Exchange current contributions in null-plane quantum models of elastic electron-deuteron scattering

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(Received 11 December 2008; revised manuscript received 8 June 2009; published 31 August 2009)

We investigate the effects of two-body currents on elastic electron-deuteron scattering in an exactly Poincaré invariant quantum mechanical model with a null-plane kinematic symmetry. While calculations using single-nucleon currents as input produce good qualitative agreement with experiment, the two-body current that we investigate produces a good quantitative agreement between theory and experiment for all three elastic scattering observables.

DOI: 10.1103/PhysRevC.80.025503 PACS number(s): 11.80.−m, 13.40.Gp, 21.45.Bc, 25.30.Bf

I. INTRODUCTION

In the one-photon-exchange approximation, elastic electron-deuteron scattering observables are functions of the current matrix elements $\langle (1,d)P_d', \mu_d' | I^\mu(0) | (1,d)P_d, \mu\rangle$, where $|(1,d)P_d, \mu\rangle$ are deuteron four-momentum eigenstates and $I^\mu(x)$ is the deuteron current density operator.

In a relativistic quantum theory, the deuteron eigenstates transform irreducibly with respect to a dynamical unitary representation of the Poincaré group. In this work, they are eigenstates of a two-nucleon mass operator constructed by adding "realistic" nucleon-nucleon interactions, such as Argonne V18 or CD Bonn [1,2], to the square of the noninteracting invariant mass operator, $M^2 = M_0^2 + 4mv_{nn}$. With this mass operator, the S-matrix elements of the resulting two-nucleon model are consistent with the nucleon-nucleon scattering data used to determine the interaction.

The representation of the dynamics can additionally be chosen so the interaction v_{nn} is invariant with respect to a noninteracting "kinematic" subgroup of the Poincaré group. Different choices of kinematic subgroup [3] are related by unitary transformations that do not change the two-body S matrix [4], but the choice of representation of the dynamics affects the representation of the current operator. In this work, the interaction is taken to be invariant with respect to a subgroup of the Poincaré group that leaves a plane tangent to the light cone invariant. This is called the null-plane kinematic subgroup. It is the largest possible kinematic subgroup, and it has well-known advantages for treating electron-scattering problems.

The hadronic current density operator is conserved and covariant with respect to the dynamical representation of the Poincaré group. It is necessarily the sum of one-body and two-body operators. Cluster properties imply that one-body current operators, which are determined by empirical nucleon form factors, are conserved and transform covariantly with respect to the noninteracting representation of the Poincaré group. In a dynamical model, the one-body parts of the current are only covariant with respect to the kinematic subgroup.

Two-body parts of the current operator are needed to restore full covariance and may include additional contributions of a dynamical origin. Deuteron matrix elements of an exactly covariant hadronic current are constrained by the Poincaré transformation properties of the current operator and deuteron state vectors. These linear relations can be used to express all of the current matrix elements in terms of three linearly independent current matrix elements, which are linearly related to invariant deuteron form factors. This current is also covariant with respect to the null-plane kinematic subgroup. The kinematic subgroup is relevant because the one and two-body parts of the current operator are separately covariant with respect to the kinematic subgroup. This more limited symmetry allows all current matrix elements to be expressed in terms of four linearly independent current matrix elements, using only null-plane kinematic transformations. The full covariance implies an additional dynamical constraint among these four kinematically independent matrix elements. In a dynamical model with a null-plane kinematic symmetry, the additional dynamical constraint is rotational covariance [5–8], which was appropriately referred to as the "angular condition" by Leutwyler and Stern.

Any kinematically covariant current operator can be used to define matrix elements of a fully covariant current by evaluating any three linearly independent current matrix elements using the kinematically covariant current operator. The remaining current matrix elements are generated by the constraints implied by covariance, current conservation, and parity. This construction ensures that the fully covariant current and kinematically covariant current operators agree on the three independent matrix elements and *all matrix elements related to them by kinematic Poincaré transformations*. The resulting covariant current depends on the choice of the three linearly independent matrix elements, which are chosen among the four kinematically independent matrix elements related by the angular condition.

Null-plane "impulse calculations" compute the three independent current matrix elements using the single-nucleon current operators. When the null plane is oriented so the "+" component of the momentum transfer, $Q^+ = Q^0 + Q^3$, is zero, the independent matrix elements can be chosen to be matrix elements of the + component, $I^+(0) = I^0(0) + I^3(0)$ of the current. When the independent matrix elements are taken to be matrix elements of the + component of the current, then (1) the one-body matrix elements are invariant with respect to null-plane boosts, (2) the single-nucleon form factors exactly factor out of the integral for the current matrix element, and (3) the momentum transfer to the nucleons is identical to the momentum transfer to the deuteron.

When used with realistic nucleon-nucleon interactions and empirical nucleon form factors, null-plane impulse calculations give a good qualitative description of all the elastic electron-deuteron scattering observables. The results are not sensitive to the choice of realistic interaction, empirical nucleon form factors, or the choice of independent matrix elements used to construct the fully covariant current. Quantitative differences between the results of null-plane impulse calculations and experiment, particularly in the observables A and T_{20} cannot be explained by experimental uncertainties or uncertainties in theoretical input.

To account for the difference between theory and experiment, it is necessary to introduce an additional two-body current operator that has a nonvanishing contribution to the set of independent current matrix elements used to generate the fully covariant current. In this paper, we investigate the consequences of including an additional two-body current that has an operator structure motivated by "pair-current" contributions in covariant formulations of elastic electron-deuteron scattering. In this work, the two-body current is constructed to be covariant with respect to the null-plane kinematic subgroup.

Null-plane quantum mechanical models use Poincaré covariant two-component spinors rather than Lorentz covariant four-component spinors. Dirac spinors, which are spinor representations of Lorentz boosts, transform between the two- and four-component spinor representations. In the four-component representation, interactions that have a γ_5 vertex include contributions that couple to the photon through a causal propagator that do not appear in the null-plane impulse calculation. In the two-component spinor representation, this contribution can be included as a two-body operator in the current. These considerations determine the structure of our model two-body current. The operator is constructed to satisfy exact null-plane kinematic covariance. It is extended to a fully covariant current in the same way that the one-body current in the "null-plane impulse" calculations is extended to a fully covariant current.

The resulting model current, when added to the current generated by the empirical nucleon form factors, leads to a good quantitative description of all the elastic electron-deuteron scattering observables. Ambiguities related to the implementation of the angular condition are investigated.

Elastic electron-deuteron scattering has been studied by many authors [6,7,9–22] using a variety of different methods and assumptions. In all cases, the problem is to construct matrix elements of a conserved covariant current operator for a range of spacelike momentum transfers Q. The calculations discussed in this work are based on quantum mechanical models in which the Poincaré group is an exact symmetry of the underlying theory and the interaction has a null-plane kinematic symmetry. Exactly Poincaré invariant quantum mechanical models using other choices of kinematic subgroups have also been applied to electron-deuteron scattering. Different choices of kinematic subgroup have dynamical consequences for the impulse calculations. Another class of calculations is based on quasipotential treatments. These calculations extract the current matrix elements from covariant amplitudes without directly using the underlying quantum theory. This is done by generalizing Mandelstam's [23] method to extract current matrix elements from Green's functions using Bethe-Salpeter wave functions. "Relativistic impulse approximations" in this formalism depend on the specific quasipotential reduction, but they generally include a pair-current contribution that is related to the two-body current used in this paper. Detailed reviews can be found in Refs. [20,24].

The Poincaré invariant dynamical models in this paper are formulated in a representation with a null-plane kinematic symmetry [5,25,26]. Unlike models based on null-plane quantum field theory [8,27,28], where few degrees of freedom truncations break rotational covariance, the class of quantum mechanical models that we consider are *exactly* rotationally invariant. The null-plane treatment of electron scattering has the following advantages:

(i) For electron scattering, the momentum transfer Q is spacelike, so the orientation of the null plane can be chosen such that the + component, $Q^+ := Q^0 + Q^3$, of the momentum transfer is zero. When $Q^+ = 0$, all current matrix elements can be constructed from three independent matrix elements of the + component, $I^+(0) := I^0(0) + I^3(0)$, of the current:

$$\begin{aligned} &\langle (1,d), \tilde{\mathbf{P}}', \mu' | I^+(0) | (1,d), \tilde{\mathbf{P}}, \mu \rangle, \\ &\tilde{\mathbf{P}} := (P^+, P_1, P_2), \end{aligned} \tag{1}$$

where we have labeled the deuteron four-momenta by their null-plane components $\tilde{\mathbf{P}}$ and the spins are null-plane spins (defined in Sec. II).

(ii) There is a three-parameter subgroup of Lorentz boosts that leaves the null plane invariant. Because these boosts form a subgroup, there are no Wigner rotations associated with null-plane boosts. It follows that matrix elements of $I^+(0)$ with $Q^+ = 0$ and *delta-function normalized initial and final states* are independent of frames related by null-plane boosts:

$$\begin{aligned} &\langle (1,d), \tilde{\mathbf{P}}''', \mu' | I^+(0) | (1,d), \tilde{\mathbf{P}}'', \mu \rangle \\ &\quad = \langle (1,d), \tilde{\mathbf{P}}', \mu' | I^+(0) | (1,d), \tilde{\mathbf{P}}, \mu \rangle, \end{aligned} \tag{2}$$

where $P''' = \Lambda P'$, $P'' = \Lambda P$, and Λ is any null-plane preserving boost. This means that matrix elements of $I^+(0)$ are equal to null-plane Breit-frame matrix elements with the same null-plane spin magnetic quantum numbers. The null-plane Breit-frame matrix elements are defined to have momentum transfer perpendicular to the unit vector ($\hat{\mathbf{z}}$), which defines the orientation of the null plane. This is consistent with the requirement that $Q^+ = 0$.

(iii) A consequence of property two is that the *one-body* current matrix elements of $I^+(0)$ with the physical momentum transfer *exactly* factor out of the corresponding nuclear current matrix elements:

$$\begin{aligned} &\langle \Psi, \tilde{\mathbf{P}} + \tilde{\mathbf{Q}} | I_i^+(0) | \Psi, \tilde{\mathbf{P}} \rangle \\ &\quad = \langle \tilde{\mathbf{p}}_{i0} + \tilde{\mathbf{Q}} | I_i^+(0) | \tilde{\mathbf{p}}_{i0} \rangle \int d\tilde{\mathbf{p}}_1 \cdots d\tilde{\mathbf{p}}_n \langle \Psi, \tilde{\mathbf{P}} \\ &\qquad + \tilde{\mathbf{Q}} | \tilde{\mathbf{p}}_1 \cdots \tilde{\mathbf{p}}_i + \tilde{\mathbf{Q}} \cdots \tilde{\mathbf{p}}_n \rangle \langle \tilde{\mathbf{p}}_1 \cdots \tilde{\mathbf{p}}_i \cdots \tilde{\mathbf{p}}_n | \Psi, \tilde{\mathbf{P}} \rangle. \end{aligned} \tag{3}$$

This follows directly from Eq. (2), because covariance with respect to null-plane boosts can be used to remove the dependence of the constituent current matrix elements on the internal momenta without changing the spin sums or normalization. A proof can be found in Ref. [26]. This means that the matrix elements of the one-body contributions to $I^+(0)$ can be expressed as sums of products of null-plane Breit-frame nucleon matrix elements multiplied by functions of Q^2 that only depend on the initial and final wave functions.

With this formalism, it is possible to satisfy exact rotational covariance, and the impulse contribution to the independent current matrix elements involves only experimentally observable on-shell single-nucleon matrix elements.

This paper is organized as follows. In Sec. II, we introduce our notation and define the null-plane kinematic subgroup. In Sec. III, we construct a dynamical representation of the Poincaré group with a null-plane kinematic symmetry that provides a realistic description of the two-nucleon system. In Sec. IV, we review the experimental observables for elastic electron-deuteron scattering and their relation to current matrix elements. In Sec. V, we discuss the nucleon currents that are used to construct the one-body part of our current operator. In Sec. VI, we use the Wigner-Eckart theorem for the Poincaré group to realize the dynamical constraints on the current, and we discuss some of the ambiguities in the implementation of the angular condition. In Sec. VII, we define our dynamical two-body current. In Sec. VIII, we summarize our results. Appendix A summarizes how we relate our two-body current to the pion-exchange part of the model interaction, and Appendix B contains additional details related to how this is done with the AV18 interaction.

II. NULL-PLANE KINEMATICS

In this section, we discuss our notation and the null-plane kinematic subgroup introduced by Dirac [3]. The null plane is the three-dimensional hyperplane of points tangent to the light cone satisfying the condition

$$\{x | x^+ := t + \mathbf{x}\cdot\hat{\mathbf{e}}_3 = 0\}. \tag{4}$$

The null-plane components $\tilde{\mathbf{x}} = (x^+, \mathbf{x}_\perp)$ of the four-vector x^μ are

$$x^\pm := t \pm \mathbf{x}\cdot\hat{\mathbf{e}}_3, \quad \mathbf{x}_\perp = (\mathbf{x}\cdot\hat{\mathbf{e}}_1, \mathbf{x}\cdot\hat{\mathbf{e}}_2). \tag{5}$$

Four-vectors x^μ can be represented by 2×2 Hermitian matrices in the null-plane components of x^μ:

$$X = \begin{pmatrix} x^+ & x_\perp^* \\ x_\perp & x^- \end{pmatrix} = x^\mu \sigma_\mu, \quad x_\perp = x_1 + i x_2, \qquad x^\mu = \frac{1}{2}\mathrm{Tr}(\sigma_\mu X), \tag{6}$$

where $\sigma_\mu = (I, \boldsymbol{\sigma})$ and $\boldsymbol{\sigma}$ are the 2×2 Pauli matrices. Since the determinant of X is the square of the proper time, $-x^2$, if Λ is a complex 2×2 matrix with unit determinant, then the transformation

$$X \to X' = \Lambda X \Lambda^\dagger \tag{7}$$

defines a real Lorentz transformation,

$$\Lambda^\mu_\nu = \tfrac{1}{2}\mathrm{Tr}(\sigma_\mu \Lambda \sigma_\nu \Lambda^\dagger). \tag{8}$$

We use the notation Λ to represent both the 2×2 and 4×4 representations of Lorentz transformations; the implied representation is easily determined. Points on the null plane can be represented by triangular matrices of the form

$$X = \begin{pmatrix} 0 & x_\perp^* \\ x_\perp & x^- \end{pmatrix}. \tag{9}$$

Poincaré transformations

$$X \to X' = \Lambda X \Lambda^\dagger + A \tag{10}$$

that leave the null plane invariant [i.e., preserve the form of Eq. (9)] have the form

$$\Lambda = \begin{pmatrix} \alpha & 0 \\ \beta & 1/\alpha \end{pmatrix}, \quad A = \begin{pmatrix} 0 & a_\perp^* \\ a_\perp & a^- \end{pmatrix} = a^\mu \sigma_\mu, \tag{11}$$

where $\alpha \neq 0$, β and $a_\perp$ are complex, and a^- is real. These transformations form a seven-parameter subgroup of the Poincaré group that leaves the null plane invariant. This subgroup is called the *kinematic subgroup* of the null plane. This subgroup includes the three-parameter subgroup of translations on the null plane, a three-parameter subgroup of null-plane-preserving boosts as well as rotations about the 3-axis.

The null-plane-preserving boost that transforms a rest four-momentum $(m, \mathbf{0})$ to a value p is the matrix-valued function of $\tilde{\mathbf{v}} = \tilde{\mathbf{p}}/m := (p^+/m, p^1/m, p^2/m)$ given by

$$\Lambda_f(\tilde{\mathbf{p}}/m) := \Lambda_f(\tilde{\mathbf{v}}) := \begin{pmatrix} \sqrt{v^+} & 0 \\ v_\perp/\sqrt{v^+} & 1/\sqrt{v^+} \end{pmatrix}. \tag{12}$$

Since the reality of α in Eq. (11) is preserved under matrix multiplication, the null-plane boosts, Eq. (12), form a *subgroup* of the Lorentz group.

The action of a null-plane boost on an arbitrary four-momentum vector is determined by using Eq. (12) in Eq. (7). The resulting transformation property of the $+$ and $\perp$ components of the four-momentum is

$$p^+ \to p^{+\prime} = v^+ p^+, \quad \mathbf{p}_\perp \to \mathbf{p}'_\perp = \mathbf{p}_\perp + \mathbf{v}_\perp p^+. \tag{13}$$

Since p^- does not appear in Eq. (13), the three components $\tilde{\mathbf{p}} := (p^+, \mathbf{p}_\perp)$ are called a "null-plane vector." The $-$ component can be calculated using the mass-shell condition

$$p^- = \frac{m^2 + \mathbf{p}_\perp^2}{p^+}. \tag{14}$$

The null-plane spin of a particle of mass m is defined [5,26,29] so that it (1) agrees with ordinary canonical spin in the particle's rest frame and (2) is invariant with respect to null-plane boosts, Eq. (12).

The null-plane boost of Eq. (12) differs from the more standard rotationless boost, which has the 2×2 matrix form

$$\Lambda_c(\mathbf{k}/m) = e^{\frac{1}{2}\boldsymbol{\rho}\cdot\boldsymbol{\sigma}}, \tag{15}$$

where $\boldsymbol{\rho}$ is the rapidity

$$\boldsymbol{\rho} = \hat{\mathbf{k}}|\boldsymbol{\rho}|, \quad \cosh(|\boldsymbol{\rho}|) = \sqrt{\mathbf{k}^2 + m^2}/m, \quad \sinh(|\boldsymbol{\rho}|) = \mathbf{k}/m. \tag{16}$$

The null-plane representation of the single-nucleon Hilbert space $\mathcal{H}_1$ is the space of square integrable functions of the null-plane vector components of the particle's four-momentum and the three-component of its null-plane spin:

$$\psi(\tilde{\mathbf{p}}, \mu) = \langle (j, m), \tilde{\mathbf{p}}, \mu|\psi\rangle, \quad \int_0^\infty dp^+ \int_{\mathbb{R}^2} d\mathbf{p}_\perp \sum_{\mu=-j}^{j} |\psi(\tilde{\mathbf{p}}, \mu)|^2 < \infty. \tag{17}$$

The Poincaré group acts irreducibly on single-nucleon states

$$\langle (j, m)\tilde{\mathbf{p}}, \mu|U_1(\Lambda, A)|\psi\rangle = \int_0^\infty dp^{+\prime}\int_{\mathbb{R}^2} d\mathbf{p}'_\perp \sum_{\mu'=-j}^{j} \mathcal{D}^{m,j}_{\tilde{\mathbf{p}},\mu;\tilde{\mathbf{p}}',\mu'}[\Lambda, A]\langle (j, m), \tilde{\mathbf{p}}', \mu'|\psi\rangle, \tag{18}$$

where the Poincaré group Wigner $\mathcal{D}$ function in the null-plane irreducible basis [26] is

$$\mathcal{D}^{m,j}_{\tilde{\mathbf{p}},\mu;\tilde{\mathbf{p}}',\mu'}[\Lambda, A] := \langle (j, m), \tilde{\mathbf{p}}, \mu|U(\Lambda, A)|(j, m), \tilde{\mathbf{p}}', \mu'\rangle = (\tilde{\mathbf{p}} - \tilde{\boldsymbol{\Lambda}}(p'))\sqrt{\frac{p^+}{p^{+\prime}}} D^j_{\mu\mu'}[R_{fw}(\Lambda, \tilde{\mathbf{p}}')]e^{ip\cdot a}, \tag{19}$$

$D^j_{\mu\mu'}[R]$ is the ordinary SU(2) Wigner D function, and

$$R_{fw}(\Lambda, \tilde{\mathbf{p}}') := \Lambda_f^{-1}(\tilde{\mathbf{p}}/m)\Lambda\Lambda_f(\tilde{\mathbf{p}}'/m) \tag{20}$$

is a null-plane Wigner rotation. It is the identity matrix when Λ is a null-plane boost.

Our model Hilbert space for the two-nucleon system is the tensor product of two single-nucleon Hilbert spaces. The kinematic representation of the Poincaré group on the two-nucleon Hilbert space is the tensor product of two single-nucleon representations of the Poincaré group:

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_1, \quad U_0(\Lambda, A) := U_1(\Lambda, A) \otimes U_1(\Lambda, A). \tag{21}$$

The tensor product of two one-body irreducible representations of the Poincaré group is reducible. It can be decomposed into an orthogonal linear superposition of irreducible representations using the Poincaré group Clebsch-Gordan coefficients in the null-plane basis,

$$\Psi((j_1, m_1), \tilde{\mathbf{p}}_1, \mu_1, (j_2, m_2), \tilde{\mathbf{p}}_2, \mu_2) = \sum \int \langle (j_1, m_1), \tilde{\mathbf{p}}_1, \mu_1; (j_2, m_2), \tilde{\mathbf{p}}_2, \mu_2|(j, k), l, s, \tilde{\mathbf{P}}, \mu\rangle \times d\tilde{\mathbf{P}}k^2\, dk\Psi((j, m), l, s, \tilde{\mathbf{P}}, \mu), \tag{22}$$

where the Poincaré group Clebsch-Gordan coefficient [26] in the null-plane basis is

$$\langle (j_1, m_1), \tilde{\mathbf{p}}_1, \mu_1; (j_2, m_2), \tilde{\mathbf{p}}_2, \mu_2|(j, k), l, s, \tilde{\mathbf{P}}, \mu\rangle = \delta(\tilde{\mathbf{P}} - \tilde{\mathbf{p}}_1 - \tilde{\mathbf{p}}_2)\frac{\delta(k - k(\tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2))}{k^2}\left|\frac{\partial(\tilde{\mathbf{P}}, \mathbf{k})}{\partial(\tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2)}\right|^{1/2} \times Y_{lm}(\hat{\mathbf{k}}(\tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2))D^{1/2}_{\mu_1,\mu_1'}[R_{fc}(\mathbf{k}/m_1)]D^{1/2}_{\mu_2,\mu_2'}[R_{fc}(-\mathbf{k}/m_2)] \times \left\langle \frac{1}{2}, \mu_1', \frac{1}{2}, \mu_2'\right| s, \mu_s\Big\rangle \langle l, m, s, \mu_s|j, \mu\rangle, \tag{23}$$

with

$$\left|\frac{\partial(\tilde{\mathbf{P}}, \mathbf{k})}{\partial(\tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2)}\right|^{1/2} = \sqrt{\frac{(p_1^+ + p_2^+)\omega_1(\mathbf{k})\omega_2(\mathbf{k})}{(\omega_1(\mathbf{k}) + \omega_2(\mathbf{k}))p_1^+ p_2^+}}, \tag{24}$$

$$R_{fc}(\mathbf{k}/m_1) := \Lambda_f^{-1}(\mathbf{k}/m_1)\Lambda_c(\mathbf{k}/m_1), \quad R_{fc}(-\mathbf{k}/m_2) := \Lambda_f^{-1}(-\mathbf{k}/m_2)\Lambda_c(-\mathbf{k}/m_2), \tag{25}$$

and $(\omega(\mathbf{k}), \mathbf{k}) = k(p_1, p_2) = \Lambda_f^{-1}(\tilde{\mathbf{P}}/M_0)p_1$. The rotation $R_{fc}(\mathbf{k}/m)$, called a Melosh rotation [26,30], transforms the null-plane spins so they rotate under a single representation of SU(2), which allows them to be coupled using ordinary SU(2) Clebsch-Gordan coefficients. The absence of Wigner rotations in Eq. (23) is a consequence of the null-plane boosts forming a subgroup. The quantum numbers l and s are degeneracy parameters that label different irreducible representations with the same mass and spin that appear in the tensor product. In Eq. (23), the two-body invariant mass M_0, which has a continuous spectrum, is replaced by $\mathbf{k}^2$, which is related to M_0 by

$$M_0 = 2\sqrt{\mathbf{k}^2 + m^2}. \tag{26}$$

In this basis, two-nucleon wave functions have the form

$$\psi((j, k), l, s\tilde{\mathbf{P}}, \mu) = \langle (j, k), l, s\tilde{\mathbf{P}}, \mu|\psi\rangle, \quad \int_0^\infty dP^+ \int_{\mathbb{R}^2} d\mathbf{P}_\perp \int_0^\infty k^2 dk \sum_{\mu=-j}^{j}\sum_{s=0}^{1}\sum_{l=|j-s|}^{|j+s|} \times |\psi((j, k), l, s, \tilde{\mathbf{P}}, \mu)|^2 < \infty \tag{27}$$

and $U_0(\Lambda, A)$ acts irreducibly on these states:

$$\langle (j, k), l, s, \tilde{\mathbf{P}}, \mu|U_0(\Lambda, A)|\psi\rangle = \int_0^\infty dP^{+\prime} \int_{\mathbb{R}^2} d\mathbf{P}'_\perp \sum_{\mu'=-j}^{j} \mathcal{D}^{M_0(k),j}_{\tilde{\mathbf{P}},\mu;\tilde{\mathbf{P}}',\mu'} \times [\Lambda, A]\langle (j, k), l, s, \tilde{\mathbf{P}}', \mu'|\psi\rangle = \sum_{\mu'=-j}^{j} e^{iP\cdot a}\sqrt{\frac{P^{+\prime}}{P^+}} D^j_{\mu\mu'}[R_{fw}(\Lambda, \tilde{\mathbf{P}}'/M_0)] \times \langle (j, k), l, s, \tilde{\mathbf{P}}', \mu'|\psi\rangle, \tag{28}$$

where $P = \Lambda P'$. This basis is used in the formulation of our dynamical model in the next section.

In the rest of this paper, we use the following notation for covariant, canonical, and null-plane basis vectors for mass-m, spin-j irreducible representation spaces, which are related by

$$|(j,m),p,\mu\rangle = |(j,m),\mathbf{p},\mu\rangle\sqrt{\frac{\omega_m(\mathbf{p})}{m}} = \sum_{\nu=-j}^{j}|(j,m),\tilde{\mathbf{p}},\nu\rangle\sqrt{\frac{p^+}{m}}D^j_{\nu\mu}[R_{fc}(\mathbf{k}/m)]. \tag{29}$$

III. DYNAMICS

Dynamical models of the two-nucleon system in Poincaré invariant quantum mechanics are defined by a dynamical unitary representation of the Poincaré group acting on the two-nucleon Hilbert space. The mass Casimir operator for this representation can be defined by adding a realistic nucleon-nucleon interaction v_{nn} [26,31] to the square of the noninteracting two-nucleon mass operator as follows:

$$M^2 = M_0^2 + 4mv_{nn}, \tag{30}$$

where m is the nucleon mass; and for a dynamical model with a null-plane kinematic symmetry, v_{nn} has the form

$$\langle (j',k'),l',s',\tilde{\mathbf{P}}',\mu'|v_{nn}|(j,k),l,s,\tilde{\mathbf{P}},\mu\rangle = \delta(\tilde{\mathbf{P}}'-\tilde{\mathbf{P}})\delta_{j'j}\delta_{\mu'\mu}\langle k',l',s'|v^j|k,l,s\rangle \tag{31}$$

in the noninteracting irreducible basis [Eq. (23)]. The interaction v_{nn} is restricted so that M^2 is a positive operator and designed so that M^2 commutes with the noninteracting operators j^2, $\tilde{\mathbf{P}}$, and j_z and is independent of the eigenvalues of $\tilde{\mathbf{P}}$ and j_z.

Simultaneous eigenstates of M^2, j^2, $\tilde{\mathbf{P}}$, j_z in the noninteracting irreducible basis have the form

$$\langle (j',k'),l',s',\tilde{\mathbf{P}}',\mu'|(j,\lambda),\tilde{\mathbf{P}},\mu\rangle = \delta(\tilde{\mathbf{P}}'-\tilde{\mathbf{P}})\delta_{j'j}\delta_{\mu'\mu}\phi_{\lambda,j}(k,l,s), \tag{32}$$

where the wave function $\phi_{\lambda,j}(k,l,s)$ is a solution of the eigenvalue equation

$$(\lambda^2-4k^2-4m^2)\phi_{\lambda,j}(k,l,s) = \sum_{s'=0}^{1}\sum_{l'=|j-s'|}^{|j+s'|}\int_0^\infty 4m\langle k,l,s|v^j_{nn}|k',l',s'\rangle \times k'^2\,dk'\phi_{\lambda,j}(k',l',s'). \tag{33}$$

The eigenfunctions $\phi_{\lambda,j}(k,s)$ of this mass operator are also solutions to the nonrelativistic Schrödinger equation with interaction v_{nn}, which can be seen by dividing both sides of Eq. (33) by $4m$. The deuteron is an even-parity bound state with $j=s=1$, and parity limits the l' sum in Eq. (33) to $l'\in\{0,2\}$.

If $\{M^2,j,\tilde{\mathbf{P}},\mu\}$ have the same interpretation as $\{M_0^2,j,\tilde{\mathbf{P}},\mu\}$, then it follows from Eq. (31) that the eigenstates of Eq. (32) transform irreducibly with respect to a *dynamical* representation of the Poincaré group defined by replacing the eigenvalues of M_0 in $\mathcal{D}^{M_0,j}_{\tilde{\mathbf{P}},\mu;\tilde{\mathbf{P}}',\mu'}[\Lambda,A]$ by the eigenvalues λ of M:

$$\begin{aligned}\langle (j',k'),l',s'\tilde{\mathbf{P}}',\mu'|U(\Lambda,A)|(j,\lambda),\tilde{\mathbf{P}},\mu\rangle &= \int\sum_{\mu''=-j}^{j}\langle (j',k'),l',s'\tilde{\mathbf{P}}',\mu'|(j,\lambda),\tilde{\mathbf{P}}'',\mu''\rangle d\tilde{\mathbf{P}}'' \\ &\quad\times\langle (j,\lambda),\tilde{\mathbf{P}}'',\mu''|U(\Lambda,A)|(j,\lambda),\tilde{\mathbf{P}},\mu\rangle \\ &= \phi_{j,\lambda}(k',l',s')\mathcal{D}^{\lambda,j'}_{\tilde{\mathbf{P}}',\mu';\tilde{\mathbf{P}},\mu}[\Lambda,A].\end{aligned} \tag{34}$$

Since the eigenstates of Eq. (32) are complete, $U(\Lambda,A)$, defined by Eq. (34), is a dynamical unitary representation of the Poincaré group on the two-nucleon Hilbert space. *When* (Λ,A) is an element of the kinematic subgroup of the null plane, the Poincaré Wigner $\mathcal{D}$ function in the null-plane basis, $\mathcal{D}^{\lambda,j}_{\tilde{\mathbf{P}}',\mu';\tilde{\mathbf{P}},\mu}[\Lambda,A]$, is independent of the mass eigenvalue λ and is thus identical to the Poincaré Wigner $\mathcal{D}$ function for the noninteracting irreducible representation. This shows that the null-plane kinematic subgroup defined by Eq. (11) is the kinematic subgroup of the dynamical representation in Eq. (34).

The rotational covariance in Eq. (34) is *exact*, but because $\mathcal{D}^{\lambda,j}_{\tilde{\mathbf{P}}',\mu';\tilde{\mathbf{P}},\mu}[\Lambda,A]$ depends explicitly on the mass eigenvalue λ when $\Lambda=R$ is a rotation, the rotations are dynamical transformations.

The eigenfunctions of the interacting mass operator M are identical functions of k,l,s to the eigenfunctions of the nonrelativistic Schrödinger equation with interaction v_{nn}. The identity

$$\Omega_\pm(H_{\rm nr},H_{0\rm nr}) = \Omega_\pm(H_r,H_{0r}), \tag{35}$$

where $\Omega_\pm(H,H_0)$ are the scattering wave operators for the nonrelativistic and relativistic system, respectively, follows from the elementary calculation

$$\begin{aligned}\Omega_\pm(H_r,H_{0r}) &= \lim_{t\to\pm\infty}e^{i(P_0^++P^-)t/2}e^{-i(P_0^++P_0^-)t/2} = \lim_{t'\to\pm\infty}e^{iP^-t'}e^{-iP_0^-t'} \\ &= \lim_{t'\to\pm\infty}e^{i(M^2+\mathbf{P}^2_{0\perp})t'/P_0^+}e^{-i(M_0^2+\mathbf{P}^2_{0\perp})t'/P_0^+} \\ &= \lim_{t''\to\pm\infty}e^{iM^2t''}e^{-iM_0^2t''} \\ &= \lim_{t''\to\pm\infty}e^{i(\mathbf{k}^2/m+v_{nn})4mt''+i4m^2t''}e^{-i(\mathbf{k}^2/m)4mt''-i4m^2t''} \\ &= \lim_{t'''\to\pm\infty}e^{i(\mathbf{P}^2/4m+\mathbf{k}^2/m+v_{nn})t'''}e^{-i(\mathbf{P}^2/4m+\mathbf{k}^2/m)t'''} \\ &= \Omega_\pm(H_{\rm nr},H_{0\rm nr}),\end{aligned} \tag{36}$$

where the limits are strong or absolute Abelian [32] limits and the times in Eq. (36) were reparametrized as follows:

$$t''' = 4mt'' = 4mt'/P_0^+ = 2mt/P_0^+. \tag{37}$$

This proof uses kinematic symmetries that follow from Eq. (31) and the spectral condition $P_0^+>0$. It follows from Eq. (36) that relativistic and nonrelativistic scattering matrices are *identical* functions of $\mathbf{k}$:

$$\begin{aligned}S(H_{\rm nr},H_{0\rm nr}) &= \Omega_+^\dagger(H_{\rm nr},H_{0\rm nr})\Omega_-(H_{\rm nr},H_{0\rm nr}) \\ &= \Omega_+^\dagger(H_r,H_{0r})\Omega_-(H_r,H_{0r}) = S(H_r,H_{0r}).\end{aligned} \tag{38}$$

Since realistic nn interactions [1,2] are constructed by correctly transforming experimental differential cross-section data to the center-of-momentum frame and then fitting the solution of the scattering problem to the transformed data, these interactions can be used in Eq. (30) *without modification*. There is a small binding energy correction of about 1 part in 2000, which we ignore. The difference in the relativistic and nonrelativistic wave function is due to the Poincaré-group Clebsch-Gordan coefficients, which are used to transform the wave function to the tensor product representation used in the computation of the current matrix elements.

We use this method to construct the deuteron eigenstates and the representation of the Poincaré group that we use to evaluate the deuteron current matrix elements.

IV. OBSERVABLES

The differential cross section for elastic electron-deuteron scattering in the one-photon-exchange approximation,

$$d\sigma = \frac{(2\pi)^4 e}{\sqrt{(p_e \cdot p_d)^2 - m_e^2 m_d^2}} \times |\langle (j_e, m_e), p'_e, \mu'_e | I_{e\mu}(0)(j_e, m_e), |p_e, \mu_e\rangle \frac{(2\pi)^3}{q^2} \times \langle (1,d), p'_d, \mu'_d | I_d^{\mu}(0) | (1,d), p_d, \mu_d\rangle|^2 \times \delta^4(p'_d - p_d + p'_e - p_e) \frac{d\mathbf{p}'_e}{\omega_e(\mathbf{p}'_e)} \frac{d\mathbf{p}'_d}{\omega_d(\mathbf{p}'_d)}, \tag{39}$$

is a quadratic function of the deuteron current matrix elements, $\langle (1,d)p'_d, \mu'_d | I_d^{\mu}(0) | (1,d), p_d, \mu_d\rangle$. Covariance, current conservation, and discrete symmetries imply that only three of the deuteron current matrix elements are linearly independent [33], which means that any deuteron elastic-scattering observable can be expressed in terms of three independent quantities for a given momentum transfer.

Standard observables are the structure functions $A(Q^2)$ and $B(Q^2)$ and the tensor polarization $T_{20}(Q^2, \theta)$ at $\theta_{\text{lab}} = 70^\circ$. The quantities $A(Q^2)$, $B(Q^2)$ are determined from the unpolarized laboratory frame differential cross section using a Rosenbluth [34] separation:

$$\frac{d\sigma}{d\Omega}(Q^2, \theta) = \frac{\alpha^2 \cos^2(\theta/2)}{4E_i^2 \sin^4(\theta/2)} \frac{E_f}{E_i} \times [A(Q^2) + B(Q^2)\tan^2(\theta/2)], \tag{40}$$

while $T_{20}(Q^2, \theta)$ is extracted from the difference in the cross sections for target deuterons having canonical spin polarizations $\mu_d = 1$ and $\mu_d = 0$ at a fixed laboratory scattering angle [35]:

$$T_{20}(Q^2, \theta) = \sqrt{2} \frac{\frac{d\sigma}{d\Omega}_1(Q^2, \theta) - \frac{d\sigma}{d\Omega}_0(Q^2, \theta)}{\frac{d\sigma}{d\Omega}(Q^2, \theta)}. \tag{41}$$

These experimental observables are related to the deuteron form factors [9,36–38]$G_0(Q^2)$, $G_1(Q^2)$, and $G_2(Q^2)$ by

$$A(Q^2) = G_0^2(Q^2) + \tfrac{2}{3}\eta G_1^2(Q^2) + G_2^2(Q^2), \tag{42}$$

$$B(Q^2) = \tfrac{4}{3}\eta(1+\eta) G_1^2(Q^2), \tag{43}$$

$$T_{20}(Q^2, \theta) = -\frac{G_2^2(Q^2) + \sqrt{8} G_0(Q^2) G_2(Q^2) + \frac{1}{3}\eta G_1^2(Q^2)[1 + 2(1+\eta)\tan^2(\theta/2)]}{\sqrt{2}[A(Q^2) + B(Q^2)\tan^2(\theta/2)]}, \tag{44}$$

where $\eta = \frac{Q^2}{4M_d^2}$, and M_d is the deuteron mass.

The form factors are Poincaré invariant quantities that are linearly related to current matrix elements. They are traditionally expressed in terms of canonical spin current matrix elements in the standard Breit frame, with the momentum transfer chosen parallel to the axis of spin quantization:

$$G_0(Q^2) = \frac{1}{3}(\langle (1,d), P'_b, 0 | I^0(0) | (1,d), P_b, 0\rangle + 2\langle (1,d), P'_b, 1 | I^0(0) | (1,d), -P_b, 1\rangle), \tag{45}$$

$$G_1(Q^2) = -\sqrt{\frac{2}{\eta}} \langle (1,d), P'_b, 1 | I^1(0) | (1,d), P_b, 0\rangle, \tag{46}$$

$$G_2(Q^2) = \frac{\sqrt{2}}{3} \langle (1,d), P'_b, 0 | I^0(0) | - (1,d), P_b, 0\rangle - \langle (1,d), P'_b, 1 | I^0(0) | (1,d), P_b, 1\rangle), \tag{47}$$

where $P'_b = (M_d\sqrt{1+\eta}, 0, 0, \frac{\mathbf{Q}}{2})$ and $P_b = (M_d\sqrt{1+\eta}, 0, 0, -\frac{\mathbf{Q}}{2})$, and the normalization of $G_2(Q^2)$ follows the conventions used in Ref. [38].

The zero-momentum transfer limit of these form factors are related to the charge, magnetic moment, and quadrupole moments [9], that is,

$$\lim_{Q^2 \to 0} G_0(Q^2) = 1, \quad \lim_{Q^2 \to 0} G_1(Q^2) = \frac{M_d}{m_n}\mu_d, \quad \lim_{Q^2 \to 0} \frac{G_2(Q^2)}{Q^2} = \frac{1}{2\sqrt{3}} Q_d. \tag{48}$$

Current covariance, current conservation, and discrete symmetries can be used to express the matrix elements in Eqs. (45)–(47) in terms of matrix elements of the plus component of the current in the null-plane Breit frame, where the momentum transfer is in the x direction and the spins are null-plane spins.

The canonical spin basis vectors in Eqs. (45)–(47) are related to the null-plane basis vectors defined in Eq. (17)

by Eq. (29), which involves a change in normalization and a Melosh rotation [Eq. (25)] of the spins.

V. NUCLEON CURRENTS

The impulse current is the sum of single-nucleon current operators for the proton and neutron. For spin-1/2 systems that are eigenstates of parity, there are two linearly independent current matrix elements [33]. They are related to the Dirac nucleon form factors $F_1(Q^2)$ and $F_2(Q^2)$ by

$$\langle (j_n, m_n), p', \nu' | I^\mu(0) | (j_n, m_n) p, \nu\rangle = \bar{u}(p')\Gamma^\mu u(p) \quad (49)$$

where

$$\Gamma^\mu = \gamma^\mu F_1(Q^2) - \frac{i}{2m}\sigma^{\mu\nu} Q_\nu F_2(Q^2), \quad (50)$$

and $u(p)$ is a canonical spin Dirac spinor.

For spacelike momentum transfers with $Q^+ = 0$, these form factors can be expressed in terms of the independent null-plane matrix elements

$$\left\langle \tfrac{1}{2}\right| I^+(0) \left|\tfrac{1}{2}\right\rangle := \left\langle (j_n, m_n), \tilde{\mathbf{p}}', \tfrac{1}{2}\right| I^+(0) \left| (j_n, m_n), \tilde{\mathbf{p}}, \tfrac{1}{2}\right\rangle = F_1(\mathbf{Q}^2), \quad (51)$$

$$\left\langle \tfrac{1}{2}\right| I^+(0) \left| -\tfrac{1}{2}\right\rangle := \left\langle (j_n, m_n), \tilde{\mathbf{p}}', \tfrac{1}{2}\right| I^+(0) \left| (j_n, m_n), \tilde{\mathbf{p}}, -\tfrac{1}{2}\right\rangle = -\sqrt{\tau} F_2(\mathbf{Q}^2), \quad (52)$$

where $\tau := \frac{\mathbf{Q}^2}{4m^2}$.

The Dirac nucleon form factors $F_1(Q^2)$ and $F_2(Q^2)$ are related to the Sachs form factors by [39]

$$F_1(\mathbf{Q}^2) = \frac{G_e(\mathbf{Q}^2) + \tau G_m(\mathbf{Q}^2)}{1+\tau}, \quad (53)$$

$$F_2(\mathbf{Q}^2) = \frac{G_m(\mathbf{Q}^2) - G_e(\mathbf{Q}^2)}{1+\tau}. \quad (54)$$

We consider recent parametrizations from Bijker and Iachello [40]; Bradford, Bodek, Budd, and Arrington [41]; Budd, Bodek, and Arrington [42]; Kelly [43]; and Lomon [44–46]. These parametrizations all determine the proton-electric form factors using the polarization experiments [47]. The input to our calculations is the isoscalar form factors, which are the sum of the proton and neutron form factors.

VI. DEUTERON CURRENTS: DYNAMICAL CONSTRAINTS

The hadronic current density operator $I^\mu(x)$ transforms as a four-vector density under the dynamical representation (34) of the Poincaré group,

$$U(\Lambda, A) I^\mu(x) U^\dagger(\Lambda, A) = (\Lambda^{-1})^\mu{}_\nu I^\nu(\Lambda x + a). \quad (55)$$

The current operator must also satisfy current conservation

$$g_{\mu\nu}[P^\mu, I^\nu(0)]_- = 0, \quad (56)$$

in addition to symmetries with respect to space reflections and time reversal. Here $g_{\mu\nu}$ is the Minkowski metric with signature $(-+++)$. Current covariance and current conservation are dynamical constraints on the current operator.

Using the identity

$$\begin{aligned}\langle (1,d), \tilde{\mathbf{P}}', \nu' | I^\mu(x) | (1,d), \tilde{\mathbf{P}}, \nu\rangle &= \langle (1,d), \tilde{\mathbf{P}}', \nu' | U^\dagger(\Lambda, A) U(\Lambda, A) I^\mu(x) U^\dagger \\ &\quad \times (\Lambda, A) U(\Lambda, A) | (1,d), \tilde{\mathbf{P}}, \nu\rangle, \end{aligned} \quad (57)$$

with the current covariance [Eq. (55)] and the transformation properties [Eq. (34)] of the deuteron eigenstates gives linear equations for each value of Λ and A that relate the different matrix elements. Current conservation leads to additional linear constraints on the current matrix elements:

$$g_{\alpha\mu}(P'^\alpha - P^\alpha)\langle (1,d), \tilde{\mathbf{P}}', \nu' | I^\mu(0) | (1,d), \tilde{\mathbf{P}}, \nu\rangle = 0. \quad (58)$$

It is well known that these constraints, when combined with space-reflection and time-reversal symmetries [33], can be used to express any deuteron current matrix element as a linear combination of three independent current matrix elements. This is the Wigner-Eckart theorem for the Poincaré group applied to a conserved current. The invariant form factors or independent current matrix elements play the role of invariant matrix elements in the Wigner-Eckart theorem.

If the coordinate axes are chosen so the + component of the momentum transfer is zero, which can always be done for electron scattering, then it is possible to choose the independent matrix elements as matrix elements of the + component of the current density, $I^+(0)$ [7]. Matrix elements of $I^+(0)$ in an irreducible null-plane basis with normalization (17) are invariant with respect to the group of null-plane boosts.

It follows from this invariance that the matrix elements of $I^+(0)$ only depend on the square of the momentum transfer and the individual null-plane spin components. There are $3 \times 3 = 9$ combinations of initial and final spins. Covariance with respect to the null-plane kinematic subgroup relates all of them to four kinematically independent current matrix elements. The one- and two-body parts of the current operator transform independently under the action of the kinematic subgroup. The kinematic subgroup does not include the full rotation group. Rotational covariance [5–7] gives one additional constraint among the kinematically covariant matrix elements, called the "angular condition," reducing the number of independent matrices to three. It relates matrix elements of the one- and two-body parts of the current operator.

Because the constraints (55) and (56) are dynamical, it is a nontrivial problem to construct fully covariant current *operators*. It is much easier to construct a set of deuteron matrix elements of a conserved covariant current operator. This can be done by first computing a set of three independent current matrix elements. The remaining matrix elements are generated from these three matrix elements using the constraints implied by Eqs. (55) and (58). It is possible to do better by computing the three independent matrix elements using a model current that is covariant with respect to the null-plane kinematic subgroup. Operators that are covariant with respect to the kinematic subgroup are easily constructed. In this case, the kinematically covariant current and fully covariant current will agree not only on the three independent current matrix elements, but also on any matrix elements generated from the independent matrix elements by the null-plane kinematic subgroup. The angular condition is needed to obtain all the current matrix elements. It is a dynamical constraint that generates the fourth kinematically independent matrix element from the three independent matrix elements in a manner consistent with rotational covariance. The remaining current matrix elements can be generated from this fourth matrix element using only kinematic covariance. This gives all the deuteron matrix elements of a conserved covariant current operator.

While it is straight forward to construct kinematically covariant current operators, there are many fully covariant current operators that agree with a given kinematically covariant current operator on different subsets of matrix elements. This leads to ambiguities when a kinematically covariant current is used to generate matrix elements of a fully covariant current. Model assumptions are needed to eliminate these ambiguities. Specifically, a fully covariant current operator can be chosen to agree with the kinematically covariant current operator on any set of three independent current matrix elements and all matrix elements generated from them using kinematic Poincaré transformations. Because kinematic covariance cannot generate all the current matrix elements, the resulting covariant current will depend on the choice of independent current matrix elements used to generate the full current, which is the model assumption.

"Null-plane impulse approximations" choose the kinematically covariant current to be the sum of the single-nucleon currents. For the deuteron, the elastic scattering observables computed in the null-plane impulse approximation are not very sensitive to the choice of independent current matrix elements used to generate the full current.

In this paper, we use the same method to compute matrix elements of a dynamical two-body current operator. A kinematically covariant model two-body current is used to compute independent current matrix elements. The remaining current matrix elements are generated using Eqs. (55) and (58). As in the case of the null-plane impulse approximation, a complete specification of a fully covariant model current requires both a kinematically covariant current and a choice of kinematically independent current matrix elements.

If the deuteron eigenstates $|(1,d),\tilde{\mathbf{P}},\mu\rangle$ are given a delta-function normalization, then the matrix elements

$$I^+_{\mu,\nu} := \langle (1,d),\tilde{\mathbf{P}}',\mu|I^+(0)|(1,d),\tilde{\mathbf{P}},\nu\rangle, \tag{59}$$

with $Q^+ = P^{+\prime} - P^+ = 0$ and null-plane spins, are invariant with respect to null-plane boosts. The four kinematically independent matrix elements of $I^+(0)$ can be taken as $I^+_{1,1}$, $I^+_{1,0}$, $I^+_{0,0}$, and $I^+_{1,-1}$. They are related by rotational covariance. The result is that only three linear combinations of these matrix elements can be taken as independent.

The three independent linear combinations of the current matrix elements used to generate the fully covariant current in Ref. [7] were

$$I^+_{1,1} + I^+_{0,0}, \quad I^+_{1,0}, \quad I^+_{1,-1}. \tag{60}$$

We refer to this choice of independent current matrix elements as choice I. These independent matrix elements can be distinguished by the number of spin flips between the initial and final states.

Rotational covariance, or equivalently the angular condition, relates the difference $I^+_{1,1} - I^+_{0,0}$ to the matrix elements in Eq. (60):

$$I^+_{1,1} - I^+_{0,0} = -\frac{1}{1+\eta}[\eta(I^+_{1,1} + I^+_{0,0}) - 2\sqrt{2\eta}I^+_{1,0} + I^+_{1,-1}], \tag{61}$$

where

$$\eta := Q^2/4M_d^2. \tag{62}$$

This relation is dynamical because η involves the deuteron mass eigenvalue M_d.

Direct computation of the difference $I^+_{1,1} - I^+_{0,0}$ using the kinematically covariant current operator and the difference determined by the constraint (61) of current covariance

$$\Delta_- := I^+_{1,1} - I^+_{0,0} + \frac{1}{1+\eta}[\eta(I^+_{1,1} + I^+_{0,0}) - 2\sqrt{2\eta}I^+_{1,0} + I^+_{1,-1}] \tag{63}$$

gives a measure of the size of the dynamical contribution to the current operator that is generated by the rotational covariance constraint. If this difference is small compared to the size of independent matrix elements, then there will not be too much sensitivity to the choice of independent matrix elements; on the other hand, if this difference is large, there will be an increased sensitivity to the choice of independent current matrix elements.

Both impulse or impulse plus an exchange current can be used to compute the independent current matrix elements. Our calculations, shown in Fig. 1, indicate that Δ_- is larger when our model exchange current contributions are added to the independent matrix elements. This means that the presence of the exchange current enhances the sensitivity of the results to the choice of independent matrix elements. Because of this increased sensitivity, we investigate the impact of using different choices of independent linear combinations of current matrix elements to generate fully covariant current matrix elements on the sensitivity of the elastic electron-deuteron scattering observables.

To motivate the other choices that we use, note that any matrix element of a conserved covariant current in a set of deuteron eigenstates can be expressed in terms of a rank 3 covariant current tensor [36]. The deuteron null-plane spin operator can be expressed in terms of the Poincaré group generators as

$$(0,\mathbf{j}_f) := -\frac{1}{2M_d}\Lambda_f^{-1}(\tilde{\mathbf{P}}/M_d)^\mu_\nu \epsilon^\mu_{\alpha\beta\gamma} P^\alpha J^{\beta\gamma}, \tag{64}$$

where P^α is the deuteron four-momentum operator, $J^{\beta\gamma}$ is the deuteron angular momentum tensor, and $\Lambda_f^{-1}(\tilde{\mathbf{P}}/M_d)^\mu_\nu$ is a 4×4 Lorentz-transform-valued matrix of the operator P^μ [25,26]. The quantity $(0,\mathbf{j}_f)$ is not a four-vector because $\Lambda_f^{-1}(\tilde{\mathbf{P}}/M_d)^\mu_\nu$ is a matrix of operators. If the operator $\Lambda_f(\tilde{\mathbf{P}}/M_d)^\mu_\nu$ is applied to both sides of Eq. (64), the result is a four-vector, which up to the factor M_d in the denominator, is the Pauli-Lubanski vector for the deuteron.

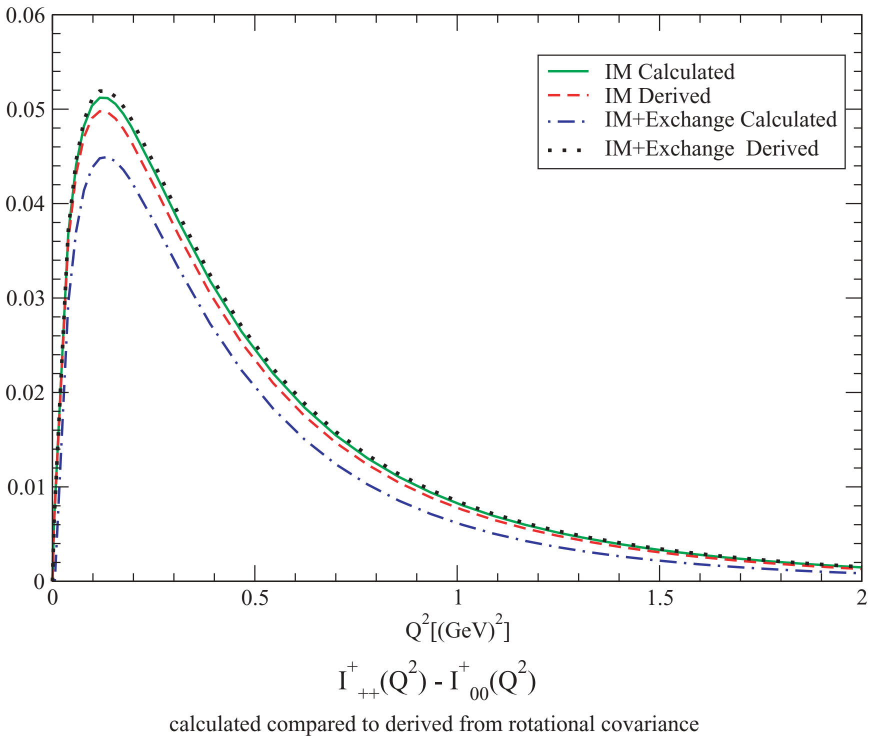

FIG. 1. (Color online) $I^{+}_{++}(0) - I^{+}_{00}(0)$ calculated directly compared with $I^{+}_{++}(0) - I^{+}_{00}(0)$ calculated by rotational covariance with and without exchange currrent I.

It follows from Eq. (64) that it is possible to replace the deuteron spins by four-vectors by multiplying the current matrix element by the inverse of the boost used in Eq. (64); the resulting vector is orthogonal to the four-momentum. If this boost is applied to the initial and final deuteron states in a current matrix element, then the result is a rank-three tensor density $T^{\mu}_{\rho\sigma}(P', P)$, satisfying [12,36]

$$P^{\rho\prime} T^{\mu}_{\rho\sigma}(P', P) = T^{\mu}_{\rho\sigma}(P', P) P^{\sigma} = (P'_{\mu} - P_{\mu}) T^{\mu}_{\rho\sigma}(P', P) = 0. \tag{65}$$

Formally this tensor density is related to the null-plane current matrix elements by

$$T^{\mu}_{\rho\sigma}(P', P) := \Lambda_f(\tilde{\mathbf{P}}'/M_d)^{i}_{\rho} O^{*}_{\nu' i} \sqrt{P^{+\prime}} \langle d, \tilde{\mathbf{P}}', \nu' | I^{\mu}(0) | d, \tilde{\mathbf{P}}, \nu \rangle \times \sqrt{P^{+}} O_{\nu j} \Lambda_f(\tilde{\mathbf{P}}/M_d)^{j}{}_{\sigma}, \tag{66}$$

where $O_{\nu j}$ is the unitary matrix

$$O_{\nu j} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \end{pmatrix} \tag{67}$$

that converts Cartesian components of vectors to spherical components, and the i, j sums go from 1 to 3. The factors $\sqrt{P^{+}}$ give $|\tilde{\mathbf{P}}, \nu\rangle \sqrt{P^{+}}$ a covariant normalization.

The general form of this Lorentz covariant tensor density $T^{\mu}_{\rho\sigma}(P', P)$ can be parametrized by the available four-vectors and three invariant form factors $F_1(Q^2)$, $F_2(Q^2)$, and, $F_3(Q^2)$:

$$T^{\mu}_{\rho\sigma}(P', P) = \frac{1}{2}(P' + P)^{\mu} E(P')_{\rho\alpha} \times \left[g^{\alpha\beta} F_1(Q^2) + \frac{Q^{\alpha} Q^{\beta}}{M_d^2} F_2(Q^2) \right] E(P)_{\beta\sigma} - \frac{1}{2} E(P')_{\rho\alpha} [Q^{\alpha} g^{\mu\beta} - g^{\mu\alpha} Q^{\beta}] F_3(Q^2) E(P)_{\beta\sigma}, \tag{68}$$

where

$$E(P)_{\alpha\beta} = g_{\alpha\beta} - \frac{P_{\alpha} P_{\beta}}{P^2} \tag{69}$$

is the covariant projector on the subspace orthogonal to P. The projectors are not needed when this tensor is contracted with vectors orthogonal to the final or initial four-momentum.

We use this covariant representation of the current matrix elements to identify preferred choices of independent current matrix elements. Independent matrix elements can be constructed by contracting this current tensor with different sets of "polarization four-vectors." These are four-momentum dependent vectors $v^{\nu}(P)$ that are orthogonal P^{μ}. The contraction has the form

$$v'_{af}(P')^{\rho} T^{\mu}_{\rho\sigma}(P', P) v_{bi}(P)^{\sigma} = \frac{1}{2}(P' + P)^{\mu} \left[v'_{af}(P') \cdot v_{bi}(P) F_1(Q^2) + \frac{(v'_{af}(P') \cdot Q)(Q \cdot v_{bi}(P))}{M_d^2} F_2(Q^2) \right] - \frac{1}{2} [(v'_{af}(P') \cdot Q) v_{bi}(P)^{\mu} - v_{af}(P')^{\mu} (Q \cdot v_{bi}(P))] F_3(Q^2). \tag{70}$$

In this case, the three invariant scalar products

$$v'_{af}(P')\cdot v_{bi}(P),\quad v'_{af}(P')\cdot Q,\quad Q\cdot v_{bi}(P),\qquad (71)$$

can be used in Eqs. (71) and (66) to relate the form factors to the current matrix elements. Three pairs of polarization vectors give three equations that can be solved to express $F_1(Q^2)$, $F_2(Q^2)$, and $F_3(Q^2)$ in terms of the current matrix elements. Given the form factors, the rest of the current matrix elements are determined by the current tensor.

The form factors $F_1(Q^2)$, $F_2(Q^2)$, and $F_3(Q^2)$ defined by Eqs. (66) and (68) are related to the form factors $G_1(Q^2)$, $G_2(Q^2)$, and $G_3(Q^2)$ [38], which are defined in terms of canonical-spin Breit-frame matrix elements with momentum transfer in the 3 direction:

$$\begin{aligned}&[(1+2\eta)F_1+4\eta(1-\eta)F_2+2\eta F_3]\\&\quad =_c\left\langle(1,d),\frac{\mathbf{Q}}{2},0\middle|I^0(0)\middle|(1,d),-\frac{\mathbf{Q}}{2},0\right\rangle_c\\&\quad =G_0(Q^2)+\sqrt{2}G_2(Q^2),\end{aligned}\qquad (72)$$

$$\begin{aligned}&F_1(Q^2)\\&\quad =_c\left\langle(1,d),\frac{\mathbf{Q}}{2},1\middle|I^0(0)\middle|(1,d),-\frac{\mathbf{Q}}{2},1\right\rangle_c\\&\quad =G_0(Q^2)-\frac{1}{\sqrt{2}}G_2(Q^2),\end{aligned}\qquad (73)$$

$$\begin{aligned}&F_3(Q^2)\\&\quad =\sqrt{\frac{2}{\eta}}\,{}_c\left\langle(1,d),\frac{\mathbf{Q}}{2},-1\middle|I^1(0)\middle|(1,d),-\frac{\mathbf{Q}}{2},0\right\rangle_c\\&\quad =G_1(Q^2).\end{aligned}\qquad (74)$$

The enhanced sensitivity to the choice of linearly independent matrix elements when the exchange current is included, as shown in Fig. 1, suggests that in addition to choice I [Eq. (60)] of linearly independent matrix elements used in Ref. [7], other choices should be considered.

Frankrurt, Frederico, and Strikman [48] defined independent current matrix elements using polarization vectors [Eq. (70)] constructed from the last three columns $v_{1c}(P_b)$, $v_{2c}(P_b)$, and $v_{3c}(P_b)$ of the canonical boost in the null-plane Breit frame

$$(v_{1c}(P_b),v_{2c}(P_b),v_{3c}(P_b))=\begin{pmatrix}-\sqrt{\eta}&0&0\\\sqrt{1+\eta}&0&0\\0&1&0\\0&0&1\end{pmatrix}.\qquad (75)$$

These are automatically orthogonal to the momentum. Three independent current matrix elements are extracted from the contractions $v_{1c}(P'_b)^\rho T^+_{\rho\sigma}(P'_b,P_b)v_{1c}(P_b)^\sigma$, $v_{2c}(P'_b)^\rho T^+_{\rho\sigma}(P'_b,P_b)v_{2c}(P_b)^\sigma$, and $v_{3c}(P'_b)^\rho T^+_{\rho\sigma}(P'_b,P_b)v_{1c}(P_b)^\sigma$. These can be computed consistently using a kinematically covariant model current in all frames related to the null-plane Breit frame by null-plane kinematic transformations. Their preference for this choice is to avoid using the $3'3$ matrix element, which is maximally suppressed in the infinite momentum frame. In Ref. [5], operators are classified as good, bad, or terrible according to their scaling properties with respect to Lorentz boosts to the infinite momentum frame. The $3'3$ matrix element is terrible in this classification. We call this choice of independent current matrix elements choice II.

We are indebted to Fritz Coester [49] for suggesting a third choice of independent current matrix elements that emphasizes the kinematic symmetries of the null plane. This choice uses columns of the null-plane boost as polarization vectors:

$$(v_{1f}(P),v_{2f}(P),v_{3f}(P))=\begin{pmatrix}P^x/M_d&P^y/M_d&P^0/M_d-M_d/P^+\\1&0&P^x/M_d\\0&1&P^y/M_d\\-P^x/M_d&-P^y/M_d&P^z/M_d+M_d/P^+\end{pmatrix}.\qquad (76)$$

The independent combinations are $v_{1f}(P')^\rho T^+_{\rho\sigma}(P',P)v_{1f}(P)^\sigma$, $v_{2f}(P')^\rho T^+_{\rho\sigma}(P',P)v_{2f}(P)^\sigma$, and $v_{3c}(P')^\rho T^+_{\rho\sigma}(P',P)v_{1c}(P)^\sigma$. When the first two terms are evaluated in the null-plane Breit frame, the scalar products of Eq. (71) that relate the contractions to the form factors are mass independent, which is preserved under kinematic transformations. This means that the boost parameters that appear in the calculations of the one-body current matrix elements are independent of the constituent masses. To construct a third independent linear combination that exploits this mass independence would require using matrix elements of $I^2(0)$. Rather than using $I^2(0)$, F_3 is extracted using polarization vectors that are mutually orthogonal, with one of them also orthogonal to Q, which gives matrix elements that are independent of F_1 and F_2. This results in the same expression for $F_3=G_1$ as used in the other two schemes. We call this choice of independent current matrix elements choice III.

These three choices lead to expressions for deuteron form factors based on three different covariant current operators generated using three different choices of independent linear combinations of matrix elements of a kinematically covariant model current $I^+(0)$. These different current operators lead to distinct expressions for the deuteron form factors in terms of the null-plane current matrix elements of $I^+(0)$. The results, expressed in terms of the standard deuteron form factors and null-plane Breit frame matrix elements of the + component of the current, are as follows:

Choice I:

$$\begin{aligned}&(1+\eta)G_0(Q^2)\\&\quad =\left(\frac{1}{2}-\frac{\eta}{3}\right)(I_{11}+I_{00})+\frac{5\sqrt{2\eta}}{3}I_{10}+\left(\frac{2\eta}{3}-\frac{1}{6}\right)I_{1-1},\end{aligned}\qquad (77)$$

$$\begin{aligned}&(1+\eta)G_1(Q^2)\\&\quad =I_{11}+I_{00}-I_{1-1}-(1-\eta)\sqrt{\frac{2}{\eta}}I_{10},\end{aligned}\qquad (78)$$

$$(1+\eta)G_2(Q^2) = -\frac{\sqrt{2}\eta}{3}(I_{11}+I_{00}) + \frac{4\sqrt{\eta}}{3}I_{10} - \frac{\sqrt{2}}{3}(2+\eta)I_{1-1}. \quad (79)$$

Choice II: $G_0(Q^2)$ of choice I is replaced by

$$(1+\eta)G_0(Q^2) = \left(\frac{2\eta}{3}+1\right)I^+_{1,1} - \frac{\eta}{3}I^+_{00} + \frac{2\sqrt{2\eta}}{3}I^+_{1,0} + \left(\frac{2\eta+1}{3}\right)I^+_{1,-1}. \quad (80)$$

Choice III: $G_0(Q^2)$ and $G_2(Q^2)$ of choice I are replaced by

$$G_0(Q^2) = \left(1+\frac{2\eta}{3}\right)I^+_{11} + \frac{1}{3}I^+_{1,-1} - \frac{2\eta}{3}G_1(Q^2), \quad (81)$$

$$G_2(Q^2) = \frac{2\sqrt{2}}{3}\left(\eta I^+_{1,1} - I^+_{1,-1} - \eta G_1(Q^2)\right). \quad (82)$$

Since $B(Q^2)$ only depends on $G_1(Q^2)$, which uses the same linear combination of current matrix elements for all three choices, the computation of $B(Q^2)$ is unchanged. These choices lead to a $B(Q^2)$ that is in good agreement with experimental data.

The different linear combinations of null-plane current matrix elements used in choices I–III have a nontrivial effect on the scattering observables $A(Q^2)$ and $T_{20}(Q^2,\theta)$. Selecting one of these choices is a model assumption.

VII. DEUTERON CURRENTS: DYNAMICAL EXCHANGE CURRENTS

The realistic interactions [1,2] used to construct the mass operator (30) include a one-pion-exchange contribution plus a short-range contribution that is designed so that the two-body cross sections fit experiment.

"Required" two-body currents, discussed in the previous section, are needed to satisfy the constraints of current covariance and current conservation. These constraints are satisfied by directly computing a set of linearly independent matrix elements and then using the constraints to determine the remaining current matrix elements. Even if the independent current matrix elements are computed using the one-body impulse current, the matrix elements generated by the constraints will have two-body contributions. The interaction dependence arises because the covariance and current conservation constraints involve the deuteron mass, which is the eigenvalue of the dynamical two-body mass operator (33).

In addition to the two-body currents directly generated by covariance and current conservation, realistic interactions contain terms that involve the exchange of charged mesons, leading to the exchange of the nucleon charges, and the possibility of two-body currents associated with these exchanges. These currents give additional contributions to the linearly independent matrix elements of $I^+(0)$ that are needed to compute electron-scattering observables in null-plane quantum mechanics. The form of the dynamical exchange current contribution to $I^+(0)$ is model dependent, but we assume that the most important contribution is motivated by one-pion-exchange physics.

To motivate the structure of our two-body current, consider the pseudoscalar pion-nucleon vertex

$$\mathcal{L}(x) = -ig_\pi : \bar{\Psi}(x)\gamma_5\Psi(x)\boldsymbol{\tau}\cdot\boldsymbol{\phi}(x) : . \quad (83)$$

When this vertex is coupled to a photon-nucleon vertex with a causal propagator, the propagator connecting these two vertices can be decomposed into a sum of two terms involving the covariant spinor projectors $\Lambda_+(\mathbf{p}) = u(\mathbf{p})\bar{u}(\mathbf{p})$ and $\Lambda_-(-\mathbf{p}) = -v(-\mathbf{p})\bar{v}(-\mathbf{p})$, where $u(\mathbf{p})$ and $v(\mathbf{p})$ are Dirac spinors. This results in a sum of two terms with the following spin structures:

$$\bar{u}(\mathbf{p})\Gamma^\mu u(\mathbf{p}')\bar{u}(\mathbf{p}')\gamma^5 u(\mathbf{p}''), \quad (84)$$

$$-\bar{u}(\mathbf{p})\Gamma^\mu v(-\mathbf{p}')\bar{v}(-\mathbf{p}')\gamma^5 u(\mathbf{p}''). \quad (85)$$

In this form, these operator structures are expressed as 2×2 matrices in the nucleon spins. In Poincaré invariant quantum models, the first term is included in the one-body part of the current. The second term, which leads to the "pair current" in covariant formulations, is not included in the one-body part of the current in Poincaré invariant quantum models. It can be included as a contribution to the two-body part of the current. To do this, it is useful to convert the v spinors to u spinors using

$$v(-\mathbf{p})\bar{v}(-\mathbf{p}) = \gamma_5\beta u(\mathbf{p})\bar{u}(\mathbf{p})\beta\gamma_5, \quad (86)$$

which replaces the operator structure in the second term by

$$\bar{u}(\mathbf{p})\Gamma^\mu\beta\gamma_5 u(\mathbf{p}')\bar{u}(\mathbf{p}')\beta u(\mathbf{p}''). \quad (87)$$

The factors β appearing in this expression are spinor space-reflection operators. Since space reflections do not leave the null plane invariant, they become dynamical operators in null-plane quantum mechanics.

The canonical Dirac spinors can be converted to null-plane spinors by multiplying the spins by Melosh rotations, $u_f(\mathbf{p}) = u(\mathbf{p})D^{1/2}[R_{cf}(p)]$. These rotations cancel in the combinations that appear in the projectors $\Lambda_\pm(\mathbf{p})$, so the projectors can be expressed directly in terms of null-plane Dirac spinors.

To construct a kinematically covariant current, we first make the model assumption that β should be replaced by an invariant operator that agrees with β in the rest frame of the initial (or final deuteron). It requires $\beta \to -P_d\cdot\gamma/M_d$.

Because the null-plane boosts do not have Wigner rotations, the null-plane Dirac spinors satisfy the following covariance condition with respect to null-plane boosts Λ:

$$u_f(\tilde{\mathbf{p}}) = S(\Lambda^{-1})u_f(\tilde{\boldsymbol{\Lambda}}p), \quad \bar{u}_f(\tilde{\mathbf{p}}) = \bar{u}_f(\tilde{\boldsymbol{\Lambda}}p)S(\Lambda).$$

This leads to the identity

$$\frac{m}{\sqrt{p^+}}\bar{u}_f(\mathbf{p})\Gamma^+\left(-P_d\cdot\frac{\gamma}{M_d}\right)\gamma_5 u_f(\mathbf{p}')\bar{u}_f(\mathbf{p}')\left(-P_d\cdot\frac{\gamma}{M_d}\right) \times u_f(\mathbf{p}'')\frac{m}{\sqrt{p^{+\prime}}} = \frac{m}{\sqrt{(\Lambda p)^+}}\bar{u}_f(\tilde{\boldsymbol{\Lambda}}p)\Gamma^+\left(-\Lambda P_d\cdot\frac{\gamma}{M_d}\right) \times \gamma_5 u_f(\tilde{\boldsymbol{\Lambda}}p')\bar{u}_f(\tilde{\boldsymbol{\Lambda}}p')\left(-\Lambda P_d\cdot\frac{\gamma}{M_d}\right) \times u_f(\tilde{\boldsymbol{\Lambda}}p'')\frac{m}{\sqrt{(\Lambda p'')^+}}, \quad (88)$$

which defines a null-plane-boost-invariant current kernel that agrees with Eq. (87) in the rest frame of the initial deuteron. For the adjoint, we use the rest frame of the final deuteron.

The operator (87) factors into a product of 2×2 spin matrices, $\bar{u}(\mathbf{p})\Gamma^\mu(-P_d \cdot \gamma/M_d)\gamma_5 u(\mathbf{p}')$ and $\bar{u}(\mathbf{p}')(-P_d \cdot \gamma/M_d)u(\mathbf{p}'')$. The term $\bar{u}(\mathbf{p}')(-P_d \cdot \gamma/M_d)u(\mathbf{p}'')$ replaces $\bar{u}(\mathbf{p}')\gamma_5 u(\mathbf{p}'')$ in one vertex of the one-pion-exchange interaction, while the factor $\bar{u}(\mathbf{p})\Gamma^\mu(-P_d \cdot \gamma/M_d)\gamma_5 u(\mathbf{p}')$ has the appearance of a modified one-body current.

To be consistent with our input two-body interaction, we modify the one-pion-exchange contribution of the model interaction by replacing the part of the phenomenological interaction that comes from $\bar{u}(\mathbf{p}')\gamma_5 u(\mathbf{p}'')$ by a modified interaction that comes from $\bar{u}(\mathbf{p}')(-\Lambda P_d \cdot \gamma/M_d)u(\mathbf{p}'') = \bar{u}(\mathbf{p}')\beta u(\mathbf{p}'')$ in the deuteron rest frame. We then apply this modified interaction to the deuteron wave function and use kinematic null-plane boosts to transform the result to an arbitrary frame.

The resulting matrix elements of the kinematically covariant two-body current has the form

$$\begin{aligned}
&\big\langle(1,d),\tilde{\mathbf{P}}',\nu'\big|I^\mu_{\text{ex}}(0)\big|(1,d),\tilde{\mathbf{P}},\nu\big\rangle\\
&:= \int \langle(1,d),\tilde{\mathbf{P}}',\nu'|(j_n,m_n),\tilde{\mathbf{p}}'_1,\nu'_1,(j_n,m_n),\tilde{\mathbf{p}}'_2,\nu'_2\rangle\\
&\quad\times\left(-\frac{1}{2m}\right)\bar{u}_{nf}(p'_1)\Gamma^\mu\gamma_5\left(\frac{P\cdot\gamma}{M_d}\right)u_{nf}(p''_1)d\tilde{\mathbf{p}}'_1 d\tilde{\mathbf{p}}''_1 d\tilde{\mathbf{p}}'_2\\
&\quad\times\langle(j_n,m_n),\tilde{\mathbf{p}}''_1,\nu''_1,(j_n,m_n),\tilde{\mathbf{p}}'_2,\nu'_2|U(\Lambda_f(P/M_d))\tilde{v}_\pi\\
&\quad\times|(1,d),\tilde{\mathbf{P}}_0,\nu\rangle + (1\to 2) + \text{h.c.}, \qquad (89)
\end{aligned}$$

where we replaced the Dirac spinors with a covariant normalization used in projection operator $\Lambda_\pm(\mathbf{p})$ by spinors with a noncovariant normalization

$$u_{nf}(p) := \sqrt{\frac{m}{p^+}}u_f(p) \qquad (90)$$

to simplify the notation. We also remark that the projectors can be expressed in terms of canonical or null-plane spinors with covariant normalization, $u_c(\mathbf{p})\bar{u}_c(\mathbf{p}) = u_f(\mathbf{p})\bar{u}_f(\mathbf{p})$. Details of the construction are discussed in Appendixes A and B.

The quantity

$$\langle(1,d),\tilde{\mathbf{P}},\nu|(j_n,m_n),\tilde{\mathbf{p}}_1,\nu_1,(j_n,m_n),\tilde{\mathbf{p}}_2,\nu_2\rangle \qquad (91)$$

is the deuteron eigenstate in the tensor product basis, $\tilde{\mathbf{P}}_0 = (M_d, 0, 0)$ is the rest frame value of the deuteron null-plane momentum, and $U(\Lambda_f(P/M_d))$ represents a kinematic null-plane boost to the Breit frame. The modified pion-exchange interaction $\tilde{v}_\pi$ has the form (see Appendix A):

$$\begin{aligned}
&\langle\tilde{\mathbf{P}}',\mathbf{k}',\nu'_1,\nu'_2|\tilde{v}_\pi|\tilde{\mathbf{P}},\mathbf{k},\nu_1,\nu_2\rangle\\
&\to \delta(\tilde{\mathbf{P}}'-\tilde{\mathbf{P}})\frac{g_\pi^2}{(2\pi)^3}\frac{2m}{2m}\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 v_\pi(\mathbf{k}-\mathbf{k}')\frac{(\mathbf{k}'-\mathbf{k})\cdot\boldsymbol{\sigma}_2}{2m}, \qquad (92)
\end{aligned}$$

where $v_\pi(\mathbf{k}-\mathbf{k}')$ is the coefficient function of the one-pion-exchange contribution to the operator

$$\delta(\tilde{\mathbf{P}}'-\tilde{\mathbf{P}})\frac{g_\pi^2}{(2\pi)^3}\frac{(\mathbf{k}-\mathbf{k}')\cdot\boldsymbol{\sigma}_1}{2m}\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 v_\pi(\mathbf{k}-\mathbf{k}')\frac{(\mathbf{k}'-\mathbf{k})\cdot\boldsymbol{\sigma}_2}{2m} \qquad (93)$$

in the phenomenological interaction. For the Argonne V18 interaction, $v_\pi(\mathbf{k}-\mathbf{k}')$ is extracted from the one-pion-exchange contribution to the tensor $\times(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)$ and spin-spin $\times(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)$ parts of the interaction using the method discussed in Refs. [50,51]. The extraction is discussed in Appendix B. The resulting interaction, $v_\pi(\mathbf{k}-\mathbf{k}')$, differs from $1/[m_\pi^2 + (\mathbf{k}-\mathbf{k}')^2]$ by the effects of the short-distance cutoff in the AV18 interaction.

This exchange current is constructed to transform covariantly with respect to the null-plane kinematic subgroup. We add this current to the impulse current when computing the three independent matrix elements. These matrix elements are then used to generate the remaining current matrix elements using the constraints of current covariance and current conservation as discussed in the previous section.

Each term in expression (89) for the exchange current matrix element can be represented as the product of $(-\frac{1}{2m})$ with the modified current $\bar{u}_{nf}(p'_1)\Gamma^\mu\gamma_5(\frac{P\cdot\gamma}{M_d})u_{nf}(p''_1)$ evaluated between a deuteron eigenstate and a pseudostate defined by applying the rotationally invariant modified interaction $\tilde{v}_\pi$ in Eq. (92) to the rest deuteron state and kinematically boosting the result to the Breit frame. This defines a kinematically covariant exchange current that we use to compute the independent matrix elements of $I^+(0)$.

VIII. RESULTS

The input to our calculation of the elastic electron-deuteron scattering observables is (1) the choice of nucleon form factors [40–43,45,46], (2) the choice of nucleon-nucleon interaction [1,2], and (3) the choice of independent linear combinations of current matrix elements used to generate the full current [7,11,48,52].

The extraction of proton-electric form factors based on polarization measurements compared with measurements based on the Rosenbluth separation were found to be inconsistent [47]; these inconsistencies have been explained [53] by including two-photon-exchange corrections in the Rosenbluth separation. This has led to a modification of the phenomenological parametrizations of the proton electric form factor. All of the nucleon form factors that we use are consistent with the extractions based on the polarization measurements.

Neutron electric form factor data are only available for a limited range of momentum transfers. The high-momentum transfer behavior of the different parametrizations is a consequence of different theoretical assumptions. This leads to some variation of the parametrizations for momentum transfers above $Q^2 \sim 1\,\text{GeV}^2$. Different parametrizations of the neutron electric form factor are compared with a dipole form factor in Fig. 2. The form factors that we compare are recent parametrizations given by Lomon [45], Budd, Bodeck, and Arrington (BBA) [42], Bradford, Budd, Bodeck, and Arrington (BBBA) [41], Kelly [43], and Bijker and Iachello (BI) [40]. All of these parametrizations agree for $Q^2 < 1$. The curves in Fig. 2 are given as ratios to dipole form factors, which emphasize differences in the parametrizations. The input to our calculations is the isoscalar linear combinations of the nucleon form factors $F_{1N}(Q^2)$ and $F_{2N}(Q^2)$. These form factors are plotted in Figs. 3 and 4 for the same parametrizations that are compared in Fig. 2. These plots show no significant variation

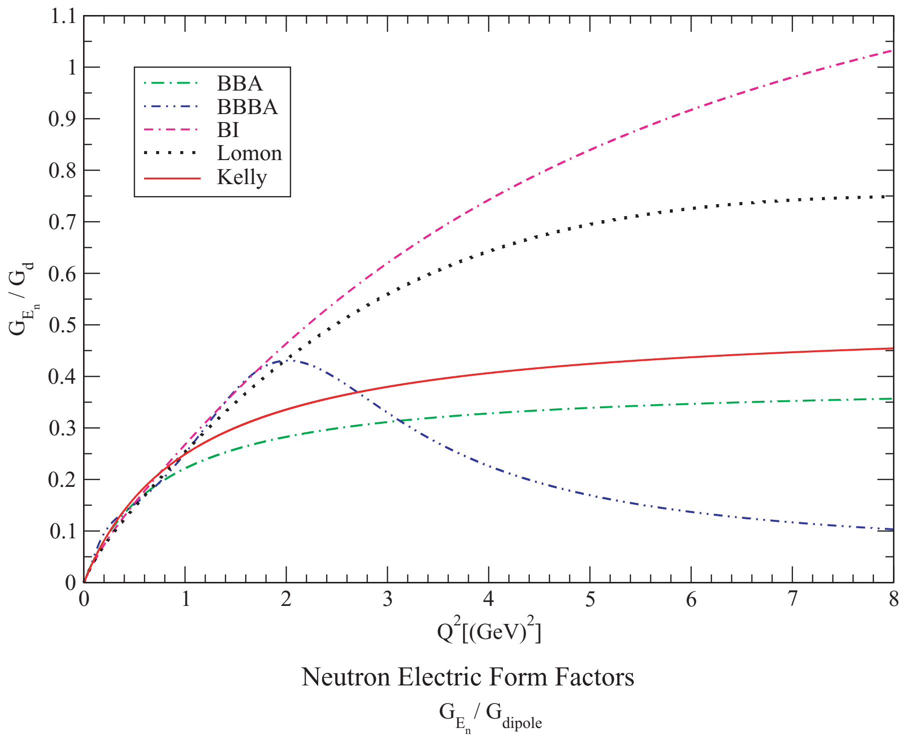

FIG. 2. (Color online) Neutron electric form factor parametrizations.

among the various nucleon form factors. Our calculations of elastic-scattering observables are not very sensitive to these differences.

As shown in Sec. VII, our model exchange current breaks up into a product of an effective "one-body" current and an interaction. The interaction is a modification of the one-pion-exchange contribution to the tensor part of the phenomenological interaction. Most of our calculations are based on the Argonne V18 interaction. We extract the one-pion-exchange contribution to the V18 potential by first discarding the short-range parts of the interaction that contribute to the spin-spin and tensor forces, then we use a method developed by Riska [50] and Schiavilla, Pandharipande, and Riska [51] to extract the one-pion-exchange contribution to the tensor force from the remaining parts of the interactions. This procedure is discussed in Appendix B. The one-pion-exchange potential

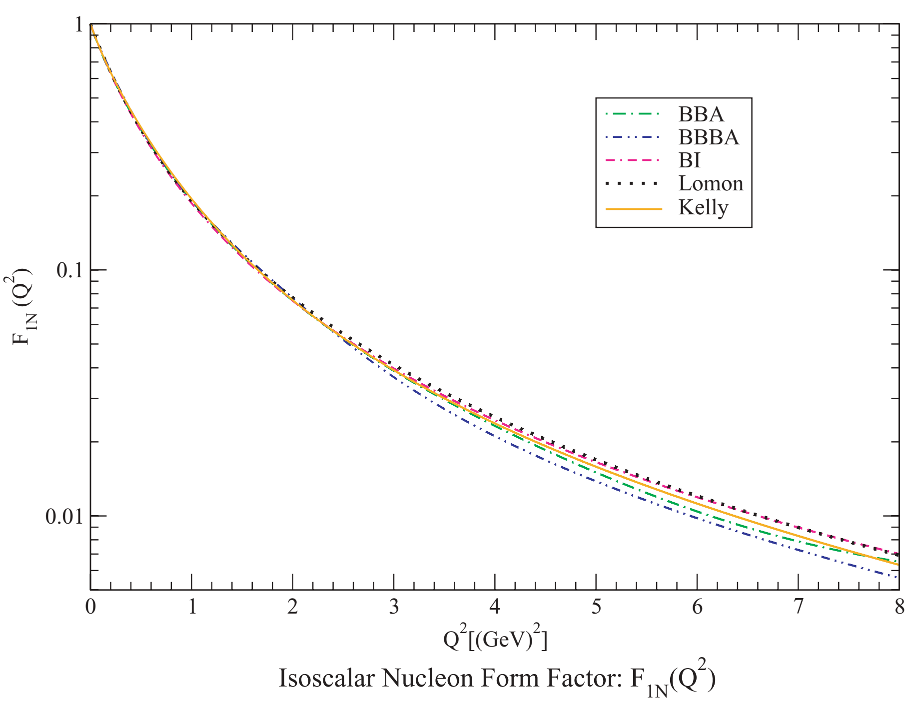

FIG. 3. (Color online) Isoscalar nucleon form factor $F_{1N}(Q^2)$.

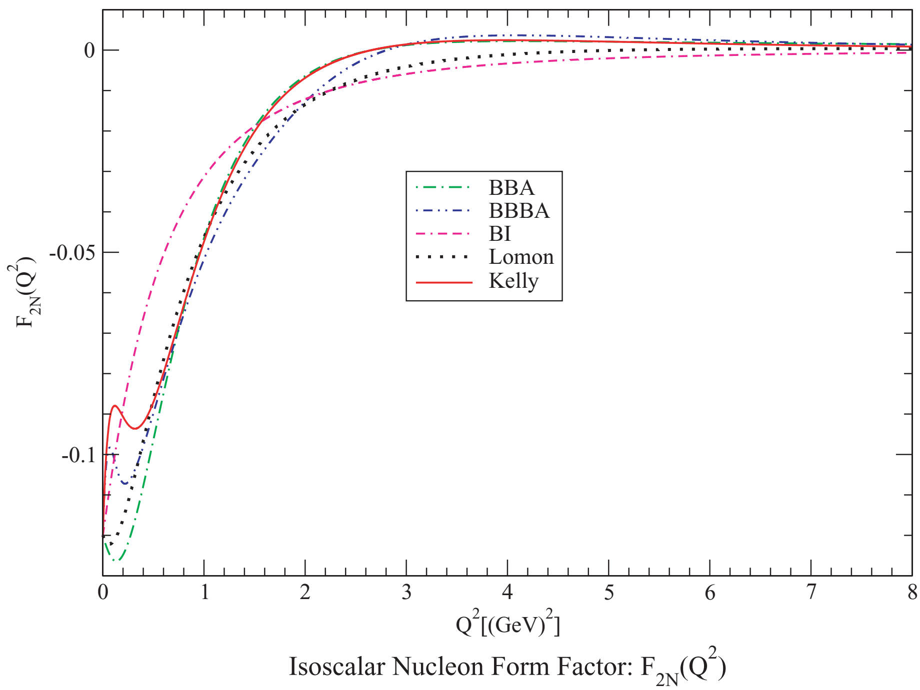

FIG. 4. (Color online) Isoscalar nucleon form factor $F_{2N}(Q^2)$.

that we extract differs from $\frac{1}{m_\pi^2+(\mathbf{k}-\mathbf{k}')^2}$ by the effects of the short distance configuration-space cutoff that appears in the Argonne V18 interaction. The Fourier transform of the extracted interaction is compared with the pion-exchange potential without the cutoff in Fig. 5. The dotted curve includes the cutoff parameters that are used in the Argonne V18 interaction. The most important differences are for momenta above 1–2 fm^{-1}.

Figures 6–8 show the three deuteron form factors, G_0, G_1, and G_2, with and without the exchange current included for the two-nucleon form factors (BI and BBBA) that have the largest high-momentum transfer difference in the neutron electric form factor. The independent matrix elements [Eq. (60)] are calculated using the one-body parts of the current and with the exchange current added. The remaining current matrix elements are determined by the constraints of current

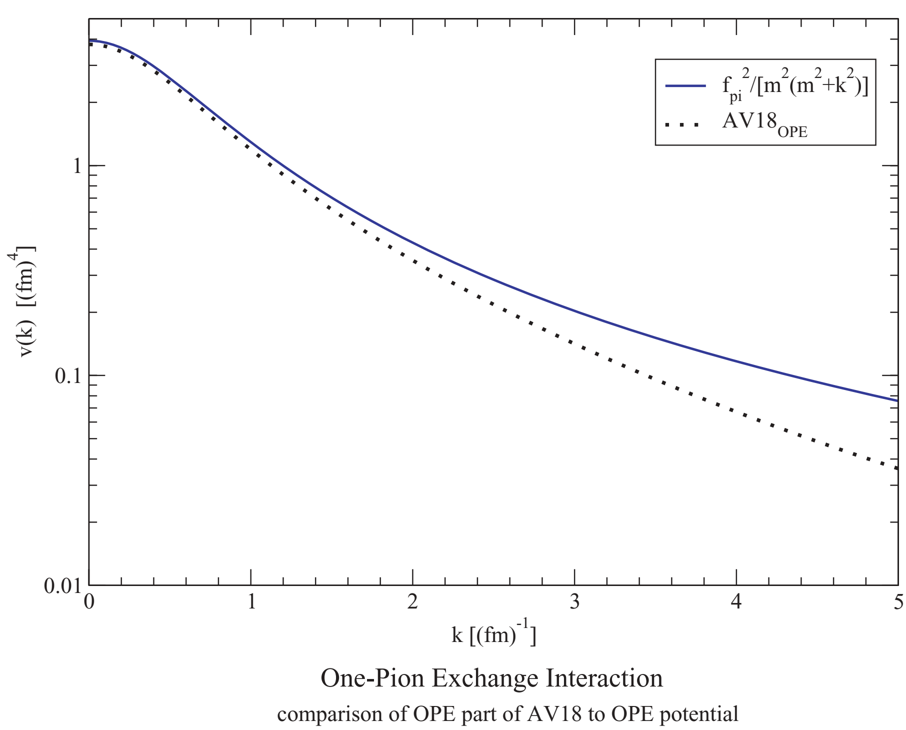

FIG. 5. (Color online) One-pion-exchange interaction compared with one-pion-exchange part of AV18 interaction.

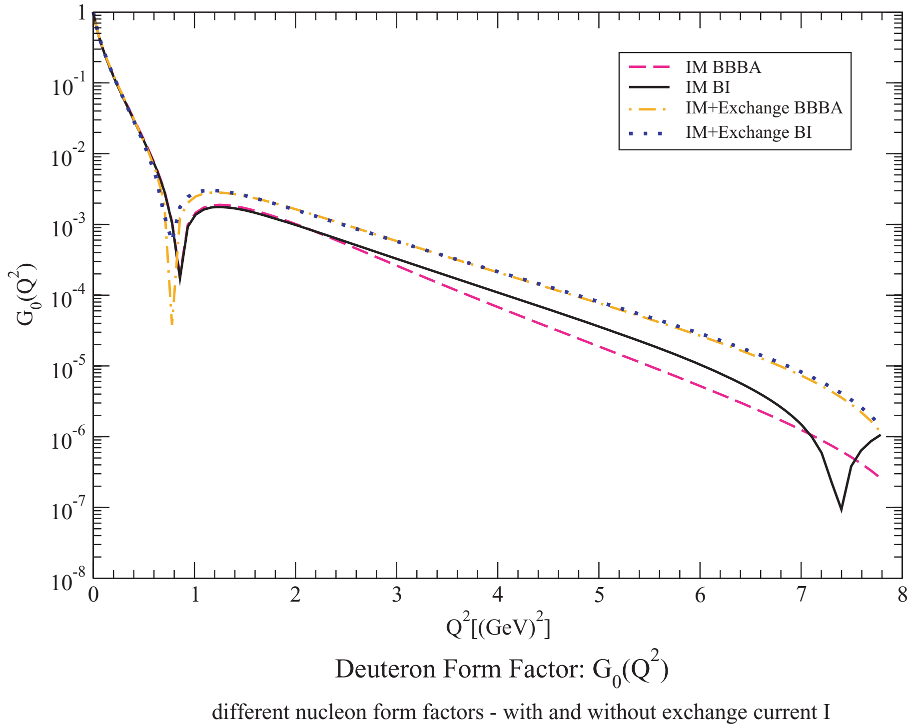

FIG. 6. (Color online) Deuteron form factor $G_0(Q^2)$; different nucleon form factors with and without exchange current I.

conservation and current covariance. The figures show that the addition of the exchange current contributions leads to an enhancement of G_0 above the minimum at $Q^2 = 1$ GeV^2. The minimum of G_1 shifts to the right, and G_2 is enhanced. Except for momentum transfers Q^2 between 6 and 7, these form factors are not very sensitive to the different assumptions made about the neutron electric form factor.

In subsequent figures, data for the observables A are labeled Stanford Mark III [54], CEA [37], Orsay [55], SLAC E101 [56], Saclay ALS [57], DESY [58], Bonn [59], Mainz [60], JLab Hall C [61], JLab Hall A [62], and Monterey [63]. Data for B are labeled SLAC NPSA NE4 [64], Martin [65], Bonn [59], Saclay ALS [66], Mainz [60], and Stanford Mark III [54]. Data for T_{20} are labeled Novosibirsk-85 [67,68],

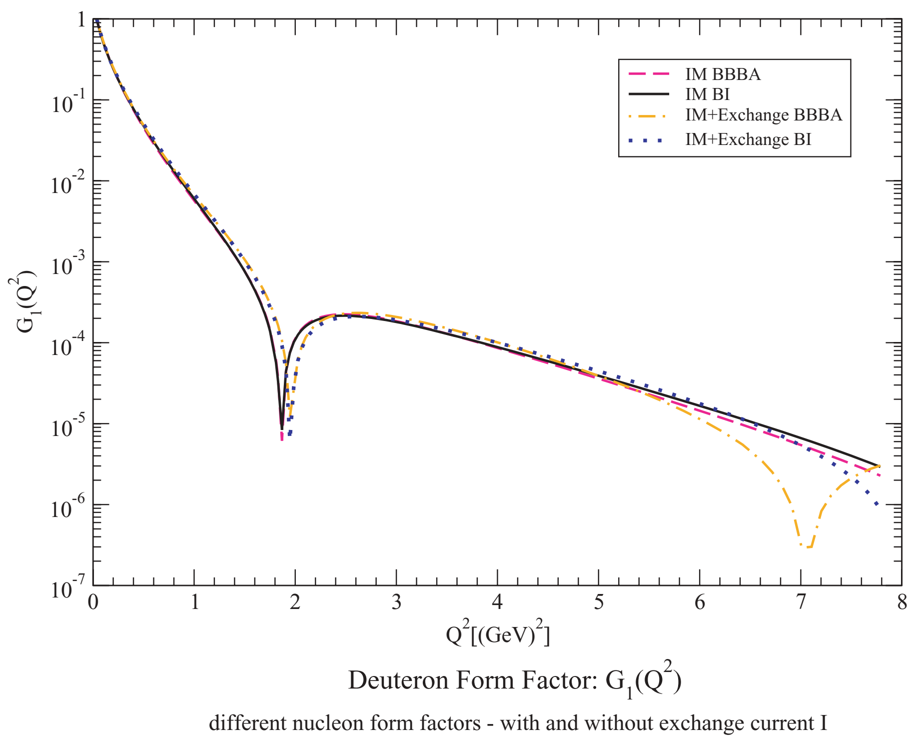

FIG. 7. (Color online) Deuteron form factor $G_1(Q^2)$; different nucleon form factors with and without exchange current I.

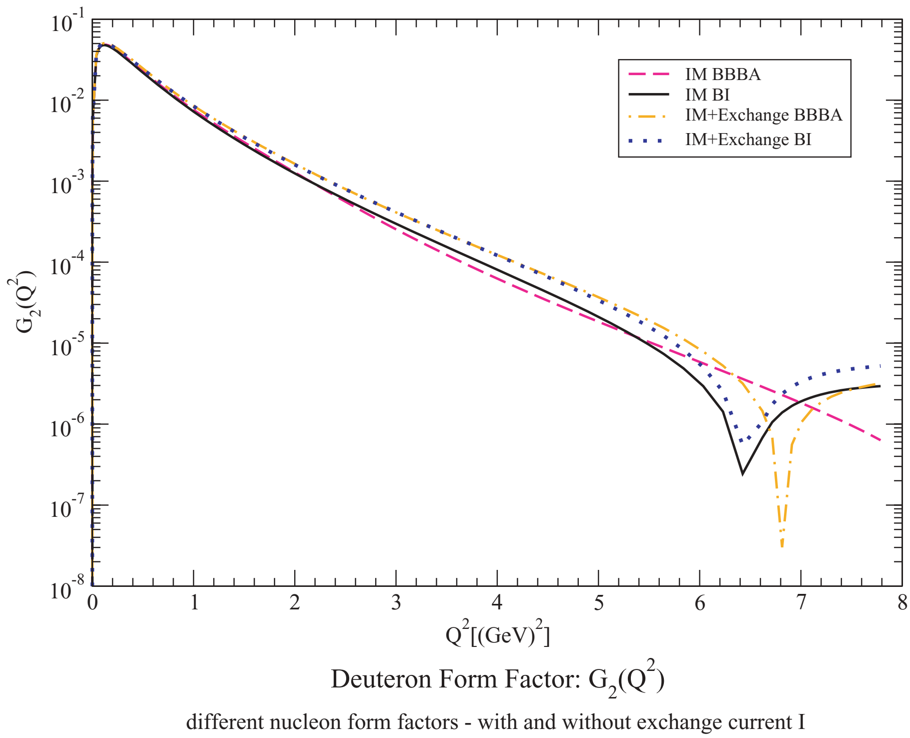

FIG. 8. (Color online) Deuteron form factor $G_2(Q^2)$; different nucleon form factors with and without exchange current I.

Novosibirsk-90 [69], Bates-84 [70], Bates-91 [71], and JLab Hall C [72].

Calculations of A, B, and T_{20} using the independent matrix elements of choice I and the five parametrizations of the nucleon form factors used in Figs. 2–4 are shown in Figs. 9–11. While the null-plane impulse calculations give a qualitative understanding of the data, it is clear from these calculations that the null-plane impulse approximation is inadequate.

Figures 12–14 show the effects of including the phenomenological pion-exchange current defined in Sec. VII. We see that A and B provide acceptable fits to the data when the pion-exchange current is included. The results are insensitive

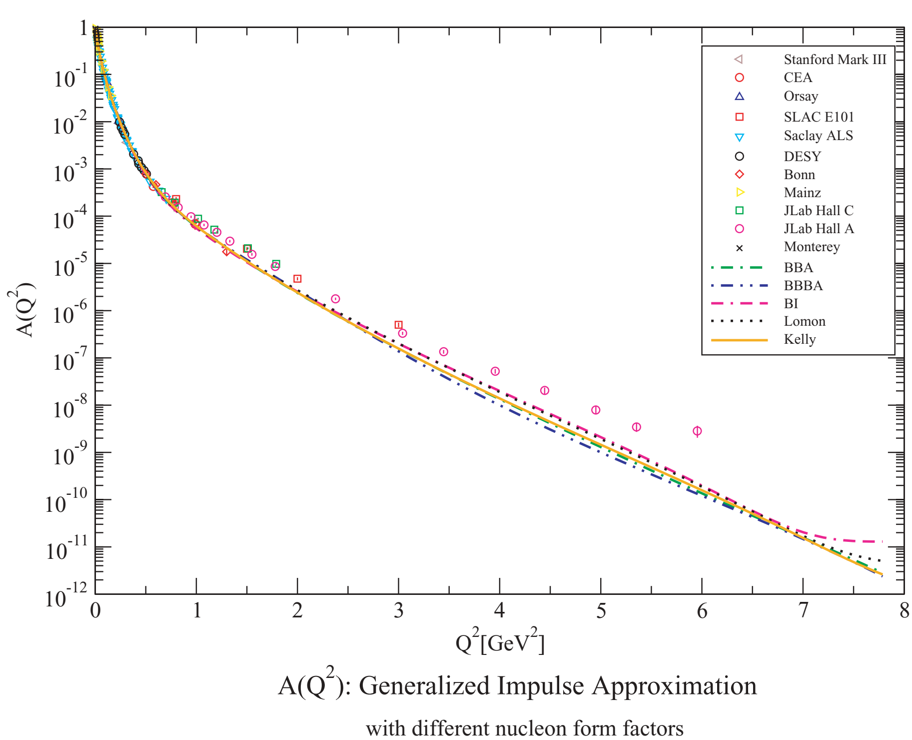

FIG. 9. (Color online) $A(Q^2)$: Generalized impulse approximation with different nucleon form factors.

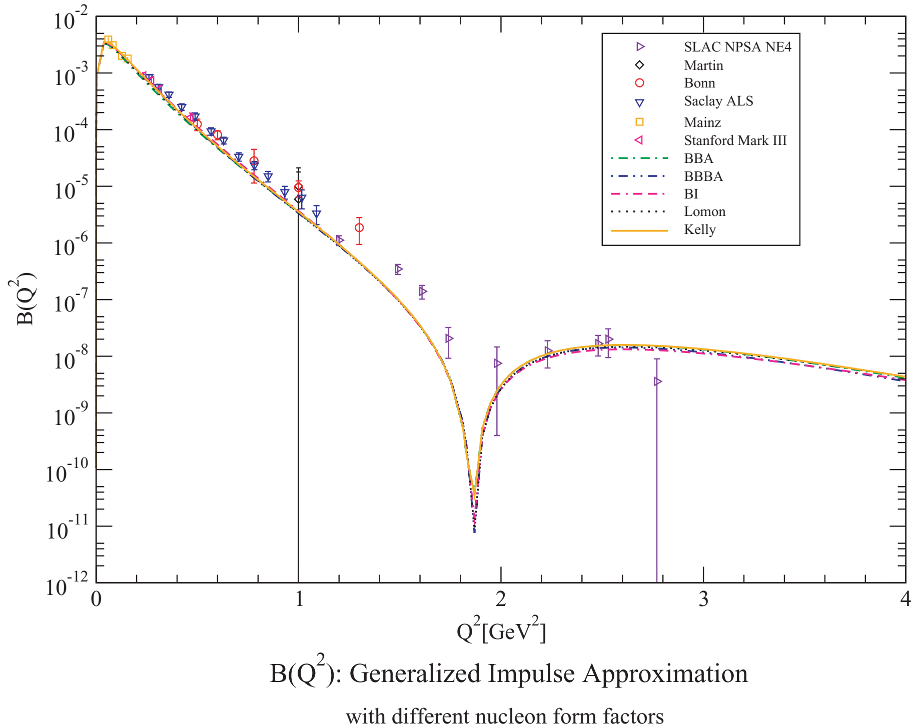

FIG. 10. (Color online) $B(Q^2)$: Generalized impulse approximation with different nucleon form factors.

to the assumptions used in parametrizing the high-momentum transfer behavior of the neutron electric form factor. T_{20} is closer to the data, but it is still below the most recent Jlab Hall C data [72] between Q^2 of 0.5 and 2 GeV^2.

The calculations displayed in Figs. 12–14 are based on choice I of independent current matrix elements given by Eq. (60). The presence of the exchange current in Eq. (89) increases the sensitivity to the choice of the independent current matrix elements. This is shown in Fig. 1, in which is plotted the difference $I_{11}^{+}(0) - I_{00}^{+}(0)$ with and without the exchange current using a direct calculation of the difference or by generating the difference using current conservation and current covariance. The difference between the dashed curve and solid curve shows that the required two-body contributions to the current in this difference is small when the independent current matrix elements are computed in the impulse

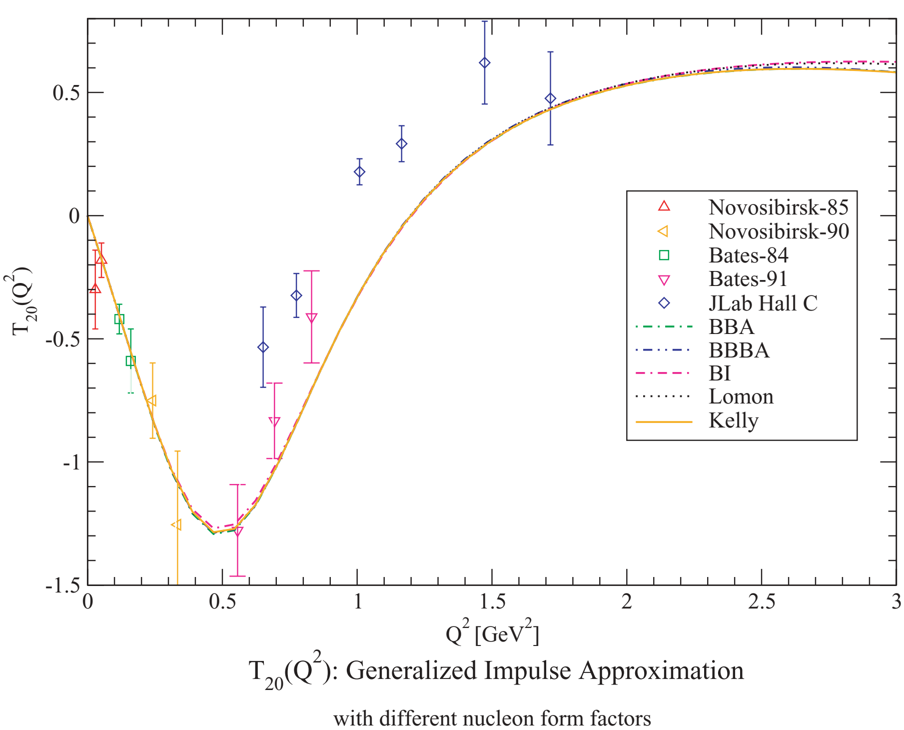

FIG. 11. (Color online) $T_{20}(Q^2, 70^\circ)$: Generalized impulse approximation with different nucleon form factors.

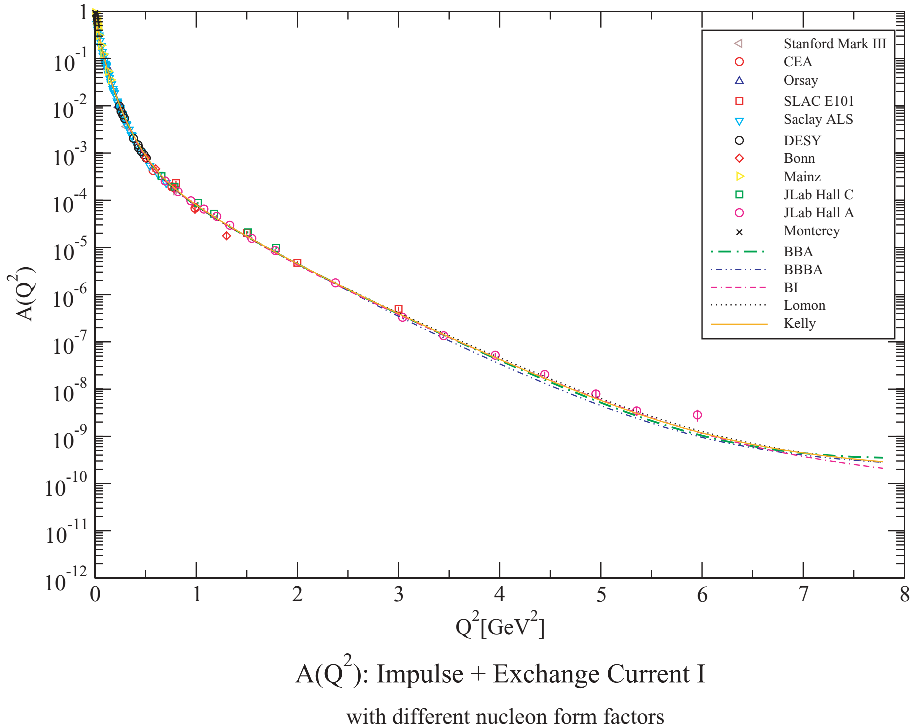

FIG. 12. (Color online) $A(Q^2)$: Generalized impulse approximation plus exchange current I.

approximation. Comparing the dotted and dash-dot curve indicates that much larger required two-body contributions to the current are needed when the exchange current contributions are included in all matrix elements. This suggests that there will be a nontrivial sensitivity to the choice of independent current matrix elements used to generate the fully covariant exchange current.

To test this, we examined the use of choices II and III of independent matrix elements discussed in Sec. VI. These methods relate form factors to independent current matrix elements by contracting different sets of polarization vectors into the current tensor. In both approaches, there are preferred polarization vectors; in one case, the vectors are chosen to minimize the dependence on matrix elements that are

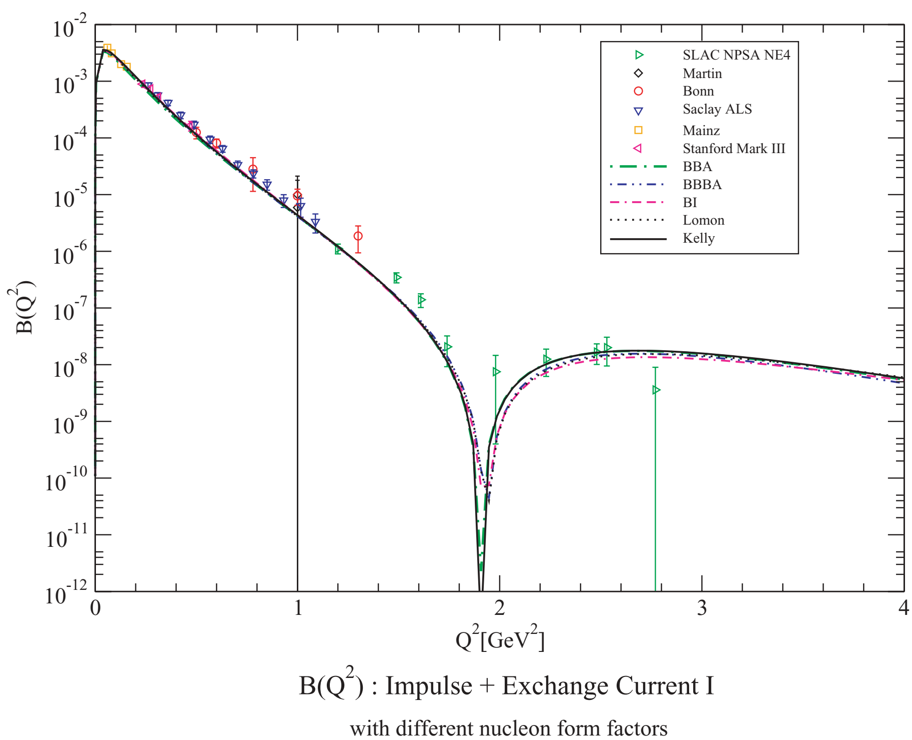

FIG. 13. (Color online) $B(Q^2)$: Generalized impulse approximation plus exchange current I with different nucleon form factors.

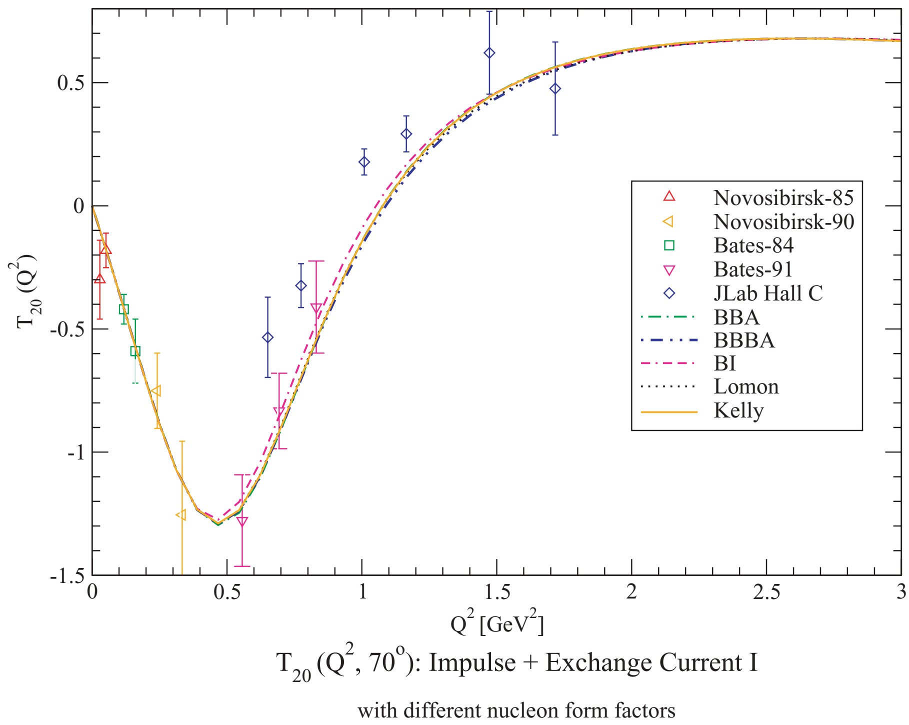

FIG. 14. (Color online) $T_{20}(Q^2, 70^\circ)$: Generalized impulse approximation plus exchange current I with different nucleon form factors.

maximally suppressed in the infinite momentum frame ($P^+ \rightarrow 0$), while the other choice minimizes the mass dependence in the contractions used to define the independent current matrix elements. Both choices were discussed in Sec. VI.

Figures 15–18 show the deuteron elastic scattering observables $A(Q^2)$ and $T_{20}(Q^2)$ for choices II and III of polarization vectors. In both cases, G_1 is computed using the same linear combination of current matrix elements used for choice I. Since B only depends on G_1, B is identical for all three choices. All three choices of independent matrix elements give different predictions for A and T_{20}. For choices II and III, there is a mild enhancement of A at higher momentum transfers compared to choice I. There is also a larger effect on the tensor polarization that brings the curve to within the experimental error bars.

The result is that both choice II and choice III of independent current matrix elements give consistent results for the

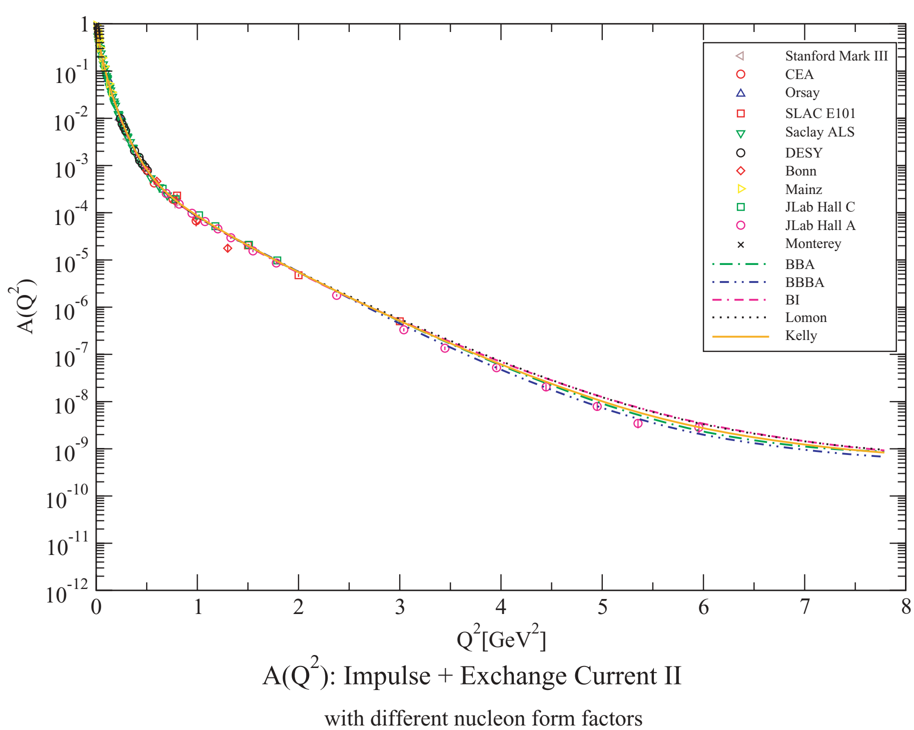

FIG. 15. (Color online) $A(Q^2)$: Generalized impulse approximation plus exchange current II with different nucleon form factors.

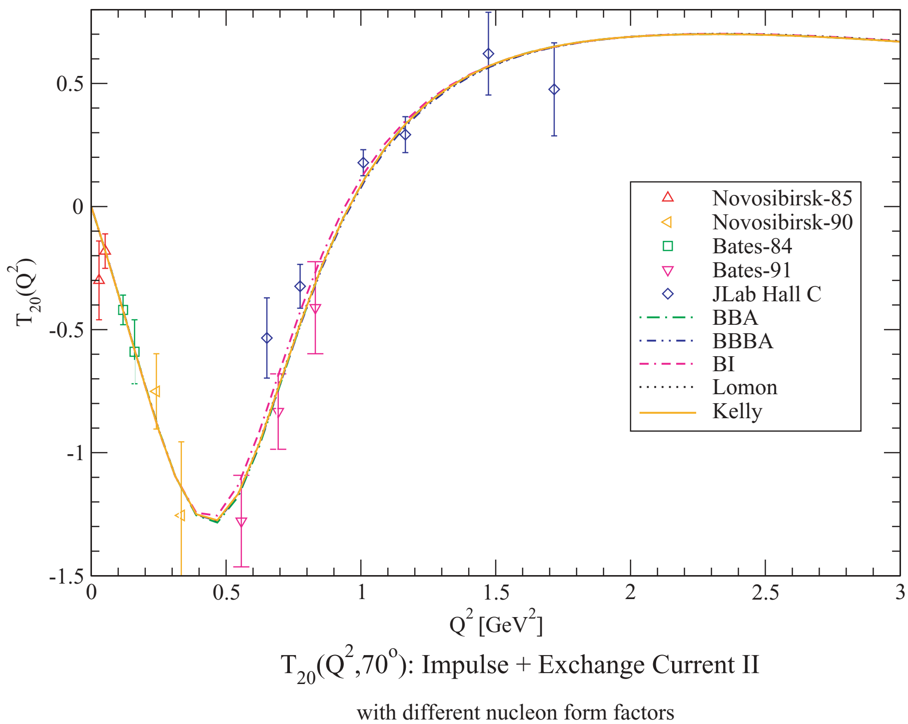

FIG. 16. (Color online) $T_{20}(Q^2,70^\circ)$: Generalized impulse approximation plus exchange current II with different nucleon form factors.

elastic scattering observables, and they both provide a good description of the existing data over a wide range of momentum transfers. It is clear that there is a nontrivial sensitivity to the choice of independent current matrix elements when these results are compared with the corresponding results based on choice I.

Another potential source of sensitivity to the input is the choice of nucleon-nucleon interaction. Any phase equivalent change in the nucleon-nucleon interaction is automatically accompanied by a corresponding change in the current operator. For interactions with a long-range meson exchange tail, one might expect that the same data could be understood by simply adjusting the cutoff parameter. For typical soft interactions that are useful in low-energy problems, one expects that a more significant modification of the current would be necessary.

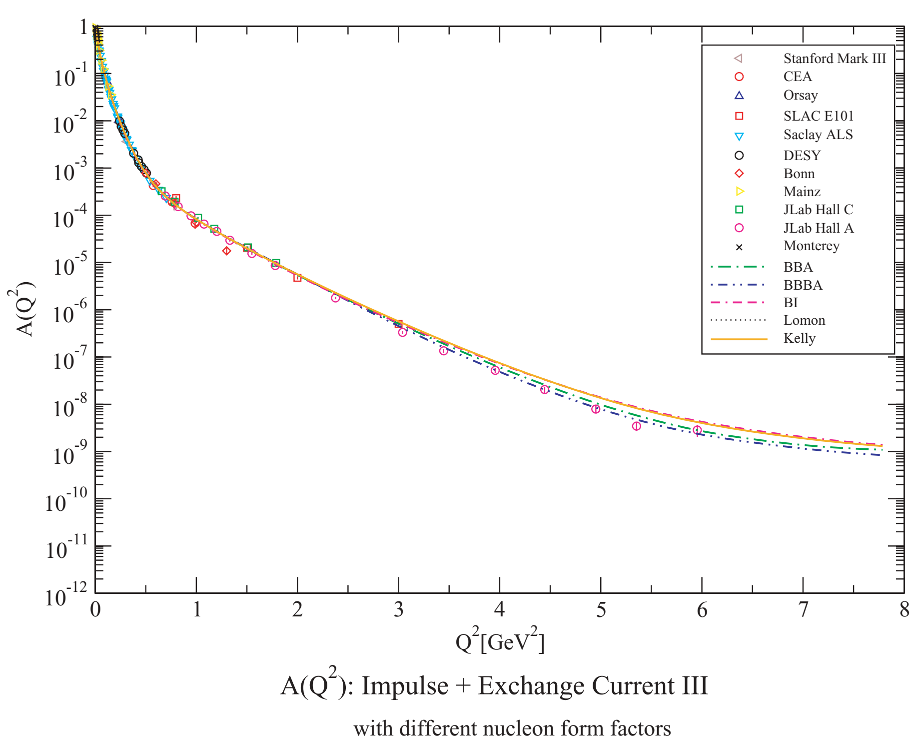

FIG. 17. (Color online) $A(Q^2)$: Generalized impulse approximation plus exchange current III with different nucleon form factors.

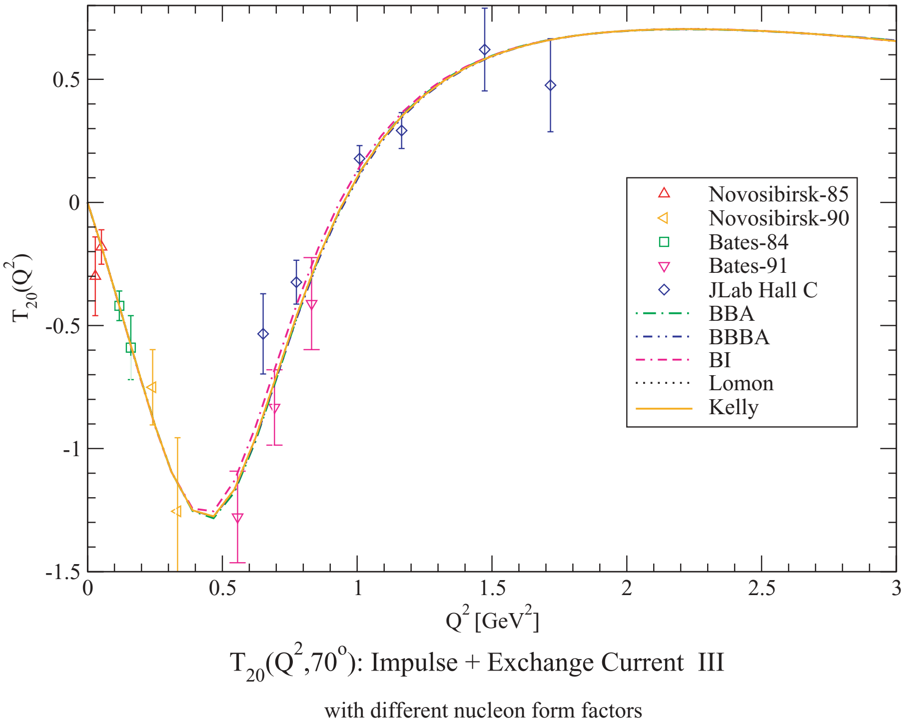

FIG. 18. (Color online) $T_{20}(Q^2,70^\circ)$: Generalized impulse approximation plus exchange current III.

In Figs. 19–21, we compare calculations of $A(Q^2)$, $B(Q^2)$, and $T_{20}(Q^2)$ using the CD Bonn wave functions with and without the exchange current. In these calculations, the exchange current is still based on the Argonne V18 cutoffs. The calculations show the generalized impulse calculations and calculations in which the exchange current is added to the impulse current, without adjusting the cutoff parameters. The calculations clearly show that there is more sensitivity to the choice of nucleon-nucleon interactions than to the choice of nucleon form factor.

Good consistency with all experimental observables is obtained using the V18 interaction with nucleon form factors [41] and the choice of independent current matrix elements suggested by Frankfrut, Frederico, and Strickman or by

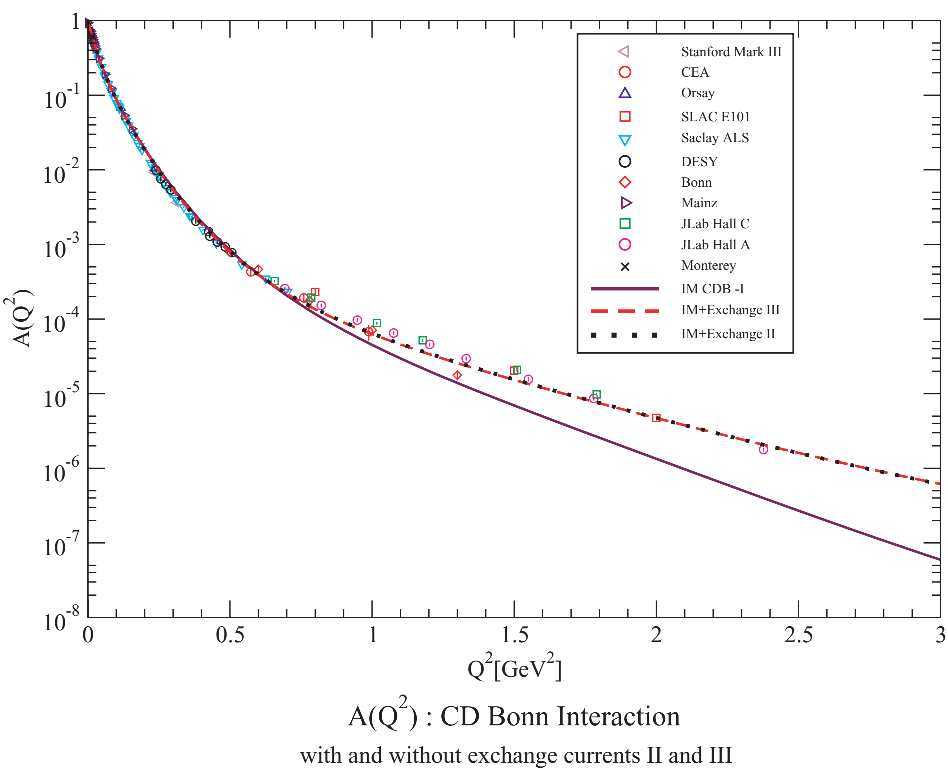

FIG. 19. (Color online) $A(Q^2)$: CD Bonn with and without exchange currents II and III.

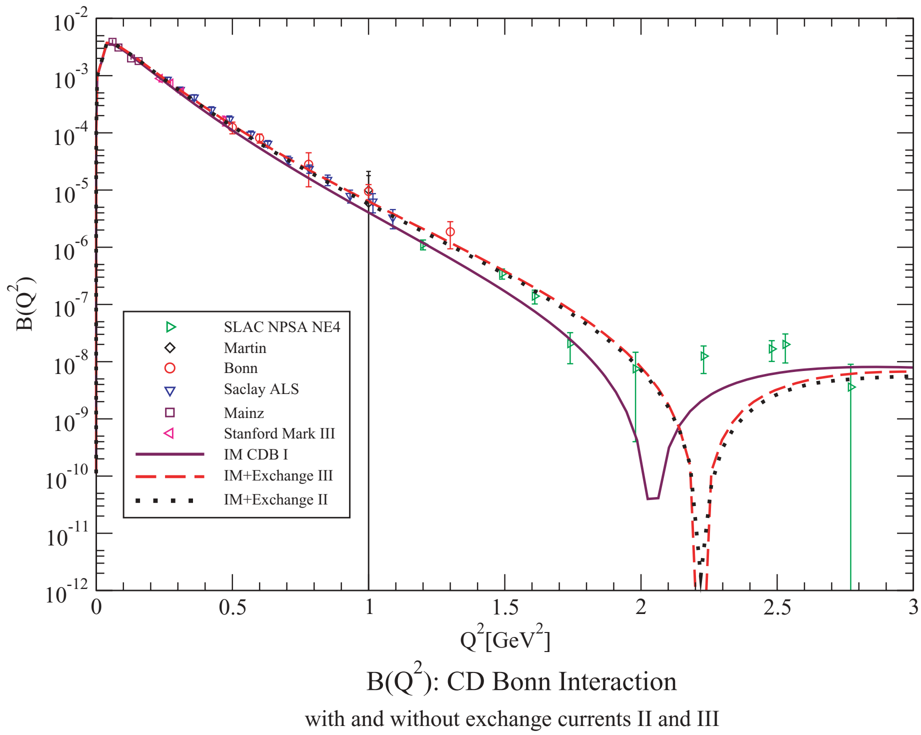

FIG. 20. (Color online) $B(Q^2)$: CD Bonn with and without exchange currents II and III.

Coester, discussed in Sec. VI. These calculations are shown in Figs. 22–24. For these choices, the model exchange current explains the difference between the generalized impulse approximation and the experimental data. Figure 25 shows the ratio of the experimental values of $A(Q^2)$ to the calculated values using exchange current II. The data are presented on a linear scale to better illustrate the comparison between theory and experiment. The solid line represents the calculation with exchange current II, while the diamonds show the ratio of exchange current III to exchange current II. There are no significant differences between the two exchange currents for momentum transfers below 4 GeV^2. While our exchange current, which used the one-pion-exchange part of the Argonne V18 interaction, required the numerical computation

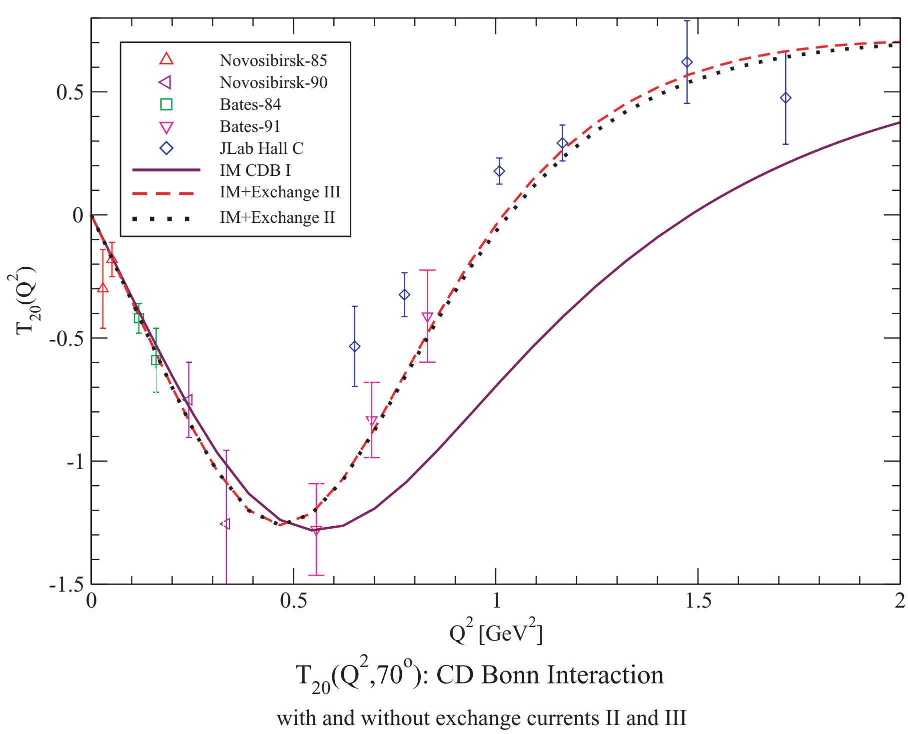

FIG. 21. (Color online) $T_{20}(Q^2, 70^\circ)$: CD Bonn with and without exchange currents II and III.

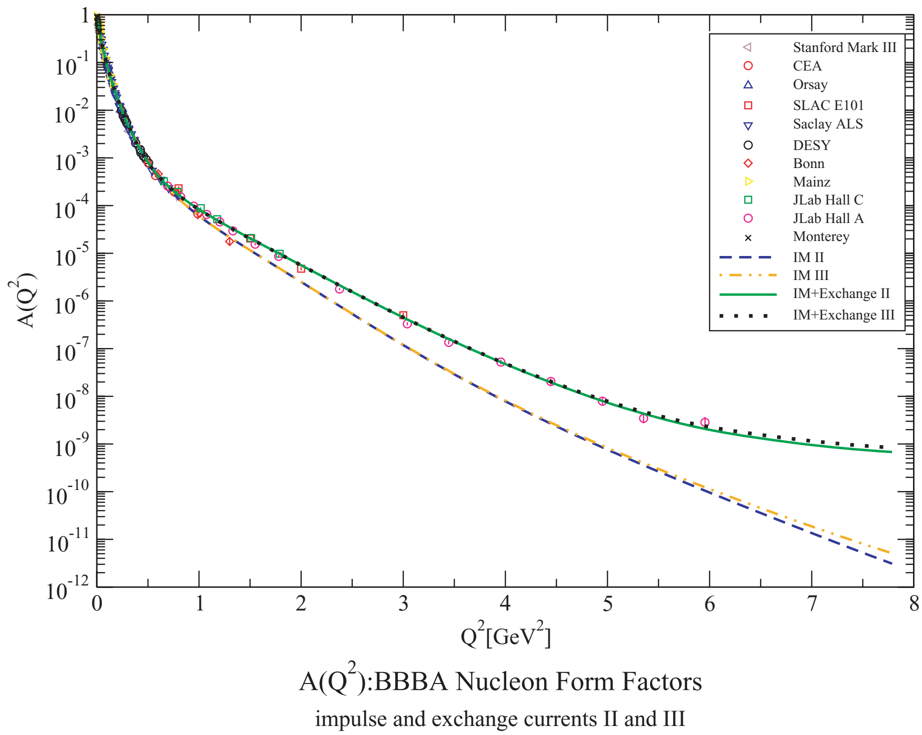

FIG. 22. (Color online) $A(Q^2)$, AV18, BBBA form factors with and without exchange currents II and III.

of Fourier transforms, the resulting interaction contribution to the exchange current differed very little from a simple static momentum-space pion-exchange interaction (Fig. 5), which indicates the simplicity of our exchange current.

Our results indicate that this simple exchange current is sufficient to provide a good quantitative understanding of elastic electron-deuteron scattering for a wide range of momentum transfers. The sensitivities to both the choice of interaction and the choice of independent matrix elements are the largest uncertainties in the calculations, and these uncertainties are all larger than the experimental uncertainties. For momentum transfers with available data, there is very little sensitivity to the uncertainties in the neutron electric form factors.

Finally we compare the magnetic and quadrupole moments of the deuteron with and without the exchange current and

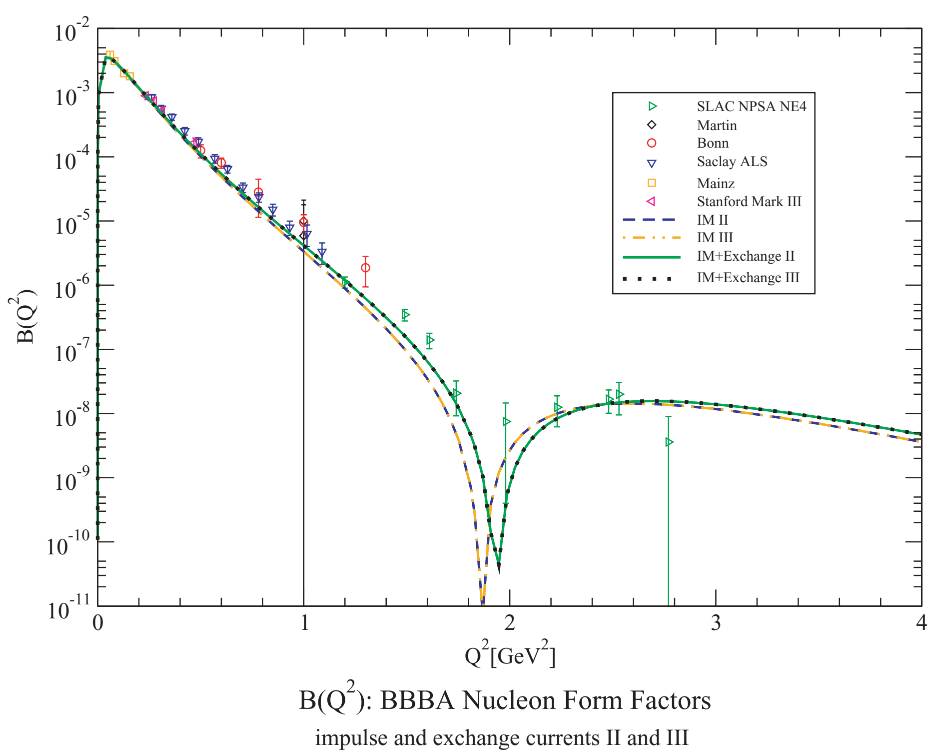

FIG. 23. (Color online) $B(Q^2)$, AV18, BBBA form factors with and without exchange currents II and III.

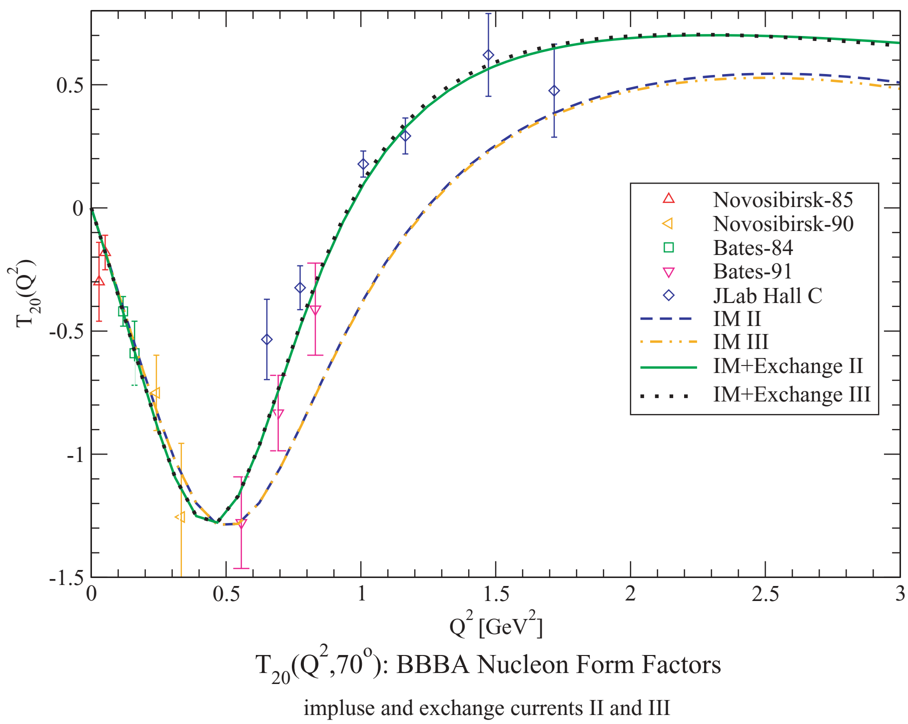

FIG. 24. (Color online) $T_{20}(Q^2, 70^\circ)$, AV18, BBBA form factors with and without exchange currents II and III.

using the three different choices of independent current matrix elements. The results are shown in Table I. Our calculations do not exhibit any sensitivity to the choice of nucleon form factor, which are sufficiently well constrained by experiment at low momentum transfers. The exchange current contributions affect the results for all of the moments. In both cases, they are closer to experiment than the moments computed using the generalized impulse approximation. The magnetic moment is in good agreement (to within computational accuracy) with experiment, while the quadrupole moment differs from the experimental result by a few percent.

The conclusion of our research is that a simple exchange current motivated by one-pion exchange and the freedom to define a conserved covariant current operator by choosing a preferred set of independent current matrix elements is sufficient to provide a good fit to all three elastic scattering observables using Poincaré invariant quantum theory with a null-plane kinematic symmetry. The model exchange current has the spin structure of a "pair" current, designed with a null-plane kinematic symmetry. The quality of the nucleon form factors has progressed to the point at which our results are insensitive to the choice of nucleon form factor.

While is it straightforward to include low-order pion-exchange physics in a more general class of models, the strategy for making the best choice of independent current matrix elements in a general class of electron-nucleus reactions requires more investigation. The principles used to derive choices I–III of independent current matrix elements all can be generalized to treat initial and final states with different spins. Whether there is one consistent set of principles that works universally for all reactions is not yet known.

A second observation is that our model, with one of the two preferred choices of independent current matrix elements, provides a better description of all three observables than methods based on truncations of null-plane field theory or instant-form relativistic quantum mechanics. Our model has features of both: unlike the instant-form model, our model has the full null-plane kinematic symmetry with all the advantages discussed at the beginning of this paper. Unlike truncations of a null-plane field theory, which emphasize cluster properties at the expense of exact Poincaré invariance, our model is exactly Poincaré invariant. We expect that the Poincaré invariance constraint to be more important for momenta near or slightly above the deuteron mass scale, since the deuteron mass scale is involved in the implementation of the symmetry.

This research provides a useful first step in trying to devise a more systematic treatment of model exchange currents in Poincaré invariant quantum mechanics with a null-plane kinematic symmetry. It leads to a simple current

TABLE I. Deuteron magnetic and quadrupole moments evaluated in the impulse approximation and including the exchange current contribution. The values are the same using all six different nucleon form factor parametrizations and three combinations of independent current matrix elements. The Argonne V18 potential is used in the calculation. The values labeled with WSS are from Ref. [1]. The experimental values are 0.2860 ± 0.0015 fm^2 [73] and $0.857406 \pm 0.000001\mu_N$ [74].

	IM	IM+Exchange	IM(WSS)	IM+MEC(WSS)
Q_d	0.2698	0.2752	0.270	0.275
μ_d	0.8535	0.8596	0.847	0.871

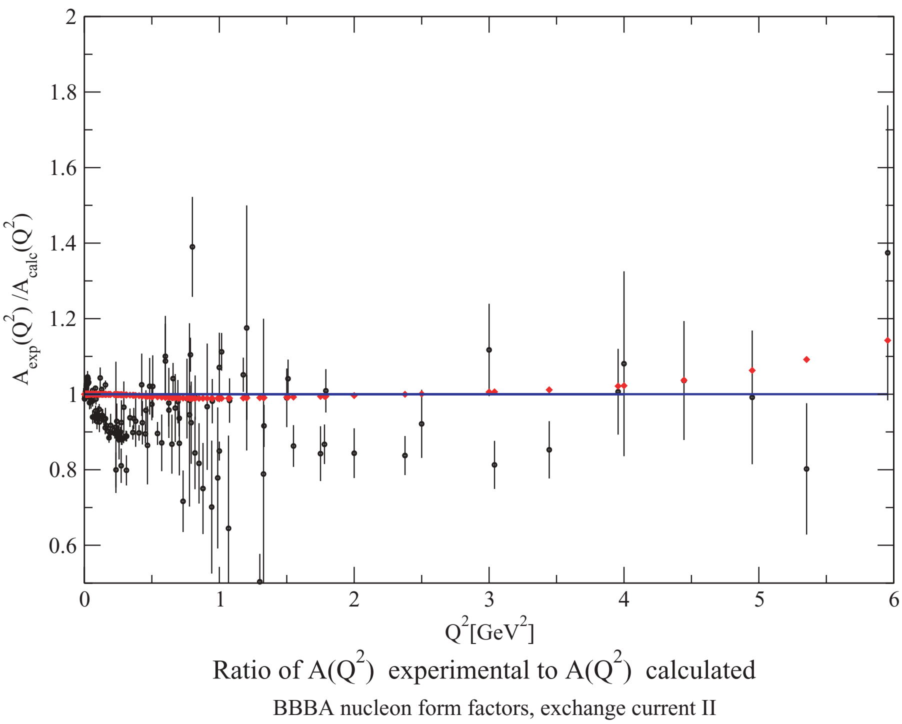

FIG. 25. (Color online) Ratio of experimental values of $A(Q^2)$ to calculated values of $A(Q^2)$ with exchange current II.

that provides a significant improvement in all three elastic scattering observables when compared with the corresponding impulse calculations; however, additional research is needed to determine if these methods can be successfully applied to a larger class of reactions.

ACKNOWLEDGMENTS

The authors are grateful to Fritz Coester, who made many material suggestions that significantly improved the quality of this work. This work was performed under the auspices of the US Department of Energy, Office of Nuclear Physics, under Contract No. DE-FG02-86ER40286.

APPENDIX A: CURRENT CONSTRUCTION

The steps motivating the structure of the model exchange current of Eq. (89) are summarized in this appendix. We consider a pseudoscalar pion-nucleon vertex

$$\mathcal{L}(x) = -ig_\pi : \bar{\Psi}(x)\gamma_5\Psi(x)\boldsymbol{\tau}\cdot\boldsymbol{\phi}(x) : \tag{A1}$$

where $g_\pi = 2m\frac{f_\pi}{m_\pi}$ is the pseudoscalar pion-nucleon coupling constant, and m is the nucleon mass. In what follows we use canonical Dirac spinors with a noncovariant normalization

$$u_{\rm nc}(\mathbf{p}) := \sqrt{\frac{m}{\omega(\mathbf{p})}}u_c(\mathbf{p}) \tag{A2}$$

and use the notation

$$v_\pi(\mathbf{k}',\mathbf{k}) := \frac{g_\pi^2}{(2\pi)^3}\frac{\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2}{m_\pi^2+(k'-k)^2-i0^+}. \tag{A3}$$

In this notation, the tree-level one-pion-exchange transition amplitude is given by the rotationally invariant kernel:

$$\begin{aligned}&\langle\mathbf{k}',\mu_1',\mu_2'|v_\pi|\mathbf{k},\mu_1,\mu_2\rangle\\&\quad:= \bar{u}_{\rm nc}(\mathbf{k}')\gamma_5 u_{\rm nc}(\mathbf{k})v_\pi(\mathbf{k}',\mathbf{k})\bar{u}_{\rm nc}(-\mathbf{k}')\gamma_5 u_{\rm nc}(-\mathbf{k}),\end{aligned} \tag{A4}$$

where we have assumed a plane-wave normalization $\langle\mathbf{k}'|\mathbf{k}\rangle = \delta(\mathbf{k}'-\mathbf{k})$.

The tree-level one-pion-exchange transition amplitude in the presence of an external electromagnetic field, using the vertex (A1) with

$$A^\mu(q) := \frac{1}{(2\pi)^4}\int e^{-iq\cdot y}A_\mu(y)d^4y, \tag{A5}$$

includes four terms, one of which is

$$\begin{aligned}&\langle\mathbf{p}_1',\mu_1',\mathbf{p}_2',\mu_2'|I^\mu(0)^\mu|\mathbf{p}_1,\mu_1,\mathbf{p}_2,\mu_2\rangle\\&\quad= \bar{u}_{\rm nc}(\mathbf{p}_1')\Gamma^\mu u_{\rm nc}(\mathbf{r})\bar{u}_{\rm nc}(\mathbf{r})\gamma_5 u_{\rm nc}(\mathbf{p}_1)\\&\qquad\times\frac{1}{E_{12}-\omega(\mathbf{r})-\omega(\mathbf{p}_2')+i0^+}v_\pi(\mathbf{p}_2',\mathbf{p}_2)\bar{u}_{\rm nc}(\mathbf{p}_2')\gamma_5 u_{\rm nc}(\mathbf{p}_2)\\&\qquad+\bar{u}_{\rm nc}(\mathbf{p}_1')\Gamma^\mu v_{\rm nc}(-\mathbf{r})\bar{v}_{\rm nc}(-\mathbf{r})\gamma_5 u_{\rm nc}(\mathbf{p}_1)\\&\qquad\times\frac{1}{E_{12}-\omega(\mathbf{p}_2')+\omega(\mathbf{r})-i0^+}\\&\qquad\times v_\pi(\mathbf{p}_2',\mathbf{p}_2)\bar{u}_{\rm nc}(\mathbf{p}_2')\gamma_5 u_{\rm nc}(\mathbf{p}_2)+\cdots,\end{aligned} \tag{A6}$$

where

$$E_{12} = \omega(\mathbf{p}_1)+\omega(\mathbf{p}_2) \tag{A7}$$

is the initial energy,

$$\mathbf{r} = \mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_2', \tag{A8}$$

and

$$\Gamma^\mu = \gamma^\mu F_1(Q^2) - \frac{i}{2m}\sigma^{\mu\nu},\quad Q_\nu F_2(Q^2) \tag{A9}$$

is the nucleon impulse current. The $+\cdots$ represents the contribution from the other three terms related by either Hermitian conjugation or exchanging the proton and neutron.

To motivate the structure of our model exchange current, we first evaluate this covariant expression in the rest frame of the initial two-body system so the pion-exchange interaction appears as a rotationally invariant function of the relative momenta. This allows us to relate the interaction part of this kernel to the interaction kernel in Eq. (A4). Next we treat the initial energy of the two-body system, E_{12}, as a parameter that can be expressed in terms of the mass of the initial state (deuteron) and the kinematically conserved momenta. We use the assumed null-plane kinematic symmetry of the interaction to express the energy denominators in terms of masses and null-plane kinematic variables; $E_{12} \to \frac{1}{2}(P^+ + \frac{M^2+\mathbf{P}_\perp}{P^+})$, $E_{120} \to \frac{1}{2}(P^+ + \frac{M_0^2+\mathbf{P}_\perp}{P^+})$. These are all model assumptions. In the rest frame of the initial deuteron, $P^- = M_d$ which gives

$$\begin{aligned} &E_{12} - \omega(\mathbf{r}) - \omega(\mathbf{p}_2') \\ &\quad \to E_{12} - E_{120} \to \frac{1}{2}\frac{M_d^2 - M_0^2(\mathbf{k}')}{P^+_{\text{rest}}} \to \frac{M_d^2 - M_0^2(\mathbf{k}')}{2M_d} \\ &\quad \approx \frac{M_d^2 - M_0^2(\mathbf{k}')}{4m}, \end{aligned} \tag{A10}$$

and

$$\begin{aligned} &E_{12} - \omega(\mathbf{p}_2') + \omega(\mathbf{r}) \\ &\quad \to E_{12} \to \frac{1}{2}\left(P^+ + \frac{M^2 + \mathbf{P}_\perp}{P^+}\right) \to \frac{1}{2}\left(M_d + \frac{M_d^2}{M_d}\right) \\ &\quad = M_d \approx 2m. \end{aligned} \tag{A11}$$

Using Eqs. (A10) and (A11) in Eq. (A6) gives

$$\begin{aligned} &\langle \mathbf{p}_1', \mu_1', -\mathbf{k}', \mu_2' | I^\mu(0)^\mu | \mathbf{k}, \mu_1, -\mathbf{k}, \mu_2 \rangle \\ &\quad = \bar{u}_{\text{nc}}(\mathbf{p}_1')\Gamma^\mu u_{\text{nc}}(\mathbf{k}') \frac{4m}{M_d^2 - M_0^2 - i0^+} \bar{u}_{\text{nc}}(\mathbf{k}')\gamma_5 u_{\text{nc}}(\mathbf{k}) \\ &\qquad \times v_\pi(\mathbf{k}', \mathbf{k}) \bar{u}_{\text{nc}}(-\mathbf{k}')\gamma_5 u_{\text{nc}}(-\mathbf{k}) \\ &\qquad - \bar{u}_{\text{nc}}(\mathbf{p}_1')\Gamma^\mu v_{\text{nc}}(-\mathbf{k}') \frac{1}{2m} \bar{v}_{\text{nc}}(-\mathbf{k}')\gamma_5 u_{\text{nc}}(\mathbf{k}) \\ &\qquad \times v_\pi(\mathbf{k}', \mathbf{k}) \bar{u}_{\text{nc}}(-\mathbf{k}')\gamma_5 u_{\text{nc}}(-\mathbf{k}) + \cdots. \end{aligned} \tag{A12}$$

The terms in this expression are easy to interpret. The first two lines represent the product of a one-body current matrix element, the propagator, $\frac{4m}{M_d^2 - M_0^2 - i0^+}$, associated with Eq. (33) and the rotationally invariant kernel of the pion-exchange interaction (A4). This term is already included in the one-body contribution to the current matrix element.

The last two lines have a similar form except one of the $u_{\text{nc}}(\mathbf{k})$ spinors in the current and the rotationally invariant interaction kernel are replaced by $v_{\text{nc}}(-\mathbf{k})$ spinors. In addition, the propagator term is replaced by the factor $1/2m$. We find it convenient for computational purposes to split the v spinor terms that give the Dirac projector $\Lambda_-(-\mathbf{k}) = -v_c(-\mathbf{k})\bar{v}_c(-\mathbf{k})$ and to use $\gamma_5\beta$ to convert the $v_{\text{nc}}(-\mathbf{k})$ to $u_{\text{nc}}(\mathbf{k})$. The two factors of β break kinematic covariance. We restore manifest kinematic covariance by using this expression to define the current matrix elements in the rest frame of the initial system, and we transform to the Breit frame by requiring kinematic covariance.

Using

$$v_{\text{nc}}(-\mathbf{p})\bar{v}_{\text{nc}}(-\mathbf{p}) = \gamma_5\beta u_{\text{nc}}(\mathbf{p})\bar{u}_{\text{nc}}(\mathbf{p})\beta\gamma_5, \tag{A13}$$

the last two lines of Eq. (A12) become

$$\begin{aligned} &\bar{u}_{\text{nc}}(\mathbf{p}_1')\Gamma^\mu \gamma_5 \beta u_{\text{nc}}(\mathbf{k}') \frac{1}{2m} \bar{u}_{\text{nc}}(\mathbf{k}')\beta u(\mathbf{k}) v_\pi(\mathbf{k}', \mathbf{k}) \\ &\qquad \times \bar{u}_{\text{nc}}(-\mathbf{k}')\gamma_5 u_{\text{nc}}(-\mathbf{k}) + \cdots. \end{aligned} \tag{A14}$$

This separates into the product of an effective one-body current,

$$\bar{u}_{\text{nc}}(\mathbf{p}_1')\Gamma^\mu \gamma_5 \beta u_{\text{nc}}(\mathbf{k}'), \tag{A15}$$

a factor $1/2m$, and a modified one-pion-exchange interaction

$$\bar{u}_{\text{nc}}(\mathbf{k}')\beta u_{\text{nc}}(\mathbf{k}) v_\pi(\mathbf{k}', \mathbf{k}) \bar{u}_{\text{nc}}(-\mathbf{k}')\gamma_5 u_{\text{nc}}(-\mathbf{k}). \tag{A16}$$

The interaction term is identical to the rotationally invariant kernel of the one-pion-exchange interaction (A4) with the replacement: $\bar{u}_{\text{nc}}(\mathbf{k}')\gamma_5 u_{\text{nc}}(\mathbf{k}) \to \bar{u}_{\text{nc}}(\mathbf{k}')\beta u_{\text{nc}}(\mathbf{k})$. This replacement preserves the rotational invariance of this kernel in the rest frame, but it is not covariant with respect to null-plane boosts.

This splitting has the advantage that the computation of the exchange current matrix element has the same structure as the computation of an impulse current matrix element with the current replaced by Eq. (A15) and the deuteron wave function $|\psi\rangle$ replaced by $\frac{1}{2m}\tilde{v}|\psi\rangle$, where $\tilde{v}$ is the modified interaction (A16). The input to this calculation is defined in the rest frame of the initial deuteron. We get the null-plane Breit frame result by requiring kinematic covariance.

The effective one-body current of Eq. (A15) can be extended to a kinematically covariant operator that agrees with Eq. (A15) in the deuteron rest frame by replacing the β in Eq. (A15) by

$$\beta \to -P \cdot \gamma / M_d. \tag{A17}$$

This leads to a kinematically covariant modified current kernel

$$\bar{u}_{\text{nc}}(\mathbf{p}_1')\Gamma^\mu \gamma_5 \frac{(P \cdot \gamma)}{M_d} u_{\text{nc}}(\mathbf{k}'). \tag{A18}$$

Kinematic covariance is all that this needed for a consistent computation of the current matrix elements.

The modified interaction (A16) is rotationally invariant; so when it is applied to the rest deuteron eigenstate, the resulting pseudostate has the same spin as the deuteron. This can be consistently defined in any other kinematically related frame using null-plane boosts. This, along with the replacement (A17), restores the manifest kinematic covariance.

The last step is to replace the modified interaction (A16) by the corresponding modified one-pion-exchange part of a realistic model interaction. In a typical realistic interaction, the pseudoscalar pion-exchange interaction is obtained by replacing the spinor terms in the rotationally invariant kernel (A4) by

$$\bar{u}_{\text{nc}}(\mathbf{k}')\gamma_5 u_{\text{nc}}(\mathbf{k}) \to \frac{\boldsymbol{\sigma} \cdot (\mathbf{k} - \mathbf{k}')}{2m}. \tag{A19}$$

This expression is obtained by retaining the leading term in a $\mathbf{k}/m$ expansion of the spinor term, which is normally justified because the one-pion-exchange interaction also includes a high-momentum or short-distance cutoff. The net effect is that the resulting interaction, when included in the full nucleon-nucleon interaction, provides a good description of the two-nucleon bound state and scattering observables.

Expanding $\bar{u}(\mathbf{k}')\beta u(\mathbf{k})$ to the same order in $\mathbf{k}/m$ gives 1. This suggests that the modified interaction (A16) can be modeled by replacing the one-pion-exchange contribution to the realistic interaction

$$\boldsymbol{\sigma}_1 \cdot (\mathbf{k} - \mathbf{k}') v_\pi(\mathbf{k} - \mathbf{k}') \boldsymbol{\sigma}_2 \cdot (\mathbf{k}' - \mathbf{k}) \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 \tag{A20}$$

by

$$2m v_\pi(\mathbf{k} - \mathbf{k}') \boldsymbol{\sigma}_2 \cdot (\mathbf{k}' - \mathbf{k}) \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2. \tag{A21}$$

The one-pion-exchange interaction (A4) only contributes to the part of the tensor force in the nucleon-nucleon interaction that multiplies the isospin exchange operator $\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2$. The tensor interaction also has contributions from the vector exchanges, which also contribute to the spin-spin interaction. Riska [50] and Schiavilla, Pandharipande, and Riska [51] introduced a method for isolating the pion-exchange contribution to the tensor force of a phenomenological interaction using linear combinations of the tensor and spin-spin interactions. The resulting interaction is

$$v_{\text{ps}}(\mathbf{k} - \mathbf{k}') = \tfrac{1}{3}(2v_t(\mathbf{k} - \mathbf{k}') - v_{\text{ss}}(\mathbf{k} - \mathbf{k}')), \tag{A22}$$

where v_t and v_{ss} are tensor and spin-spin contributions to the charge-exchange part of the Argonne V18 interaction. We extract this from the pion-exchange contribution to the Argonne V18 potential. This is compared with $\frac{1}{m_\pi^2 + (\mathbf{k}-\mathbf{k}')^2}$ in Fig. 5. The difference between these curves is due to the short-distance cutoff used in the AV18 interaction.

When this interaction is applied to the deuteron bound state vector, the result is a spin 1 "pseudo wave function." The resulting "pseudovector" can be defined in the Breit frame by requiring that it transforms covariantly with respect to the null-plane kinematic subgroup. This kinematic covariance ensures that the current kernel is kinematically covariant provided the pseudocurrent is modified following Eq. (A17). The kinematic covariance of the current is needed for consistent calculation matrix elements of $I^+(0)$.

We can now write the form of the Breit frame matrix elements of this one-pion-exchange contribution to the exchange current as

$$\begin{aligned}
&\langle (1,d), \tilde{\mathbf{P}}', \mu' | I^+_{\text{ex}}(0) | (1,d), \tilde{\mathbf{P}}, \mu\rangle \\
&= \sum \int \langle (1,d), \tilde{\mathbf{P}}', \mu' | (j',k'), \tilde{\mathbf{P}}''', l', s', \mu'''\rangle k'^2 dk'\, d\tilde{\mathbf{P}}''' \langle (j',k'), \tilde{\mathbf{P}}''', l', s', \mu''' | (j_p, m_p), \tilde{\mathbf{p}}'_p, \mu'_p, (j_n, m_n), \tilde{\mathbf{p}}'_n, \mu'_n\rangle\, d\tilde{\mathbf{p}}'_p d\tilde{\mathbf{p}}'_n \\
&\quad \times \langle \tilde{\mathbf{p}}'_p, \mu'_p, \tilde{\mathbf{p}}'_n, \mu'_n | I^+_{\text{ex-eff}}(0) | (j_p, m_p)\tilde{\mathbf{p}}_p, \mu_p, (j_n, f_n)\tilde{\mathbf{p}}_n, \mu_n\rangle\, d\tilde{\mathbf{p}}_p\, d\tilde{\mathbf{p}}_n \\
&\quad \times \langle (j_p, m_p), \tilde{\mathbf{p}}_p, \mu_p, (j_n, m_n), \tilde{\mathbf{p}}_n, \mu_n | (j,k)\tilde{\mathbf{P}}'', l, s, \mu''\rangle k^2 dk\, d\tilde{\mathbf{P}}'' \langle (j,k)\tilde{\mathbf{P}}'', l, s, \mu'' | \tilde{\mathbf{P}}, \mu, \chi\rangle,
\end{aligned} \tag{A23}$$

where the terms in this expression are the deuteron wave function in the free-particle irreducible null-plane basis

$$\begin{aligned}
&\langle (1,d)\tilde{\mathbf{P}}', \mu', d | (1,k'), l', s', \tilde{\mathbf{P}}''', \mu'''\rangle \\
&= \delta(\tilde{\mathbf{P}}' - \tilde{\mathbf{P}}''')\delta_{j'j'''}\delta_{\mu'\mu'''}\phi^*_{j'}(k', l', s'), \quad (j' = s' = 1),
\end{aligned} \tag{A24}$$

the Poincaré group Clebsch-Gordan coefficients in the null-plane basis

$$\begin{aligned}
&\langle \tilde{\mathbf{P}}''', k', j', l', s', \mu''' | \tilde{\mathbf{p}}'_p, \mu'_p, \tilde{\mathbf{p}}'_n, \mu'_n\rangle \\
&= \delta(\tilde{\mathbf{P}}''' - \tilde{\mathbf{p}}'_p - \tilde{\mathbf{p}}'_n)\frac{\delta(k' - k(\tilde{\mathbf{p}}'_p, \tilde{\mathbf{p}}'_n))}{k'^2} \\
&\quad \times \sqrt{\frac{\partial(\tilde{\mathbf{P}}, \mathbf{k})}{\partial(\tilde{\mathbf{p}}'_p, \tilde{\mathbf{p}}'_n)}} \langle j, \mu | l, m_l, s, m_s\rangle \\
&\quad \times \left\langle s, m_s \middle| \frac{1}{2}, \mu_p, \frac{1}{2}, \mu_n \right\rangle Y^*_{lm_l}(\hat{\mathbf{k}}(\tilde{\mathbf{p}}'_p \tilde{\mathbf{p}}'_n)) \\
&\quad \times D^{1/2}_{\mu_p \mu'_p}\left[\Lambda_c^{-1}(\mathbf{k}/m)\Lambda_f(\mathbf{k}/m)\right] D^{1/2}_{\mu_n \mu'_n} \\
&\quad \times \left[\Lambda_c^{-1}(-\mathbf{k}/m)\Lambda_f(-\mathbf{k}/m)\right],
\end{aligned} \tag{A25}$$

the proton effective current

$$\begin{aligned}
&\langle \tilde{\mathbf{p}}'_p, \mu'_p, \tilde{\mathbf{p}}'_n, \mu'_n | I^+_{\text{ex-eff}}(0) | \tilde{\mathbf{p}}_p, \mu_p, \tilde{\mathbf{p}}_n, \mu_n\rangle \\
&= \delta(\tilde{\mathbf{p}}_n - \tilde{\mathbf{p}}_n)\bar{u}_{nf}(\tilde{\mathbf{p}}'_p, \mu'_p)\Gamma^\mu_p(q)\gamma^5(\eta^0\gamma^0 - \eta^1\gamma^1) \\
&\quad \times u_{nf}(\tilde{\mathbf{p}}_p, \mu_p),
\end{aligned} \tag{A26}$$

where the null-plane Dirac spinors are related to the canonical Dirac spinors by a Melosh rotation

$$\begin{aligned}
&u_f(\tilde{\mathbf{p}}_p, \mu_p)_\mu \\
&= u_c(\tilde{\mathbf{p}}_p, \mu_p)_{\mu'} D^{1/2}_{\mu'\mu}\left[\Lambda_c^{-1}(\mathbf{p}_p/m)\Lambda_f(-\mathbf{p}_p/m)\right],
\end{aligned} \tag{A27}$$

another Poincaré Clebsch-Gordan coefficient

$$\begin{aligned}
&\langle \tilde{\mathbf{p}}_p, \mu_p, \tilde{\mathbf{p}}_n, \mu_n | \tilde{\mathbf{P}}'', k, j, l, s, \mu''\rangle \\
&= \delta(\tilde{\mathbf{P}}''' - \tilde{\mathbf{p}}'_p - \tilde{\mathbf{p}}'_n)\frac{\delta(k' - k(\tilde{\mathbf{p}}'_p, \tilde{\mathbf{p}}'_n))}{k'^2}\sqrt{\frac{\partial(\tilde{\mathbf{P}}, \mathbf{k})}{\partial(\tilde{\mathbf{p}}'_p, \tilde{\mathbf{p}}'_n)}} \\
&\quad \times D^{1/2}_{\mu_p \mu'_p}\left[\Lambda_f^{-1}(\mathbf{k}/m)\Lambda_c(\mathbf{k}/m)\right] \\
&\quad \times D^{1/2}_{\mu_n \mu'_n}\left[\Lambda_f^{-1}(-\mathbf{k}/m)\Lambda_c(-\mathbf{k}/m)\right] \\
&\quad \times \left\langle s, m_s \middle| \frac{1}{2}, \mu'_p, \frac{1}{2}, \mu'_n \right\rangle \langle j, \mu | l, m_l, s, m_s\rangle \\
&\quad \times Y_{lm_l}(\hat{\mathbf{k}}(\tilde{\mathbf{p}}'_p, \tilde{\mathbf{p}}'_n)),
\end{aligned} \tag{A28}$$

and the pseudo wave function

$$\langle (k,j),l,s,\tilde{\mathbf{P}}'',\mu''|\tilde{\mathbf{P}},\mu,\chi\rangle = \delta(\tilde{\mathbf{P}}''-\tilde{\mathbf{P}})\int\sum\langle j,\mu''|l,m_l,s,m_s\rangle \times \left\langle s,m_s\left|\frac{1}{2},\mu_p,\frac{1}{2},\mu_n\right.\right\rangle \frac{1}{(2\pi)^{3/2}}Y^*_{lm_l}(\hat{\mathbf{k}})\,d\hat{\mathbf{k}} \times(\mathbf{k}-\mathbf{k}')\cdot\boldsymbol{\sigma}_{\mu_n\mu_n'}v_{\rm ps}(\mathbf{k}'-\mathbf{k})\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 d\mathbf{k}'Y_{l'm_l'}(\hat{\mathbf{k}}') \times\left\langle s',m_s',\left|\frac{1}{2},\mu_p',\frac{1}{2},\mu_n'\right.\right\rangle\langle j,\mu,|,s',m_s',l',m_l'\rangle\phi_{lsj}(k'), \tag{A29}$$

where $v_{\rm ps}$ is given by Eq. (A19). The full exchange current is the sum of the above quantity and the three other terms related by taking Hermitian conjugates or exchanging the particle that couples to the photon. The computation of the pseudo wave function is discussed in Appendix B.

APPENDIX B: CALCULATION OF PSEUDO WAVE FUNCTION

The calculation of the pseudo wave function of Eq. (A29) requires the one-pion-exchange part of the tensor interaction. Riska [50] and Schiavilla, Pandharipande, and Riska [51] outlined a method for extracting the pseudoscalar contribution to the tensor force as a linear combination of the radial coefficient functions in the Argonne V18 interaction. The method is based on the observation that both pseudoscalar and vector meson exchange contribute to both the tensor and spin-spin interaction in the static limit [50].

The Fourier transform of the interaction has the structure

$$v_{nn}(\boldsymbol{\kappa}) = \Omega_t v_t(\boldsymbol{\kappa}^2) + \Omega_{\rm ss}v_{\rm ss}(\boldsymbol{\kappa}^2)+\cdots, \tag{B1}$$

where $\boldsymbol{\kappa} := \mathbf{k}'-\mathbf{k}$,

$$\Omega_t := (\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\boldsymbol{\kappa}^2 - 3\boldsymbol{\sigma}_1\cdot\boldsymbol{\kappa}\boldsymbol{\sigma}_2\cdot\boldsymbol{\kappa})(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2), \tag{B2}$$

$$\Omega_{\rm ss} := \boldsymbol{\kappa}^2(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2), \tag{B3}$$

and we have explicitly exhibited the tensor and spin-spin contributions to the interaction.

The coefficient functions in Eq. (B1) are Fourier transforms of the coefficient functions of the corresponding terms in the Argonne V18 interaction,

$$v_{\rm ss}(\boldsymbol{\kappa}^2) = \frac{4\pi}{\boldsymbol{\kappa}^2}\int_0^\infty dr\, r^2 v^{\sigma\tau}(r)(j_0(\kappa r)-1), \tag{B4}$$

$$v_t(\boldsymbol{\kappa}^2) = \frac{4\pi}{\boldsymbol{\kappa}^2}\int_0^\infty dr\, r^2 v^{t\tau}(r) j_2(\kappa r), \tag{B5}$$

where the factor $1/\boldsymbol{\kappa}^2$ is due to the difference in the conventions used to define the tensor operator in momentum and configuration space.

These coefficient functions have contributions from both pseudoscalar and vector meson exchange. To separate them, following Ref. [50], define the tensor and spinor operators and note that

$$(\boldsymbol{\sigma}_1\cdot\boldsymbol{\kappa}\boldsymbol{\sigma}_2\cdot\boldsymbol{\kappa})(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2) = -\tfrac{1}{3}(\Omega_t-\Omega_{\rm ss}), \tag{B6}$$

while vector meson exchange gives the combination [50]

$$[(\boldsymbol{\sigma}_1\times\boldsymbol{\kappa})\cdot(\boldsymbol{\sigma}_2\times\boldsymbol{\kappa})](\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2) = \tfrac{1}{3}[2\Omega_{\rm ss}+\Omega_t]. \tag{B7}$$

This implies that

$$v_t\Omega_t + v_{\rm ss}\Omega_{\rm ss} = v_{\rm ps}\tfrac{1}{3}(\Omega_{\rm ss}-\Omega_t) + v_v\tfrac{1}{3}(2\Omega_{\rm ss}+\Omega_t) = \tfrac{1}{3}(v_v - v_{\rm ps})\Omega_t + \tfrac{1}{3}(v_{\rm ps}+2v_v)\Omega_{\rm ss}. \tag{B8}$$

Using these expressions, we can isolate the pseudoscalar and vector contribution to the interaction using the following linear combinations of the tensor and spin-spin interactions:

$$v_{\rm ps} = v_{\rm ss} - 2v_t, \tag{B9}$$

$$v_v = v_{\rm ss} + v_t. \tag{B10}$$

For the Argonne V18 interaction, which is used in our calculations, there remains a small difference between the pseudoscalar interaction calculated using Eq. (B9) and the contribution that comes directly from terms that can be directly identified with the one-pion-exchange contribution. We only retain the one-pion-exchange contribution to these terms; this still includes the short-distance cutoff used in the Argonne V18 r-space potential.

The interactions $v^{\sigma\tau}(r)$ and $v^{t\tau}(r)$ have the form

$$v^{\sigma\tau}(r) \equiv \frac{f^2}{9}\left\{\left(\frac{m_0}{m_\pm}\right)^2 m_0 Y(\mu_0,r) + 2m_\pm Y(\mu_\pm,r)\right\} + v^c(r), \tag{B11}$$

$$v^{t\tau}(r) \equiv \frac{f^2}{9}\left\{\left(\frac{m_0}{m_\pm}\right)^2 m_0 T(\mu_0,r) + 2m_\pm T(\mu_\pm,r)\right\} + v^t(r). \tag{B12}$$

Here $Y(\mu,r)$ and $T(\mu,r)$ are the Yukawa and tensor functions with the exponential cutoff of the Urbana and Argonne V_{14} models:

$$Y(\mu,r) = \frac{e^{-\mu r}}{\mu r}(1-e^{-cr^2}), \tag{B13}$$

$$T(\mu,r) = \left(1+\frac{3}{\mu r}+\frac{3}{(\mu r)^2}\right)Y(\mu r)(1-e^{-cr^2}), \tag{B14}$$

The terms $v^c(r)$ and $v^t(r)$ are short-range phenomenological interactions that we discard for the computation of the exchange current.

To carry out the angular part of the integral in Eq. (A29), we expand the pseudoscalar potential $v_{\rm ps}(\mathbf{k}-\mathbf{k}')$ in partial waves:

$$v_{\rm ps}(|\mathbf{k}-\mathbf{k}'|) = \sum_{lm} v_l(\mathbf{k}^2,\mathbf{k}'^2)Y^*_{lm}(\hat{\mathbf{k}})Y_{lm}(\hat{\mathbf{k}}'), \tag{B15}$$

where

$$v_l(\mathbf{k}^2,\mathbf{k}'^2) = 2\pi\int_{-1}^1 P_l(u)v_{\rm ps}(\sqrt{\mathbf{k}^2+\mathbf{k}'^2-2kk''u})\,du. \tag{B16}$$

We expand the vector $\mathbf{k}$ and $\mathbf{k}'$ in spherical harmonics

$$\mathbf{k} = k\left(\sqrt{\frac{2\pi}{3}}(Y_{1-1}(\hat{\mathbf{k}}) - Y_{11}(\hat{\mathbf{k}})), i\sqrt{\frac{2\pi}{3}}(Y_{1-1}(\hat{\mathbf{k}}) + Y_{11}(\hat{\mathbf{k}})), \sqrt{\frac{4\pi}{3}}Y_{10}(\hat{\mathbf{k}})\right) \tag{B17}$$

to obtain the following expression for the pseudo wave function:

$$\begin{aligned}\chi^{(c)}_{\tilde{P}_d,\mu_d}(\vec{k}'', \mu''_p, \mu'_n) &\equiv \frac{1}{(2\pi)^3}\int d\hat{\mathbf{k}}k^2dk\sum_{l=0,2}\sum_{\mu_l=-l}^{l}\sum_{l'=0}^{\infty}\sum_{\mu'_l=-l'}^{l'} v_{l'}(\mathbf{k}^2, \mathbf{k}''^2)\\ &\times Y_{l'\mu'_l}(\hat{\mathbf{k}}'')Y^*_{l'\mu'_l}(\hat{\mathbf{k}})Y_{l\mu_l}(\hat{\mathbf{k}})\left[k''\left(\sqrt{\frac{2\pi}{3}}(Y_{1-1}(\hat{\mathbf{k}}'') - Y_{11}(\hat{\mathbf{k}}'')), i\sqrt{\frac{2\pi}{3}}(Y_{1-1}(\hat{\mathbf{k}}'') + Y_{11}(\hat{\mathbf{k}})), \sqrt{\frac{4\pi}{3}}Y_{10}(\hat{\mathbf{k}})\right)\right.\\ &\left. -\ k\left(\sqrt{\frac{2\pi}{3}}(Y_{1-1}(\hat{\mathbf{k}}) - Y_{11}(\hat{\mathbf{k}})), i\sqrt{\frac{2\pi}{3}}(Y_{1-1}(\hat{\mathbf{k}}) + Y_{11}(\hat{\mathbf{k}})), \sqrt{\frac{4\pi}{3}}Y_{10}(\hat{\mathbf{k}})\right)\right]\\ &\times \langle\mu'_n|\vec{\sigma}|\mu_n\rangle\langle s_p, \mu''_p, s_n, \mu_n|s, \mu_s\rangle\\ &\times \langle s, \mu_s, l, \mu_l|j, \mu\rangle\ .\ Y^l_{\mu_l}(\hat{\mathbf{k}})u_l(k),\end{aligned} \tag{B18}$$

The angular integrals are evaluated using

$$\int d\hat{\mathbf{k}}Y^{*l'}_{\mu'_l}(\hat{\mathbf{k}})Y^l_{\mu_l}(\hat{\mathbf{k}}) = \delta_{l'l}\delta_{\mu'_l\mu_l}, \tag{B19}$$

$$\int d\hat{\mathbf{k}}Y^{*l'}_{\mu'_l}(\hat{\mathbf{k}})Y^l_{\mu_l}(\hat{\mathbf{k}})Y^1_{\mu''_l}(\hat{\mathbf{k}}) = \sqrt{\frac{(2\cdot 1+1)(2\cdot l+1)}{4\pi(2\cdot l'+1)}}\langle 1, \mu''_l, l, \mu_l|l', \mu'_l\rangle\langle 1, 0, l, 0|l', 0\rangle. \tag{B20}$$

After carrying out the integral on $\hat{\mathbf{k}}$: the expression for the pseudo wave function becomes

$$\begin{aligned}\chi^{(c)}_{\tilde{P}_d,\mu_d}(\vec{k}'', \mu''_p, \mu'_n) &\equiv \frac{1}{(2\pi)^3}\int k^2dk\sum_{l=0,2}\sum_{\mu_l=-l}^{l}\sum_{l'}\sum_{\mu'_l=-l'}^{l'} v_{l'}(\mathbf{k}^2, \mathbf{k}''^2,)\\ &\times Y_{l'\mu_{l'}}(\hat{\mathbf{k}}'')u_l(k)\langle s_p, \mu''_p, s_n, \mu_n|s, \mu_s\rangle\langle s, \mu_s, l, \mu_l|j, \mu\rangle\\ &\times\left\{\sqrt{\frac{2\pi}{3}}\langle\mu'_n|\sigma_x|\mu_n\rangle\left[k''(Y_{1-1}(\hat{\mathbf{k}}'') - Y_{11}(\hat{\mathbf{k}}''))\delta_{l'l}\delta_{\mu_{l'}\mu_l}\right.\right.\\ &+ k\sqrt{\frac{(2\cdot 1+1)(2\cdot l+1)}{4\pi(2\cdot l'+1)}}\langle 1, 0, l, 0|l', 0\rangle\\ &\left.\times(-\langle 1, -1, l, \mu_l|l', \mu_{l'}\rangle + \langle 1, 1, l, \mu_l|l', \mu_{l'}\rangle)\right]\\ &+ i\sqrt{\frac{2\pi}{3}}\langle\mu'_n|\sigma_y|\mu_n\rangle\left[k''(Y_{1-1}(\hat{\mathbf{k}}'') + Y_{11}(\hat{\mathbf{k}}''))\delta_{l'l}\delta_{\mu_{l'}\mu_l}\right.\\ &+ k\sqrt{\frac{(2\cdot 1+1)(2\cdot l+1)}{4\pi(2\cdot l'+1)}}\langle 1, 0, l, 0|l', 0\rangle\\ &\left.\times(-\langle 1, -1, l, \mu_l|l', \mu_{l'}\rangle - \langle 1, 1, l, \mu_l|l', \mu_{l'}\rangle)\right]\\ &+ \sqrt{\frac{4\pi}{3}}\langle\mu'_n|\sigma_z|\mu_n\rangle\left[k''Y_{10}(\hat{\mathbf{k}}'')\delta_{l'l}\delta_{\mu_{l'}\mu_l}\right.\\ &- k\sqrt{\frac{(2\cdot 1+1)(2\cdot l+1)}{4\pi(2\cdot l'+1)}}\langle 1, 0, l, 0|l', 0\rangle\\ &\left.\left.\times\langle 1, 0, l, \mu_l|l', \mu_{l'}\rangle\right]\right\}.\end{aligned} \tag{B21}$$

The sum over l' includes only a finite number of values. It can only be 0 or 1 for $l = 0$ and 1, 2, or 3 when $l = 2$. This limits the sum on l' to the first four partial waves, $l' = 0, 1, 2, 3$. The final expression of the pseudo wave function can be written in the form

$$\chi^{(c)}_{\tilde{P}_d,\mu_d}(\vec{k}'', \mu''_p, \mu'_n) = \sum_{ll'} I_{ll'}(k'')f_{ll'}(\hat{k}'', \mu''_p, \mu'_n), \tag{B22}$$

where $f_{ll'}(\hat{k}'', \mu''_p, \mu'_n)$ are angle-dependent coefficients, and $I_{ll'}(k'')$ are the relevant scalar quantities having the form

$$I_{ll'}(k'') \equiv \int k^2dk\ v_l(k, k'')u_{l'}(k) \tag{B23}$$

or

$$I_{ll'}(k'')\int k'^3\ dk\ v_l(k, k'')u_{l'}(k). \tag{B24}$$

The allowed combinations of l and l' pairs requires the following integrals $I_{ll'}(k'')$:

$$I_{00}(k'') \equiv \int v_0(k, k'')u_0(k)k^2dk, \tag{B25}$$

$$I_{10}(k'') \equiv \int v_1(k, k'')u_0(k)k^3dk, \tag{B26}$$

$$I_{12}(k'') \equiv \int v_1(k, k'')u_2(k)k^3\ dk, \tag{B27}$$

$$I_{22}(k'') \equiv \int v_2(k, k'')u_2(k)k^2\ dk, \tag{B28}$$

$$I_{32}(k'') \equiv \int v_3(k, k'')u_2(k)k^3dk. \tag{B29}$$

The total exchange current contribution is

$$\langle \tilde{\mathbf{P}}'_d, \mu'_d, d|I^+_{\rm ex}(0)|\tilde{\mathbf{P}}_d, \mu_d, d\rangle = \sum \int d\mathbf{k}' J_i \big\{ \Psi^*_{\tilde{\mathbf{P}}'_d,\mu'_d}(\mathbf{k}', \mu'_p, \mu_n)\big[\bar{u}_f(\tilde{\mathbf{p}}'_p, \mu'_p)\Gamma^\mu_p(q) \times \gamma^5(-P\cdot\gamma/M_d)u_f(\tilde{\mathbf{p}}_p, \mu_p)\big]\chi^{(f)}_{\tilde{\mathbf{P}}_d,\mu_d}(\mathbf{k}, \mu_p, \mu_n) + \chi^{(f)*}_{\tilde{\mathbf{P}}'_d,\mu'_d}(\mathbf{k}, \mu'_p, \mu_n)\big[\bar{u}_f(\tilde{\mathbf{p}}'_p, \mu'_p)(-P\cdot\gamma/M_d)\gamma^5 \times \Gamma^\mu_p(q)u_f(\tilde{p}_p, \mu_p)\big]\Psi_{\tilde{\mathbf{P}}_d,\mu_d}(\mathbf{k}, \mu_p, \mu_n)\big\} + [p \leftrightarrow n], \tag{B30}$$

where J_i is the Jacobian

$$J_i = \left|\frac{\partial(\tilde{\mathbf{p}}''_p, \tilde{\mathbf{p}}''_n)}{\partial(\tilde{\mathbf{P}}'', \mathbf{k}'')}\right|^{\frac{1}{2}} \left|\frac{\partial(\tilde{\mathbf{P}}', \mathbf{k}')}{\partial(\tilde{\mathbf{p}}'_p, \tilde{\mathbf{p}}'_n)}\right|^{\frac{1}{2}}. \tag{B31}$$

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