned} \mathcal{G}[\bar{v}_\rho(\tau,\rho)] = \frac{1}{\nu(1-\nu)\big(1-\bar{v}_\rho^2\big)^2} \Big\{ &\big[\big(\nu\bar{v}_\rho^2 + (\nu-1)\big]\bar{v}_{\rho,\tau} \\ &+ \big[(-1+4\nu-2\nu^2) + 2\nu(\nu-1)\bar{v}_\rho^2\big]\bar{v}_\rho\bar{v}_{\rho,\rho} \\ &+ \nu\bar{v}_\rho\big[1-\bar{v}_\rho^2\big]/\tau \\ &+ \nu\big[1-\bar{v}_\rho^2\big]\big[(1-\nu)+\nu\bar{v}_\rho^2\big]/\rho\Big\}. \end{aligned} \tag{27}$$

In Eq. (25), the function $\ln \mathcal{K}$ depends (via $\bar{v}_\rho$) on two variables τ and ρ, and we have two equations for the two first-order derivatives $\frac{\partial \ln \mathcal{K}}{\partial \tau}$ and $\frac{\partial \ln \mathcal{K}}{\partial \tau}$. For $\ln \mathcal{K}$ as a single function of two variables, τ and ρ, the two equations can be consistent only if the following constraint on second-order derivatives is satisfied $\frac{\partial^2 Ln \mathcal{K}}{\partial\tau\partial\rho} = \frac{\partial^2 Ln \mathcal{K}}{\partial\rho\partial\tau}$, that is,

$$\frac{\partial}{\partial\rho}\mathcal{F}[\bar{v}_\rho(\tau,\rho)] - \frac{\partial}{\partial\tau}\mathcal{G}[\bar{v}_\rho(\tau,\rho)] = 0. \tag{28}$$

Thus we only need to solve the above single equation for the velocity field $\bar{v}_\rho(\tau,\rho)$. Because $\mathcal{F}$ and $\mathcal{G}$ already involve the first derivatives of $\bar{v}_{\rho,\tau}$ and $\bar{v}_{\rho,\rho}$, the reduced velocity Eq. (28) is a second-order partial differential equation for the velocity field. As a minor caveat, the method applies to the case/region in which $\ln \mathcal{K}$ is at least second-order differentiable. This reduction method can be demonstrated in the more explicit case of $(1+1)$D hydrodynamics; see Appendices B and C.

Given the above constraints, we can then solve from Eq. (25) the matter field S directly,

$$\mathcal{K} = \mathcal{K}_0 \cdot e^{[\int_{\tau_0}^{\tau} d\tau' \mathcal{F}[\tau',\rho] + \int_{\rho_0}^{\rho} d\rho' \mathcal{G}[\tau,\rho']]}, \tag{29}$$

with $\mathcal{K}_0$ being the value at arbitrary reference point τ_0, ρ_0.

Finally let us summarize our approach: after including into the flow field ansatz the physically desired longitudinal and transverse flows, we have reduced the hydrodynamic equations into a single Eq. (28) involving *ONLY* the transverse velocity field $\bar{v}_\rho$, and any solution to this equation automatically leads to the pressure field which together with the velocity field forms a solution to the original hydrodynamics equations:

$$p = \text{constant} \times \frac{\rho\tau}{\kappa\bar{\gamma}^2} \times e^{[\int^{\tau} d\tau' \mathcal{F}[\tau',\rho] + \int^{\rho} d\rho' \mathcal{G}[\tau,\rho']]}. \tag{30}$$

C. Examination of the method

We now examine the correctness of the reduced Eq. (28) and the solution (30), using the two known simple analytic solutions, Eqs. (10) and (11), as both of them are special cases of our embedding with longitudinal and transverse radial flows.

For the 1D Bjorken expansion, we have $\bar{v}_{\rho \mathrm{Bj.}} = 0$, which leads to

$$\mathcal{F}_{\mathrm{Bj.}} = \frac{\nu}{\nu-1}\frac{1}{\tau}, \quad \mathcal{G}_{\mathrm{Bj.}} = \frac{1}{\rho}. \tag{31}$$

One can easily verify that the above $\mathcal{F}_{\mathrm{Bj}}$. and $\mathcal{G}_{\mathrm{Bj}}$. satisfy the reduced Eq. (28). Furthermore by inserting $\mathcal{F}_{\mathrm{Bj}}$. and $\mathcal{G}_{\mathrm{Bj}}$. into the solution (30) one finds exactly the pressure in Eq. (10).

For the 3D Hubble expansion, we have $\bar{v}_{\rho \mathrm{Hu.}} = \rho/\tau$, which leads to

$$\begin{aligned} \mathcal{F}_{\mathrm{Hu.}} &= \frac{3}{\tau} + \frac{\nu - 5/2}{1-\nu}\frac{2\tau}{\tau^2-\rho^2}, \\ \mathcal{G}_{\mathrm{Hu.}} &= \frac{1}{\rho} + \frac{\nu - 5/2}{1-\nu}\frac{-2\rho}{\tau^2-\rho^2}. \end{aligned} \tag{32}$$

Again it can be easily shown that the above $\mathcal{F}_{\mathrm{Hu}}$. and $\mathcal{G}_{\mathrm{Hu}}$. satisfy the reduced Eq. (28). Furthermore by inserting $\mathcal{F}_{\mathrm{Hu}}$. and $\mathcal{G}_{\mathrm{Hu}}$. into the solution (30), one finds exactly the pressure in Eq. (11).

IV. APPLICATION OF THE METHOD

As an example of an application of the embedding-reduction method in the previous section, we show how to find all possible solutions with the following ansatz for the radial velocity field:

$$\bar{v}_\rho = A \cdot \tau^B \cdot \rho^C, \tag{33}$$

with A, B, and C being arbitrary real numbers. We note that the two known exact solutions we mentioned are special cases of the above form: the 1D Bjorken expansion corresponds to $A = 0$ while the 3D Hubble expansion corresponds to $A = 1$, $B = -1$, and $C = 1$. The velocity field (33), when put into Eqs. (26) and (27), gives the following:

$$\begin{aligned} \mathcal{F}[\tau,\rho] = \frac{1-2B}{\tau} &- \frac{B + (1-4B)\nu + 2B\nu^2}{\nu(1-\nu)} \cdot \frac{1}{\tau\big[1-\bar{v}_\rho^2\big]} \\ &- \frac{C + (1-C)\nu}{\nu(1-\nu)} \cdot \frac{\bar{v}_\rho}{\rho\big[1-\bar{v}_\rho^2\big]} + \frac{B(1-2\nu)}{\nu(1-\nu)} \\ &\cdot \frac{1}{\tau\big[1-\bar{v}_\rho^2\big]^2} + \frac{C(1-2\nu)}{\nu(1-\nu)} \cdot \frac{\bar{v}_\rho}{\rho\big[1-\bar{v}_\rho^2\big]^2}, \end{aligned} \tag{34}$$

$$\mathcal{G}[\tau,\rho] = \frac{-2C-\nu/(1-\nu)}{\rho} - \frac{2C\nu^2-\nu-C}{\nu(1-\nu)}\cdot\frac{1}{\rho\left[1-\bar{v}_\rho^2\right]} - \frac{B-1}{1-\nu}\cdot\frac{\bar{v}_\rho}{\tau\left[1-\bar{v}_\rho^2\right]} + \frac{C(2\nu-1)}{\nu(1-\nu)}\cdot\frac{1}{\rho\left[1-\bar{v}_\rho^2\right]^2} + \frac{B(2\nu-1)}{\nu(1-\nu)}\cdot\frac{\bar{v}_\rho}{\tau\left[1-\bar{v}_\rho^2\right]^2}. \quad (35)$$

We have used $\bar{v}_{\rho,\rho} = \bar{v}_\rho \cdot C/\rho$ and $\bar{v}_{\rho,\tau} = \bar{v}_\rho \cdot B/\tau$. As a check of the above result, one can verify that by setting $A = 0$ they reduce to Eq. (31), whereas by setting $A = 1$, $B = -1$, and $C = 1$ they reduce to Eq. (32), as they should.

By inserting Eqs. (34) and (35) into Eq. (28), one obtains a rather complicated constraint equation for the constants A, B, and C. However, after lengthy calculations, all possible combinations of A, B, and C for solving the equation can actually be exhausted. Leaving the detailed (and technical) derivations to Appendix A, we only list the final results here.

(i) Solution I. $A = 0$, with $0 < \nu \leqslant \frac{1}{2}$ and $|z| < t$ (1D Bjorken)—see Eq. (10).

(ii) Solution II. $A = 1$, $B = -1$, and $C = 1$, with $0 < \nu \leqslant \frac{1}{2}$ and $\sqrt{x^2+y^2+z^2} < t$ (3D Hubble)—see Eq. (11).

(iii) Solution III. $A = 1$, $B = 1$, and $C = -1$, with $0 < \nu \leqslant \frac{1}{2}$, $|z| < t$ and $\sqrt{x^2+y^2+z^2} > t$—the solutions are

$$v_x = \frac{x}{t}\cdot\frac{t^2-z^2}{x^2+y^2},\quad v_y = \frac{y}{t}\cdot\frac{t^2-z^2}{x^2+y^2},\quad v_z = \frac{z}{t},$$
$$p = \frac{\text{constant}}{(\tau\rho)^{1/(1-\nu)}(\rho^2-\tau^2)^{(1-3\nu)/(2\nu-2\nu^2)}}. \quad (36)$$

(iv) Solution IV. $A = 1$, $B = 1/3$, and $C = -1/3$, with $\nu = 1/4$, $|z| < t$, and $\sqrt{x^2+y^2+z^2} > t$—the solutions are

$$v_x = \frac{x}{t}\cdot\left(\frac{t^2-z^2}{x^2+y^2}\right)^{2/3},\quad v_y = \frac{y}{t}\cdot\left(\frac{t^2-z^2}{x^2+y^2}\right)^{2/3},$$
$$v_z = \frac{z}{t},\quad p = \text{constant}\times\frac{\left(\rho^{2/3}-\tau^{2/3}\right)^{2/3}}{(\rho\tau)^{4/3}}. \quad (37)$$

(v) Solution V. $A = -1$, $B = -1$, and $C = 1$, with $\nu = 1/2$ and $\sqrt{x^2+y^2+z^2} < t$—the solutions are

$$v_x = \frac{-x}{t},\quad v_y = \frac{-y}{t},\quad v_z = \frac{z}{t},$$
$$p = \text{constant}\times(\tau^2-\rho^2). \quad (38)$$

It can be verified that these solutions obtained by the method introduced here are indeed solutions of the original hydrodynamics equations (1). One should notice the different applicable kinematic regions in each of the above solutions that come from the constraint that the flow velocity shall be less than the speed of light. For a detailed discussion about solutions in different regions with respect to the kinematic light cone, see, for example, the Appendices of Ref. [23].

We notice that all the solutions (except the trivial Solution I with $A = 0$) satisfy two features: (1) $B = -C$ and (2) $|A| = 1$. The first feature may be the result of dimensional reasons. The second feature, $|A| = 1$, may be heuristically understood in the following way. We first consider the case $B = -C < 0$, that is, $\bar{v}_\rho = A(\rho/\tau)^C$ with $C > 0$: in this case the solution exists in the region $\rho < \tau\cdot|A|^{-1/C}$, and in particular $\rho = 0$ for $\tau = 0$. Thus for any $\tau > 0$, the flow front, which travels with the speed of light, $|\bar{v}_\rho| = 1$, is located at $\rho = \tau$, and hence, $|A| = 1$. Next we consider the case $B = -C > 0$, that is, $\bar{v}_\rho = A(\tau/\rho)^B$ with $B > 0$: in this case the solution exists in the region $\rho > \tau\cdot A^{1/B}$ with $A > 0$, separated from an empty region by the boundary at $\rho = \tau\cdot A^{1/B}$. At this boundary, the flow velocity approaches the speed of light $\bar{v}_\rho \to 1$, which enforces the matter density to drop to zero to avoid an infinite T^{mn} [due to the γ-factor in Eq. (2)]. We imagine that at time $\tau = 0$ the matter fills the whole space and then starts to flow outward; thus the boundary also moves outward from the origin with the speed of light $\bar{v}_\rho = 1$. This again implies the boundary should lie at $\rho = \tau$, requiring $A = 1$.

The above example of the proposed embedding-reduction method demonstrates the advantage of analytical solutions. Not only could we find some solutions of the specific type (33) but we actually were able to *exhaust* all solutions of this type. This also implies that for parametrization of the flow velocity field, like in the blast wave model for a RHIC fireball, there are only very limited choices for the flow profile ansatz.

V. SUMMARY AND DISCUSSION

In summary, a general framework for the analytical treatment of RIHD equations has been developed. The method features a separation of longitudinal and transverse expansions, as inspired by RHIC phenomenology. After the separation, the longitudinal and transverse radial flows are embedded utilizing lower-dimensional solutions. The remaining equations are found to be reducible to a single constraint equation for the transverse radial flow velocity field only, which can be solved completely for a certain ansatz for the velocity field. All solutions with power-law dependence on the proper time and transverse radius have been found.

We now discuss various possible extensions of the present approach.

Nontrivial longitudinal embedding. In the current work the longitudinal flow is embedded with the Hwa-Bjorken solution. It would be very interesting to try embedding the newly found (1 + 1)D solutions in, for example, Refs. [23–26] with more realistic longitudinal expansion for RHIC which would be useful for studying elliptic flow in the forward/backward rapidity and their correlation [34].

Solutions with non-power-law transverse expansion. It would also be interesting to test a more nontrivial ansatz for the embedded transverse flow. For example, we know from numerical calculations of radial flow [35] in central collisions at the RHIC that the radial velocity field may be parametrized as $v_\rho \approx f(\tau)r/\tau$, with $f(\tau\to 0)\to 0$ and $f(\tau \gg 1)\to 1$. Such parametrization can be cast into the derived velocity Eq. (28) to find possible solutions.

Small deformation and elliptic flow. The analytic treatment of transverse elliptic flow is difficult. One approximate method may be to introduce a parametrically small deformation of the matter field (with a certain eccentricity parameter ϵ_2) on top of an exact solution with transverse radial flow and use linearized hydrodynamics equations to investigate possible universal relations between the finally developed velocity field anisotropy v_2 and the initial ϵ_2 [36].

Transverse elliptic flow embedding. Another possibility to seek exact solutions with transverse elliptic flow is to use instead of (ρ,ϕ) certain hyperbolic coordinates that by definition incorporate elliptic anisotropy; see the Appendix of Ref. [37] for an example of such curved coordinates that may be used to develop a similar embedding-reduction procedure describing transverse elliptic flow. Another possibility will be to combine certain conformal transformations with hydrodynamics equations to degrade the elliptic geometry back to a spherical one.

2D Hubble embedding. In all the previously discussed options, we have chosen to embed $(1+1)$D Hubble flow for the $(t,z)\to(\tau,\eta)$ part, due to an emphasis on RHIC evolution. Theoretically, one can also embed a $(1+2)$D Hubble flow for the $(t,x,y)\to(t,\rho,\phi)\to(\tau_\rho,\eta_\rho,\phi)$ part and can eventually reduce the equations to a velocity equation with two variables (τ_ρ,z) in exactly the same manner as before.

ACKNOWLEDGMENTS

JL is grateful to Edward Shuryak for very helpful discussions. This work is supported by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of Nuclear Physics of the US Department of Energy under Contract DE-AC02-05CH11231.

APPENDIX A

In this Appendix we give the detailed derivations leading to the solutions in Sec. IV with the transverse velocity ansatz (33).

We first evaluate the derivatives $\frac{\partial\mathcal{F}}{\partial\rho}$ and $\frac{\partial\mathcal{G}}{\partial\tau}$ with $\mathcal{F}$ and $\mathcal{G}$ given in Eqs. (34) and (35). Again we will make use of $\bar{v}_{\rho,\rho}=\bar{v}_\rho\cdot C/\rho$ and $\bar{v}_{\rho,\tau}=\bar{v}_\rho\cdot B/\tau$ for the velocity ansatz (33).

The result for $\frac{\partial\mathcal{F}}{\partial\rho}$ is

$$\begin{aligned}\frac{\partial\mathcal{F}}{\partial\rho} &= \frac{\bar{v}_\rho}{\nu(1-\nu)\rho^2\tau^2\left(1-\bar{v}_\rho^2\right)^3}\times\left\{f_1\rho\tau\bar{v}_\rho\left(1-\bar{v}_\rho^2\right)\right.\\ &\quad+f_2\tau^2\left[(1+C)\bar{v}_\rho^2+(C-1)\right]\left(1-\bar{v}_\rho^2\right)\\ &\quad\left.+f_3\tau\rho\bar{v}_\rho+f_4\tau^2\left[(3C+1)\bar{v}_\rho^2+(C-1)\right]\right\}\\ &= \frac{\bar{v}_\rho}{\nu(1-\nu)\rho^2\tau^2\left(1-\bar{v}_\rho^2\right)^3}\times\left\{[-f_2(C+1)]\tau^2\bar{v}_\rho^4\right.\\ &\quad+[-f_1]\rho\tau\bar{v}_\rho^3+[2f_2+f_4(3C+1)]\tau^2\bar{v}_\rho^2\\ &\quad\left.+[f_1+f_3]\rho\tau\bar{v}_\rho+[(f_2+f_4)(C-1)]\tau^2\right\},\end{aligned} \tag{A1}$$

with coefficients $f_{1,2,3,4}$ given by

$$\begin{aligned}f_1&=-[B+(1-4B)\nu+2B\nu^2]\times(2C),\\ f_2&=-[C+(1-C)\nu],\\ f_3&=B\times(4C)\times(1-2\nu),\\ f_4&=C\times(1-2\nu).\end{aligned} \tag{A2}$$

The result for $\frac{\partial\mathcal{G}}{\partial\tau}$ is

$$\begin{aligned}\frac{\partial\mathcal{G}}{\partial\tau} &= \frac{\bar{v}_\rho}{\nu(1-\nu)\rho^2\tau^2\left(1-\bar{v}_\rho^2\right)^3}\times\left\{g_1\rho\tau\bar{v}_\rho\left(1-\bar{v}_\rho^2\right)\right.\\ &\quad+g_2\rho^2\left[(1+B)\bar{v}_\rho^2+(B-1)\right]\left(1-\bar{v}_\rho^2\right)\\ &\quad\left.+g_3\tau\rho\bar{v}_\rho+g_4\rho^2\left[(3B+1)\bar{v}_\rho^2+(B-1)\right]\right\}\\ &= \frac{\bar{v}_\rho}{\nu(1-\nu)\rho^2\tau^2\left(1-\bar{v}_\rho^2\right)^3}\times\left\{[-g_2(B+1)]\rho^2\bar{v}_\rho^4\right.\\ &\quad+[-g_1]\rho\tau\bar{v}_\rho^3+[2g_2+g_4(3B+1)]\rho^2\bar{v}_\rho^2\\ &\quad\left.+[g_1+g_3]\rho\tau\bar{v}_\rho+[(g_2+g_4)(B-1)]\rho^2\right\},\end{aligned} \tag{A3}$$

with coefficients $g_{1,2,3,4}$ given by

$$\begin{aligned}g_1&=-[-C-\nu+2C\nu^2]\times(2B),\\ g_2&=-[(B-1)\nu],\\ g_3&=(4B)\times C\times(2\nu-1),\\ g_4&=B\times(2\nu-1).\end{aligned} \tag{A4}$$

Now combining the results into Eq. (28) we obtain the following (with $\bar{v}_\rho=A\tau^B\rho^C$ substituted in):

$$\frac{\partial\mathcal{F}}{\partial\rho}-\frac{\partial\mathcal{G}}{\partial\tau}=\frac{\bar{v}_\rho}{\nu(1-\nu)\rho^2\tau^2\left(1-\bar{v}_\rho^2\right)^3}\times\mathcal{I}[\tau,\rho]=0,$$

with

$$\begin{aligned}\mathcal{I}[\tau,\rho]= &\;[-f_2(C+1)A^4]\tau^{4B+2}\rho^{4C}\\ &+[g_2(B+1)A^4]\tau^{4B}\rho^{4C+2}\\ &+[(g_1-f_1)A^3]\tau^{3B+1}\rho^{3C+1}\\ &+[(2f_2+f_4(3C+1))A^2]\tau^{2B+2}\rho^{2C}\\ &+[(-2g_2-g_4(3B+1))A^2]\tau^{2B}\rho^{2C+2}\\ &+[(f_1+f_3-g_1-g_3)A]\tau^{B+1}\rho^{C+1}\\ &+[(f_2+f_4)(C-1)]\tau^2\\ &+[-(g_2+g_4)(B-1)]\rho^2.\end{aligned} \tag{A5}$$

Clearly the solutions are

$$\bar{v}_\rho=0 \tag{A6}$$

or

$$\mathcal{I}[\tau,\rho]=0. \tag{A7}$$

The former solution is just the Hwa-Bjorken one. Any solution with nonvanishing transverse velocity then has to satisfy the latter condition, Eq. (A7), which we now focus on. Again one can test the correctness of the above equation by using the 3D Hubble ($A=1$, $B=-1$, and $C=1$) solution.

In Eq. (A7), terms with various powers of τ,ρ (and only power terms) appear in $\mathcal{I}[\tau,\rho]$: to make all of them either mutually cancel (among terms with exactly the same τ,ρ powers) or vanish by respective coefficients to eventually zero is quite nontrivial. A thorough sorting of the sequences of τ,ρ powers can exhaust all possibilities to satisfy the algebraic equation (A5).

To see how this actually works, we give one concrete example. Let's consider the case when $B>0$ and

$C \neq 0$: this implies that for the exponents of τ we have $4B+2 > 4B > 2B$, $4B+2 > 3B+1 > B+1$, $4B+2 > 2B+2 > 2B$, $2B+2 > 2$, and $2B+2 > B+1$. Therefore the term $[-f_2(C+1)A^4]\tau^{4B+2}\rho^{4C}$ in Eq. (A5) *cannot* be canceled by any other one and has to vanish by itself. This leads to

$$f_2(C+1)A^4 = 0, \tag{A8}$$

which in turn gives three possibilities, $f_2 = 0$, $C = -1$, or $A = 0$. In this example we follow $C = -1$ (other choices lead to other solutions). With $C = -1$ we notice again that the term $[-(g_2+g_4)(B-1)]\rho^2$ *cannot* be canceled by any other remaining terms and thus shall vanish by itself. This leads to

$$-(g_2+g_4)(B-1) = 0, \tag{A9}$$

which again has two possibilities $B = 1$ or $g_2 + g_4 = 0$. Now we choose to follow $B = 1$: with this choice the remaining terms are significantly simplified and finally lead to two equations about the coefficients:

$$\begin{aligned} 2g_2A^4 + (g_1 - f_1)A^3 + 2(f_2 - f_4)A^2 = 0,\\ -2(g_2+2g_4)A^2 + (f_1+f_3-g_1-g_3)A - 2(f_2+f_4) = 0. \end{aligned} \tag{A10}$$

It can then be verified that the only solution is $A = 1$ for arbitrary ν.

Of course there are many combinations to be checked. Nevertheless the number of possibilities is finite, which one can examine one by one. Note not all possibilities appearing initially can finally lead to a solution: there are only four variables, A, B, C, and ν, and in most cases it turns out contradiction occurs at the end, which means no solution. After a tedious examination we have found all possible solutions as listed in Sec. IV, and there is no more solution of the power-law ansatz type as in Eq. (33).

APPENDIX B

In this Appendix we use $(1+1)$D ideal relativistic hydrodynamics to demonstrate the reduction method in a more explicit manner. The hydrodynamics equations are [in (t,z) coordinates]

$$\frac{[\partial_t + v\partial_z]\epsilon}{\epsilon + p} = -\partial_z v - \gamma_v^2[\partial_t + v\partial_z]\left(\frac{v^2}{2}\right), \tag{B1}$$

$$\gamma_v^2[\partial_t + v\partial_z]v = -\frac{\partial_z p}{\epsilon+p} - \frac{v\partial_t p}{\epsilon+p}. \tag{B2}$$

In the above v is the spatial velocity dz/dt and $\gamma_v \equiv 1/\sqrt{1-v^2}$. The energy density ε and pressure p shall be related by the E.o.S, which we use in a slightly different way. We introduce the enthalpy density $w = \epsilon + p$ and the speed of sound $c_s \equiv \sqrt{\partial p/\partial\epsilon}$ (which can be deduced from the E.o.S) and use the following relations,

$$d\epsilon = \frac{1}{1+c_s^2}dw, \quad dp = \frac{c_s^2}{1+c_s^2}dw, \tag{B3}$$

to rewrite the hydrodynamics equations into

$$\partial_t[\ln(w)] + v\partial_z[\ln(w)] = -\frac{\gamma_v^2}{1-\xi}[\partial_x v + v\partial_t v], \tag{B4}$$

$$v\partial_t[\ln(w)] + \partial_z[\ln(w)] = -\frac{\gamma_v^2}{\xi}[v\partial_x v + \partial_t v], \tag{B5}$$

with $\xi \equiv c_s^2/(1+c_s^2)$. From these two equations we can obtain $\partial_t[\ln(w)]$ and $\partial_z[\ln(w)]$:

$$\begin{aligned} \partial_t[\ln(w)] &= \mathcal{X}[v, \partial_t v, \partial_z v] = \mathcal{X}[t,z]\\ &= \frac{-\gamma_v^4}{\xi(1-\xi)}\{[\xi - (1-\xi)v^2]\partial_z v + [2\xi - 1]v\,\partial_t v\}, \end{aligned} \tag{B6}$$

$$\begin{aligned} \partial_z[\ln(w)] &= \mathcal{Y}[v, \partial_t v, \partial_z v] = \mathcal{Y}[t,z]\\ &= \frac{-\gamma_v^4}{\xi(1-\xi)}\{[(1-\xi) - \xi v^2]\partial_t v + [1-2\xi]v\,\partial_z v\}. \end{aligned} \tag{B7}$$

The necessary and sufficient condition for the above set of equations to be soluble is the following:

$$\partial_z\mathcal{X}[t,z] - \partial_t\mathcal{Y}[t,z] = 0. \tag{B8}$$

Thus we have reduced the original hydrodynamics equations into a single but second-order differential equation for the velocity field $v(t,z)$ only. With the above satisfied, the matter field is given by

$$w(t,z) = w_0 \cdot e^{\int_{t_0}^{t} dt'\mathcal{X}[t',z] + \int_{z_0}^{z} dz'\mathcal{Y}[t,z']}, \tag{B9}$$

with w_0 its value at arbitrary reference point (t_0,z_0).

In the case of a linear E.o.S like the one in Eq. (3), the speed of sound c_s and thus ξ are constants independent of ε or p, and we can further simplify the reduced equation (B8) for velocity field into the following:

$$\begin{aligned} &[(1-\xi) - \xi v^2](1-v^2)\left(\partial_t^2 v\right) + [(2-3\xi) - \xi v^2](2v)(\partial_t v)^2\\ &\quad - [\xi - (1-\xi)v^2](1-v^2)\left(\partial_z^2 v\right) - [(3\xi - 1)\\ &\quad + (\xi - 1)v^2](2v)\left(\partial_z v\right)^2 + 2(1-2\xi)(1-v^2)v(\partial_t\partial_z v)\\ &\quad + 2(1-2\xi)(1+3v^2)(\partial_t v)(\partial_z v) = 0\ . \end{aligned} \tag{B10}$$

A similar scheme can be carried out for curved coordinates like (τ,η) in a straightforward way. We notice a similar implementation using light-cone variables $z_\pm = t \pm z$ in Ref. [24].

We emphasize that while the above procedure seems somewhat trivial in $(1+1)$D, its realization is much more nontrivial and involved in $(1+3)$D. We also point out that the reduced equation (B8) for the velocity field (or the simplified one in the case of linear E.o.S) must be satisfied by *all* solutions to the $(1+1)$D hydrodynamics equations. In Appendix C we give a nontrivial and involved example from the recently found Nagy-Csörgő-Csanád (NCC) solutions [23] (which also include $(1+1)$D Hwa-Bjroken as a special case) to show the correctness and usefulness of the derived velocity equation.

APPENDIX C

In this Appendix we show that the NCC family of analytic solutions in Ref. [23] for 1D ideal hydrodynamics equations with a linear E.o.S can also be deduced by subjecting their velocity field ansatz to the reduced equations (B10) we derived. With the resulting velocity field we also show the matter field of NCC solutions is indeed given by Eq. (B9).

The velocity field ansatz of NCC solutions is the following (for inside-light-cone region, i.e., $|z| < |t|$):

$$v = \tanh[\lambda\eta] = \frac{(t+z)^\lambda - (t-z)^\lambda}{(t+z)^\lambda + (t-z)^\lambda}, \quad \eta = \frac{1}{2}\ln\left[\frac{t+z}{t-z}\right], \tag{C1}$$

with λ being some constant. With the above, we obtain the following relations for the derivatives:

$$\begin{aligned} \partial_t v &= \lambda(1-v^2)\partial_t\eta, \\ \partial_z v &= \lambda(1-v^2)\partial_z\eta, \\ \partial_t^2 v &= \lambda(1-v^2)\big[\big(\partial_t^2\eta\big) - 2\lambda v(\partial_t\eta)^2\big], \\ \partial_z^2 v &= \lambda(1-v^2)\big[\big(\partial_z^2\eta\big) - 2\lambda v(\partial_z\eta)^2\big], \\ \partial_t\partial_z v &= \lambda(1-v^2)[(\partial_t\partial_z\eta) - 2\lambda v(\partial_t\eta)(\partial_z\eta)]. \end{aligned} \tag{C2}$$

Substituting the above derivatives into our velocity Eq. (B10), we obtain the following:

$$\begin{aligned} 0 = \lambda(1-v^2)^2\big\{&[(1-\xi) - \xi v^2]\big(\partial_t^2\eta\big) \\ &+ [-\xi + (1-\xi)v^2]\big(\partial_z^2\eta\big) + 2(1-2\xi)v(\partial_t\partial_z\eta) \\ &+ 2(1-2\xi)\lambda(1+v^2)(\partial_t\eta)(\partial_z\eta) \\ &+ 2(1-2\xi)\lambda v[(\partial_t\eta)^2 + (\partial_z\eta)^2]\big\}. \end{aligned} \tag{C3}$$

After evaluating the derivatives of η in the above, we obtain

$$\frac{2\lambda(1-v^2)^2}{(t^2-z^2)^2}\cdot(1-2\xi)\cdot(1-\lambda) \cdot[(tz)v^2 - (t^2+z^2)v + (tz)] = 0. \tag{C4}$$

We find three classes of solutions:

(i) $\xi = 1/2$ with arbitrary λ;
(ii) $\lambda = 1$ with arbitrary ξ (which is nothing but the Hwa-Bjorken solution);
(iii) $[(tz)v^2 - (t^2+z^2)v + (tz)] = 0$, which yields only one causal solution in the forward light-cone with $v = z/t$, but this is just the $\lambda = 1$ solution.

These cover the (1 + 1)D NCC solutions found in Ref. [23]. Note their parameter κ from E.o.S $\epsilon = \kappa p$ is related to our E.o.S parameter $\xi \equiv c_s^2/(1+c_s^2)$ by $\kappa = (1-\xi)/\xi$. It should be mentioned that the last Appendix of Ref. [23] gives the general solution of the (1 + 1)D relativistic hydrodynamical solutions for $\xi = 1/2$ with arbitrary initial conditions in the forward light-cone, and it also gives various spherically symmetric solutions in an arbitrary number of spatial dimensions. These aspects are not discussed in the present article.

Next we examine the matter field corresponding to the velocity field solutions:

(i) for the $\xi = 1/2$ case, we have $\mathcal{X} = (-\lambda)[2t/(t^2-z^2)]$ and $\mathcal{Y} = (-\lambda)[-2z/(t^2-z^2)]$, which via our Eq. (B9) gives $w = w_0[\frac{t_0^2-z_0^2}{t^2-z^2}]^\lambda$;
(ii) for the $\lambda = 1$ case, we have $v = z/t$ and thus $\mathcal{X} = \frac{-1}{2(\xi-1)}[2t/(t^2-z^2)]$ and $\mathcal{Y} = \frac{-1}{2(\xi-1)}[-2z/(t^2-z^2)]$, which via our Eq. (B9) gives $w = w_0[\frac{t_0^2-z_0^2}{t^2-z^2}]^{1/(2-2\xi)}$.

The two cases can be combined into a single form:

$$w = w_0 \times \left[\frac{t_0^2-z_0^2}{t^2-z^2}\right]^{\frac{\lambda}{2(1-\xi)}}, \tag{C5}$$

which is the same as that obtained in Ref. [23].

[1] S. Borgini, A. Diaferio, K. Dolag, and S. Schindler, arXiv:0801.1032 [astro-ph]; O. Regev, New Astron. Rev. **51**, 819 (2008); J. E. Horvath, arXiv:astro-ph/0703233; L. Mayer, S. Kazantzidis, and A. Escala, arXiv:0807.3329 [astro-ph].
[2] U. W. Heinz, arXiv:0901.4355 [nucl-th]; P. F. Kolb and U. W. Heinz, arXiv:nucl-th/0305084.
[3] D. A. Teaney, arXiv:0905.2433 [nucl-th]; P. Romatschke, arXiv:0902.3663 [hep-ph].
[4] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Butterworth-Heinemann, Oxford, UK, 1987), 2nd ed.
[5] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley & Sons, New York, 1972).
[6] J. Y. Ollitrault, Eur. J. Phys. **29**, 275 (2008).
[7] E. Gourgoulhon, EAS Publ. Ser. **21**, 43 (2006).
[8] S. A. Voloshin, A. M. Poskanzer, and R. Snellings, arXiv:0809.2949 [nucl-ex].
[9] D. Teaney, J. Lauret, and E. V. Shuryak, Phys. Rev. Lett. **86**, 4783 (2001); arXiv:nucl-th/0110037.
[10] P. F. Kolb, P. Huovinen, U. W. Heinz, and H. Heiselberg, Phys. Lett. **B500**, 232 (2001); P. F. Kolb, J. Sollfrank, and U. W. Heinz, Phys. Rev. C **62**, 054909 (2000).
[11] T. Hirano and K. Tsuda, Phys. Rev. C **66**, 054905 (2002); T. Hirano and M. Gyulassy, Nucl. Phys. **A769**, 71 (2006).
[12] P. Romatschke and U. Romatschke, Phys. Rev. Lett. **99**, 172301 (2007); H. Song and U. W. Heinz, Phys. Lett. **B658**, 279 (2008); K. Dusling and D. Teaney, Phys. Rev. C **77**, 034905 (2008).
[13] E. V. Shuryak, Prog. Part. Nucl. Phys. **53**, 273 (2004); Nucl. Phys. **A750**, 64 (2005); Prog. Part. Nucl. Phys. **62**, 48 (2009).
[14] M. Gyulassy and L. McLerran, Nucl. Phys. **A750**, 30 (2005).
[15] D. E. Kharzeev, Nucl. Phys. **A827**, 118c (2009).
[16] T. Schaefer and D. Teaney, arXiv:0904.3107 [hep-ph].
[17] J. Liao and E. Shuryak, arXiv:0809.2419 [hep-ph].
[18] J. Liao and E. Shuryak, Phys. Rev. C **75**, 054907 (2007); Phys. Rev. Lett. **101**, 162302 (2008); Phys. Rev. D **73**, 014509 (2006); Nucl. Phys. **A775**, 224 (2006); Phys. Rev. C **77**, 064905 (2008); Phys. Rev. Lett. **102**, 202302 (2009).
[19] M. N. Chernodub and V. I. Zakharov, Phys. Rev. Lett. **98**, 082002 (2007); M. N. Chernodub, H. Verschelde, and V. I. Zakharov, arXiv:0905.2520 [hep-ph].
[20] A. D'Alessandro and M. D'Elia, Nucl. Phys. **B799**, 241 (2008).
[21] C. Ratti and E. Shuryak, arXiv:0811.4174 [hep-ph].
[22] T. Csörgő, F. Grassi, Y. Hama, and T. Kodama, Heavy Ion Phys. A **21**, 63 (2004) [Acta Phys. Hung. A **21**, 63 (2004)]; M. Csanád, M. I. Nagy, and T. Csörgő, Eur. Phys. J. Spec. Top. **155**, 19 (2008).

[23] M. I. Nagy, T. Csörgő, and M. Csanád, Phys. Rev. C **77**, 024908 (2008).
[24] A. Bialas, R. A. Janik, and R. B. Peschanski, Phys. Rev. C **76**, 054901 (2007).
[25] S. Pratt, Phys. Rev. C **75**, 024907 (2007).
[26] C. Y. Wong, Phys. Rev. C **78**, 054902 (2008); arXiv:0809.0517 [nucl-th].
[27] Yu. M. Sinyukov and I. A. Karpenko, Acta Phys. Hung. A **25**, 141 (2006).
[28] M. S. Borshch and V. I. Zhdanov, SIGMA **3**, 116 (2007).
[29] T. Koide and T. Kodama, Phys. Rev. E **78**, 051107 (2008); T. Koide, *ibid.* **75**, 060103(R) (2007).
[30] L. D. Landau, Izv. Akad. Nauk SSSR Ser. Fiz. **17**, 51 (1953); I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. **27**, 529 (1954).
[31] R. C. Hwa, Phys. Rev. D **10**, 2260 (1974).
[32] J. D. Bjorken, Phys. Rev. D **27**, 140 (1983).
[33] T. S. Biró, Phys. Lett. **B474**, 21 (2000); **B487**, 133 (2000).
[34] J. Liao and V. Koch, Phys. Rev. Lett. **103**, 042302 (2009).
[35] P. F. Kolb, Heavy Ion Phys. **21**, 243 (2004).
[36] J. Y. Ollitrault, Phys. Rev. D **46**, 229 (1992).
[37] J. Liao and E. Shuryak, arXiv:0804.4890 [hep-ph].