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Three-particle model of the pion-nucleon system

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A relativistic, three-particle model of the pion-nucleon system is constructed in the instant form of relativistic quantum mechanics using the Bakamjian-Thomas procedure. The model space includes subspaces for the nucleon (N), the resonances $\Delta = P_{33}(1232)$, $P_{11}(1440)$, $D_{13}(1520)$, $S_{11}(1535)$, and $S_{31}(1620)$, as well as πN, $\pi\Delta$, ηN, and $\pi\pi N$ subspaces. The model specifies a Poincaré invariant mass operator that includes vertex interactions that couple the various subspaces, as well as renormalization terms. Projection operator techniques are used to reduce the equations arising from this mass operator to a set of three-dimensional Lippmann-Schwinger integral equations that couple only the πN, $\pi\Delta$, and ηN channels. After a partial wave analysis these three-dimensional equations simplify to three coupled, one-dimensional integral equations for each partial wave. The mass operator interactions are derived from effective, hadronic Lagrangians that introduce a set of coupling constants. Cutoff functions are introduced to take into account the nonelementary nature of the particles in the model. These cutoff functions introduce a set of cutoff masses. The coupling constants, cutoff masses, and bare baryon masses are determined by fitting to a partial wave analysis of pion-nucleon elastic scattering up to a c.m. energy of $W = 1550$ MeV.

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I. INTRODUCTION

For some time now people have been constructing and analyzing relativistic models of few-particle systems based on the direct construction of the generators of the proper Poincaré group. The original motivation for this approach can be traced back to an important paper by Dirac [1] in which he pointed out that there are various possibilities for incorporating interactions in the Poincaré generators. Dirac called these possibilities the instant form, the point form, and the front form. Each form is associated with a three-dimensional hypersurface in Minkowski space that is invariant under a subgroup of the Poincaré transformations, $x' = ax + b$, and intersects every world line just once. For the instant, point, and front forms the hypersurfaces can be taken to be $t =$ const., $c^2t^2 - \mathbf{x}^2 = a^2 > 0$ with $t > 0$, and $ct + z = 0$, respectively. In Dirac's approach the generators associated with these hypersurfaces are taken to be noninteracting, and interactions are put into the remaining generators. In the instant form, for example, the three-momentum $\mathbf{P}$ and the angular momentum $\mathbf{J}$ are noninteracting, whereas the Hamiltonian H and the generator of rotationless boosts $\mathbf{K}$ contain interactions.

A practical scheme for constructing the instant form generators $\{H, \mathbf{P}, \mathbf{J}, \mathbf{K}\}$ was developed some time ago by Bakamjian and Thomas [2]. In their approach the ten generators are expressed in terms of another set of ten operators, $\{M, \mathbf{P}, \mathbf{S}, \mathbf{X}\}$, where M is the mass operator, $\mathbf{S}$ is a spin operator, and $\mathbf{X}$ is the so-called Newton-Wigner position operator [3]. This second set of operators satisfies much simpler commutation rules than the generators, which facilitates the construction of models. In the Bakamjian-Thomas scheme $\mathbf{P}$, $\mathbf{S}$, and $\mathbf{X}$ are taken to be noninteracting, and an interaction is put only into the mass operator, M. It follows from the relations between the sets $\{H, \mathbf{P}, \mathbf{J}, \mathbf{K}\}$ and $\{M, \mathbf{P}, \mathbf{S}, \mathbf{X}\}$ that this leads to an instant form of relativistic quantum mechanics. Bakamjian-Thomas schemes have also been developed for the point and front forms [4].

It is relatively straightforward to carry out a Bakamjian-Thomas construction for a system in which there are coupled one- and two-particle channels. In particular, models of the pion-nucleon system have been constructed in which there are single-baryon and meson-baryon channels [5–7]. Such models contain vertex interactions that couple single-baryon states to meson-baryon states, as well as interactions that couple meson-baryon states directly to each other. The vertex interactions lead to renormalization effects.

With systems of three or more particles another ingredient enters into consideration, that is, cluster separability. This is the requirement that when parts of the system are separated by large spacelike separations the subsystems should be dynamically independent [4,8–10]. Coester [11] showed that for three-particle systems cluster separability of the S matrix can be achieved for a Bakamjian-Thomas construction; however, it was subsequently shown that the Hamiltonian does not become additive for separated subsystems [12]. Fortunately, for any number of particles cluster separability for the Poincaré generators can be achieved by a unitary transformation of the generators obtained from a Bakamjian-Thomas construction [4,8–10,13]. This unitary operator is often referred to as a packing operator. Bakker *et al.* [14] have shown how the front form can be used to construct three-particle models that satisfy cluster separability when the particle number is fixed.

A relativistic model of the NN-πNN system has been developed by Betz and Coester [15] within the Bakamjian-Thomas framework. Their model describes NN scattering, πN scattering, πd scattering, and pion production and absorption. The elementary degrees of freedom are the nucleon, the Δ isobar, and the pion. The practicality of the model has been demonstrated by calculations carried out by Betz and Lee [16].

Relativistic effects in three-body bound states have been investigated within the framework of a simplified model of the three-nucleon system, constructed using the Bakamjian-Thomas scheme [17]. The relativistic effects were found to be rather small (i.e., approximately a 3% reduction in the three-body binding energy). Recently, the Bakamjian-Thomas approach has been used to construct and analyze realistic, relativistic models of neutron-deuteron scattering [18].

The three-body front form equations of Bakker *et al.* [14] have been used to study proton-deuteron scattering at a proton laboratory energy of 800 MeV [19]. The relativistic effects were found to be most noticeable for the spin observables.

The Bakamjian-Thomas construction has also been used to formulate relativistic quark models of the baryons. In particular Szczepaniak *et al.* [20] have used this construction to derive a Poincaré invariant formulation of the Isgur-Karl quark model for the baryons [21,22]. Coester *et al.* [23] have presented a simply solvable quark model for the baryons within the Bakamjian-Thomas framework. Their model reproduces the empirical spectra of the baryons in all flavor sectors to an accuracy of a few percent. Their model also involves the construction of current density operators that are consistent with empirical nucleon form factors at low and medium momenta.

Pichowsky *et al.* [24] have used the Bakamjian-Thomas construction to develop a model that describes $\pi\pi$ scattering from threshold up to 1400 MeV. Their model properly includes unitarity cuts for one-, two, and three-hadron states.

Most relevant to the present work is Klink's [25] Bakamjian-Thomas construction of point form mass operators from vertex interactions. He considered a simplified model in which a scalar "nucleon" interacts with a scalar "pion." A truncated Hilbert space consisting of the direct sum of N and πN states leads to an eigenvalue problem for the physical nucleon mass as well as a Lippmann-Schwinger equation for πN scattering. Another truncation consisting of NN and πNN states leads to an eigenvalue problem for the "deuteron," along with a model for NN scattering with pion production. A more realistic application of this approach has been given by Krassnigg *et al.* [26], who have considered vector mesons within the chiral constituent quark model in which the hyperfine interaction between the confined quark-antiquark pair is generated by Goldstone boson exchange.

Recently, the Bakamjian-Thomas approach has been used to develop a method for constructing models of the πN system, which include $\pi\pi N$ states [27,28]. A limited model space was used to illustrate the method. This space consisted of an N subspace, a πN subspace, a $\pi\pi N$ subspace, and a $\pi\sigma N$ subspace. Here we will extend the space so as to construct a realistic model of the πN system. This extended space includes subspaces for the $N = N(938)\,P_{11}$, the resonances $\Delta = \Delta(1232)\,P_{33}$, $R = N(1440)\,P_{11}$, $D = N(1520)\,D_{13}$, $S = N(1535)\,S_{11}$, and $S' = \Delta(1620)\,S_{31}$, as well as πN, $\pi\Delta$, ηN, and $\pi\pi N$ subspaces. A mass operator is developed that includes coupling within and between these various subspaces. Vertex interactions couple the N and Δ subspaces to the πN subspace, couple the R, D, and S' to the πN and $\pi\Delta$ subspaces, and couple the S to the πN and ηN subspaces. Renormalization terms are included along with these $1 \Leftrightarrow 2$ vertex interactions to take into account the renormalizations of the baryon masses. Vertex interactions are also included and couple the πN and $\pi\Delta$ subspaces to the $\pi\pi N$ subspace. These interactions describe the processes $N \Leftrightarrow \pi N$ and $\Delta \Leftrightarrow \pi N$ in the presence of a spectator pion. Renormalization terms are included along with these $2 \Leftrightarrow 3$ vertex interactions. Traditional potentials that couple the πN subspace to itself are also included. These potentials are based on $\overline{N}$, Δ, σ, and ρ exchange processes.

Projection operator techniques are used to develop a set of integral equations that only involve couplings among the states of the $\pi N, \pi\Delta$, and ηN subspaces. Elimination of the single-baryon subspaces and the $\pi\pi N$ subspace leads to effective, energy-dependent $2 \Leftrightarrow 2$ potentials that produce couplings among the πN, $\pi\Delta$, and ηN states. The elimination of the single-baryon states leads to the so-called direct or s-channel potentials that couple meson (μ)-baryon (β) to meson (μ')-baryon (β') states through an intermediate baryon (β'') state according to $\mu\beta \Leftrightarrow \beta'' \Leftrightarrow \mu'\beta'$. The intermediate baryons have bare masses that are renormalized by the interactions. The elimination of the $\pi\pi N$ states leads to the crossed or u-channel potentials that couple pion-baryon to pion-baryon states through intermediate $\pi\pi N$ states according to $\pi\beta \Leftrightarrow \pi\pi N \Leftrightarrow \pi\beta'$, where β and β' are an N or a Δ. The final equations whose solutions are the various πN scattering and reaction amplitudes are a set of three-dimensional, coupled Lippmann-Schwinger equations, which after a partial wave analysis involve a single continuous variable. These Lippmann-Schwinger integral equations have singular kernels that have to be handled with care. We employ a contour deformation technique to deal with these singularities.

The various interactions that we develop are derived from effective hadronic Lagrangians that introduce a collection of coupling constants. To take into account the fact that we are not dealing with elementary particles we introduce phenomenological cutoff functions or form factors that contain cutoff masses. We fit the coupling constants, cutoff masses, and the baryons' bare masses to a partial wave analysis of the πN elastic scattering data. The phenomenological nature of the interactions we employ makes it clear that the calculations carried out here are not fundamental in character. The claim here is that the success of the calculations helps to establish at the hadronic level the mechanisms that account for the structure of the pion-nucleon scattering amplitudes. The hope is that advances in understanding and analyzing quantum chromodynamics will lead to fundamental explanations for these mechanisms, which are essentially various exchange processes between composite particles.

The outline of the paper is as follows. Section II gives a brief summary of the Bakamjian-Thomas approach. Some kinematics and a description of the basis states that span the model space are given in Sec. III. Section IV presents the mass operator. The reduction of the system of equations that the mass operator leads to is carried out in Sec. V. The πN and $\pi\Delta$ propagators are derived in Sec. VI. Section VII presents the $\mu\beta \Leftrightarrow \beta'$ vertex functions. The details of the crossed potentials are worked out in Sec. VIII. The $\overline{N}$, Δ, σ, and ρ exchange potentials are presented in Sec. IX. Isospin symmetry and rotational invariance are used in Sec. X to carry out a partial wave analysis of the three-dimensional equations derived in Sec. V, thereby leading to a system of coupled equations in one continuous variable. The method for constructing the partial wave matrix elements of the various $\mu\beta \Leftrightarrow \mu'\beta'$ potentials is given in Sec. XI. The method used for solving the final coupled, one-dimensional integral equations is described in Sec. XII. The results of solving these equations and fitting the parameters to the data are given in Sec. XIII. A discussion of the results and suggestions for future work is presented in Sec. XIV.

Throughout we work in units in which $\hbar = c = 1$.

II. GENERAL BACKGROUND

In a satisfactory relativistic quantum mechanics there exists a unitary operator $U(a, b)$, corresponding to the Poincaré transformation $x' = ax + b$, that maps a quantum mechanical state vector $|\psi\rangle$ associated with the x frame to the vector $|\psi'\rangle$ associated with the x' frame according to

$$|\psi'\rangle = U(a, b)|\psi\rangle. \tag{2.1}$$

For proper transformations the unitary operator can be parametrized in the form

$$U(a, b) = \exp(ib \cdot P)\exp[i(\boldsymbol{\theta} \cdot \mathbf{J} + \boldsymbol{\omega} \cdot \mathbf{K})], \tag{2.2}$$

$$P = (H, \mathbf{P}). \tag{2.3}$$

Here $\mathbf{J}$ is the angular momentum operator, $\mathbf{K}$ is the boost operator, H is the Hamiltonian of the system, and $\mathbf{P}$ is the three-momentum operator. Since the law of combination for the Poincaré transformations is $(a', b') \circ (a, b) = (a'a, a'b + b')$, the unitary operators must combine according to the relation

$$U(a', b')U(a, b) = U(a'a, a'b + b') \tag{2.4}$$

so as to provide a representation of the Poincaré group. This implies a set of commutation rules for the generators $\{H, \mathbf{P}, \mathbf{K}, \mathbf{J}\}$, which is commonly referred to as the Poincaré algebra.

In constructing the ten generators $\{H, \mathbf{P}, \mathbf{K}, \mathbf{J}\}$ it is convenient to work with another set of ten Hermitian operators, that is, $\{M, \mathbf{P}, \mathbf{S}, \mathbf{X}\}$, where M is the mass operator, $\mathbf{S}$ is a spin operator, and $\mathbf{X}$ is the so-called Newton-Wigner position operator [3]. This second set of operators satisfies a much simpler set of commutation rules than the Poincaré algebra; in fact the only nonzero commutators of this set are [2,4]

$$[P^m, X_n] = -i\delta_{mn}, \quad [S_l, S_m] = i\varepsilon_{lmn}S_n. \tag{2.5}$$

The three-momentum operator $\mathbf{P}$ is common to both sets, whereas the other generators can be expressed in terms of the operators of the second set by the relations [2,4]

$$H = (\mathbf{P}^2 + M^2)^{1/2}, \tag{2.6a}$$

$$\mathbf{J} = \mathbf{X} \times \mathbf{P} + \mathbf{S}, \tag{2.6b}$$

$$\mathbf{K} = -\frac{1}{2}(\mathbf{X}H + H\mathbf{X}) - \frac{\mathbf{P} \times \mathbf{S}}{M + H}. \tag{2.6c}$$

It can be shown that if the commutators of the set $\{M, \mathbf{P}, \mathbf{S}, \mathbf{X}\}$ are zero, except for those given by Eqs. (2.5), then the generators given by Eqs. (2.6), in combination with $\mathbf{P}$, satisfy the Poincaré algebra.

In the Bakamjian-Thomas construction [2,4] of the set $\{M, \mathbf{P}, \mathbf{S}, \mathbf{X}\}$ the operators $\mathbf{P}$, $\mathbf{S}$, and $\mathbf{X}$ are chosen to be the same as those for the system of particles without interactions, whereas the mass operator M contains interactions. The commutation rules for $\mathbf{P}$, $\mathbf{S}$, and $\mathbf{X}$ are then automatically satisfied, and it is only necessary to ensure that

$$[M, \mathbf{P}] = [M, \mathbf{S}] = [M, \mathbf{X}] = 0. \tag{2.7}$$

With this procedure the generators $\mathbf{P}$ and $\mathbf{J}$ are noninteracting, whereas H and $\mathbf{K}$ contain interactions. This defines an *instant form* of relativistic quantum mechanics, since the Poincaré transformations constructed from the noninteracting generators map a Minkowski space, t = constant hypersurface into itself.

It should be noted that the Lippmann-Schwinger equations that are solved to obtain S-matrix elements from a Bakamjian-Thomas mass operator are three dimensional in character, and hence not manifestly covariant; therefore it is not obvious that the S-matrix elements transform properly in passing from one inertial frame to another. Fortunately, a number of authors [4,9,11,29–31] have shown that the S-matrix elements do transform properly.

III. THE MODEL SPACE

The model constructed here describes the nucleon, $N = N(938)P_{11}$, the resonances $\Delta = \Delta(1232)P_{33}$, $R = N(1440)P_{11}$, $D = N(1520)D_{13}$, $S = N(1535)S_{11}$, and $S' = \Delta(1620)S_{31}$, as well as two pions, π_1 and π_2, and the η meson. The possible types of states are given by $|\beta\rangle$, where β is any of the baryons, two-particle states $|\pi_1 N\rangle$, $|\pi_2 N\rangle$, $|\pi_1 \Delta\rangle$, $|\pi_2 \Delta\rangle$, and $|\eta N\rangle$, and the three-particle state $|\pi_1\pi_2 N\rangle$. A state of any type is orthogonal to any state of another type (e.g., $\langle N|N\pi_2\rangle = 0$).

The various energies that are encountered are given by

$$E_a(\mathbf{p}) = \left(\mathbf{p}^2 + m_a^2\right)^{1/2}, \tag{3.1a}$$

$$W_{ab}(\mathbf{q}) = E_a(\mathbf{q}) + E_b(\mathbf{q}), \tag{3.1b}$$

$$E_{ab}(\mathbf{p}, \mathbf{q}) = \left[\mathbf{p}^2 + W_{ab}^2(\mathbf{q})\right]^{1/2}, \tag{3.1c}$$

$$W_{abc}(\mathbf{k}, \mathbf{q}) = E_a(\mathbf{k}) + E_{bc}(\mathbf{k}, \mathbf{q}), \tag{3.1d}$$

$$E_{abc}(\mathbf{p}, \mathbf{k}, \mathbf{q}) = \left[\mathbf{p}^2 + W_{abc}^2(\mathbf{k}, \mathbf{q})\right]^{1/2}. \tag{3.1e}$$

In general W indicates a c.m. energy. For example, $W_{\pi N}(\mathbf{q})$ is the energy of a pion and a nucleon in a c.m. frame in which the pion has three-momentum $\mathbf{q}$ and the nucleon has three-momentum $-\mathbf{q}$, and $E_{\pi N}(\mathbf{p}, \mathbf{q})$ is their energy in a frame in which their total three-momentum is $\mathbf{p}$. The total four-momentum of a set of particles with total energy E and total three-momentum $\mathbf{p}$ is given by

$$p = (p^0, \mathbf{p}) = (E, \mathbf{p}), \tag{3.2}$$

where for convenience we let $\omega = E_\pi$, $\varepsilon = E_N$, $\varepsilon_\beta = E_\beta$ with $\beta \neq N$, and $\omega_\eta = E_\eta$.

States of total four-momentum p are obtained by boosting a c.m. or rest frame state using the unitary operator that corresponds to a so-called canonical boost $l_c(p)$ [27]. This particular boost is defined by

$$x = l_c(p)x_{\rm cm}, \tag{3.3a}$$

$$x^0 = \frac{p^0 x^0_{\rm cm} + \mathbf{p}\cdot\mathbf{x}_{\rm cm}}{W}, \tag{3.3b}$$

$$\mathbf{x} = \mathbf{x}_{\rm cm} + \left(x^0_{\rm cm} + \frac{\mathbf{p}\cdot\mathbf{x}_{\rm cm}}{p^0+W}\right)\frac{\mathbf{p}}{W}, \tag{3.3c}$$

$$W = +(p\cdot p)^{1/2}. \tag{3.3d}$$

The inverse boost is obtained by interchanging x and $x_{\rm cm}$ and letting $\mathbf{p} \to -\mathbf{p}$.

The state of a baryon with three-momentum $\mathbf{p}$, isospin component i, and spin component m is denoted by $|\mathbf{p}im\rangle_\beta$. The state of pion π_a with isospin component u and a nucleon, N, or delta, Δ, with components i and m is given by $|\mathbf{p}(\mathbf{k}u)im\rangle_{a\beta}$, where $a = 1, 2$ and $\beta = N, \Delta$. Here $\mathbf{p}$ is the total three-momentum and $\mathbf{k}$ is the c.m. three-momentum of the pion. The parentheses around $\mathbf{k}u$ will play a role when the mass operator interactions are defined. The ηN states are given by $|\mathbf{pk}im\rangle_{\eta N}$.

The $\pi_a\pi_b N$ states are denoted by $|\mathbf{p}(\mathbf{k}u)\boldsymbol{\rho}tim\rangle_a$ with $a = 1, 2; b = 1, 2; a \neq b$. Here $\mathbf{k}$ is the three-momentum of π_a in the $\pi_a\pi_b N$ c.m. frame, and $\boldsymbol{\rho}$ is the three-momentum of π_b in the $\pi_b N$ c.m. frame obtained by an inverse canonical boost from the $\pi_a\pi_b N$ c.m. frame. It should be noted that the $a = 1$ states are related to the $a = 2$ states since both are $\pi\pi N$ states [27].

All of the states have delta function, Kronecker delta normalizations—for example,

$$\begin{aligned} &{}_{a\beta}\langle \mathbf{p}(\mathbf{k}u)im|\mathbf{p}'(\mathbf{k}'u')i'm'\rangle_{a\beta} \\ &\quad = \delta^3(\mathbf{p}-\mathbf{p}')\delta^3(\mathbf{k}-\mathbf{k}')\delta_{uu'}\delta_{ii'}\delta_{mm'}. \end{aligned} \tag{3.4}$$

IV. THE MASS OPERATOR

The mass operator that acts in the space spanned by the states described in Sec. III is of the form

$$M = M_0 + V, \tag{4.1a}$$

$$V = \sum_\beta V_\beta + \sum_{\beta=N,\Delta} V_{\pi N\beta} + V_{\rm pot}, \tag{4.1b}$$

where M_0 is the noninteracting mass operator, and V contains the interactions. Here, and in what follows, unless indicated otherwise, $\sum_\beta$ indicates a sum over all the baryons (i.e., N, Δ, R, D, S, and S'). The noninteracting mass operator is defined by its action on our basis states, that is,

$$M_0|\mathbf{p}im\rangle_\beta = m_\beta|\mathbf{p}im\rangle_\beta, \tag{4.2a}$$

$$M_0|\mathbf{p}(\mathbf{k}u)im\rangle_{a\beta} = W_{\pi\beta}(\mathbf{k})|\mathbf{p}(\mathbf{k}u)im\rangle_{a\beta}, \quad \beta = N, \Delta, \tag{4.2b}$$

$$M_0|\mathbf{pk}im\rangle_{\eta N} = W_{\eta N}(\mathbf{k})|\mathbf{pk}im\rangle_{\eta N}, \tag{4.2c}$$

$$M_0|\mathbf{p}(\mathbf{k}u)\boldsymbol{\rho}tim\rangle_a = W_{\pi\pi N}(\mathbf{k},\boldsymbol{\rho})|\mathbf{p}(\mathbf{k}u)\boldsymbol{\rho}tim\rangle_a. \tag{4.2d}$$

The interaction V_β acts in the $\{|\beta\rangle, |\pi_1 N\rangle, |\pi_2 N\rangle\}$ subspace when $\beta = N$ or Δ, in the $\{|\beta\rangle, |\pi_1 N\rangle, |\pi_2 N\rangle, |\pi_1\Delta\rangle, |\pi_2\Delta\rangle\}$ subspace when $\beta = R, D$, or S', and in the $\{|S\rangle, |\pi_1 N\rangle, |\pi_2 N\rangle, |\eta N\rangle\}$ subspace when $\beta = S$. It is given by

$$\begin{aligned} V_\beta = &\sum_{im}\int |\mathbf{p}im\rangle_\beta\, d^3p\big[m^{(0)}_\beta - m_\beta\big]_\beta\langle\mathbf{p}im| \\ &+ \frac{1}{\sqrt{2}}\sum_{a=1}^{2}\sum_{\beta'}\Bigg\{\sum_{uim}\sum_{i'm'}\int |\mathbf{p}(\mathbf{k}u)im\rangle_{a\beta'} \\ &\times U_{\pi\beta',\beta}(\mathbf{k}uim, i'm')\, d^3p\, d^3k\, {}_\beta\langle\mathbf{p}i'm'| + (\dagger)\Bigg\}, \quad \beta \neq S, \end{aligned} \tag{4.3a}$$

$$\begin{aligned} V_S = &\sum_{im}\int |\mathbf{p}im\rangle_S\, d^3p\big[m^{(0)}_S - m_S\big]\, {}_S\langle\mathbf{p}im| \\ &+ \frac{1}{\sqrt{2}}\sum_{a=1}^{2}\Bigg\{\sum_{uim}\sum_{i'm'}\int |\mathbf{p}(\mathbf{k}u)im\rangle_{aN} \\ &\times U_{\pi N,S}(\mathbf{k}uim, i'm')\, d^3p\, d^3k\, {}_S\langle\mathbf{p}i'm'| + (\dagger)\Bigg\} \\ &+ \Bigg\{\sum_{im}\sum_{i'm'}\int |\mathbf{pk}im\rangle_{\eta N} \\ &\times U_{\eta N,S}(\mathbf{k}im, i'm')\, d^3p\, d^3k\, {}_S\langle\mathbf{p}i'm'| + (\dagger)\Bigg\}. \end{aligned} \tag{4.3b}$$

The first terms on the right-hand sides are mass renormalization terms with $m^{(0)}_\beta$ and m_β the bare β mass and physical β mass, respectively. The other terms describe vertex interactions, for example, $\pi_a + \beta' \Leftrightarrow \beta$, where the $U_{\pi\beta',\beta}(\mathbf{k}uim, i'm') = U^*_{\beta,\pi\beta'}(i'm', \mathbf{k}uim)$ are vertex functions that we will derive from effective Lagrangians.

The interaction $V_{\pi N\beta}$ describes the vertex interactions $\pi_b + N \Leftrightarrow \beta$ with π_a, where $b \neq a$, playing the role of a spectator, and it also includes a renormalization term. This interaction is defined by

$$\begin{aligned} V_{\pi N\beta} = &\sum_{a=1}^{2}\sum_{uim}\int |\mathbf{p}(\mathbf{k}u)im\rangle_{a\beta}\, d^3p\, d^3k\, V^\pi_\beta(k)\, {}_{a\beta}\langle\mathbf{p}(\mathbf{k}u)im| \\ &+ \sum_{a=1}^{2}\Bigg\{\sum_{utim}\sum_{i'm'}\int |\mathbf{p}(\mathbf{k}u)\boldsymbol{\rho}tim\rangle_a\, d^3p\, d^3k\, d^3\rho \\ &\times V_{\pi N,\beta}(\boldsymbol{\rho}tim, i'm'; \mathbf{k})\, {}_{a\beta}\langle\mathbf{p}(\mathbf{k}u)i'm'| + (\dagger)\Bigg\}. \end{aligned} \tag{4.4}$$

We note that the parentheses around $\mathbf{k}$ and u draw attention to the variables of the spectator pion π_a. Clearly this interaction is diagonal in these variables, which is consistent with the fact that they describe a spectator particle. The first term on the right-hand side has to do with the renormalization of the N and Δ in the presence of the spectator pion. We will see subsequently how the functions $V^\pi_\beta(k)$ are determined. The second term describes the vertex interactions.

TABLE I. Interactions.

V_β	$\beta = N, \Delta;\ \beta \Leftrightarrow \beta$ (renormalization), $\beta \Leftrightarrow \pi N$ $\beta = R, D, S';\ \beta \Leftrightarrow \beta$ (renormalization), $\beta \Leftrightarrow \pi N, \pi\Delta$ $\beta = S;\ S \Leftrightarrow S$ (renormalization), $S \Leftrightarrow \pi N, \eta N$
$V_{\pi N\beta}$	$\beta = N, \Delta;\ \pi\beta \Leftrightarrow \pi\beta$ (renormalization), $\pi\beta \Leftrightarrow \pi\pi N$
$V_{\rm pot}$	$\pi N \Leftrightarrow \pi N$

The interaction $V_{\rm pot}$ is a traditional potential that couples $\pi_a N$ to $\pi_a N$ and is given by

$$V_{\rm pot} = \sum_{a=1}^{2}\sum_{uim}\sum_{u'i'm'}\int |\mathbf{p}(\mathbf{k}u)im\rangle_{aN}\, d^3p\, d^3k d^3k' \times V_{\pi N,\pi N}(\mathbf{k}uim, \mathbf{k}'u'i'm')_{aN}\langle \mathbf{p}(\mathbf{k}'u')i'm'|. \quad (4.5)$$

This potential includes $\overline{N}$, Δ, σ, and ρ exchange interactions.

The various interactions are summarized in Table I.

The Poincaré invariance of the mass operator can be established using the procedures outlined in Sec. V of Ref. [27].

V. THREE-PARTICLE EQUATIONS

To derive integral equations for the various amplitudes we introduce the following projection operators for the subspaces of our model:

$$\begin{aligned} P_\beta &: \{|\mathbf{p}im\rangle_\beta\}; \beta = N, \Delta, R, D, S, S', \\ P_{a\beta} &: \{|\mathbf{p}(\mathbf{k}u)im\rangle_{a\beta}\}; a = 1, 2;\ \beta = N, \Delta, \\ P_{\eta N} &: \{|\mathbf{pk}im\rangle_{\eta N}\}, \\ P_{\pi\pi N} &: \{|\mathbf{p}(\mathbf{k}u)\boldsymbol{\rho}tim\rangle_a\}; a = 1 \text{ or } 2. \end{aligned} \quad (5.1)$$

These projection operators are mutually orthogonal, that is,

$$P_\mu P_\nu = P_\mu \delta_{\mu\nu}. \quad (5.2)$$

Since the states that define the projection operators are eigenstates of the noninteracting mass operator M_0 we have

$$M_0 P_\mu = P_\mu M_0. \quad (5.3)$$

We let $|\Psi\rangle$ be a state vector for the system that satisfies

$$(W - M_0)|\Psi\rangle = V|\Psi\rangle, \quad (5.4)$$

and we indicate projections onto the various subspaces by

$$|\psi_\mu\rangle = P_\mu|\Psi\rangle. \quad (5.5)$$

We can write, with the help of Eqs. (4.3)–(4.5),

$$\left[W - m_\beta^{(0)}\right]|\psi_\beta\rangle = \sum_{\mu\beta'} P_\beta V_\beta |\psi_{\mu\beta'}\rangle, \quad (5.6)$$

where for $\beta = N, \Delta; \mu\beta' = \pi_1 N, \pi_2 N$, for $\beta = R,\ D, S'$; $\mu\beta' = \pi_1 N,\ \pi_2 N,\ \pi_1\Delta,\ \pi_2\Delta$, and for $\beta = S;\ \mu\beta' = \pi_1 N$, $\pi_2 N$, ηN (see Table I). Using Eqs. (4.3)–(4.5) we obtain

$$[W - M_0 - P_{aN}V_{\pi NN}]|\psi_{aN}\rangle = \sum_\beta P_{aN}V_\beta|\psi_\beta\rangle + P_{aN}V_{\rm pot}|\psi_{aN}\rangle + P_{aN}V_{\pi NN}|\psi_{\pi\pi N}\rangle, \quad (5.7)$$

$$[W - M_0 - P_{a\Delta}V_{\pi N\Delta}]|\psi_{a\Delta}\rangle = \sum_{\beta=R,D,S'} P_{a\Delta}V_\beta|\psi_\beta\rangle + P_{a\Delta}V_{\pi N\Delta}|\psi_{\pi\pi N}\rangle, \quad (5.8)$$

$$[W - M_0]|\psi_{\eta N}\rangle = P_{\eta N}V_S|\psi_S\rangle, \quad (5.9)$$

$$[W - M_0]|\psi_{\pi\pi N}\rangle = \sum_{a=1}^{2}\sum_{\beta=N,\Delta} P_{\pi\pi N}V_{\pi N\beta}|\psi_{a\beta}\rangle. \quad (5.10)$$

If we put Eqs. (5.6) and (5.10) into Eq. (5.7), and rearrange, we find an equation that only involves the two-particle channels:

$$\begin{aligned} &\left[W - M_0 - P_{aN}V_{\pi NN} - P_{aN}V_{\pi NN}\frac{P_{\pi\pi N}}{z - M_0}V_{\pi NN}\right]|\psi_{aN}\rangle \\ &= \sum_\beta\sum_{\mu\beta'} P_{aN}V_\beta\frac{P_\beta}{z - m_\beta^{(0)}}V_\beta|\psi_{\mu\beta'}\rangle \\ &\quad + \sum_{b\neq a}\sum_{\beta=N,\Delta} P_{aN}V_{\pi NN}\frac{P_{\pi\pi N}}{z - M_0}V_{\pi N\beta}|\psi_{b\beta}\rangle \\ &\quad + P_{aN}V_{\rm pot}|\psi_{aN}\rangle + P_{aN}V_{\pi NN}\frac{P_{\pi\pi N}}{z - M_0}V_{\pi N\Delta}|\psi_{a\Delta}\rangle. \end{aligned} \quad (5.11)$$

Here, and in what follows, $z = W + i\varepsilon$, where ε is a positive infinitesimal parameter. We shall see that the last term on the right side of Eq. (5.11) vanishes.

If we put Eqs. (5.6) and (5.10) into Eq. (5.8), and rearrange, we find another equation that only involves the two-particle channels:

$$\begin{aligned} &\left[W - M_0 - P_{a\Delta}V_{\pi N\Delta} - P_{a\Delta}V_{\pi N\Delta}\frac{P_{\pi\pi N}}{z - M_0}V_{\pi N\Delta}\right]|\psi_{a\Delta}\rangle \\ &= \sum_{\beta=R,D,S'}\sum_{\mu\beta'} P_{a\Delta}V_\beta\frac{P_\beta}{z - m_\beta^{(0)}}V_\beta|\psi_{\mu\beta'}\rangle \\ &\quad + \sum_{b\neq a}\sum_{\beta=N,\Delta} P_{a\Delta}V_{\pi N\Delta}\frac{P_{\pi\pi N}}{z - M_0}V_{\pi N\beta}|\psi_{b\beta}\rangle \\ &\quad + P_{a\Delta}V_{\pi N\Delta}\frac{P_{\pi\pi N}}{z - M_0}V_{\pi NN}|\psi_{aN}\rangle. \end{aligned} \quad (5.12)$$

We shall see that the last term on the right side of Eq. (5.12) vanishes.

Finally, if we put Eq. (5.6) into Eq. (5.9) we complete a closed system of coupled two-particle equations with the result

$$[W - M_0]|\psi_{\eta N}\rangle = \sum_{a=1}^{2} P_{\eta N}V_S\frac{P_S}{z - m_S^{(0)}}V_S|\psi_{aN}\rangle + P_{\eta N}V_S\frac{P_S}{z - m_S^{(0)}}V_S|\psi_{\eta N}\rangle. \quad (5.13)$$

We project Eqs. (5.11)–(5.13) onto the basis states described in Sec. III, and define the wave function components

$$ {}_{\beta}\langle \mathbf{p}im|\Psi\rangle = \delta^3(\mathbf{p}-\mathbf{p}')\psi_\beta(im), \tag{5.14a}$$

$$ {}_{a\beta}\langle \mathbf{p}(\mathbf{k}u)im|\Psi\rangle = \delta^3(\mathbf{p}-\mathbf{p}')\psi_{\pi\beta}(\mathbf{k}uim)/\sqrt{2}, \tag{5.14b}$$

$$ {}_{\eta N}\langle \mathbf{pk}im|\Psi\rangle = \delta^3(\mathbf{p}-\mathbf{p}')\psi_{\eta N}(\mathbf{k}im), \tag{5.14c}$$

$$ {}_{a}\langle \mathbf{p}(\mathbf{k}u)\boldsymbol{\rho}tim|\Psi\rangle = \delta^3(\mathbf{p}-\mathbf{p}')\psi_{\pi\pi N}(\mathbf{k}u\boldsymbol{\rho}tim). \tag{5.14d}$$

Here $\mathbf{p}'$ is the total three-momentum of the state $|\Psi\rangle$.

We now construct the matrix elements of the various interaction terms that occur in Eqs. (5.11)–(5.13). With the help of Eqs. (4.4) and (4.2d), we can derive

$$\begin{aligned} &{}_{a\beta}\langle \mathbf{p}(\mathbf{k}u)im|V_{\pi N\beta}\frac{P_{\pi\pi N}}{z-M_0}V_{\pi N\beta'}|\mathbf{p}'(\mathbf{k}'u')i'm'\rangle_{a\beta'} \\ &= \delta^3(\mathbf{p}-\mathbf{p}')\delta^3(\mathbf{k}-\mathbf{k}')\delta_{uu'} \\ &\times \sum_{tjn}\int V_{\beta,\pi N}(im,\boldsymbol{\rho}tjn;\mathbf{k})\frac{d^3\rho}{z-W_{\pi\pi N}(k,\rho)} \\ &\times V_{\pi N,\beta'}(\boldsymbol{\rho}tjn,i'm';\mathbf{k}). \end{aligned}\tag{5.15}$$

Since the vertex functions $V_{\pi N,\beta}(\boldsymbol{\rho}tjn,im;\mathbf{k})$ must conserve isospin, we find $\sum_{tj}\langle 1,1/2,t,j|TM\rangle V_{\pi N,\beta}(\boldsymbol{\rho}tjn,im;\mathbf{k}) = \delta_{TT_\beta}\delta_{Mi}V_{\pi N,\beta}(\boldsymbol{\rho}n,m;\mathbf{k})$, which in turn implies $V_{\pi N,\beta}(\boldsymbol{\rho}tjn, im;\mathbf{k}) = \langle 1,\ 1/2,t,\ j|T_\beta i\rangle V_{\pi N,\beta}(\boldsymbol{\rho}n,\ m;\mathbf{k})$. It then follows from the orthogonality relation for the Clebsch-Gordon coefficients that the last terms on the right sides of Eqs. (5.11) and (5.12) vanish.

We shall now show that combining Eq. (5.15) with $\beta=\beta'$ with the other contributions from the left-hand sides of Eqs. (5.11) and (5.12) leads to propagators for the πN and $\pi\Delta$ systems, respectively. From Eq. (6.11) of Ref. [27] we have

$$\begin{aligned} &\sum_{tjn}\int V_{N,\pi N}(im,\boldsymbol{\rho}tjn;\mathbf{k})\frac{d^3\rho}{z-W_{\pi\pi N}(k,\rho)} \\ &\quad\times V_{\pi N,N}(\boldsymbol{\rho}tjn,i'm';\mathbf{k}) \\ &= \delta_{ii'}\delta_{mm'}\int d^3\rho\frac{F_{\pi NN}(\rho;k)}{z-W_{\pi\pi N}(k,\rho)}, \end{aligned}\tag{5.16a}$$

where

$$F_{\pi NN}(\rho;k) = \int\frac{d\Omega(\boldsymbol{\rho})}{4\pi}\sum_{tjn}|V_{\pi N,N}(\boldsymbol{\rho}tjn,im;\mathbf{k})|^2. \tag{5.16b}$$

Combining these results with the other terms on the left side of Eq. (5.11), and using Eq. (4.4), we can write

$$\begin{aligned} &{}_{aN}\langle \mathbf{p}(\mathbf{k}u)im|z-M_0-V_{\pi NN} \\ &\quad -V_{\pi NN}\frac{P_{\pi\pi N}}{z-M_0}V_{\pi NN}|\mathbf{p}'(\mathbf{k}'u')i'm'\rangle_{aN} \\ &= \delta^3(\mathbf{p}-\mathbf{p}')\delta^3(\mathbf{k}-\mathbf{k}')\delta_{uu'}\delta_{ii'}\delta_{mm'}Z_{\pi N}^{-1}(k)d_{\pi N}(k,z), \end{aligned}\tag{5.17}$$

where the πN propagator $d_{\pi N}^{-1}(k,z)$ is defined by

$$\begin{aligned} d_{\pi N}(k,z) = Z_{\pi N}(k)\bigg[&z-W_{\pi N}(k)-V_N^\pi(k) \\ &-\int d^3\rho\frac{F_{\pi NN}(\rho;k)}{z-W_{\pi\pi N}(k,\rho)}\bigg]. \end{aligned}\tag{5.18}$$

We have introduced the function $Z_{\pi N}(k)$ so that

$$d_{\pi N}(k,z)\underset{z\to W_{\pi N}(k)}{\longrightarrow} z-W_{\pi N}(k), \tag{5.19}$$

which leads to

$$V_N^\pi(k) = -\int d^3\rho\frac{F_{\pi NN}(\rho;k)}{W_{\pi N}(k)-W_{\pi\pi N}(k,\rho)}, \tag{5.20}$$

which in turn leads to

$$\begin{aligned} d_{\pi N}(k,z) = Z_{\pi N}(k)[z-W_{\pi N}(k)]\bigg\{&1+\int d^3\rho \\ &\times\frac{F_{\pi NN}(\rho;k)}{[z-W_{\pi\pi N}(k,\rho)][W_{\pi N}(k)-W_{\pi\pi N}(k,\rho)]}\bigg\}, \end{aligned}\tag{5.21}$$

along with [see Eqs. (3.1b) and (3.1d)]

$$Z_{\pi N}^{-1}(k) = 1+\int d^3\rho\frac{F_{\pi NN}(\rho;k)}{[\varepsilon(k)-E_{\pi N}(k,\rho)]^2}. \tag{5.22}$$

Using the simple identity $1/ab = 1/a^2+(a-b)/a^2b$ we can rewrite Eq. (5.21) as

$$\begin{aligned} d_{\pi N}(k,z) = [z-W_{\pi N}(k)]\bigg\{&1-[z-W_{\pi N}(k)] \\ &\times\int d^3\rho\frac{Z_{\pi N}(k)F_{\pi NN}(\rho;k)}{[z-W_{\pi\pi N}(k,\rho)][\varepsilon(k)-E_{\pi N}(k,\rho)]^2}\bigg\}. \end{aligned}\tag{5.23}$$

To construct the $\pi\Delta$ propagator we begin by defining [see Eq. (5.15)] J_Δ by

$$\begin{aligned} \delta_{ii'}J_\Delta(m,m';\mathbf{k},z) = &\sum_{tjn}\int V_{\Delta,\pi N}(im,\boldsymbol{\rho}tjn;\mathbf{k}) \\ &\times\frac{d^3\rho}{z-W_{\pi\pi N}(k,\rho)}V_{\pi N,\Delta}(\boldsymbol{\rho}tjn,i'm';\mathbf{k}) \end{aligned}\tag{5.24}$$

If we combine the terms from the left-hand side of Eq. (5.12) and use Eqs. (4.4) and (5.15), we can write

$$\begin{aligned} &{}_{a\Delta}\langle \mathbf{p}(\mathbf{k}u)im|z-M_0-V_{\pi N\Delta} \\ &\quad -V_{\pi N\Delta}\frac{P_{\pi\pi N}}{z-M_0}V_{\pi N\Delta}|\mathbf{p}'(\mathbf{k}'u')i'm'\rangle_{a\Delta} \\ &= \delta^3(\mathbf{p}-\mathbf{p}')\delta^3(\mathbf{k}-\mathbf{k}')\delta_{uu'}\delta_{ii'}d_{\pi\Delta}(m,m';\mathbf{k},z), \end{aligned}\tag{5.25}$$

where the $\pi\Delta$ propagator is defined by the inverse of the matrix function whose matrix elements are given by

$$\begin{aligned} d_{\pi\Delta}(m,m';\mathbf{k},z) = [&z-W_{\pi\Delta}(k)-V_\Delta^\pi(k)]\delta_{mm'} \\ &-J_\Delta(m,m';\mathbf{k};z). \end{aligned}\tag{5.26}$$

Unlike $d_{\pi N}(k,z)$ we cannot have $d_{\pi\Delta}(m,m';\mathbf{k},z)$ vanish when $z\to W_{\pi\Delta}(k)$. This is because according to Eq. (5.24) J_Δ has a right-hand cut for $z>2m_\pi+m_N$ and picks up an imaginary part in this range. Since $W_{\pi\Delta}(k)\geqslant m_\pi+m_\Delta > 2m_\pi+m_N$, $W_{\pi\Delta}(k)$ falls in this range.

We now consider the direct or pole terms in Eqs. (5.11)–(5.13). These are the terms with simple poles at $z=m_\beta^{(0)}$. As an example, we can show, with the help of Table I and

Eqs. (4.3a) and (5.14b), that the pole term in Eq. (5.12) is given by

$$
\begin{aligned}
&{}_{a\Delta}\langle \mathbf{p}(\mathbf{k}u)im| \sum_{\beta=R,D}\sum_{a'=1}^{2}\sum_{\beta'=N,\Delta} V_\beta \frac{P_\beta}{z-m_\beta^{(0)}} V_\beta|\psi_{a'\beta'}\rangle \\
&= \delta^3(\mathbf{p}-\mathbf{p}') \sum_{\beta'=N,\Delta}\sum_{u'i'm'}\int \sum_{\beta=R,D}\sum_{i''m''} \\
&\times \frac{U_{\pi\Delta,\beta}(\mathbf{k}uim, i''m'')U_{\beta,\pi\beta'}(i''m'', \mathbf{k}'u'i'm')}{z-m_\beta^{(0)}} \\
&\times d^3k'\psi_{\pi\beta'}(\mathbf{k}'u'i'm')/\sqrt{2}. \qquad (5.27)
\end{aligned}
$$

This result and similar results for the pole terms in Eqs. (5.11) and (5.13) lead us to define

$$
\begin{aligned}
&B^d_{\pi\beta,\pi\beta'}(\mathbf{k}uim, \mathbf{k}'u'i'm'; z) \\
&= Z^{1/2}_{\pi\beta}(k) \sum_{\beta''i''m''} \\
&\times \frac{U_{\pi\beta,\beta''}(\mathbf{k}uim, i''m'')U_{\beta'',\pi\beta'}(i''m'', \mathbf{k}'u'i'm')}{z-m^{(0)}_{\beta''}} Z^{1/2}_{\pi\beta'}(k'), \\
&\beta = N,\Delta;\ \beta' = N,\Delta, \qquad (5.28a)
\end{aligned}
$$

where the sum on β'' is determined by (see Table I)

$$
\begin{aligned}
&\beta'' = N,\Delta,R,D,S,S' \quad \text{if} \quad \beta=\beta'=N, \\
&\beta'' = R,D,S' \quad \text{if} \quad \beta,\beta' = N,\Delta \quad \text{or} \quad \Delta,N \quad \text{or} \quad \Delta,\Delta. \qquad (5.28b)
\end{aligned}
$$

In Eq. (5.28a), and subsequently, $Z_{\pi\Delta}(k) \equiv 1$. $Z^{1/2}_{\pi N}(k)$ has been introduced into Eq. (5.28a), and will be introduced into other interactions, so as to renormalize the vertex functions.

The direct interactions that involve the ηN channel are defined by

$$
\begin{aligned}
&B^d_{\pi N,\eta N}(\mathbf{k}uim, \mathbf{k}'i'm'; z) \\
&= B^{d*}_{\eta N,\pi N}(\mathbf{k}'i'm', \mathbf{k}uim; z^*) \\
&= Z^{1/2}_{\pi N}(k) \sum_{i''m''} \frac{U_{\pi N,S}(\mathbf{k}uim, i''m'')U_{S,\eta N}(i''m'', \mathbf{k}'i'm')}{z-m^{(0)}_S} \qquad (5.29)
\end{aligned}
$$

and

$$
\begin{aligned}
&B^d_{\eta N,\eta N}(\mathbf{k}im, \mathbf{k}'i'm'; z) \\
&= \sum_{i''m''} \frac{U_{\eta N,S}(\mathbf{k}im, i''m'')U_{S,\eta N}(i''m'', \mathbf{k}'i'm')}{z-m^{(0)}_S}. \qquad (5.30)
\end{aligned}
$$

We now consider the second terms on the right sides of Eqs. (5.11) and (5.13). These are the so-called crossed or nucleon exchange terms. In working these out we need the following results from Ref. [27]:

$$
\begin{aligned}
&{}_a\langle \mathbf{p}(\mathbf{k}u)\boldsymbol{\rho}tim|\mathbf{p}'(\mathbf{k}'u')\boldsymbol{\rho}'t'i'm'\rangle_b \\
&= \delta^3(\mathbf{p}-\mathbf{p}')\delta_{ut'}\delta_{tu'}\delta_{ii'}\delta^3[\boldsymbol{\rho}-\mathbf{f}_{\pi N}(\mathbf{k}', -\mathbf{k}-\mathbf{k}')] \\
&\times Q_{mm'}(\mathbf{k},\mathbf{k}')\delta^3[\boldsymbol{\rho}'-\mathbf{f}_{\pi N}(\mathbf{k}, -\mathbf{k}-\mathbf{k}')],\ a \neq b, \qquad (5.31a) \\
&Q_{mm'}(\mathbf{k},\mathbf{k}') = B_{\pi N}(\mathbf{k}', -\mathbf{k}-\mathbf{k}')B_{\pi N}(\mathbf{k}, -\mathbf{k}-\mathbf{k}') \\
&\times D^{(1/2)}_{mm'}[r_{\pi N}(\mathbf{k}', -\mathbf{k}-\mathbf{k}')r^{-1}_{\pi N}(\mathbf{k}, -\mathbf{k}-\mathbf{k}')]. \qquad (5.31b)
\end{aligned}
$$

Here $\mathbf{f}_{\pi N}(\mathbf{p}_\pi, \mathbf{p}_N)$ is the three-momentum of π in the πN c.m. frame, which according to the inverse of Eq. (3.3) is given by

$$
\begin{aligned}
\mathbf{f}_{\pi N}(\mathbf{p}_\pi, \mathbf{p}_N) &= \frac{1}{2}(\mathbf{p}_\pi - \mathbf{p}_N) - \frac{1}{2}\left[\omega(\mathbf{p}_\pi) - \varepsilon(\mathbf{p}_N) + \frac{m_\pi^2 - m_N^2}{\sqrt{p\cdot p}}\right]\frac{\mathbf{p}}{p^0+\sqrt{p\cdot p}}, \qquad (5.32a) \\
p &= p_\pi + p_N = (\omega(\mathbf{p}_\pi)+\varepsilon(\mathbf{p}_N), \mathbf{p}_\pi + \mathbf{p}_N). \qquad (5.32b)
\end{aligned}
$$

The function $B_{\pi N}(\mathbf{p}_\pi, \mathbf{p}_N)$ is defined by

$$
B_{\pi N}(\mathbf{p}_\pi, \mathbf{p}_N) = \left\{\frac{\omega(\mathbf{p}_\pi)+\varepsilon(\mathbf{p}_N)}{\omega(\mathbf{p}_\pi)\varepsilon(\mathbf{p}_N)} \times \frac{\omega[\mathbf{f}_{\pi N}(\mathbf{p}_\pi,\mathbf{p}_N)]\varepsilon[\mathbf{f}_{\pi N}(\mathbf{p}_\pi,\mathbf{p}_N)]}{W_{\pi N}[\mathbf{f}_{\pi N}(\mathbf{p}_\pi,\mathbf{p}_N)]}\right\}^{1/2}. \qquad (5.33)
$$

The argument of the SU(2) representative $D^{(1/2)}$ is determined by

$$
\begin{aligned}
&r_{\pi N}(\mathbf{p}_\pi, \mathbf{p}_N) = r_c\big[l_c^{-1}(p_\pi + p_N), p_N\big], \\
&p_\pi = (\omega(\mathbf{p}_\pi), \mathbf{p}_\pi), \quad p_N = (\varepsilon(\mathbf{p}_N), \mathbf{p}_N), \qquad (5.34)
\end{aligned}
$$

where r_c is a so-called Wigner rotation [4], which for canonical boosts and a general Lorentz transformation a is defined by

$$
r_c(a,p) = l_c^{-1}(ap)al_c(p). \qquad (5.35)
$$

Now with the help of Eqs. (4.4) and (4.2d) and the identity (see Eq. (4.52) of Ref. [27])

$$
W_{\pi\pi N}[\mathbf{k}, \mathbf{f}_{\pi N}(\mathbf{k}', -\mathbf{k}-\mathbf{k}')] = \omega(\mathbf{k}) + \varepsilon(-\mathbf{k}-\mathbf{k}') + \omega(\mathbf{k}'), \qquad (5.36)
$$

we find

$$
\begin{aligned}
&{}_{a\beta}\langle \mathbf{p}(\mathbf{k}u)im|V_{\pi N\beta}\frac{P_{\pi\pi N}}{z-M_0}V_{\pi N\beta'}|\mathbf{p}'(\mathbf{k}'u')i'm'\rangle_{b\beta'} \\
&= \delta^3(\mathbf{p}-\mathbf{p}')Z^{-1/2}_{\pi\beta}(k)B^c_{\pi\beta,\pi\beta'}(\mathbf{k}uim, \mathbf{k}'u'i'm'; z)Z^{-1/2}_{\pi\beta'}(k'), \qquad (5.37)
\end{aligned}
$$

where

$$
\begin{aligned}
&B^c_{\pi\beta,\pi\beta'}(\mathbf{k}uim, \mathbf{k}'u'i'm'; z) \\
&= \sum_{jnn'} V_{\beta,\pi N}(im, \boldsymbol{\rho}u'jn; \mathbf{k})Z^{1/2}_{\pi\beta}(k) \\
&\times \frac{Q_{nn'}(\mathbf{k},\mathbf{k}')}{z-\omega(\mathbf{k})-\varepsilon(-\mathbf{k}-\mathbf{k}')-\omega(\mathbf{k}')} Z^{1/2}_{\pi\beta'}(k') \\
&\times V_{\pi N,\beta'}(\boldsymbol{\rho}'ujn', i'm'; \mathbf{k}'), \quad \beta \text{ and } \beta' = N,\Delta, \qquad (5.38a) \\
&\boldsymbol{\rho} = \mathbf{f}_{\pi N}(\mathbf{k}', -\mathbf{k}-\mathbf{k}'), \quad \boldsymbol{\rho}' = \mathbf{f}_{\pi N}(\mathbf{k}, -\mathbf{k}-\mathbf{k}'), \qquad (5.38b)
\end{aligned}
$$

The V_{pot} term in Eq. (5.11) leads us to define [see Eq. (4.5)]

$$
\begin{aligned}
&B^{\text{pot}}_{\pi N,\pi N}(\mathbf{k}uim, \mathbf{k}'u'i'm') \\
&= Z^{1/2}_{\pi N}(k)V_{\pi N,\pi N}(\mathbf{k}uim, \mathbf{k}'u'i'm')Z^{1/2}_{\pi N}(k'). \qquad (5.39)
\end{aligned}
$$

In writing out the details of Eqs. (5.11) and (5.12) it is convenient to define the following combinations:

$$\begin{aligned} B_{\pi N,\pi N}(\mathbf{k}uim, \mathbf{k}'u'i'm';z) &= B^d_{\pi N,\pi N}(\mathbf{k}uim, \mathbf{k}'u'i'm';z) \\ &\quad + B^c_{\pi N,\pi N}(\mathbf{k}uim, \mathbf{k}'u'i'm';z) \\ &\quad + B^{\text{pot}}_{\pi N,\pi N}(\mathbf{k}uim, \mathbf{k}'u'i'm'), \end{aligned} \quad (5.40a)$$

$$\begin{aligned} B_{\pi N,\pi\Delta}(\mathbf{k}uim, \mathbf{k}'u'i'm';z) &= B^*_{\pi\Delta,\pi N}(\mathbf{k}'u'i'm', \mathbf{k}uim;z^*) \\ &= B^d_{\pi N,\pi\Delta}(\mathbf{k}uim, \mathbf{k}'u'i'm';z) \\ &\quad + B^c_{\pi N,\pi\Delta}(\mathbf{k}uim, \mathbf{k}'u'i'm';z), \end{aligned} \quad (5.40b)$$

$$\begin{aligned} B_{\pi\Delta,\pi\Delta}(\mathbf{k}uim, \mathbf{k}'u'i'm';z) &= B^d_{\pi\Delta,\pi\Delta}(\mathbf{k}uim, \mathbf{k}'u'i'm';z) \\ &\quad + B^c_{\pi\Delta,\pi\Delta}(\mathbf{k}uim, \mathbf{k}'u'i'm';z). \end{aligned} \quad (5.40c)$$

If we now contract Eqs. (5.11), (5.12), and (5.13) with ${}_{aN}\langle \mathbf{p}(\mathbf{k}u)im|$, ${}_{a\Delta}\langle \mathbf{p}(\mathbf{k}u)im|$, and ${}_{\eta N}\langle \mathbf{pk}im|$, respectively, and use Eqs. (5.14b), (5.14c), (5.15), (5.17), (5.24), (5.26), (5.27), (5.37), and (5.39), we find the equations

$$\begin{aligned} d_{\pi N}(k,z)Z^{-1/2}_{\pi N}(k)\psi_{\pi N}(\mathbf{k}uim) &= \sum_{\beta'=N,\Delta}\sum_{u'i'm'}\int B_{\pi N,\pi\beta'}(\mathbf{k}uim, \mathbf{k}'u'i'm';z) \\ &\quad \times d^3k' Z^{-1/2}_{\pi\beta'}(k')\psi_{\pi\beta'}(\mathbf{k}'u'i'm') \\ &\quad + \sum_{i'm'}\int B^d_{\pi N,\eta N}(\mathbf{k}uim, \mathbf{k}'i'm';z)d^3k'\psi_{\eta N}(\mathbf{k}'i'm'), \end{aligned} \quad (5.41)$$

$$\begin{aligned} \sum_{m'} d_{\pi\Delta}(m,m';\mathbf{k},z)Z^{-1/2}_{\pi\Delta}(k)\psi_{\pi\Delta}(\mathbf{k}uim') &= \sum_{\beta'=N,\Delta}\sum_{u'i'm'}\int B_{\pi\Delta,\pi\beta'}(\mathbf{k}uim, \mathbf{k}'u'i'm';z) \\ &\quad \times d^3k' Z^{-1/2}_{\pi\beta'}(k')\psi_{\pi\beta'}(\mathbf{k}'u'i'm'), \end{aligned} \quad (5.42)$$

$$\begin{aligned} [z - W_{\eta N}(k)]\psi_{\eta N}(\mathbf{k}im) &= \sum_{u'i'm'}\int B^d_{\eta N,\pi N}(\mathbf{k}im, \mathbf{k}'u'i'm';z) \\ &\quad \times d^3k' Z^{-1/2}_{\pi N}(k')\psi_{\pi N}(\mathbf{k}'u'i'm') \\ &\quad + \sum_{i'm'}\int B^d_{\eta N,\eta N}(\mathbf{k}im, \mathbf{k}'i'm';z)d^3k'\psi_{\eta N}(\mathbf{k}'i'm'). \end{aligned} \quad (5.43)$$

Assigning along with $\mathbf{p}'$ the quantum numbers $\mathbf{k}'$, u', i', and m' to the pion-nucleon "initial state," we add these quantum numbers to our two-particle wave functions [e.g., $\psi_{\pi N}(\mathbf{k}uim) \to \psi_{\pi N}(\mathbf{k}uim, \mathbf{k}'u'i'm')$]. We let $W = W_{\pi N}(k')$ and note that since [see Eq. (5.19)] $d_{\pi N}[k, W_{\pi N}(k') + i\varepsilon]\delta^3(\mathbf{k}-\mathbf{k}') = 0$, we can replace $\psi_{\pi N}$ on the left side of Eq. (5.41) with $\psi_{\pi N}(\mathbf{k}uim, \mathbf{k}'u'i'm') - Z^{-1/2}_{\pi N}(k)\delta^3(\mathbf{k}-\mathbf{k}')\delta_{uu'}\delta_{ii'}\delta mm'$. We also add these quantum numbers to the arguments of $\psi_{\pi\Delta}$ and $\psi_{\eta N}$. After making this replacement for $\psi_{\pi N}$ in Eq. (5.41), we can divide through by $d_{\pi N}(k,z)$.

We will show in Sec. VI that there exists a matrix-function $\Gamma(\mathbf{n},m,\mathbf{n}',m';k,z)$, where $\mathbf{n}$ and $\mathbf{n}'$are unit vectors, that has the property

$$\begin{aligned} \sum_{m''}\int \Gamma(\widehat{\mathbf{k}}, m, \mathbf{n}, m'';k,z)d\Omega(\mathbf{n}) &\times \sum_{m'} d_{\pi\Delta}(m'',m';k\mathbf{n},z)Z^{-1/2}_{\pi\Delta}(k)\psi_{\pi\Delta}(k\mathbf{n},uim') \\ &= Z^{-1/2}_{\pi\Delta}(k)\psi_{\pi\Delta}(\mathbf{k}uim), \end{aligned} \quad (5.44)$$

which enables us to solve Eq. (5.42) for $Z^{-1/2}_{\pi\Delta}(k)\psi_{\pi\Delta}(\mathbf{k}uim)$.

We now can write

$$\begin{aligned} Z^{-1/2}_{\pi N}(k)\psi_{\pi N}(\mathbf{k}uim, \mathbf{k}'u'i'm') &= \delta^3(\mathbf{k}-\mathbf{k}')\delta_{uu'}\delta_{ii'}\delta_{mm'} \\ &\quad + d^{-1}_{\pi N}(k,z)X_{\pi N,\pi N}(\mathbf{k}uim, \mathbf{k}'u'i'm';z), \end{aligned} \quad (5.45a)$$

$$\begin{aligned} X_{\pi N,\pi N}(\mathbf{k}uim, \mathbf{k}'u'i'm';z) &= \sum_{\beta''=N,\Delta}\sum_{u''i''m''}\int B_{\pi N,\pi\beta''}(\mathbf{k}uim, \mathbf{k}''u''i''m'';z) \\ &\quad \times d^3k'' Z^{-1/2}_{\pi\beta''}(k'')\psi_{\pi\beta''}(\mathbf{k}''u''i''m'', \mathbf{k}'u'i'm') \\ &\quad + \sum_{i''m''}\int B^d_{\pi N,\eta N}(\mathbf{k}uim, \mathbf{k}''i''m'';z)d^3k'' \\ &\quad \times \psi_{\eta N}(\mathbf{k}''i''m'', \mathbf{k}'u'i'm'), \end{aligned} \quad (5.45b)$$

$$\begin{aligned} Z^{-1/2}_{\pi\Delta}(k)\psi_{\pi\Delta}(\mathbf{k}uim, \mathbf{k}'u'i'm') &= \sum_{m''}\int \Gamma(\widehat{\mathbf{k}}, m, \mathbf{n}, m'';k,z)d\Omega(\mathbf{n}) \\ &\quad \times X_{\pi\Delta,\pi N}(k\mathbf{n},uim'', \mathbf{k}'u'i'm';z), \end{aligned} \quad (5.46a)$$

$$\begin{aligned} X_{\pi\Delta,\pi N}(\mathbf{k}uim, \mathbf{k}'u'i'm';z) &= \sum_{\beta''=N,\Delta}\sum_{u''i''m''}\int B_{\pi\Delta,\pi\beta''}(\mathbf{k}uim, \mathbf{k}''u''i''m'';z) \\ &\quad \times d^3k'' Z^{-1/2}_{\pi\beta''}(k'')\psi_{\pi\beta''}(\mathbf{k}''u''i''m'', \mathbf{k}'u'i'm'), \end{aligned} \quad (5.46b)$$

$$\begin{aligned} \psi_{\eta N}(\mathbf{k}im, \mathbf{k}'u'i'm') &= [z - W_{\eta N}(k)]^{-1}X_{\eta N,\pi N}(\mathbf{k}im, \mathbf{k}'u'i'm';z), \end{aligned} \quad (5.47a)$$

$$\begin{aligned} X_{\eta N,\pi N}(\mathbf{k}im, \mathbf{k}'u'i'm';z) &= \sum_{u''i''m''}\int B^d_{\eta N,\pi N}(\mathbf{k}im, \mathbf{k}''u''i''m'';z)d^3k'' Z^{-1/2}_{\pi N}(k'') \\ &\quad \times \psi_{\pi N}(\mathbf{k}''u''i''m'', \mathbf{k}'u'i'm') \\ &\quad + \sum_{i''m''}\int B^d_{\eta N,\eta N}(\mathbf{k}im, \mathbf{k}''i''m'';z)d^3k'' \\ &\quad \times \psi_{\eta N}(\mathbf{k}''i''m'', \mathbf{k}'i'm'). \end{aligned} \quad (5.47b)$$

Upon putting Eqs. (5.45a), (5.46a), and (5.47a) into Eqs. (5.45b), (5.46b), and (5.47b), respectively, we arrive at the following closed set of coupled Lippmann-Schwinger equations for the scattering amplitudes:

$$\begin{aligned}
&X_{\pi N,\pi N}(\mathbf{k}uim,\mathbf{k}'u'i'm';z)\\
&= B_{\pi N,\pi N}(\mathbf{k}uim,\mathbf{k}'u'i'm';z)\\
&\quad+\sum_{u''i''m''}\int B_{\pi N,\pi N}(\mathbf{k}uim,\mathbf{k}''u''i''m'';z)\\
&\quad\times\frac{d^3k''}{d_{\pi N}(k'',z)}X_{\pi N,\pi N}(\mathbf{k}''u''i''m'',\mathbf{k}'u'i'm';z)\\
&\quad+\sum_{u''i''m''}\int B_{\pi N,\pi\Delta}(\mathbf{k}uim,\mathbf{k}''u''i''m'';z)d^3k''\\
&\quad\times\sum_{m'''}\int \Gamma(\widehat{\mathbf{k}}'',m'',\mathbf{n},m''';k'',z)d\Omega(\mathbf{n})X_{\pi\Delta,\pi N}\\
&\quad\times(k''\mathbf{n},u''i''m''',\mathbf{k}'u'i'm';z)\\
&\quad+\sum_{i''m''}\int B^d_{\pi N,\eta N}(\mathbf{k}uim,\mathbf{k}''i''m'';z)\\
&\quad\times\frac{d^3k''}{z-W_{\eta N}(k'')}X_{\eta N,\pi N}(\mathbf{k}''i''m'',\mathbf{k}'u'i'm';z),
\end{aligned}\tag{5.48}$$

$$\begin{aligned}
&X_{\pi\Delta,\pi N}(\mathbf{k}uim,\mathbf{k}'u'i'm';z)\\
&= B_{\pi\Delta,\pi N}(\mathbf{k}uim,\mathbf{k}'u'i'm';z)\\
&\quad+\sum_{u''i''m''}\int B_{\pi\Delta,\pi N}(\mathbf{k}uim,\mathbf{k}''u''i''m'';z)\\
&\quad\times\frac{d^3k''}{d_{\pi N}(k'',z)}X_{\pi N,\pi N}(\mathbf{k}''u''i''m'',\mathbf{k}'u'i'm';z)\\
&\quad+\sum_{u''i''m''}\int B_{\pi\Delta,\pi\Delta}(\mathbf{k}uim,\mathbf{k}''u''i''m'';z)d^3k''\\
&\quad\times\sum_{m'''}\int \Gamma(\widehat{\mathbf{k}}'',m'',\mathbf{n},m''';k'',z)d\Omega(\mathbf{n})\\
&\quad\times X_{\pi\Delta,\pi N}(k''\mathbf{n},u''i''m''',\mathbf{k}'u'i'm';z),
\end{aligned}\tag{5.49}$$

$$\begin{aligned}
&X_{\eta N,\pi N}(\mathbf{k}im,\mathbf{k}'u'i'm';z)\\
&= B^d_{\eta N,\pi N}(\mathbf{k}im,\mathbf{k}'u'i'm';z)\\
&\quad+\sum_{u''i''m''}\int B^d_{\eta N,\pi N}(\mathbf{k}im,\mathbf{k}''u''i''m'';z)\\
&\quad\times\frac{d^3k''}{d_{\pi N}(k'',z)}X_{\pi N,\pi N}(\mathbf{k}''u''i''m'',\mathbf{k}'u'i'm';z)\\
&\quad+\sum_{i''m''}\int B^d_{\eta N,\eta N}(\mathbf{k}im,\mathbf{k}''i''m'';z)\\
&\quad\times\frac{d^3k''}{z-W_{\eta N}(k'')}X_{\eta N,\pi N}(\mathbf{k}''i''m'',\mathbf{k}'u'i'm';z).
\end{aligned}\tag{5.50}$$

In Secs. VI–IX we develop explicit models for our various interactions starting with effective Lagrangians that describe hadronic vertices.

VI. THE PROPAGATORS

In Sec. VII of Ref. [27] the vertex function $V_{\pi N,N}(\boldsymbol{\rho}tim,i'm';\mathbf{k})$, which appears in the interaction $V_{\pi N\beta}$ defined by Eq. (4.4), is derived starting with the effective Lagrangian

$$\mathcal{L}_{\pi NN}(x)=\frac{s_{\pi NN}}{2m_N}\overline{N}(x)[\gamma^\mu\partial_\mu\boldsymbol{\tau}\cdot\boldsymbol{\pi}(x)]\gamma_5N(x).\tag{6.1}$$

In Ref. [27] we considered a mix of pseudoscalar and pseudovector coupling; here we consider only pseudovector coupling. Here $s_{\pi NN}$ is a pion-nucleon coupling constant. Usually such a coupling constant is designated by $g_{\pi NN}$, but we will see that the coupling constants will be renormalized, and we will consistently use s's for unrenormalized coupling constants and g's for renormalized coupling constants.

The Lagrangian [Eq. (6.1)] leads to the following $\pi\pi N$-πN matrix element of the corresponding Hamiltonian [27]:

$$\begin{aligned}
&\langle\mathbf{p}_1u_1,\mathbf{p}_2u_2,\mathbf{p}_Nim|H_{\pi NN}|\mathbf{p}'_\pi u',\mathbf{p}'_Ni'm'\rangle\\
&\quad=\delta^3(\mathbf{p}-\mathbf{p}')\delta^3(\mathbf{p}_1-\mathbf{p}'_\pi)\delta_{u_1u'}H_{\pi NN}(\mathbf{p}_2u_2,\mathbf{p}_Nim;\mathbf{p}'_Ni'm')\\
&\qquad+(\mathbf{p}_1u_1\Leftrightarrow\mathbf{p}_2u_2),\\
&\mathbf{p}=\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_N,\quad \mathbf{p}'=\mathbf{p}'_\pi+\mathbf{p}'_N,
\end{aligned}\tag{6.2}$$

where

$$\begin{aligned}
&H_{\pi NN}(\mathbf{p}_\pi t,\mathbf{p}_Nim;\mathbf{p}'_Ni'm')\\
&\quad=-i\frac{s_{\pi NN}}{2m_N}(\boldsymbol{\varepsilon}^*_t\cdot\boldsymbol{\tau})_{ii'}C_{\pi NN}(\mathbf{p}_\pi,\mathbf{p}_N,\mathbf{p}'_N)\\
&\qquad\times\overline{u}(p_N,m)\gamma_\mu p^\mu_\pi\gamma_5u(p'_N,m'),
\end{aligned}\tag{6.3a}$$

$$\boldsymbol{\varepsilon}_\pm=\mp(1/\sqrt{2})\,(1,\pm i,0)\,,\quad \boldsymbol{\varepsilon}_0=(0,0,1)\,,\tag{6.3b}$$

$$C_{abc}(\mathbf{p}_a,\mathbf{p}_b,\mathbf{p}'_c)=\left[\frac{m_bm_c}{(2\pi)^3\,2E_a(\mathbf{p}_a)\,E_b(\mathbf{p}_b)\,E_c(\mathbf{p}'_c)}\right]^{1/2}.\tag{6.4}$$

When Eq. (6.3a) is transformed to the πN c.m. frame it leads to the vertex function [27]

$$\begin{aligned}
&V_{\pi N,N}(\boldsymbol{\rho}tim,i'm';\mathbf{k})\\
&\quad=V^*_{N,\pi N}(i'm',\boldsymbol{\rho}tim;\mathbf{k})\\
&\quad=is_{\pi NN}G_{\pi NN}(\rho)(\boldsymbol{\varepsilon}^*_t\cdot\boldsymbol{\tau})_{ii'}\left[\frac{W_{\pi N}(\boldsymbol{\rho})}{E_{\pi N}(-\mathbf{k},\boldsymbol{\rho})}\right]^{1/2}\\
&\qquad\times C_{\pi NN}(\boldsymbol{\rho},\boldsymbol{\rho},-\mathbf{k})\chi^{N\dagger}_m\boldsymbol{\sigma}\cdot\mathbf{Q}(\boldsymbol{\rho},\mathbf{k})\chi^N_{m'},
\end{aligned}\tag{6.5a}$$

where χ^N_m is a two-component column matrix with matrix elements $(\chi^N_m)_n=\delta_{mn}$ and

$$\begin{aligned}
\mathbf{Q}(\boldsymbol{\rho},\mathbf{k})&=\left[\frac{\varepsilon(\boldsymbol{\rho})+m_N}{2m_N}\right]^{1/2}\left[\frac{\varepsilon(\boldsymbol{\rho}')+m_N}{2m_N}\right]^{1/2}\\
&\quad\times\left[\frac{\boldsymbol{\rho}}{\varepsilon(\boldsymbol{\rho})+m_N}\frac{W_{\pi N}(\boldsymbol{\rho})+m_N}{2m_N}\right.\\
&\quad\left.+\frac{\boldsymbol{\rho}'}{\varepsilon(\boldsymbol{\rho}')+m_N}\frac{W_{\pi N}(\boldsymbol{\rho})-m_N}{2m_N}\right],
\end{aligned}\tag{6.5b}$$

$$\begin{aligned}
\varepsilon(\boldsymbol{\rho}')&=[E_{\pi N}(-\mathbf{k},\boldsymbol{\rho})\varepsilon(-\mathbf{k})-\mathbf{k}^2]/W_{\pi N}(\boldsymbol{\rho}),\\
\boldsymbol{\rho}'&=\mathbf{k}[E_{\pi N}(-\mathbf{k},\boldsymbol{\rho})-\varepsilon(-\mathbf{k})]/W_{\pi N}(\boldsymbol{\rho}).
\end{aligned}\tag{6.5c}$$

The vertex function contains a form factor, $G_{\pi NN}(\rho)$, which takes into account the extension of our particles and gives convergence to integrals and the integral equations. This form factor is given by

$$G_{\pi NN}(\rho) = \frac{\left(\Lambda_{\pi NN}^2 - m_N^2\right)^2 + \Lambda_{\pi NN}^4}{\left[\Lambda_{\pi NN}^2 - 2m_N^2 + W_{\pi N}^2(\rho)\right]^2 + \Lambda_{\pi NN}^4}, \quad (6.6)$$

which is normalized to one when $W_{\pi N}(\boldsymbol{\rho}) = m_N$. We can easily work out $F_{\pi NN}$, which is defined by Eq. (5.16b), by using the identity

$$(\boldsymbol{\varepsilon}_t^* \cdot \boldsymbol{\tau})_{ii'} = -\sqrt{3}\langle 1, 1/2, t, i|1/2, i'\rangle, \quad (6.7)$$

where $\langle 1, 1/2, t, i|1/2, i'\rangle$ is a Clebsch-Gordon coefficient. The result is

$$\begin{aligned} F_{\pi NN}(\rho; k) &= 3s_{\pi NN}^2 G_{\pi NN}^2(\rho)\frac{W_{\pi N}(\rho)}{E_{\pi N}(k,\rho)}C_{\pi NN}^2(\rho,\rho,k) \\ &\times \left\{\frac{\varepsilon(\rho) - m_N}{2m_N}\frac{\varepsilon(\rho') + m_N}{2m_N}\left[\frac{W_{\pi N}(\rho) + m_N}{2m_N}\right]^2\right. \\ &\left. + \frac{\varepsilon(\rho) + m_N}{2m_N}\frac{\varepsilon(\rho') - m_N}{2m_N}\left[\frac{W_{\pi N}(\rho) - m_N}{2m_N}\right]^2\right\}. \end{aligned} \quad (6.8)$$

We now derive the $\pi N\Delta$ vertex function $V_{\pi N,\Delta}(\boldsymbol{\rho} tjn, i'm'; \mathbf{k})$ that appears in the interaction of Eq. (4.4) and is needed to construct the $\pi\Delta$ propagator through Eqs. (5.26) and (5.24). We start with the effective Lagrangian [7]

$$\mathcal{L}_{\pi N\Delta}(x) = -\frac{s_{\pi N\Delta}}{m_\pi}\overline{\Delta}^\mu(x)[\partial_\mu \mathbf{T}_{N\Delta}^\dagger \cdot \boldsymbol{\pi}(x)]N(x) + (\dagger), \quad (6.9)$$

where $\mathbf{T}_{N\Delta}$ is an isospin transition operator. In general, spin and isospin transition operators are defined by [7]

$$\mathbf{X}_{\beta\beta'} = \sum_{mnn'} \boldsymbol{\varepsilon}_m \chi_n^\beta \langle 1\beta mn|\beta' n'\rangle \chi_{n'}^{\beta'\dagger}, \ \mathbf{X} = \mathbf{S}, \mathbf{T}, \quad (6.10)$$

where β stands for a baryon or its spin or isospin. Here χ_n^β is a $(2\beta+1)$-component column matrix with matrix elements $(\chi_n^\beta)_m = \delta_{nm}$. The Lagrangian [Eq. (6.9)] leads to the following $\pi\pi N$-$\pi\Delta$ matrix element of the corresponding Hamiltonian:

$$\begin{aligned} &\langle \mathbf{p}_1 u_1, \mathbf{p}_2 u_2, \mathbf{p}_N im|H_{\pi N\Delta}|\mathbf{p}'_\pi u', \mathbf{p}'_\Delta i'm'\rangle \\ &= \delta^3(\mathbf{p} - \mathbf{p}')\delta^3(\mathbf{p}_1 - \mathbf{p}'_\pi)\delta_{u_1 u'} H_{\pi N\Delta}(\mathbf{p}_2 u_2, \mathbf{p}_N im; \mathbf{p}'_\Delta i'm') \\ &\quad + (\mathbf{p}_1 u_1 \Leftrightarrow \mathbf{p}_2 u_2), \\ &\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_N, \quad \mathbf{p}' = \mathbf{p}'_\pi + \mathbf{p}'_\Delta, \end{aligned} \quad (6.11)$$

where

$$\begin{aligned} &H_{\pi N\Delta}(\mathbf{p}_\pi t, \mathbf{p}_N im; \mathbf{p}'_\Delta i'm') \\ &= is_{\pi N\Delta}(\boldsymbol{\varepsilon}_t^* \cdot \mathbf{T}_{N\Delta})_{ii'} C_{\pi N\Delta}(\mathbf{p}_\pi, \mathbf{p}_N, \mathbf{p}'_\Delta)\overline{u}(p_N, m) \\ &\quad \times \frac{p_{\pi\mu}}{m_\pi}u_\Delta^\mu(p'_\Delta, m'). \end{aligned} \quad (6.12)$$

The Δ spinor that appears here is given by [7]

$$u_\Delta^\mu(p, m) = \sum_{m'} u_\Delta(p, m') \times \left\{\frac{\mathbf{p}\cdot\mathbf{S}_{N\Delta}}{m_\Delta}, \mathbf{S}_{N\Delta} + \frac{(\mathbf{p}\cdot\mathbf{S}_{N\Delta})\mathbf{p}}{m_\Delta\left[\varepsilon_\Delta(\mathbf{p}) + m_\Delta\right]}\right\}_{m'm}^\mu. \quad (6.13)$$

By using the same techniques as those used in Sec. VII of Ref. [27] to derive $V_{\pi N,N}(\boldsymbol{\rho} tim, i'm'; \mathbf{k})$ it can be shown that

$$\begin{aligned} &V_{\pi N,\Delta}(\boldsymbol{\rho} tim, i'm'; \mathbf{k}) \\ &= V_{\Delta,\pi N}^*(i'm', \boldsymbol{\rho} tim; \mathbf{k}) \\ &= is_{\pi N\Delta}G_{\pi N\Delta}(\rho)(\boldsymbol{\varepsilon}_t^* \cdot \mathbf{T}_{N\Delta})_{ii'}\left[\frac{W_{\pi N}(\boldsymbol{\rho})}{E_{\pi N}(-\mathbf{k},\boldsymbol{\rho})}\right]^{1/2} \\ &\quad \times C_{\pi N\Delta}(\boldsymbol{\rho}, \boldsymbol{\rho}, -\mathbf{k})\,\overline{u}(\rho_N, m)\,\frac{\rho_{\pi\mu}}{m_\pi}u_\Delta^\mu(\rho'_\Delta, m'), \end{aligned} \quad (6.14a)$$

with

$$\begin{aligned} \rho_\pi &= (\omega(\boldsymbol{\rho}), \boldsymbol{\rho}), \quad \rho_N = (\varepsilon(\boldsymbol{\rho}), -\boldsymbol{\rho}), \\ \rho'_\Delta &= (\varepsilon_\Delta(\boldsymbol{\rho}'), -\boldsymbol{\rho}'), \\ \varepsilon_\Delta(\boldsymbol{\rho}') &= [E_{\pi N}(-\mathbf{k},\boldsymbol{\rho})\varepsilon_\Delta(-\mathbf{k}) - \mathbf{k}^2]/W_{\pi N}(\boldsymbol{\rho}), \\ \boldsymbol{\rho}' &= \mathbf{k}[E_{\pi N}(-\mathbf{k},\boldsymbol{\rho}) - \varepsilon_\Delta(-\mathbf{k})]/W_{\pi N}(\boldsymbol{\rho}). \end{aligned} \quad (6.14b)$$

Substituting Eq. (6.14a) into Eq. (5.24) and using Eq. (6.10) we find

$$\begin{aligned} &J_\Delta(m, m'; \mathbf{k}; z) \\ &= \frac{s_{\pi N\Delta}^2}{m_\pi^2}\int\frac{d^3\rho\, G_{\pi N\Delta}^2(\rho)}{z - W_{\pi\pi N}(k,\rho)}\frac{W_{\pi N}(\rho)}{E_{\pi N}(k,\rho)}C_{\pi N\Delta}^2(\rho,\rho,k) \\ &\quad \times \overline{u}_\Delta(\rho'_\Delta, m)\cdot\rho_\pi\frac{\gamma_\mu\rho_N^\mu + m_N}{2m_N}\rho_\pi\cdot u_\Delta(\rho'_\Delta, m'). \end{aligned} \quad (6.15)$$

The integrals over the direction of $\boldsymbol{\rho}$ can be done analytically using the identity

$$\widehat{\boldsymbol{\rho}} = \sqrt{4\pi/3}\sum_m Y_1^m(\widehat{\boldsymbol{\rho}})\,\boldsymbol{\varepsilon}_m^*, \quad (6.16)$$

where Y_1^m is a spherical harmonic. We find

$$\int\frac{d\Omega(\boldsymbol{\rho})}{4\pi}\rho_\pi^\mu\rho_\pi^\nu = g_0^\mu\, g_0^\nu\left[\omega^2(\rho) + \frac{1}{3}\rho^2\right] - \frac{1}{3}g^{\mu\nu}\rho^2, \quad (6.17a)$$

$$\begin{aligned} \int\frac{d\Omega(\boldsymbol{\rho})}{4\pi}\rho_\pi^\mu\gamma_\mu\rho_N^\mu\rho_\pi^\nu &= g_0^\mu\, g_0^\nu\varepsilon(\rho)\gamma^0\left[\omega^2(\rho) + \frac{1}{3}\rho^2\right] \\ &\quad - \frac{1}{3}g^{\mu\nu}\rho^2\varepsilon(\rho)\gamma^0 + \frac{1}{3}\omega(\rho)\rho^2 \\ &\quad \times (g_0^\mu\gamma^\nu + \gamma^\mu g_0^\nu - 2g_0^\mu\, g_0^\nu\gamma^0). \end{aligned} \quad (6.17b)$$

With the help of these integrals and Eq. (6.13) we can show that Eq. (6.15) becomes

$$\begin{aligned} &J_\Delta(m, m'; \mathbf{k}; z) \\ &= s_{\pi N\Delta}^2\int_0^\infty\frac{d\rho\rho^2 G_{\pi N\Delta}^2(\rho)}{z - W_{\pi\pi N}(k,\rho)}\frac{W_{\pi N}(\rho)}{E_{\pi N}(k,\rho)}C_{\pi N\Delta}^2(\rho,\rho,k) \\ &\quad \times \chi_m^{\Delta\dagger}[A(\rho,k) + B(\rho,k)(\mathbf{S}_{N\Delta}^\dagger\cdot\widehat{\mathbf{k}})(\widehat{\mathbf{k}}\cdot\mathbf{S}_{N\Delta})]\chi_{m'}^\Delta, \end{aligned} \quad (6.18)$$

where

$$A(\rho, k) = \frac{2\pi\rho^2}{3m_N m_\Delta m_\pi^2}[\varepsilon(\rho)\varepsilon_\Delta(\rho') + m_N m_\Delta], \tag{6.19a}$$

$$B(\rho, k) = \frac{2\pi\rho'^2}{3m_N m_\Delta^3 m_\pi^2}\{[\varepsilon(\rho)\varepsilon_\Delta(\rho') + m_N m_\Delta][3\omega^2(\rho) + \rho^2] - 2\omega(\rho)\rho^2\varepsilon_\Delta(\rho')\}. \tag{6.19b}$$

We define a 4×4 matrix function $J_\Delta(\mathbf{k}, z)$ by

$$J_\Delta(\mathbf{k}, z) = \sum_{m,m'} \chi_m^\Delta J_\Delta(m, m'; \mathbf{k}, z)\chi_{m'}^{\Delta\dagger}, \tag{6.20}$$

and consider the matrix elements $\int d\Omega(\mathbf{k}) Y_{l,3/2,j}^{m\dagger}(\widehat{\mathbf{k}}) J_\Delta(\mathbf{k}, z) Y_{l',3/2,j'}^{m'}(\widehat{\mathbf{k}})$, where

$$Y_{lsj}^m(\widehat{\mathbf{k}}) = \sum_{m_l m_s} Y_l^{m_l}(\widehat{\mathbf{k}})\chi_{m_s}^s \langle lsm_l m_s \mid jm\rangle . \tag{6.21}$$

According to Eq. (4.10) of Ref. [7]

$$(\widehat{\mathbf{k}}\cdot\mathbf{S}_{us})Y_{lsj}^m(\widehat{\mathbf{k}}) = (-1)^{j-u-2s}\sum_L Y_{Luj}^m(\widehat{\mathbf{k}})\sqrt{(2l+1)(2s+1)(2L+1)} \times \begin{pmatrix} l & 1 & L\\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix} u & L & j\\ l & s & 1\end{Bmatrix}, \tag{6.22}$$

where () and {} are $3j$ and $6j$ symbols, respectively. Specializing this relation to $u = 1/2$ and $s = 3/2$, we find

$$\begin{aligned}
(\widehat{\mathbf{k}}\cdot\mathbf{S}_{N\Delta})Y_{j-3/2,3/2,j}^m(\widehat{\mathbf{k}}) &= \frac{1}{2}\left(\frac{2j-1}{j}\right)^{1/2} Y_{j-1/2,1/2,j}^m(\widehat{\mathbf{k}}),\\
(\widehat{\mathbf{k}}\cdot\mathbf{S}_{N\Delta})Y_{j+1/2,3/2,j}^m(\widehat{\mathbf{k}}) &= -\frac{1}{2}\left(\frac{2j+3}{3j}\right)^{1/2} Y_{j-1/2,1/2,j}^m(\widehat{\mathbf{k}}),\\
(\widehat{\mathbf{k}}\cdot\mathbf{S}_{N\Delta})Y_{j-1/2,3/2,j}^m(\widehat{\mathbf{k}}) &= \frac{1}{2}\left[\frac{2j-1}{3(j+1)}\right]^{1/2} Y_{j+1/2,1/2,j}^m(\widehat{\mathbf{k}}),\\
(\widehat{\mathbf{k}}\cdot\mathbf{S}_{N\Delta})Y_{j+3/2,3/2,j}^m(\widehat{\mathbf{k}}) &= -\frac{1}{2}\left(\frac{2j+3}{j+1}\right)^{1/2} Y_{j+1/2,1/2,j}^m(\widehat{\mathbf{k}}),
\end{aligned} \tag{6.23}$$

We now define linear combinations of the $Y_{l,3/2,j}^m(\widehat{\mathbf{k}})$ with the same parity, $(-1)^l$, by means of the relations

$$\begin{bmatrix} Z_{j-3/2,j}^m\\ Z_{j+1/2,j}^m\\ Z_{j-1/2,j}^m\\ Z_{j+3/2,j}^m\end{bmatrix} = \begin{bmatrix} U^{(1,j)} & 0\\ 0 & U^{(2,j)}\end{bmatrix}\begin{bmatrix} Y_{j-3/2,3/2,j}^m\\ Y_{j+1/2,3/2,j}^m\\ Y_{j-1/2,3/2,j}^m\\ Y_{j+3/2,3/2,j}^m\end{bmatrix}, \tag{6.24a}$$

$$U^{(1,j)} = \frac{1}{\sqrt{8j}}\begin{bmatrix}\sqrt{2j+3} & \sqrt{3(2j-1)}\\ \sqrt{3(2j-1)} & -\sqrt{2j+3}\end{bmatrix}, \tag{6.24b}$$

$$U^{(2,j)} = \frac{1}{\sqrt{8(j+1)}}\begin{bmatrix}\sqrt{3(2j+3)} & \sqrt{2j-1}\\ \sqrt{2j-1} & -\sqrt{3(2j+3)}\end{bmatrix} \tag{6.24c}$$

We note that the matrices $U^{(i,j)}$ are real, orthogonal, symmetric, and traceless. The zeros in Eq. (6.24a) are actually 2×2 blocks of zeros. The $Z_{\lambda j}^m(\widehat{\mathbf{k}})$ are an orthonormal set of functions with the convenient properties

$$\begin{aligned}
(\widehat{\mathbf{k}}\cdot\mathbf{S}_{N\Delta})Z_{\lambda j}^m(\widehat{\mathbf{k}}) &= 0, \quad \lambda = j-3/2,\ j-1/2,\\
(\widehat{\mathbf{k}}\cdot\mathbf{S}_{N\Delta})Z_{\lambda j}^m(\widehat{\mathbf{k}}) &= \sqrt{\frac{2}{3}}\,Y_{\lambda-1,1/2,j}^m(\widehat{\mathbf{k}}), \quad \lambda = j+1/2,\ j+3/2,
\end{aligned} \tag{6.25}$$

We now define $J_\lambda(k, z)$ through the relation

$$\int d\Omega(\mathbf{k}) Z_{\lambda j}^{m\dagger}(\widehat{\mathbf{k}}) J_\Delta(\mathbf{k}, z) Z_{\lambda' j'}^{m'}(\widehat{\mathbf{k}}) = \delta_{\lambda\lambda'}\delta_{jj'}\delta_{mm'} J_\lambda(k, z), \tag{6.26}$$

where according to Eqs. (6.18) and (6.20)

$$J_\lambda(k, z) = s_{\pi N\Delta}^2 \int_0^\infty \frac{d\rho\rho^2 G_{\pi N\Delta}^2(\rho)}{z - W_{\pi\pi N}(k,\rho)}\frac{W_{\pi N}(\rho)}{E_{\pi N}(k,\rho)} \times C_{\pi N\Delta}^2(\rho,\rho,k)[A(\rho,k) + B(\rho,k)\eta_\lambda], \tag{6.27}$$

with

$$\eta_\lambda = \begin{cases} 0, & \lambda = j-3/2, \quad j-1/2,\\ 2/3, & \lambda = j+1/2, \quad j+3/2.\end{cases} \tag{6.28}$$

We can write, for the $\pi\Delta$ propagator defined by Eq. (5.26),

$$d_{\pi\Delta}(m, m'; \mathbf{k}, z) = \chi_m^{\Delta\dagger} d_{\pi\Delta}(\mathbf{k}, z)\chi_{m'}^\Delta, \tag{6.29}$$

where

$$d_{\pi\Delta}(\mathbf{k}, z) = [z - W_{\pi\Delta}(k) - V_\Delta^\pi(k)]I^{(\Delta)} - J_\Delta(\mathbf{k}, z), \tag{6.30}$$

with $I^{(\Delta)}$ a 4×4 unit matrix. Using Eq. (6.26) and the orthonormality of the $Z_{\lambda j}^m(\widehat{\mathbf{k}})$, we find

$$\int d\Omega(\mathbf{k}) Z_{\lambda j}^{m\dagger}(\widehat{\mathbf{k}}) d_{\pi\Delta}(\mathbf{k}, z) Z_{\lambda' j'}^{m'}(\widehat{\mathbf{k}}) = \delta_{\lambda\lambda'}\delta_{jj'}\delta_{mm'} d_{\pi\Delta}(k, \lambda, z), \tag{6.31}$$

where

$$d_{\pi\Delta}(k, \lambda, z) = z - W_{\pi\Delta}(k) - V_\Delta^\pi(k) - J_\lambda(k, z). \tag{6.32}$$

We are now in a position to "solve" Eq. (5.42), that is, to express $\psi_{\pi\Delta}(\mathbf{k}uim)$ in terms of the right-hand side of the equation. To do this we have to solve an equation of the form

$$\sum_{m'} d_{\pi\Delta}(m, m'; \mathbf{k}, z)\phi(\mathbf{k}, m') = F(\mathbf{k}, m, z). \tag{6.33}$$

If we put Eq. (6.29) in Eq. (6.33), multiply from the left by χ_m^Δ, and sum on m, we find the equation

$$d_{\pi\Delta}(\mathbf{k}, z)\phi(\mathbf{k}) = F(\mathbf{k}, z), \tag{6.34}$$

where

$$\phi(\mathbf{k}) = \sum_m \chi_m^\Delta \phi(\mathbf{k}, m), \quad F(\mathbf{k}, z) = \sum_m \chi_m^\Delta F(\mathbf{k}, m, z). \tag{6.35}$$

To solve Eq. (6.34) we expand $\phi(\mathbf{k})$ according to

$$\phi(\mathbf{k}) = \sum_{\lambda jm} Z_{\lambda j}^m(\widehat{\mathbf{k}})\phi_{\lambda j}^m(k). \tag{6.36}$$

Using Eq. (6.31) we find from Eq. (6.34)

$$\phi^m_{\lambda j}(k) = d^{-1}_{\pi\Delta}(k,\lambda,z)\int d\Omega(\mathbf{k}) Z^{m\dagger}_{\lambda j}(\widehat{\mathbf{k}}) F(\mathbf{k},z). \qquad (6.37)$$

Combining Eqs. (6.34), (6.36), and (6.37) we find

$$\phi(\mathbf{k}) = \int d\Omega(\mathbf{n})\Gamma(\widehat{\mathbf{k}},\mathbf{n};k,z)F(k\mathbf{n},z)$$
$$= \int d\Omega(\mathbf{n})\Gamma(\widehat{\mathbf{k}},\mathbf{n};k,z)d_{\pi\Delta}(k\mathbf{n},z)\phi(k\mathbf{n}), \qquad (6.38)$$

where

$$\Gamma(\mathbf{n},\mathbf{n}';k,z) = \sum_{\lambda jm} Z^m_{\lambda j}(\mathbf{n}) d^{-1}_{\pi\Delta}(k,\lambda,z) Z^{m\dagger}_{\lambda j}(\mathbf{n}'), \qquad (6.39)$$

with $\mathbf{n}$ and $\mathbf{n}'$ unit vectors. If we contract Eq. (6.38) with $\chi^{\Delta\dagger}_m$, and insert the completeness relation for the χ^{Δ}_m between Γ and $d_{\pi\Delta}$, and between $d_{\pi\Delta}$ and $\phi(k\mathbf{n})$, we find with the help of Eq. (6.29)

$$\phi(\mathbf{k}m) = \sum_{m''}\int d\Omega(\mathbf{n})\Gamma(\widehat{\mathbf{k}},m,\mathbf{n},m'';k,z)$$
$$\times \sum_{m'} d_{\pi\Delta}(m'',m';k\mathbf{n},z)\phi(k\mathbf{n},m'), \qquad (6.40)$$

where

$$\Gamma(\mathbf{n},m,\mathbf{n}',m';k,z) = \chi^{\Delta\dagger}_m \Gamma(\mathbf{n},\mathbf{n}';k,z)\chi^{\Delta}_{m'}. \qquad (6.41)$$

This justifies Eq. (5.44).

We shall see that in our final equations for the various scattering and production amplitudes the $d^{-1}_{\pi\Delta}(k,\lambda,z)$ play the same role as the pion-nucleon propagator $d^{-1}_{\pi N}(k,z)$, and therefore we shall refer to the $d^{-1}_{\pi\Delta}(k,\lambda,z)$ as the pion-delta propagators. According to Eq. (6.24a) the possible values of the parameter λ are determined by angular momentum coupling rules and parity considerations. For $j=1/2, \lambda=1,2$, whereas for $j>1/2, \lambda=j-3/2, j-1/2, j+1/2, j+3/2$. The orbital parity is given by $(-1)^{\lambda}$, and since parity is a good quantum number, for $j=1/2$ we have either $\lambda=1$ or $\lambda=2$, whereas for $j>1/2$ we have either $\lambda=j-3/2$ and $j+1/2$ or $\lambda=j-1/2$ and $j+3/2$.

For the cutoff function that appears in Eq. (6.27) we take

$$G_{\pi N\Delta}(\rho) = \frac{\left(\Lambda^2_{\pi N\Delta}-m^2_N\right)^2+\Lambda^4_{\pi N\Delta}}{\left[\Lambda^2_{\pi N\Delta}-m^2_N-m^2_\Delta+W^2_{\pi N}(\rho)\right]^2+\Lambda^4_{\pi N\Delta}}, \qquad (6.42)$$

which is normalized to one when $W_{\pi N}(\boldsymbol{\rho}) = m_\Delta$.

We will consider two choices for the function $V^{\pi}_{\Delta}(k)$ that appears in Eq. (6.30). Either we will simply set it equal to zero or, by analogy to Eq. (5.19), we will determine it by the requirement

$$\mathrm{Re}\,[d_{\pi\Delta}(k,\lambda,z)] \underset{z\to W_{\pi\Delta}(k)}{\longrightarrow} 0, \qquad (6.43)$$

which leads to

$$V^{\pi}_{\Delta}(k) = -\mathrm{Re}\,\{J_\lambda\,[k, W_{\pi\Delta}(k)+i\varepsilon]\}\,. \qquad (6.44)$$

VII. DIRECT INTERACTIONS

The direct interactions or pole terms are given by Eqs. (5.28)–(5.30). We now determine the various vertex functions $U_{\mu\beta,\beta'}$ that are needed for their construction.

The $\pi N \Longleftrightarrow N$ vertex function can be obtained from Eq. (6.5) by "turning off" the spectator pion. To this end we let $\mathbf{k}\to\mathbf{0}$ and relabel by letting $\boldsymbol{\rho}\to\mathbf{k}$ and $t\to u$ to obtain

$$U_{\pi N,N}(\mathbf{k}uim,i'm') = \frac{i}{\sqrt{12\pi}}(\boldsymbol{\varepsilon}^*_u\cdot\boldsymbol{\tau})_{ii'}(\boldsymbol{\sigma}\cdot\widehat{\mathbf{k}})_{mm'}U_{\pi NN}(k), \qquad (7.1a)$$

where

$$U_{\pi NN}(k) = \sqrt{12\pi}\, s^{(0)}_{\pi NN} G^{(0)}_{\pi NN}(k)\left[\frac{m_N}{(2\pi)^3\,2\omega(k)\varepsilon(k)}\right]^{1/2}$$
$$\times\left[\frac{\varepsilon(k)+m_N}{2m_N}\right]^{1/2}\frac{k}{\varepsilon(k)+m_N}\frac{W_{\pi N}(k)+m_N}{2m_N}. \qquad (7.1b)$$

Justification for using different coupling constants and cutoff functions in Eqs. (6.5) and (7.1) is given in earlier work by Fuda [32] and by Pearce and Afnan [33].

For all of our pole term cutoff functions we assume the form

$$G^{(0)}_{\mu\beta\beta'}(k) = \left[\frac{k^2_{\mu\beta\beta'}+\Lambda^{(0)2}_{\mu\beta\beta'}}{k^2+\Lambda^{(0)2}_{\mu\beta\beta'}}\right]^{n^{(0)}_{\mu\beta\beta'}}, \qquad (7.2a)$$

$$k^2_{\mu\beta\beta'} = \left[m^2_{\beta'}-(m_\beta-m_\mu)^2\right]\left[m^2_{\beta'}-(m_\beta+m_\mu)^2\right]/(2m_{\beta'})^2. \qquad (7.2b)$$

These cutoff functions are normalized so that $G^{(0)}_{\mu\beta\beta'}(k)=1$ when $W_{\mu\beta}(k)=m_{\beta'}$.

The $\pi N \Leftrightarrow \Delta$ vertex function can be obtained by "turning off" the spectator pion in Eq. (6.14). With the help of Eq. (6.13) we find

$$U_{\pi N,\Delta}(\mathbf{k}uim,i'm')$$
$$= -i\sqrt{\frac{3}{4\pi}}(\boldsymbol{\varepsilon}^*_u\cdot\mathbf{T}_{N\Delta})_{ii'}(\widehat{\mathbf{k}}\cdot\mathbf{S}_{N\Delta})_{mm'}U_{\pi N\Delta}(k), \qquad (7.3a)$$

where

$$U_{\pi N\Delta}(k) = \sqrt{\frac{4\pi}{3}}\, s^{(0)}_{\pi N\Delta} G^{(0)}_{\pi N\Delta}(k)\left[\frac{m_N}{(2\pi)^3\,2\omega(k)\varepsilon(k)}\right]^{1/2}$$
$$\times\left[\frac{\varepsilon(k)+m_N}{2m_N}\right]^{1/2}\frac{k}{m_\pi}. \qquad (7.3b)$$

The expressions for the other pole term vertex functions can be obtained from Ref. [7] Since the basis states in Ref. [7] have different normalizations than the ones used here it is necessary to "translate" the results of Ref. [7]. By comparing Eq. (3.8) with Eqs. (3.14) and (3.19) of Ref. [7] we find that the vertex functions to be used here are related to those of Elmessiri and Fuda [7] by

$$U_{\mu\beta,\beta'}(\mathbf{k}uim,i'm') = \frac{U^{EF}_{\mu\beta,\beta'}(\mathbf{k}uim,i'm')}{2[(2\pi)^3 2\omega_\mu(k)\varepsilon_\beta(k)m_{\beta'}]^{1/2}}. \qquad (7.4)$$

The vertex functions for the processes $\pi N \Leftrightarrow R$ and $\pi\Delta \Leftrightarrow R$ are given by

$$U_{\pi N,R}(\mathbf{k}uim, i'm') = \frac{i}{\sqrt{12\pi}}(\boldsymbol{\varepsilon}^*_u \cdot \boldsymbol{\tau})_{ii'}(\boldsymbol{\sigma}\cdot\widehat{\mathbf{k}})_{mm'}U_{\pi NR}(k), \quad (7.5a)$$

$$U_{\pi NR}(k) = \sqrt{12\pi}\, s^{(0)}_{\pi NR} G^{(0)}_{\pi NR}(k)\left[\frac{m_N}{(2\pi)^3\, 2\omega(k)\varepsilon(k)}\right]^{1/2} \times \left[\frac{\varepsilon(k)+m_N}{2m_N}\right]^{1/2} \frac{k}{\varepsilon(k)+m_N}\,\frac{W_{\pi N}(k)+m_N}{m_R+m_N}, \quad (7.5b)$$

and

$$U_{\pi\Delta,R}(\mathbf{k}uim, i'm') = -\frac{i}{4}\sqrt{\frac{3}{\pi}}(\boldsymbol{\varepsilon}^*_u\cdot\mathbf{T}^\dagger_{R\Delta})_{ii'}(\widehat{\mathbf{k}}\cdot\mathbf{S}^\dagger_{R\Delta})_{mm'}U_{\pi\Delta R}(k), \quad (7.6a)$$

$$U_{\pi\Delta R}(k) = 4\sqrt{\frac{\pi}{3}}\, s^{(0)}_{\pi\Delta R}G^{(0)}_{\pi\Delta R}(k)\left[\frac{m_\Delta}{(2\pi)^3 2\omega(k)\varepsilon_\Delta(k)}\right]^{1/2} \times\left[\frac{\varepsilon_\Delta(k)+m_\Delta}{2m_\Delta}\right]^{1/2}\frac{k}{m_\pi}\,\frac{W_{\pi\Delta}(k)}{m_\Delta}. \quad (7.6b)$$

The vertex functions for the processes $\pi N \Leftrightarrow D$ and $\pi\Delta \Leftrightarrow D$ are given by

$$U_{\pi N,D}(\mathbf{k}uim, i'm') = -\frac{i}{\sqrt{4\pi}}(\boldsymbol{\varepsilon}^*_u\cdot\boldsymbol{\tau})_{ii'}(\boldsymbol{\sigma}\cdot\widehat{\mathbf{k}}\mathbf{S}_{ND}\cdot\widehat{\mathbf{k}})_{mm'}U_{\pi ND}(k), \quad (7.7a)$$

$$U_{\pi ND}(k) = \sqrt{4\pi}\, s^{(0)}_{\pi ND}G^{(0)}_{\pi ND}(k)\left[\frac{m_N}{(2\pi)^3 2\omega(k)\varepsilon(k)}\right]^{1/2} \times\left[\frac{\varepsilon(k)+m_N}{2m_N}\right]^{1/2}\frac{\varepsilon(k)-m_N}{m_\pi}, \quad (7.7b)$$

$$U_{\pi\Delta,D}(\mathbf{k}uim, i'm') = \frac{i}{\sqrt{4\pi}}(\boldsymbol{\varepsilon}^*_u\cdot\mathbf{T}^\dagger_{D\Delta})_{ii'} \times\left[1+\frac{\varepsilon_\Delta(k)-m_\Delta}{m_\Delta}(\widehat{\mathbf{k}}\cdot\mathbf{S}^\dagger_{N\Delta})(\widehat{\mathbf{k}}\cdot\mathbf{S}_{ND})\right]_{mm'}U_{\pi\Delta D}(k), \quad (7.8a)$$

$$U_{\pi\Delta D}(k) = \sqrt{4\pi}\, s^{(0)}_{\pi\Delta D}G^{(0)}_{\pi\Delta D}(k)\left[\frac{m_\Delta}{(2\pi)^3\, 2\omega(k)\varepsilon_\Delta(k)}\right]^{1/2} \times\left[\frac{\varepsilon_\Delta(k)+m_\Delta}{2m_\Delta}\right]^{1/2}\frac{W_{\pi\Delta}(k)-m_\Delta}{m_\pi}. \quad (7.8b)$$

The vertex functions for the processes $\pi N \Leftrightarrow S$ and $\eta N \Leftrightarrow S$ are given by

$$U_{\pi N,S}(\mathbf{k}uim, i'm') = -\frac{1}{\sqrt{12\pi}}(\boldsymbol{\varepsilon}^*_u\cdot\boldsymbol{\tau})_{ii'}\delta_{mm'}U_{\pi NS}(k), \quad (7.9a)$$

$$U_{\pi NS}(k) = \sqrt{12\pi}\, s^{(0)}_{\pi NS}G^{(0)}_{\pi NS}(k)\left[\frac{m_N}{(2\pi)^3\, 2\omega(k)\varepsilon(k)}\right]^{1/2} \times\left[\frac{\varepsilon(k)+m_N}{2m_N}\right]^{1/2}\frac{W_{\pi N}(k)-m_N}{m_S-m_N}, \quad (7.9b)$$

$$U_{\eta N,S}\left(\mathbf{k}im, i'm'\right) = \frac{1}{\sqrt{4\pi}}\delta_{ii'}\delta_{mm'}U_{\eta NS}(k), \quad (7.10a)$$

$$U_{\eta NS}(k) = -\sqrt{4\pi}\, s^{(0)}_{\eta NS}G^{(0)}_{\eta NS}(k)\left[\frac{m_N}{(2\pi)^3\, 2\omega_\eta(k)\varepsilon(k)}\right]^{1/2} \times\left[\frac{\varepsilon(k)+m_N}{2m_N}\right]^{1/2}\frac{W_{\eta N}(k)-m_N}{m_S-m_N}. \quad (7.10b)$$

The vertex functions for the processes $\pi N \Leftrightarrow S'$ and $\pi\Delta \Leftrightarrow S'$ are given by

$$U_{\pi N,S'}(\mathbf{k}uim, i'm') = \frac{1}{\sqrt{4\pi}}(\boldsymbol{\varepsilon}^*_u\cdot\mathbf{T}_{NS'})_{ii'}\delta_{mm'}U_{\pi NS'}(k), \quad (7.11a)$$

$$U_{\pi NS'}(k) = -\sqrt{4\pi}\, s^{(0)}_{\pi NS'}G^{(0)}_{\pi NS'}(k)\left[\frac{m_N}{(2\pi)^3\, 2\omega(k)\varepsilon(k)}\right]^{1/2} \times\left[\frac{\varepsilon(k)+m_N}{2m_N}\right]^{1/2}\frac{W_{\pi N}(k)-m_N}{m_{S'}-m_N}, \quad (7.11b)$$

$$U_{\pi\Delta,S'}(\mathbf{k}uim, i'm') = -\sqrt{\frac{3}{8\pi}}(\boldsymbol{\varepsilon}^*_u\cdot\mathbf{T}_{\Delta S'})_{ii'} \times[(\widehat{\mathbf{k}}\cdot S^\dagger_{S'\Delta})(\boldsymbol{\sigma}\cdot\widehat{\mathbf{k}})]_{mm'}U_{\pi\Delta S'}(k), \quad (7.12a)$$

$$U_{\pi\Delta S'}(k) = \sqrt{\frac{8\pi}{3}}\, s^{(0)}_{\pi\Delta S'}G^{(0)}_{\pi\Delta S'}(k)\left[\frac{m_\Delta}{(2\pi)^3\, 2\omega(k)\varepsilon_\Delta(k)}\right]^{1/2} \times\left[\frac{\varepsilon_\Delta(k)+m_\Delta}{2m_\Delta}\right]^{1/2}\frac{W_{\pi\Delta}(k)}{m_\pi}\,\frac{\varepsilon_\Delta(k)-m_\Delta}{m_\Delta}. \quad (7.12b)$$

VIII. CROSSED POTENTIALS

The crossed potentials are defined by Eqs. (5.38) with $V_{\pi N,N}$ and $V_{\pi N,\Delta}$ given by Eqs. (6.5) and (6.14). By following the method used in Sec. VII of Ref. [27] to relate $V_{\pi N,N}$ to $H_{\pi NN}$ we can show that the crossed potentials are also given by

$$B^c_{\pi\beta,\pi\beta'}(\mathbf{k}uim, \mathbf{k}'u'i'm'; z) = \frac{Z^{1/2}_{\pi\beta}(k)G_{\pi N\beta}(\rho)G_{\pi N\beta}(\rho')Z^{1/2}_{\pi\beta'}(k')}{z-\omega(\mathbf{k})-\varepsilon(-\mathbf{k}-\mathbf{k}')-\omega(\mathbf{k}')} \times\sum_{jn}H^*_{\pi N\beta}(\mathbf{k}'u', -\mathbf{k}-\mathbf{k}', jn; -\mathbf{k}, im) \times H_{\pi N\beta'}(\mathbf{k}u, -\mathbf{k}-\mathbf{k}', jn; -\mathbf{k}', i'm'), \quad (8.1)$$

where

$$\boldsymbol{\rho} = \mathbf{f}_{\pi N}(\mathbf{k}', -\mathbf{k}-\mathbf{k}'), \quad \boldsymbol{\rho}' = \mathbf{f}_{\pi N}(\mathbf{k}, -\mathbf{k}-\mathbf{k}'), \tag{8.2}$$

with $\mathbf{f}_{\pi N}$ defined by Eq. (5.32). The Hamiltonian matrix elements, $H_{\pi N\beta}$, are given by Eqs. (6.3), (6.4), (6.12), and (6.13). It is interesting to note that the amplitude [Eq. (8.1)] is also a direct consequence of time-ordered perturbation theory.

From Eq. (6.3a) it follows almost immediately that

$$\begin{aligned} &B^c_{\pi N,\pi N}(\mathbf{k}uim, \mathbf{k}'u'i'm'; z) \\ &= s^2_{\pi NN}\frac{Z^{1/2}_{\pi N}(k)G_{\pi NN}(\rho)G_{\pi NN}(\rho')Z^{1/2}_{\pi N}(k')}{z-\omega(\mathbf{k})-\varepsilon(-\mathbf{k}-\mathbf{k}')-\omega(\mathbf{k}')} \\ &\quad \times (\boldsymbol{\varepsilon}_{u'}\cdot\boldsymbol{\tau}\;\boldsymbol{\varepsilon}^*_u\cdot\boldsymbol{\tau})_{ii'}C_{\pi NN}(\mathbf{k}', -\mathbf{k}-\mathbf{k}', -\mathbf{k}) \\ &\quad \times C_{\pi NN}(\mathbf{k}, -\mathbf{k}-\mathbf{k}', -\mathbf{k}')\chi^{N\dagger}_m a(\mathbf{k},\mathbf{k}')a^\dagger(\mathbf{k}',\mathbf{k})\chi^N_{m'}, \end{aligned} \tag{8.3}$$

where

$$\begin{aligned} a(\mathbf{k},\mathbf{k}') &= \left[\frac{\varepsilon(-\mathbf{k})+m_N}{2m_N}\right]^{1/2}\left[\frac{\varepsilon(-\mathbf{k}-\mathbf{k}')+m_N}{2m_N}\right]^{1/2} \\ &\quad \times\left\{[1-\Lambda(\mathbf{k},\mathbf{k}')]\frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{\varepsilon(-\mathbf{k})+m_N}\right. \\ &\quad \left.+[1+\Lambda(\mathbf{k},\mathbf{k}')]\frac{\boldsymbol{\sigma}\cdot(-\mathbf{k}-\mathbf{k}')}{\varepsilon(-\mathbf{k}-\mathbf{k}')+m_N}\right\}, \end{aligned} \tag{8.4a}$$

with

$$\Lambda(\mathbf{k},\mathbf{k}') = \frac{\omega(\mathbf{k}')+\varepsilon\left(-\mathbf{k}-\mathbf{k}'\right)-\varepsilon(-\mathbf{k})}{2m_N}. \tag{8.4b}$$

Using Eqs. (6.12) and (6.13) we can show that

$$\begin{aligned} &B^c_{\pi N,\pi\Delta}(\mathbf{k}uim, \mathbf{k}'u'i'm'; z) \\ &= s_{\pi NN}s_{\pi N\Delta}\frac{Z^{1/2}_{\pi N}(k)G_{\pi NN}(\rho)G_{\pi N\Delta}(\rho')}{z-\omega(\mathbf{k})-\varepsilon(-\mathbf{k}-\mathbf{k}')-\omega(\mathbf{k}')} \\ &\quad \times (\boldsymbol{\varepsilon}_{u'}\cdot\boldsymbol{\tau}\;\boldsymbol{\varepsilon}^*_u\cdot\mathbf{T}_{N\Delta})_{ii'}C_{\pi NN}(\mathbf{k}', -\mathbf{k}-\mathbf{k}', -\mathbf{k}) \\ &\quad \times C_{\pi N\Delta}(\mathbf{k}, -\mathbf{k}-\mathbf{k}', -\mathbf{k}') \\ &\quad \times \chi^{N\dagger}_m a(\mathbf{k},\mathbf{k}')w^\dagger(\mathbf{k}',\mathbf{k})\chi^\Delta_{m'}, \end{aligned} \tag{8.5}$$

where

$$\begin{aligned} w(\mathbf{k},\mathbf{k}') &= \left[\frac{\varepsilon_\Delta(-\mathbf{k})+m_\Delta}{2m_\Delta}\right]^{1/2}\left[\frac{\varepsilon(-\mathbf{k}-\mathbf{k}')+m_N}{2m_N}\right]^{1/2} \\ &\quad \times\left\{\frac{\mathbf{k}'\cdot S^\dagger_{N\Delta}}{m_\pi}\right. \\ &\quad \left.+\frac{[\varepsilon_\Delta(-\mathbf{k})+m_\Delta]\,\omega(\mathbf{k}')+\mathbf{k}\cdot\mathbf{k}'}{m_\Delta\,[\varepsilon_\Delta(-\mathbf{k})+m_\Delta]}\;\frac{\mathbf{k}\cdot\mathbf{S}^\dagger_{N\Delta}}{m_\pi}\right\} \\ &\quad \times\left[1+\frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{\varepsilon_\Delta(-\mathbf{k})+m_\Delta}\,\frac{\boldsymbol{\sigma}\cdot(-\mathbf{k}-\mathbf{k}')}{\varepsilon(-\mathbf{k}-\mathbf{k}')+m_N}\right]. \end{aligned} \tag{8.6}$$

Once again using Eqs. (6.12) and (6.13) we find

$$\begin{aligned} &B^c_{\pi\Delta,\pi\Delta}(\mathbf{k}uim, \mathbf{k}'u'i'm'; z) \\ &= s^2_{\pi N\Delta}\frac{G_{\pi N\Delta}(\rho)G_{\pi N\Delta}(\rho')}{z-\omega(\mathbf{k})-\varepsilon(-\mathbf{k}-\mathbf{k}')-\omega(\mathbf{k}')} \\ &\quad \times (\boldsymbol{\varepsilon}_{u'}\cdot\mathbf{T}^\dagger_{N\Delta}\boldsymbol{\varepsilon}^*_u\cdot\mathbf{T}_{N\Delta})_{ii'} \\ &\quad \times C_{\pi N\Delta}(\mathbf{k}', -\mathbf{k}-\mathbf{k}', -\mathbf{k})C_{\pi N\Delta}(\mathbf{k}, -\mathbf{k}-\mathbf{k}', -\mathbf{k}') \\ &\quad \times \chi^{\Delta\dagger}_m w(\mathbf{k},\mathbf{k}')w^\dagger(\mathbf{k}',\mathbf{k})\chi^\Delta_{m'}. \end{aligned} \tag{8.7}$$

IX. ANTINUCLEON, DELTA, SIGMA, AND RHO EXCHANGE

Antinucleon exchange does not play an important role in our model, but we have found that it does have some effect on the S_{11} and S_{31} so we include it.

A straightforward application of time-ordered perturbation theory shows that the off-shell amplitude for πN elastic scattering when the intermediate state is $NN\overline{N}$ is given by equations similar to Eqs. (8.3) and (8.4), but with the replacements $z-\omega(\mathbf{k})-\varepsilon(-\mathbf{k}-\mathbf{k}')-\omega(\mathbf{k}') \longrightarrow z-\varepsilon(-\mathbf{k})-\varepsilon(\mathbf{k}+\mathbf{k}')-\varepsilon(-\mathbf{k}')$ in Eq. (8.3) and $\varepsilon(-\mathbf{k}-\mathbf{k}') \longrightarrow -\varepsilon(\mathbf{k}+\mathbf{k}')$ in Eq. (8.4). Our model does not include $|NN\overline{N}\rangle$ basis states so we use the Okubo method [7,34] to construct an effective πN-πN energy-independent potential. This amounts to choosing W to be either the "initial" energy $\omega(\mathbf{k}')+\varepsilon(-\mathbf{k}')$ or the "final" energy $\omega(\mathbf{k})+\varepsilon(-\mathbf{k})$, adding the two resulting expressions together, and dividing by 2. We find

$$\begin{aligned} &B^{\overline{N}}_{\pi N,\pi N}(\mathbf{k}uim, \mathbf{k}'u'i'm') \\ &= s^2_{\pi NN}Z^{1/2}_{\pi N}(k)G_{\pi NN}(\rho)G_{\pi NN}(\rho')Z^{1/2}_{\pi N}(k') \\ &\quad \times\frac{1}{2}\left[\frac{1}{\omega(\mathbf{k})-\varepsilon(\mathbf{k}+\mathbf{k}')-\varepsilon(-\mathbf{k}')}\right. \\ &\quad \left.+\frac{1}{\omega(\mathbf{k}')-\varepsilon(\mathbf{k}+\mathbf{k}')-\varepsilon(-\mathbf{k})}\right] \\ &\quad \times(\boldsymbol{\varepsilon}_{u'}\cdot\boldsymbol{\tau}\;\boldsymbol{\varepsilon}^*_u\cdot\boldsymbol{\tau})_{ii'}C_{\pi NN}(\mathbf{k}', \mathbf{k}+\mathbf{k}', -\mathbf{k}) \\ &\quad \times C_{\pi NN}(\mathbf{k}, \mathbf{k}+\mathbf{k}', -\mathbf{k}')\chi^{N\dagger}_m b(\mathbf{k},\mathbf{k}') \\ &\quad \times(-1)b^\dagger(\mathbf{k}',\mathbf{k})\chi^N_{m'}, \end{aligned} \tag{9.1a}$$

where

$$\begin{aligned} b(\mathbf{k},\mathbf{k}') &= \left[\frac{\varepsilon(-\mathbf{k})+m_N}{2m_N}\right]^{1/2}\left[\frac{\varepsilon(\mathbf{k}+\mathbf{k}')-m_N}{2m_N}\right]^{1/2} \\ &\quad \times\left\{[1-\Gamma(\mathbf{k},\mathbf{k}')]\frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{\varepsilon(-\mathbf{k})+m_N}\right. \\ &\quad \left.+[1+\Gamma(\mathbf{k},\mathbf{k}')]\frac{\boldsymbol{\sigma}\cdot(\mathbf{k}+\mathbf{k}')}{\varepsilon(\mathbf{k}+\mathbf{k}')-m_N}\right\}, \end{aligned} \tag{9.1b}$$

with

$$\Gamma(\mathbf{k},\mathbf{k}') = \frac{\omega(\mathbf{k}')-\varepsilon(\mathbf{k}+\mathbf{k}')-\varepsilon(-\mathbf{k})}{2m_N}. \tag{9.1c}$$

We take our Δ, σ, and ρ exchange potentials from Eqs. (A.19), (A.34), and (A.36), respectively, of Ref. [7]. To take into account the difference in the normalization of the basis states used here and in Ref. [7] we have to multiply the Elmessiri-Fuda results [7] by a factor $F(\mathbf{k},\mathbf{k}')$, where

$$F(\mathbf{k},\mathbf{k}') = \frac{1}{4(2\pi)^3[\omega(\mathbf{k})\varepsilon(-\mathbf{k})\omega(\mathbf{k}')\varepsilon(-\mathbf{k}')]^{1/2}}. \tag{9.2}$$

For the Δ exchange potential we find

$$\begin{aligned}
&B^{\Delta}_{\pi N,\pi N}(\mathbf{k}uim,\mathbf{k}'u'i'm') \\
&= s^2_{\pi N\Delta}(\boldsymbol{\varepsilon}_{u'}\cdot\mathbf{T}_{N\Delta}\boldsymbol{\varepsilon}^*_u\cdot\mathbf{T}^{\dagger}_{N\Delta})_{ii'} \\
&\quad\times\left[\frac{\varepsilon+m_N}{2m_N}\right]^{1/2}\left[\frac{\varepsilon'+m_N}{2m_N}\right]^{1/2}2m_N F(\mathbf{k},\mathbf{k}') \\
&\quad\times\left[F_\Delta(\mathbf{k},\mathbf{k}';m_N,m_\Delta)+\frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{\varepsilon+m_N}\right. \\
&\quad\left.\times\frac{\boldsymbol{\sigma}\cdot\mathbf{k}'}{\varepsilon'+m_N}F_\Delta(\mathbf{k},\mathbf{k}';-m_N,-m_\Delta)\right],
\end{aligned}\tag{9.3a}$$

$$\begin{aligned}
&F_\Delta(\mathbf{k},\mathbf{k}';m_N,m_\Delta) \\
&= Z^{1/2}_{\pi N}(k)G_{\pi\Delta N}(\rho)G_{\pi\Delta N}(\rho')Z^{1/2}_{\pi N}(k') \\
&\quad\times\frac{G_\Delta(\mathbf{k},\mathbf{k}';m_N,m_\Delta)}{4m_\pi^2\varepsilon''_\Delta} \\
&\quad\times\left[\frac{1}{\varepsilon-\varepsilon''_\Delta-\omega'}+\frac{1}{\varepsilon'-\varepsilon''_\Delta-\omega}\right],
\end{aligned}\tag{9.3b}$$

$$\begin{aligned}
&G_\Delta(\mathbf{k},\mathbf{k}';m_N,m_\Delta) \\
&= [\varepsilon''_\Delta-\varepsilon-\varepsilon'+m_\Delta+2m_N] \\
&\quad\times\frac{1}{3m_\Delta^2}\{[(2\omega+\varepsilon''_\Delta)(2\omega'+\varepsilon''_\Delta)+k^2+k'^2 \\
&\quad-m_\Delta(\omega+\omega'+\varepsilon''_\Delta-m_\Delta)]\mathbf{k}\cdot\mathbf{k}' \\
&\quad+2[k^2k'^2+\omega k'^2(\varepsilon''_\Delta+\omega')+\omega' k^2(\varepsilon''_\Delta+\omega)]\} \\
&\quad-[\varepsilon''_\Delta-\varepsilon-\varepsilon'-m_\Delta-2m_N] \\
&\quad\times\frac{1}{3m_\Delta}(\omega+\omega'+\varepsilon''_\Delta)(\varepsilon-m_N)(\varepsilon'-m_N),
\end{aligned}\tag{9.3c}$$

$$\begin{aligned}
&G_{\pi\Delta N}(\rho) \\
&= \frac{\left(\Lambda^2_{\pi\Delta N}-m^2_\Delta\right)^2+\Lambda^4_{\pi\Delta N}}{\left[\Lambda^2_{\pi\Delta N}-m^2_\Delta+W^2_{\pi\Delta}(\rho)-m^2_N\right]^2+\Lambda^4_{\pi\Delta N}},
\end{aligned}\tag{9.3d}$$

$$\begin{aligned}
&\varepsilon=\varepsilon(-\mathbf{k}),\quad \varepsilon'=\varepsilon(-\mathbf{k}'),\quad \varepsilon''_\Delta=\varepsilon_\Delta(-\mathbf{k}-\mathbf{k}'), \\
&\omega=\omega(\mathbf{k}),\quad \omega'=\omega(\mathbf{k}').
\end{aligned}\tag{9.3e}$$

The $\pi+\Delta\Longleftrightarrow N$ vertex function $G_{\pi\Delta N}(\rho)=1$ when $W_{\pi\Delta}=m_N$.

For the σ exchange potential we find

$$\begin{aligned}
&B^{\sigma}_{\pi N,\pi N}(\mathbf{k}uim,\mathbf{k}'u'i'm') \\
&= \delta_{uu'}\delta_{ii'}s_{\sigma\pi\pi}s_{\sigma NN}\,G[(k_\pi-k'_\pi)^2,m_\sigma,\Lambda_{\sigma\pi\pi}]Z^{1/2}_{\pi N}(k) \\
&\quad\times Z^{1/2}_{\pi N}(k')G[(k_N-k'_N)^2,m_\sigma,\Lambda_{\sigma NN}] \\
&\quad\times\frac{k_\pi\cdot k'_\pi}{m_\pi^2}m_\pi m_N\,F(\mathbf{k},\mathbf{k}')\left[\frac{\varepsilon(-\mathbf{k})+m_N}{2m_N}\right]^{1/2} \\
&\quad\times\left[\frac{\varepsilon(-\mathbf{k}')+m_N}{2m_N}\right]^{1/2} \\
&\quad\times\left[\frac{1}{(k_\pi-k'_\pi)^2-m_\sigma^2}+\frac{1}{(k_N-k'_N)^2-m_\sigma^2}\right] \\
&\quad\times\chi^{N\dagger}_m\left[1-\frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{\varepsilon(-\mathbf{k})+m_N}\frac{\boldsymbol{\sigma}\cdot\mathbf{k}'}{\varepsilon(-\mathbf{k}')+m_N}\right]\chi^N_{m'},
\end{aligned}\tag{9.4}$$

where

$$G[t^2,m,\Lambda]=\left[\frac{(\Lambda^2-m^2)^2+\Lambda^4}{(\Lambda^2-t^2)^2+\Lambda^4}\right]^n\tag{9.5}$$

and

$$k_\pi=(\omega(\mathbf{k}),\mathbf{k}),\quad k_N=(\varepsilon(-\mathbf{k}),-\mathbf{k}).\tag{9.6}$$

For the ρ exchange potential we find

$$\begin{aligned}
&B^{\rho}_{\pi N,\pi N}(\mathbf{k}uim,\mathbf{k}'u'i'm') \\
&= s_{\rho\pi\pi}s_{\rho NN}\,G[(k_\pi-k'_\pi)^2,m_\rho,\Lambda_{\rho\pi\pi}]Z^{1/2}_{\pi N}(k)Z^{1/2}_{\pi N}(k') \\
&\quad\times G[(k_N-k'_N)^2,m_\rho,\Lambda_{\rho NN}]\left[\frac{1}{2}\boldsymbol{\varepsilon}^*_u\cdot\boldsymbol{\tau},\frac{1}{2}\boldsymbol{\varepsilon}_{u'}\cdot\boldsymbol{\tau}\right]_{ii'} \\
&\quad\times\left[\frac{\varepsilon(-\mathbf{k})+m_N}{2m_N}\right]^{1/2}\left[\frac{\varepsilon(-\mathbf{k}')+m_N}{2m_N}\right]^{1/2}m_N F(\mathbf{k},\mathbf{k}') \\
&\quad\times\chi^{N\dagger}_m\left[R(\mathbf{k},\mathbf{k}',m_N)+\frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{\varepsilon(-\mathbf{k})+m_N}\right. \\
&\quad\left.\times\frac{\boldsymbol{\sigma}\cdot\mathbf{k}'}{\varepsilon(-\mathbf{k}')+m_N}R(\mathbf{k},\mathbf{k}',-m_N)\right]\chi^N_{m'},
\end{aligned}\tag{9.7a}$$

where

$$\begin{aligned}
R(\mathbf{k},\mathbf{k}',m_N) &= \frac{S(\mathbf{k},\mathbf{k}',m_N)}{(k_N-k'_N)^2-m_\rho^2} \\
&\quad+\frac{S(\mathbf{k},\mathbf{k}',m_N)+(\kappa_\rho/2m_N)[W_{\pi N}(\mathbf{k})-W_{\pi N}(\mathbf{k}')][\varepsilon(-\mathbf{k})-\varepsilon(-\mathbf{k}')]}{(k_\pi-k'_\pi)^2-m_\rho^2}
\end{aligned}\tag{9.7b}$$

and

$$\begin{aligned}
&S(\mathbf{k},\mathbf{k}',m_N) \\
&= W_{\pi N}(\mathbf{k})+W_{\pi N}(\mathbf{k}')-2m_N \\
&\quad-\frac{\kappa_\rho}{2m_N}\{[W_{\pi N}(\mathbf{k})+W_{\pi N}(\mathbf{k}')][\varepsilon(-\mathbf{k}) \\
&\quad+\varepsilon(-\mathbf{k}')-2m_N]-2\left(k_N\cdot k'_N-m_N^2\right)\}.
\end{aligned}\tag{9.7c}$$

We now take for the interaction defined by Eq. (5.39)

$$\begin{aligned}
&B^{\text{pot}}_{\pi N,\pi N}(\mathbf{k}uim,\mathbf{k}'u'i'm') \\
&= B^{\overline{N}}_{\pi N,\pi N}(\mathbf{k}uim,\mathbf{k}'u'i'm')+B^{\Delta}_{\pi N,\pi N}(\mathbf{k}uim,\mathbf{k}'u'i'm') \\
&\quad+B^{\sigma}_{\pi N,\pi N}(\mathbf{k}uim,\mathbf{k}'u'i'm')+B^{\rho}_{\pi N,\pi N}(\mathbf{k}uim,\mathbf{k}'u'i'm').
\end{aligned}\tag{9.8}$$

This complete the specification of the propagators and effective potentials, that is, the B's that appear in the Lippmann-Schwinger integral equations, Eqs. (5.48)–(5.50), that we need to solve to determine the various amplitudes of the pion-nucleon system. We now turn our attention to simplifying these integral equations by carrying out a partial wave analysis.

X. PARTIAL WAVE EQUATIONS

We can simplify the coupled Lippmann-Schwinger equations, Eqs. (5.48)–(5.50), by coupling the particles' isospins to a total isospin T and component M, and by taking into account that the dependence of the effective potentials (i.e., the B's) on the particles' spin components, m, are through χ_m^N and χ_m^Δ. We write

$$\begin{aligned}&\sum_{ui}\sum_{u'i'}\langle 1T_\beta ui|TM\rangle\\&\quad\times A_{\pi\beta,\pi\beta'}(\mathbf{k}uim,\mathbf{k}'u'i'm';z)\langle 1T_{\beta'}u'i'|T'M'\rangle\\&=\delta_{TT'}\delta_{MM'}\chi_m^{\beta\dagger}A^T_{\pi\beta,\pi\beta'}(\mathbf{k},\mathbf{k}';z)\chi_{m'}^{\beta'},\\&A=X,B,\quad \beta \quad\text{and}\quad \beta'=N,\Delta.\end{aligned} \tag{10.1}$$

We must treat the amplitudes that involve the η meson differently since this meson has no isospin or spin. We write, for the effective potentials,

$$\sum_{ui}\langle 1,1/2,ui|TM\rangle B^d_{\pi N,\eta N}(\mathbf{k}uim,\mathbf{k}'i'm';z)$$
$$=\delta_{T,1/2}\delta_{Mi'}\chi_m^{N\dagger}B^{1/2}_{\pi N,\eta N}(\mathbf{k},\mathbf{k}';z)\chi_{m'}^N, \tag{10.2a}$$
$$\sum_{u'i'}B^d_{\eta N,\pi N}(\mathbf{k}im,\mathbf{k}'u'i'm';z)\langle 1,1/2,u'i'|T'M'\rangle$$
$$=\delta_{1/2,T'}\delta_{iM'}\chi_m^{N\dagger}B^{1/2}_{\eta N,\pi N}(\mathbf{k},\mathbf{k}';z)\chi_{m'}^N, \tag{10.2b}$$
$$B^d_{\eta N,\eta N}(\mathbf{k}im,\mathbf{k}'i'm';z)=\delta_{ii'}\chi_m^{N\dagger}B^{1/2}_{\eta N,\eta N}(\mathbf{k},\mathbf{k}';z)\chi_{m'}^N, \tag{10.2c}$$

and for the ηN-πN scattering amplitude

$$\sum_{u'i'}X_{\eta N,\pi N}(\mathbf{k}im,\mathbf{k}'u'i'm';z)\langle 1,1/2,u'i'|T'M'\rangle$$
$$=\delta_{1/2,T'}\delta_{iM'}\chi_m^{N\dagger}X^{1/2}_{\eta N,\pi N}(\mathbf{k},\mathbf{k}';z)\chi_{m'}^N. \tag{10.3}$$

With the help of these expressions, as well as Eq. (6.41), we can rewrite Eqs. (5.48)–(5.50) as

$$\begin{aligned}&X^T_{\pi N,\pi N}(\mathbf{k},\mathbf{k}';z)\\&=B^T_{\pi N,\pi N}(\mathbf{k},\mathbf{k}';z)+\int B^T_{\pi N,\pi N}(\mathbf{k},\mathbf{q};z)\\&\quad\times\frac{d^3q}{d_{\pi N}(q,z)}X^T_{\pi N,\pi N}(\mathbf{q},\mathbf{k}';z)+\int B^T_{\pi N,\pi\Delta}(\mathbf{k},\mathbf{q};z)d^3q\\&\quad\times\int\Gamma(\widehat{\mathbf{q}},\mathbf{n};q,z)d\Omega(\mathbf{n})X^T_{\pi\Delta,\pi N}(q\mathbf{n},\mathbf{k}';z)\\&\quad+\delta_{T,1/2}\int B^{1/2}_{\pi N,\eta N}(\mathbf{k},\mathbf{q};z)\\&\quad\times\frac{d^3q}{z-W_{\eta N}(q)}X^{1/2}_{\eta N,\pi N}(\mathbf{q},\mathbf{k}';z),\end{aligned} \tag{10.4}$$

$$\begin{aligned}&X^T_{\pi\Delta,\pi N}(\mathbf{k},\mathbf{k}';z)\\&=B^T_{\pi\Delta,\pi N}(\mathbf{k},\mathbf{k}';z)+\int B^T_{\pi\Delta,\pi N}(\mathbf{k},\mathbf{q};z)\\&\quad\times\frac{d^3q}{d_{\pi N}(q,z)}X^T_{\pi N,\pi N}(\mathbf{q},\mathbf{k}';z)+\int B^T_{\pi\Delta,\pi\Delta}(\mathbf{k},\mathbf{q};z)d^3k''\\&\quad\times\int\Gamma(\widehat{\mathbf{q}},\mathbf{n};q,z)d\Omega(\mathbf{n})X^T_{\pi\Delta,\pi N}(q\mathbf{n},\mathbf{k}';z),\end{aligned} \tag{10.5}$$

$$\begin{aligned}&X^{1/2}_{\eta N,\pi N}(\mathbf{k},\mathbf{k}';z)\\&=B^{1/2}_{\eta N,\pi N}(\mathbf{k},\mathbf{k}';z)+\int B^{1/2}_{\eta N,\pi N}(\mathbf{k},\mathbf{q};z)\\&\quad\times\frac{d^3q}{d_{\pi N}(q,z)}X^{1/2}_{\pi N,\pi N}(\mathbf{q},\mathbf{k}';z)\\&\quad+\int B^{1/2}_{\eta N,\eta N}(\mathbf{k},\mathbf{q};z)\frac{d^3q}{z-W_{\eta N}(q)}X^{1/2}_{\eta N,\pi N}(q\mathbf{n},\mathbf{k}';z).\end{aligned} \tag{10.6}$$

Our partial wave amplitudes are constructed using the angular momentum eigenstates

$$Y^M_{LNJ}(\widehat{\mathbf{k}})=Y^M_{L,1/2,J}(\widehat{\mathbf{k}}),\quad Y^M_{L\Delta J}(\widehat{\mathbf{k}})=Z^M_{LJ}(\widehat{\mathbf{k}}), \tag{10.7}$$

where $Y^M_{L,1/2,J}(\widehat{\mathbf{k}})$ is defined by Eq. (6.21) and $Z^M_{LJ}(\widehat{\mathbf{k}})$ is defined by Eq. (6.24). For those amplitudes that involve only nucleons we write

$$\begin{aligned}&\int d\Omega(\mathbf{k})d\Omega(\mathbf{k}')Y^{M\dagger}_{LNJ}(\widehat{\mathbf{k}})A^T_{\mu N,\mu' N}(\mathbf{k},\mathbf{k}';z)Y^{M'}_{L'NJ'}(\widehat{\mathbf{k}}')\\&\quad=\delta_{JJ'}\delta_{MM'}\delta_{LL'}A^{TJ}_{\mu N,\mu' N}(k,k',L;z),\\&A=X,B,\ \mu\quad\text{and}\quad\mu'=\pi,\eta,\end{aligned} \tag{10.8}$$

where we have used the fact that conservation of total angular momentum and parity require that the orbital angular momentum is conserved. For the amplitudes involving the Δ we can write

$$\begin{aligned}&\int d\Omega(\mathbf{k})d\Omega(\mathbf{k}')Y^{M\dagger}_{L\beta J}(\widehat{\mathbf{k}})A^T_{\pi\beta,\pi\beta'}(\mathbf{k},\mathbf{k}';z)Y^{M'}_{L'\beta'J'}(\widehat{\mathbf{k}}')\\&\quad=\delta_{JJ'}\delta_{MM'}A^{TJ}_{\pi\beta,\pi\beta'}(k,L,k',L';z),\\&A=X,B,\ \beta,\beta'=N,\Delta,\quad\text{or}\ \Delta,N,\quad\text{or}\ \Delta,\Delta.\end{aligned} \tag{10.9}$$

TABLE II. πN-$\pi\Delta$ couplings.

J	L	L''
1/2	0	2
1/2	1	1
$\geqslant 3/2$	$J-1/2$	$J-1/2,\ J+3/2$
$\geqslant 3/2$	$J+1/2$	$J-3/2,\ J+1/2$

TABLE III. $\pi\Delta$-$\pi\Delta$ couplings.

J	L	L''
1/2	1	1
1/2	2	2
$\geqslant 3/2$	$J-1/2,\ J+3/2$	$J-1/2,\ J+3/2$
$\geqslant 3/2$	$J-3/2,\ J+1/2$	$J-3/2,\ J+1/2$

Using these expressions along with Eq. (6.39) for $\Gamma(\mathbf{n},\mathbf{n}';k,z)$ we find that Eqs. (10.4)–(10.6) lead to

$$\begin{aligned} X^{TJ}_{\pi N,\pi N}&(k,k',L;z)\\ &= B^{TJ}_{\pi N,\pi N}(k,k',L;z)+\int_0^\infty B^{TJ}_{\pi N,\pi N}(k,k'',L;z)\\ &\quad\times\frac{k''^2dk''}{d_{\pi N}(k'',z)}X^{TJ}_{\pi N,\pi N}(k'',k',L;z)\\ &\quad+\sum_{L''}\int_0^\infty B^{TJ}_{\pi N,\pi\Delta}(k,L,k'',L'';z)\\ &\quad\times\frac{k''^2dk''}{d_{\pi\Delta}(k'',L'',z)}X^{TJ}_{\pi\Delta,\pi N}(k'',L'',k',L;z)\\ &\quad+\delta_{T,1/2}\delta_{J,1/2}\int_0^\infty B^{1/2,1/2}_{\pi N,\eta N}(k,k'',0;z)\\ &\quad\times\frac{k''^2dk''}{z-W_{\eta N}(k'')}X^{1/2,1/2}_{\eta N,\pi N}(k'',k',0;z), \end{aligned}\tag{10.10}$$

$$\begin{aligned} X^{TJ}_{\pi\Delta,\pi N}&(k,L,k',L';z)\\ &= B^{TJ}_{\pi\Delta,\pi N}(k,L,k',L';z)+\int_0^\infty B^{TJ}_{\pi\Delta,\pi N}(k,L,k'',L';z)\\ &\quad\times\frac{k''^2dk''}{d_{\pi N}(k'',z)}X^{TJ}_{\pi N,\pi N}(k'',k',L';z)\\ &\quad+\sum_{L''}\int_0^\infty B^{TJ}_{\pi\Delta,\pi\Delta}(k,L,k'',L'';z)\\ &\quad\times\frac{k''^2dk''}{d_{\pi\Delta}(k'',L'',z)}X^{TJ}_{\pi\Delta,\pi N}(k'',L'',k',L';z), \end{aligned}\tag{10.11}$$

$$\begin{aligned} X^{1/2,1/2}_{\eta N,\pi N}&(k,k',0;z)\\ &= B^{1/2,1/2}_{\eta N,\pi N}(k,k',0;z)+\int_0^\infty B^{1/2,1/2}_{\eta N,\pi N}(k,k'',0;z)\\ &\quad\times\frac{k''^2dk''}{d_{\pi N}(k'',z)}X^{1/2,1/2}_{\pi N,\pi N}(k'',k',0;z)\\ &\quad+\int_0^\infty B^{1/2,1/2}_{\eta N,\eta N}(k,k'',0;z)\\ &\quad\times\frac{k''^2dk''}{z-W_{\eta N}(k'')}X^{1/2,1/2}_{\eta N,\pi N}(k'',k',0;z). \end{aligned}\tag{10.12}$$

In deriving Eqs. (10.10) and (10.12) we have used the fact that the πN-ηN and ηN-ηN interactions act only in $T=J=1/2$ states with $L_{\pi N}=L_{\eta N}=0$. The limits on the sums on L'' in Eqs. (10.10) and (10.11) are determined by conservation of total angular momentum and parity. In Eq. (10.10) the limits are given by Table II; for Eq. (10.11) they are given by Table III.

XI. PARTIAL WAVE MATRIX ELEMENTS

The direct interactions $B^d_{\mu\beta,\mu'\beta'}$ are given by Eqs. (5.28)–(5.30) with the various vertex functions $U_{\mu\beta,\beta'}$ given by Eqs. (7.1), (7.3), and (7.5)–(7.12). We couple the isospins according to Eqs. (10.1) and (10.2). Making these couplings is easily carried out with the help of the identities

$$\sum_{ui}\langle 1T_\beta ui|TM\rangle(\boldsymbol{\varepsilon}^*_u\cdot\mathbf{T}_{\beta\beta'})_{ii'}=\delta_{T,T_{\beta'}}\delta_{M,i'},\tag{11.1}$$

$$\boldsymbol{\tau}=-\sqrt{3}\mathbf{T}_{NN}=-\sqrt{3}\mathbf{T}_{1/2,1/2},\tag{11.2}$$

$$\mathbf{X}^\dagger_{\beta\beta'}=(-1)^{\beta-\beta'}\sqrt{\frac{2\beta'+1}{2\beta+1}}\mathbf{X}_{\beta'\beta},\tag{11.3}$$

$$\mathbf{X}=\mathbf{S},\mathbf{T}.$$

These relations follow from Eq. (6.10). In Eq. (11.3) β and β' denote the baryons or their spins or isospins. We note that Eq. (11.2) is also valid for $\boldsymbol{\tau}\to\boldsymbol{\sigma}$ and $\mathbf{T}\to\mathbf{S}$.

According to Eqs. (10.8) and (10.9) we need to calculate the integrals

$$\int d\Omega(\mathbf{k})d\Omega(\mathbf{k}')Y^{M\dagger}_{L\beta J}(\widehat{\mathbf{k}})B^d_{\mu\beta,\mu'\beta'}(\mathbf{k},\mathbf{k}';z)Y^{M'}_{L'\beta'J'}(\widehat{\mathbf{k}}'),$$

which are diagonal in J and M and, for $\beta=\beta'=N$, also diagonal in L. With the help of Eqs. (6.21), (6.22), (6.24), and (6.25) it is reasonably straightforward to carry out these integrations.

The crossed potentials are given by Eqs. (8.1)–(8.7). Coupling the isospins as shown in Eq. (10.1) can be carried out by using the identity

$$\begin{aligned}\sum_{ui}\sum_{u'i'}&\langle 1\beta ui|TM\rangle(\boldsymbol{\varepsilon}_{u'}\cdot\mathbf{T}_{\beta\beta''}\boldsymbol{\varepsilon}^*_u\cdot\mathbf{T}^\dagger_{\beta'\beta''})_{ii'}\langle 1\beta'u'i'|T'M'\rangle\\ &=-\delta_{TT'}\delta_{MM'}(2\beta''+1)\begin{Bmatrix}1&\beta'&T\\1&\beta&\beta''\end{Bmatrix}.\end{aligned}\tag{11.4}$$

This identity follows from Eq. (6.10) and the fact that the $6-j$ symbol can be expressed as a sum of products of Clebsch-Gordon coefficients. This identity leads to the results

$$\begin{aligned}\sum_{ui}\sum_{u'i'}&\langle 1,1/2,ui|TM\rangle(\boldsymbol{\varepsilon}_{u'}\cdot\boldsymbol{\tau}\ \boldsymbol{\varepsilon}^*_u\cdot\boldsymbol{\tau})_{ii'}\langle 1,1/2,u'i'|T'M'\rangle\\ &=\delta_{TT'}\delta_{MM'}(-\delta_{T,1/2}+2\delta_{T,3/2}),\end{aligned}\tag{11.5}$$

$$\sum_{ui}\sum_{u'i'}\langle 1,1/2,ui|TM\rangle(\boldsymbol{\varepsilon}_{u'}\cdot\boldsymbol{\tau}\ \boldsymbol{\varepsilon}_u^*\cdot\mathbf{T}_{N\Delta})_{ii'}\times\langle 1,3/2,u'i'|T'M'\rangle = \delta_{TT'}\delta_{MM'}\left(\sqrt{\frac{8}{3}}\delta_{T,1/2}-\sqrt{\frac{5}{3}}\delta_{T,3/2}\right), \tag{11.6}$$

$$\sum_{ui}\sum_{u'i'}\langle 1,3/2,ui|TM\rangle(\boldsymbol{\varepsilon}_{u'}\cdot\mathbf{T}_{N\Delta}^\dagger\boldsymbol{\varepsilon}_u^*\cdot\mathbf{T}_{N\Delta})_{ii'}\times\langle 1,3/2,u'i'|T'M'\rangle = \delta_{TT'}\delta_{MM'}\left(\frac{1}{3}\delta_{T,1/2}-\frac{2}{3}\delta_{T,3/2}+\delta_{T,5/2}\right). \tag{11.7}$$

In carrying out the partial wave integrations complications arise from the matrix operators $\mathbf{S}_{\beta\beta'}$ that couple the spin states of the baryons β and β'. To carry out the integrals in Eqs. (10.8) and (10.9) it is convenient to reorder the matrix operators $\widehat{\mathbf{k}}\cdot\mathbf{S}_{\beta\beta'}$ and $\widehat{\mathbf{k}}'\cdot\mathbf{S}_{\beta''\beta'''}$ that appear in the crossed potentials so that the $\widehat{\mathbf{k}}\cdot\mathbf{S}_{\beta\beta'}$ operators are to the left of the $\widehat{\mathbf{k}}'\cdot\mathbf{S}_{\beta''\beta'''}$ operators. This can be done by using the identity given by Eq. (4.12) of Ref. [7], that is,

$$(\widehat{\mathbf{k}}'\cdot\mathbf{S}_{\alpha\beta})(\widehat{\mathbf{k}}\cdot\mathbf{S}_{\beta\gamma}) = \sum_\lambda(-1)^{\beta+\lambda+1-2\gamma}\sqrt{(2\beta+1)(2\lambda+1)}\times\begin{Bmatrix}1 & \gamma & \lambda\\ 1 & \alpha & \beta\end{Bmatrix}(\widehat{\mathbf{k}}\cdot\mathbf{S}_{\alpha\lambda})(\widehat{\mathbf{k}}'\cdot\mathbf{S}_{\lambda\gamma}). \tag{11.8}$$

Some simplifications can by carried out with the help of the "sum rule"

$$\sum_{\beta'=|1-\beta|}^{1+\beta}(2\beta'+1)(\widehat{\mathbf{k}}\cdot\mathbf{S}_{\beta'\beta}^\dagger)(\widehat{\mathbf{k}}'\cdot\mathbf{S}_{\beta'\beta}) = (2\beta+1)(\widehat{\mathbf{k}}\cdot\widehat{\mathbf{k}}')I^{(\beta)}. \tag{11.9}$$

The action of the matrix operators on the matrix-functions, $Y_{L\beta J}^M(\widehat{\mathbf{k}})$, defined by Eq. (10.7), can then be worked out with the help of Eqs. (6.22) and (6.24). The results for the final integrals that are encountered all follow from the identity

$$\int d\Omega(\mathbf{k})d\Omega(\mathbf{k}')Y_{LSJ}^{M\dagger}(\widehat{\mathbf{k}})F(\mathbf{k},\mathbf{k}';z)Y_{L'SJ'}^{M'}(\widehat{\mathbf{k}}') = 2\pi\int_{-1}^1 dx\ P_L(x)F(\mathbf{k},\mathbf{k}';z),$$
$$x=\widehat{\mathbf{k}}\cdot\widehat{\mathbf{k}}',\quad S=1/2,3/2. \tag{11.10}$$

The $\overline{N}$ exchange potential is given by Eq. (9.1) and has the same isospin factor as the N exchange potential, so coupling its isospins also leads to Eq. (11.5).

The Δ exchange potential is given by Eq. (9.3) and coupling its isospins leads to the result

$$\sum_{ui}\sum_{u'i'}\langle 1,1/2,ui|TM\rangle(\boldsymbol{\varepsilon}_{u'}\cdot\mathbf{T}_{N\Delta}\boldsymbol{\varepsilon}_u^*\cdot\mathbf{T}_{N\Delta}^\dagger)_{ii'}\times\langle 1,1/2,u'i'|T'M'\rangle = \delta_{TT'}\delta_{MM'}\left(\frac{4}{3}\delta_{T,1/2}+\frac{1}{3}\delta_{T,3/2}\right). \tag{11.11}$$

The σ exchange potential is given by Eq. (9.4) and coupling its isospins simply leads to the factor $\delta_{TT'}\delta_{MM'}$.

The ρ exchange potential is given by Eq. (9,7). Its isospins can be coupled by first using the anticommutation rules for the components of $\boldsymbol{\tau}$ to derive the identity

$$\left[\tfrac{1}{2}\boldsymbol{\varepsilon}_u^*\cdot\boldsymbol{\tau},\,\tfrac{1}{2}\boldsymbol{\varepsilon}_{u'}\cdot\boldsymbol{\tau}\right]_{ii'} = \tfrac{1}{2}[(\boldsymbol{\varepsilon}_u^*\cdot\boldsymbol{\tau}\ \boldsymbol{\varepsilon}_{u'}\cdot\boldsymbol{\tau})_{ii'}-\delta_{uu'}\delta_{ii'}], \tag{11.12}$$

which in turn leads to

$$\sum_{ui}\sum_{u'i'}\langle 1,1/2,ui|TM\rangle\left[\frac{1}{2}\boldsymbol{\varepsilon}_u^*\cdot\boldsymbol{\tau},\,\frac{1}{2}\boldsymbol{\varepsilon}_{u'}\cdot\boldsymbol{\tau}\right]_{ii'}\times\langle 1,1/2,u'i'|T'M'\rangle = \delta_{TT'}\delta_{MM'}\left(\delta_{T,1/2}-\frac{1}{2}\delta_{T,3/2}\right). \tag{11.13}$$

The spin structure of the $\overline{N}$, Δ, σ, and ρ exchange potentials is essentially the same as that of the crossed N exchange potential given by Eqs. (8.3) and (8.4), so the partial wave integrations go through in the same way.

XII. SOLVING THE EQUATIONS

The function $Z_{\pi N}(k)$ is defined by Eq. (5.22) with $F_{\pi NN}$ given by Eq. (6.8). $Z_{\pi N}^{1/2}(k)$ appears in the following combinations: in $Z_{\pi N}^{1/2}(k)s_{\pi N\beta}^{(0)}$ with β any of the baryons, in $Z_{\pi N}^{1/2}(k)s_{\pi N\beta}$ with $\beta=N,\Delta$, and in $Z_{\pi N}^{1/2}(k)s_{\mu\pi\pi}$ and $Z_{\pi N}^{1/2}(k)s_{\mu NN}$ with $\mu=\sigma,\rho$. We will interpret a combination such as $Z_{\pi N}^{1/2}(k)s_{\pi N\beta}$ as a renormalized coupling constant $g_{\pi N\beta}$, which is independent of k. This implies that $s_{\pi N\beta}(k)=g_{\pi N\beta}/Z_{\pi N}^{1/2}(k)$ depends on k. This k dependence does not affect the Poincaré invariance of the model, and as a practical matter we shall see that the dependence of $Z_{\pi N}(k)$ on k is very weak. A summary of the renormalized coupling constants is given here:

$$\begin{aligned}
g_{\pi N\beta}^{(0)} &= Z_{\pi N}^{1/2}(k)s_{\pi N\beta}^{(0)}(k),\quad \beta=N,\Delta,R,D,S,S',\\
g_{\pi N\beta} &= Z_{\pi N}^{1/2}(k)s_{\pi N\beta}(k),\quad \beta=N,\Delta,\\
g_{\mu\pi\pi} &= Z_{\pi N}^{1/2}(k)s_{\mu\pi\pi}(k),\\
g_{\mu NN} &= Z_{\pi N}^{1/2}(k)s_{\mu NN}(k),\quad \mu=\sigma,\rho.
\end{aligned} \tag{12.1}$$

For any coupling constant that does not appear here we will replace s with g, so that from now on all coupling constants will be denoted by g's.

We can express $Z_{\pi N}(k)$ in terms of $g_{\pi NN}$ rather than $s_{\pi NN}$. It follows from Eqs. (5.22) and (6.8) that

$$\begin{aligned}
Z_{\pi N}(k) &= 1-g_{\pi NN}^2\,I_{\pi N}(k),\\
I_{\pi N}(k) &= \int d^3\rho\frac{s_{\pi NN}^{-2}\,F_{\pi NN}(\rho;k)}{[\varepsilon(k)-E_{\pi N}(k,\rho)]^2},
\end{aligned} \tag{12.2}$$

where $I_{\pi N}(k)$ does not depend on $g_{\pi NN}$ or $s_{\pi NN}$. It should be noted that if $g_{\pi NN}^2\,I_{\pi N}(k)>1$ then $Z_{\pi N}^{1/2}(k)$ is pure imaginary, which in turn implies that the s's in Eqs. (12.1) are pure imaginary. This destroys the hermiticity of the mass operator and thereby leads to unphysical results. Fortunately this does not turn out to be the case here.

We now consider the singularities that appear in the kernels of our integral equations (5.48)–(5.50), (10.4)–(10.6), and (10.10)–(10.12). There are two sources of these singularities. They appear in the meson-baryon propagators, $d_{\pi N}^{-1}(q,z)$, $d_{\pi\Delta}^{-1}(q,L,z)$, and $[z - W_{\eta N}(q)]^{-1}$, and in the crossed potentials, $B^c_{\pi\beta,\pi\beta'}$. The propagator $d_{\pi\Delta}^{-1}(q,L,z)$ appears in $\Gamma(\mathbf{n},\mathbf{n}';q,z)$, which is defined by Eq. (6.39). The crossed potentials, which are given by Eqs. (8.3)–(8.7), contribute to the potentials $B^T_{\pi\beta,\pi\beta'}(\mathbf{k},\mathbf{q};z)$ that appear in Eqs. (10.4) and (10.5) and are defined by Eqs. (5.40) and (10.1).

The pion-nucleon propagator, $d_{\pi N}^{-1}(q,z)$, is given by Eq. (5.23) and has singularities resulting from the vanishing of $z - W_{\pi N}(q)$ and $z - W_{\pi\pi N}(q,\rho)$, which appears in the integral over ρ. If we write

$$z = W + i\varepsilon = W_{\pi N}(k') + i\varepsilon, \tag{12.3}$$

where ε is an infinitesimal positive parameter and k' is the momentum of the incident pion, we can easily show that there is a simple pole in the πN propagator at $q = k' + i\eta$, where η is an infinitesimal positive parameter.

The vanishing of $z - W_{\pi\pi N}(q,\rho)$ leads to a branch cut in q. Writing

$$\begin{aligned} W_{\pi\pi N}(q,\rho) &= \omega(q) + E_{\pi N}(q,\rho) \\ &= \omega(q) + \left[q^2 + W_{\pi N}^2(\rho)\right]^{1/2}, \end{aligned} \tag{12.4}$$

we can easily solve the equation $z - W_{\pi\pi N}(q,\rho) = 0$ to show that

$$\omega(q) = \frac{z^2 - W_{\pi N}^2(\rho) + m_\pi^2}{2z}. \tag{12.5}$$

Letting $z = W + i\varepsilon$, and expanding to first order in ε, we find

$$\omega(q) = \frac{W^2 - W_{\pi N}^2(\rho) + m_\pi^2}{2W} + i\varepsilon\frac{W^2 + W_{\pi N}^2(\rho) - m_\pi^2}{W}. \tag{12.6}$$

Since $W_{\pi N}(\rho) = \omega(\rho) + \varepsilon(\rho) \geqslant m_\pi + m_N$ we have

$$\frac{W^2 - W_{\pi N}^2(\rho) + m_\pi^2}{2W} \leqslant \omega_{\max}, \tag{12.7}$$

where

$$\omega_{\max} = \frac{W^2 - 2m_\pi m_N - m_N^2}{2W}. \tag{12.8}$$

Using Eq. (12.8) in Eq. (12.6) we find that the branch cut in the ω plane is given by

$$\omega(q) = \omega_{\max} - \frac{W_{\pi N}^2(\rho) - (m_\pi + m_N)^2}{2W} + i\eta. \tag{12.9}$$

We see that the branch cut in the ω plane begins at the branch point $\omega_{\max}$ and runs to the left just above the the real axis. In the q plane the branch point is at $k_{\max}$, which is defined by $\omega(k_{\max}) = \omega_{\max}$ and is given by

$$k_{\max} = \frac{\left(W^2 - m_N^2\right)^{1/2}[W^2 - (2m_\pi + m_N)^2]^{1/2}}{2W}. \tag{12.10}$$

In the q plane the branch cut starts at $k_{\max}$ and runs just above the real axis until $k = 0$, at which point it continues along the positive imaginary axis.

The interpretation of $\omega_{\max}$ and $k_{\max}$ is quite simple. Since $W_{\pi\pi N}(q,\rho)$ is the c.m. energy of two pions and a nucleon, and since $\omega(q)$ solves the equation $W_{\pi\pi N}(q,\rho) = W$, then $\omega_{\max}$ and $k_{\max}$ are the maximum energy and momentum a pion can have in the c.m. frame of the $\pi\pi N$ system, given that the total c.m. energy of the system is W. Of course for the $\pi\pi N$ system we must have $W \geqslant 2m_\pi + m_N$, which is reflected in the formula for $k_{\max}$.

To determine the location of the pole in $d_{\pi N}^{-1}(q,z)$ relative to the branch cut we need to solve

$$W = W_{\pi N}(k') = \omega(k') + \varepsilon(k') \tag{12.11}$$

for $\omega(k')$ or k'. If we define $k'_\pi = (\omega(k'),\mathbf{k}')$, $k'_N = (\varepsilon(k'),-\mathbf{k}')$, and $k'_{\pi N} = (\omega(k') + \varepsilon(k'),\mathbf{0})$, then using the trivial identity $k'_\pi \cdot k'_{\pi N} = [(k'_\pi + k'_N) + (k'_\pi - k'_N)]\cdot k'_{\pi N}/2$ leads to the result

$$\omega(k') = \frac{W^2 + m_\pi^2 - m_N^2}{2W}. \tag{12.12}$$

From here and Eq. (12.8) we find

$$\omega(k') - \omega(k_{\max}) = \frac{2m_\pi m_N + m_\pi^2}{2W}. \tag{12.13}$$

Therefore $k' > k_{\max}$, and the pole does not lie on the cut. We note that Eq. (12.12) can be solved for k' to give

$$k' = \frac{[W^2 - (m_\pi + m_N)^2]^{1/2}[W^2 - (m_\pi - m_N)^2]^{1/2}}{2W}. \tag{12.14}$$

The $\pi\Delta$ propagator, $d_{\pi\Delta}^{-1}(q,L,z)$, defined by Eqs. (6.32) and (6.27), does not have a simple pole just above the real q axis resulting from the vanishing of $d_{\pi\Delta}(q,L,z)$, but it does have the branch cut given by Eq. (12.9). The ηN propagator, $[z - W_{\eta N}(q)]^{-1}$, does have a simple pole just above the real q axis resulting from the vanishing of $z - W_{\eta N}(q)$ when $W \geqslant m_\eta + m_N$.

According to Eqs. (8.3), (8.5), and (8.7) the singularities in $B^c_{\pi\beta,\pi\beta'}$ arise when

$$W - \omega(\mathbf{k}) - \varepsilon(-\mathbf{k}-\mathbf{q}) - \omega(\mathbf{q}) = 0. \tag{12.15}$$

For now we ignore the $i\varepsilon$ term in $z = W + i\varepsilon$. We note that according to the interpretation of $\omega_{\max}$ given here that for real momenta we must have $\omega(\mathbf{q}) \leqslant \omega_{\max}$ if Eq. (12.15) is to be satisfied. As a result of this the Born or inhomogeneous terms in Eqs. (10.4) and (10.5) do not suffer from the singularities given by Eq. (12.15), since in those terms $q = k'$ and $k' > k_{\max}$. These singularities only play a role in the kernels of the integral equations.

Solving Eq. (12.15) for q comes down to solving a quadratic equation to show that

$$q_\pm(k,x,W) = \frac{-kxf(k,W) \pm [W - \omega(k)]\,g(k,x,W)}{d(k,x,W)}, \tag{12.16}$$

where

$$f(k,W) = W^2 - 2W\omega(k) + 2m_\pi^2 - m_N^2, \tag{12.17a}$$

$$d(k,x,W) = 2\{[W - \omega(k)]^2 - k^2x^2\}, \tag{12.17b}$$

$$\begin{aligned} g(k,x,W) = \{&\left[W^2 - 2W\omega(k) - m_N^2\right]^2 \\ &- 4m_\pi^2\left[m_N^2 + k^2(1-x^2)\right]\}^{1/2}. \end{aligned} \tag{12.17c}$$

Instead of solving Eq. (12.15) for q it is possible to solve it for $\omega(q)$ to obtain

$$\omega_{\pm}(k, x, W) = \frac{[W - \omega(k)]f(k, W) \pm kxg(k, x, W)}{d(k, x, W)}. \quad (12.18)$$

We note that $g^2(k, x, W)$ can be positive or negative; therefore g can be real or pure imaginary, respectively. For fixed k and W the minimum value of $g^2(k, x, W)$ occurs when $x = 0$. We can find the values of $\omega(k)$ at which this minimum value is zero by solving

$$\left(W^2 - 2W\omega - m_N^2\right)^2 - 4m_\pi^2\left(m_N^2 - m_\pi^2 + \omega^2\right) = 0. \quad (12.19)$$

The results are

$$\omega(k_\pm) = \frac{W^2 \pm 2Wm_\pi + 2m_\pi^2 - m_N^2}{2(W \pm m_\pi)}, \quad (12.20)$$

which can be solved for $k_\pm$ to yield

$$k_\pm = \frac{W}{W \pm m_\pi}\left(\frac{W \pm 2m_\pi \mp m_N}{W \mp 2m_\pi \mp m_N}\right)^{1/2} k_{\max}. \quad (12.21)$$

With the roots of Eq. (12.29) in hand we can rewrite $g(k, x, W)$ as

$$g(k, x, W) = 2\left\{\left(W^2 - m_\pi^2\right)[\omega(k_+) - \omega(k)] \times [\omega(k_-) - \omega(k)] + m_\pi^2 k^2 x^2\right\}^{1/2}. \quad (12.22)$$

It is of interest to compare $\omega(k_+)$ and $\omega(k_-)$ with each other and with $\omega_{\max}$, which is given by Eq. (12.8). Straightforward algebra shows that

$$\frac{2W(W - m_\pi)}{m_\pi}[\omega_{\max} - \omega(k_-)] = W^2 - 2(m_\pi + m_N)W + m_N(2m_\pi + m_N). \quad (12.23)$$

The right-hand side of this equation is a monotonically increasing function of W for $W \geqslant m_\pi + m_N$. Since this equation is meaningful for $W \geqslant 2m_\pi + m_N$ we see that the minimum value of the right-hand side occurs when $W = 2m_\pi + m_N$, which yields the value 0; therefore $\omega_{\max} - \omega(k_-) \geqslant 0$. Straightforward algebra also shows that

$$\frac{2W(W + m_\pi)}{m_\pi}[\omega(k_+) - \omega_{\max}] = W^2 + 2(m_\pi + m_N)W + m_N(2m_\pi + m_N) > 0. \quad (12.24)$$

Therefore $\omega(k_+) - \omega_{\max} > 0$. Summarizing we have

$$\omega(k_-) \leqslant \omega_{\max} < \omega(k_+), \quad (12.25a)$$

$$k_- \leqslant k_{\max} < k_+. \quad (12.25b)$$

According to Eqs. (12.22) and (12.25) $g^2(k, x, W) \geqslant 0$ for $0 \leqslant k \leqslant k_-$, so the branch cuts given by Eq. (12.16) lie near the real q axis. An example of this is given in Fig. 1.

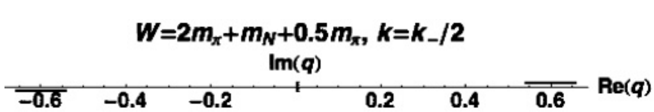

FIG. 1. $\pi\pi N$ branch cuts for $W = 1288$ MeV and $k = 71$ MeV/c.

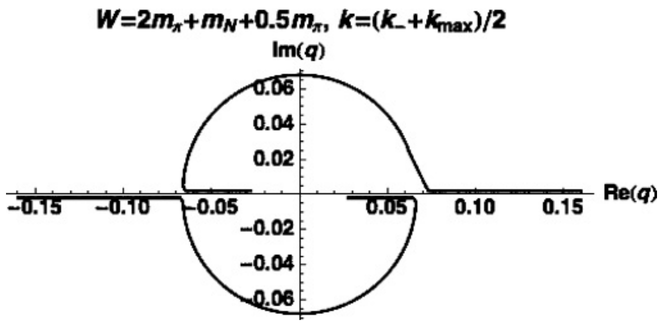

FIG. 2. $\pi\pi N$ branch cuts for $W = 1288$ MeV and $k = 142$ MeV/c.

For $k_- \leqslant k \leqslant k_{\max}$ we have $[\omega(k_+) - \omega(k)][\omega(k_-) - \omega(k)] \leqslant 0$, so $g^2(k, x, W) \leqslant 0$ for $x^2 \leqslant x_0^2$, where

$$x_0^2 = \left(W^2 - m_\pi^2\right)[\omega(k_+) - \omega(k)] \times [\omega(k) - \omega(k_-)]\big/\left(m_\pi^2 k^2\right), \quad k_- \leqslant k \leqslant k_{\max}. \quad (12.26)$$

Under these conditions the branch cuts appear as in Fig. 2.

In our integral equations the integration path runs along the positive, real q axis. As a result of this the pole in the πN propagator at $q = k' + i\eta$, the pole in the ηN propagator when $W + i\varepsilon - W_{\eta N}(q) = 0$, the πN and $\pi\Delta$ propagators' branch cuts, given by Eq. (12.9), and the cuts given by Eq. (12.16), lead to singular integrals. To deal with the singularities we deform the contour of integration. In a very abbreviated notation our integral equations are given by [see Eqs. (10.10)–(10.12)]

$$X_{aa'}(k, k'; z) = B_{aa'}(k, k'; z) + \sum_b \int_0^\infty B_{ab}(k, q; z) \times \frac{q^2 dq}{d_b(q, z)} X_{ba'}(q, k'; z), \quad (12.27)$$

where a, a', and b are cover indices for particle names and angular momentum quantum numbers. We deform the q integration path from the positive, real q axis to a path such as that shown in Fig. 3. Our deformed contour is given by

$$\zeta(q) = q - isq\exp(-q/q_0). \quad (12.28)$$

The slope of the contour at $q = 0$ is $-s$, whereas the minimum value of $\mathrm{Im}[\zeta(q)]$ occurs at $q = q_0$. Having let $q \to \zeta(q)$ in Eq. (12.27) we see that we need to determine our unknown amplitudes, $X_{ba'}(q, k'; z)$, along the deformed contour. To deal with this we let $k \to \zeta(k)$ in Eq. (12.27), which gives us integral equations along the deformed contour. Of course we must verify that these "deformed" integral equations are nonsingular. Straightforward numerical calculations show that the new locations for the branch cuts are not near the integration path shown in Fig. 3. The poles in the πN and

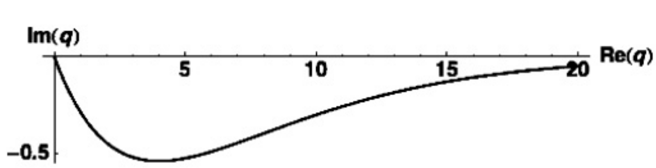

FIG. 3. Deformed integration contour.

TABLE IV. Baryon and meson masses in MeV.

$m_N = 938.92$	$m_\Delta = 1232.0$	$m_R = 1440.0$
$m_D = 1520.0$	$m_S = 1535.0$	$m_{S'} = 1620.0$
$m_\pi = 139.57$	$m_\sigma = 418.12$	$m_\eta = 547.51$
$m_\rho = 769.0$		

ηN propagators and the πN and $\pi\Delta$ propagators' branch cuts are just above the positive, real q axis, so these singularities are also not near the integration path.

The physical amplitudes for elastic, πN scattering are given by the on-shell amplitude $X^{TJ}_{\pi N,\pi N}[k', k', L; W_{\pi N}(k' + i\varepsilon)]$. According to Eq. (12.27) calculating this amplitude entails evaluating $B_{ab}(k', q; z)$ with q on the deformed contour. It is straightforward to show that the branch cuts in $B_{ab}(k', q; z)$ are far removed from the deformed contour, so there is no problem.

The integral equations are solved numerically by replacing the integrations by quadrature rules, which transforms the equations into linear matrix equations. Such equations can readily be solved by using standard matrix routines. The quadrature rules we use are obtained by mapping the standard Gauss-Legendre quadrature rules on the $-1 < x < 1$ interval to the $0 < q < \infty$ interval by means of the mapping

$$q = c\frac{1+x}{1-x}, \quad -1 < x < 1. \tag{12.29}$$

In general we have chosen $c = 0.5$.

In Sec. XIII we present the results obtained using the just described method for solving our singular integral equations.

XIII. RESULTS

Our model contains a number of parameters that at the present time can only be obtained by fitting the model to the data. These parameters consist of the coupling constants and cutoff masses associated with the various vertices, the bare masses associated with the direct interactions, as well as the mass of the σ meson. The data on the pion-nucleon system are available from the Center for Nuclear Studies (CNS), which is associated with George Washington University (GWU). The Web address for CNS is http://www.gwu.edu/~cns/; the information on the pion-nucleon system that is of direct interest to us is located at http://gwdac.phys.gwu.edu/analysis/pin_analysis.html. A recent paper by the group that maintains the data on the πN system is available [35].

TABLE VI. Vertex parameters. The particles are designated according to $P_{11}(938) - N$, $P_{33}(1232) - \Delta$, $P_{11}(1440) - R$, $D_{13}(1520) - D$, $S_{11}(1535) - S$, and $S_{31}(1620) - S'$.

Vertices	Parameters (masses m and cutoff masses Λ in MeV)
$\pi N \Leftrightarrow N$	$g^{(0)}_{\pi NN} = 2.9985,\ \Lambda^{(0)}_{\pi NN} = 950.15,\ m^{(0)}_N = 1087.2$
$\pi N \Leftrightarrow \Delta$	$g^{(0)}_{\pi N\Delta} = 1.7965,\ \Lambda^{(0)}_{\pi N\Delta} = 1585.9,\ m^{(0)}_\Delta = 1311.8$
$\pi N \Leftrightarrow R$	$g^{(0)}_{\pi NR} = 7.7777,\ \Lambda^{(0)}_{\pi NR} = 4041.5,\ m^{(0)}_R = 1562.0,$
$\pi\Delta \Leftrightarrow R$	$g^{(0)}_{\pi\Delta R} = 2.1530,\ \Lambda^{(0)}_{\pi\Delta R} = 1732.8$
$\pi N \Leftrightarrow D$	$g^{(0)}_{\pi ND} = 97.987,\ \Lambda^{(0)}_{\pi ND} = 2585.2,\ m^{(0)}_D = 1028.4,$
$\pi\Delta \Leftrightarrow D$	$g^{(0)}_{\pi\Delta D} = -6.9157,\ \Lambda^{(0)}_{\pi\Delta D} = 4831.4$
$\pi N \Leftrightarrow S$	$g^{(0)}_{\pi NS} = 0.96627,\ \Lambda^{(0)}_{\pi NS} = 6096.0,\ m^{(0)}_S = 2572.7,$
$\eta N \Leftrightarrow S$	$g^{(0)}_{\eta NS} = 3.3981,\ \Lambda^{(0)}_{\eta NS} = 2816.3$
$\pi N \Leftrightarrow S'$	$g^{(0)}_{\pi NS'} = 13.707,\ \Lambda^{(0)}_{\pi NS'} = 2248.0,\ m^{(0)}_{S'} = 1550.0,$
$\pi\Delta \Leftrightarrow S'$	$g^{(0)}_{\pi\Delta S'} = 74.364,\ \Lambda^{(0)}_{\pi\Delta S'} = 1156.3$

For the function $V^\pi_\Delta(k)$ that appears in the Δ propagator [see Eq. (6.30)] we considered two choices, $V^\pi_\Delta(k) = 0$ and $V^\pi_\Delta(k)$ given by (6.44). The results reported here are for $V^\pi_\Delta(k) = 0$. Using the other choice leads to only small changes in the parameters.

The masses of our baryons and mesons are given in Table IV. Only m_σ was obtained by adjusting to the πN data. Table V gives the parameters for the interactions from N, $\overline{N}$, Δ, σ, and ρ exchange.

The vertex parameters are given in Table VI. These parameters occur in the vertex functions $U_{\mu\beta\beta'}(k)$ defined in Sec. VII. In the cutoff functions defined by Eq. (7.2) we chose $n^{(0)}_{\mu\beta\beta'} = 10$.

As was pointed out in Sec. VII, the parameters $g_{\pi NN}$ and $\Lambda_{\pi NN}$ must be such that $Z_{\pi N}(k)$, defined by Eq. (12.2), satisfies $Z_{\pi N}(k) > 0$. Table VII gives values of $Z_{\pi N}(k)$ over a wide range of values of k. We see that the constraint is satisfied and that $Z_{\pi N}(k)$ varies very slowly.

The coupling constants, cutoff masses, bare masses, and the σ mass given in Tables IV–VI were determined by fitting to the GWU partial wave analysis. A partial wave amplitude

TABLE V. Exchange potential parameters.

Exchange potentials	Parameters (cutoff masses Λ in MeV)
N exchange, $\pi N \Leftrightarrow \pi N$	$g_{\pi NN} = 13.490,\ \Lambda_{\pi NN} = 842.24$
N exchange, $\pi N \Leftrightarrow \pi\Delta$	$g_{\pi NN} = 13.490,\ \Lambda_{\pi NN} = 842.24$
	$g_{\pi N\Delta} = 1.1828,\ \Lambda_{\pi N\Delta} = 1858.5$
N exchange, $\pi\Delta \Leftrightarrow \pi\Delta$	$g_{\pi N\Delta} = 1.1828,\ \Lambda_{\pi N\Delta} = 1858.5$
$\overline{N}$ exchange, $\pi N \Leftrightarrow \pi N$	$g_{\pi NN} = 13.490,\ \Lambda_{\pi NN} = 842.24$
Δ exchange, $\pi N \Leftrightarrow \pi N$	$g_{\pi N\Delta} = 1.1828,\ \Lambda_{\pi\Delta N} = 1455.0$
σ exchange, $\pi N \Leftrightarrow \pi N$	$g_{\sigma\pi\pi}g_{\sigma NN} = -17.658,\ \Lambda_{\sigma\pi\pi} = 2515.6,\ \Lambda_{\sigma NN} = 2600.5$
ρ exchange, $\pi N \Leftrightarrow \pi N$	$g_{\rho\pi\pi}g_{\rho NN} = 58.961,\ \kappa_\rho = 7.1693,$
	$\Lambda_{\rho\pi\pi} = 3436.3,\ \Lambda_{\rho NN} = 5885.7$

TABLE VII. Values of $Z_{\pi N}(k)$.

k (fm^{-1})	$Z_{\pi N}(k)$
0.0	0.906
1.0	0.905
2.0	0.905
3.0	0.905
4.0	0.905
5.0	0.905
6.0	0.904
7.0	0.904
8.0	0.904
9.0	0.904
10.0	0.903

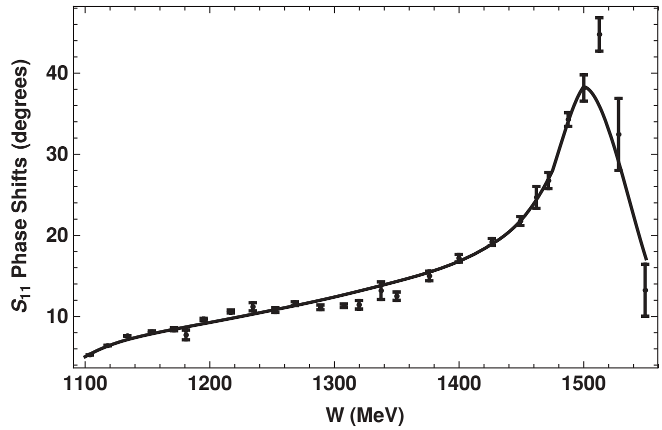

FIG. 4. S_{11} phase shifts. The solid line is theory. Points with error bars are from the CNS analysis.

can be parametrized in terms of a phase shift δ_L^{TJ} and an inelasticity η_L^{TJ}. The dependence of a partial wave amplitude on these parameters is determined by unitarity considerations. By following the development in Sec. XIII of Ref. [27] it can be shown that our amplitudes satisfy three-particle unitarity exactly. From Eq. (8.10) of Ref. [27] it follows that our elastic scattering amplitude satisfies the unitarity constraint

$$\begin{aligned}\mathrm{Im}\{X^{TJ}_{\pi N,\pi N}[k,k,L;W_{\pi N}(k)+i\varepsilon]\}\\ = -\pi k\frac{\omega(k)\varepsilon(k)}{W_{\pi N}(k)}\left|X^{TJ}_{\pi N,\pi N}[k,k,L;W_{\pi N}(k)+i\varepsilon]\right|^2,\\ \times m_\pi + m_N < W_{\pi N}(k) < 2m_\pi + m_N. \qquad (13.1)\end{aligned}$$

This constraint is satisfied by the form

$$\begin{aligned}X^{TJ}_{\pi N,\pi N}[k,k,L;W_{\pi N}(k)+i\varepsilon]\\ = -\frac{W_{\pi N}(k)}{2\pi i k\omega(k)\varepsilon(k)}\{\eta_L^{TJ}(k)\exp[2i\delta_L^{TJ}(k)]-1\}, \qquad (13.2a)\\ \eta_L^{TJ}(k)\\ = 1,\ m_\pi + m_N < W_{\pi N}(k) < 2m_\pi + m_N. \qquad (13.2b)\end{aligned}$$

The partial wave S-matrix element is now given by

$$\begin{aligned}S_L^{TJ}(k) = 1 - 2\pi i k\frac{\omega(k)\varepsilon(k)}{W_{\pi N}(k)}X^{TJ}_{\pi N,\pi N}[k,k,L;W_{\pi N}(k)+i\varepsilon]\\ = \eta_L^{TJ}(k)\exp[2i\delta_L^{TJ}(k)]. \qquad (13.3)\end{aligned}$$

We see that $2\delta_L^{TJ}(k)$ is the phase of the S-matrix element and $\eta_L^{TJ}(k)$ is its modulus.

Our fits to the GWU partial wave analysis are shown in Figs. 4–17. Partial waves are designated by $\Sigma_{2T,2J}$, where Σ is the spectroscopic symbol for the relative πN orbital angular momentum L, and T and J are the total isotopic spin and total angular momentum, respectively.

XIV. SUMMARY AND DISCUSSION

Our existing model gives reasonably good fits to the πN phase shifts and inelasticities from the elastic threshold at $W = m_\pi + m_N = 1078.5$ MeV up to $W = 1550$ MeV, which is well above the threshold for single pion production at $W = 2m_\pi + m_N = 1218.1$ MeV. The threshold for double pion production is at $W = 3m_\pi + m_N = 1357.6$ MeV, so our energy range goes above this threshold. The cross sections for double pion production are known to be small [35], and our results bear this out, since our model does not include $\pi\pi\pi N$ states. The one place where our model does poorly is in its description of the S_{31} inelasticity. This might indicate that there are other interactions that need to be included.

Extensions of the model that are being considered involve including σN and ρN channels coupled to the πN channel. Since the σ and ρ mesons are essentially $\pi\pi$ resonances these channels will contribute to the $\pi\pi N$ final state through the processes $\pi N \Rightarrow \sigma N \Rightarrow \pi\pi N$ and $\pi N \Rightarrow \rho N \Rightarrow \pi\pi N$.

The model of Matsuyama, Sato, and Lee [36] does contain the σN and ρN channels. Their model is based on a Hamiltonian that was partially constructed from a set of effective Lagrangians using a unitary transformation method [37] somewhat similar to the Okubo method [34]. Unlike here, they did not construct the direct interactions involving the baryon resonances from effective Lagrangians, but rather they assume plausible resonance amplitudes, similar to the Breit-Wigner form. Their treatment of the πN and $\pi\Delta$ propagators is also quite different from ours. Our propagators involve the complete dressing of the N and Δ due to the vertices, $N \Leftrightarrow \pi N$ and $\Delta \Leftrightarrow \pi N$, in the presence of a spectator pion. As a result of this our $\pi\Delta$ propagator, in particular, is much more complicated than theirs. Also, our model places much more emphasis on Poincaré invariance than they do in that we explicitly construct a Poincaré invariant mass operator. Since they work with an energy-independent Hamiltonian in the c.m. frame it could probably be established that their model is also Poincaré invariant. We have also been more careful in establishing that two- and three-particle unitarity is satisfied [27]. They assume that since they work with an energy-independent Hamiltonian the standard results of formal scattering theory guarantee that unitarity is satisfied. We feel that since both of our models involve particle production and absorption, unitarity really has to be established through a "model-dependent" proof, as we have done in Ref. [27].

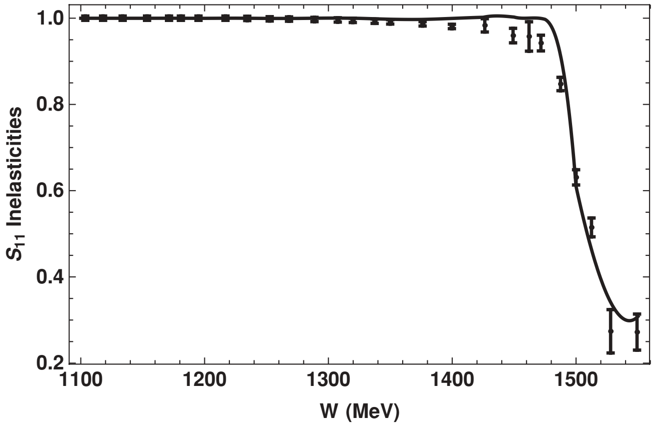

FIG. 5. S_{11} inelasticities. The solid line is theory. Points with error bars are from the CNS analysis.

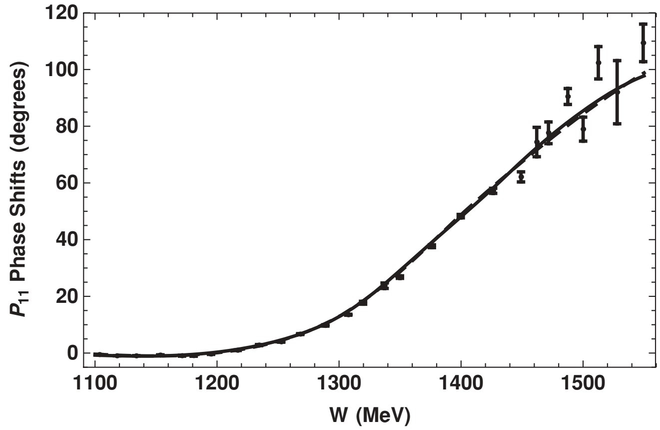

FIG. 8. P_{11} phase shifts. The curves and points have the same meaning as in Fig. 4.

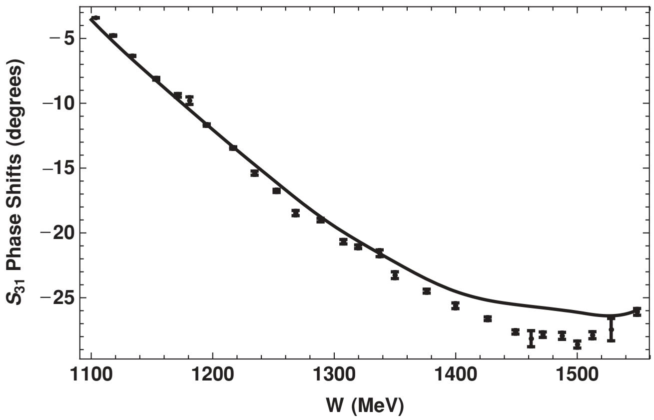

FIG. 6. S_{31} phase shifts. The curves and points have the same meaning as in Fig. 4.

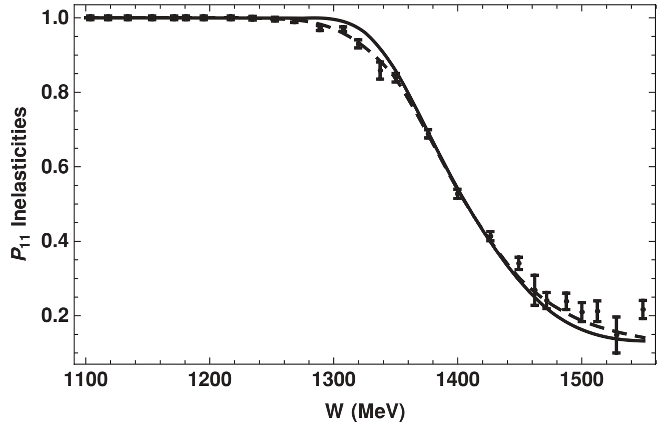

FIG. 9. P_{11} inelasticities. The curves and points have the same meaning as in Fig. 4.

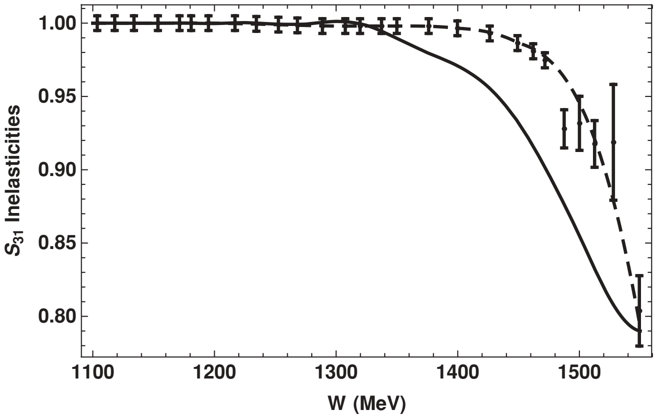

FIG. 7. S_{31} inelasticities. The curves and points have the same meaning as in Fig. 4.

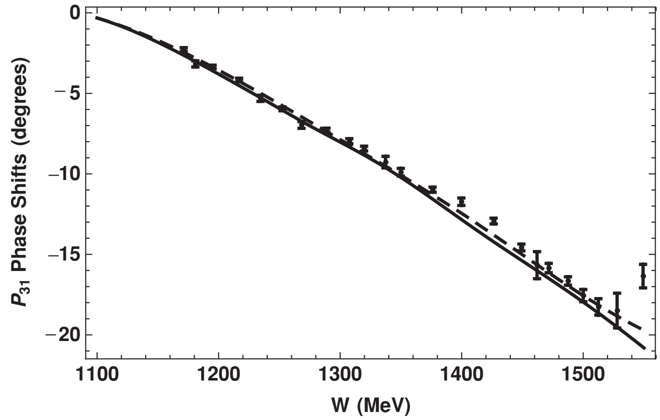

FIG. 10. P_{31} phase shifts. The curves and points have the same meaning as in Fig. 4.

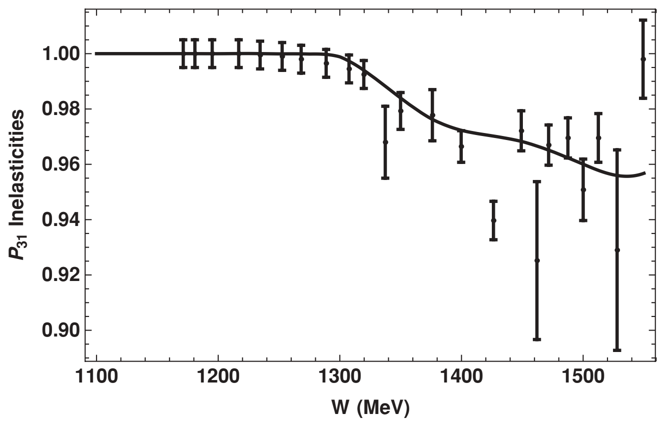

FIG. 11. P_{31} inelasticities. The curves and points have the same meaning as in Fig. 4.

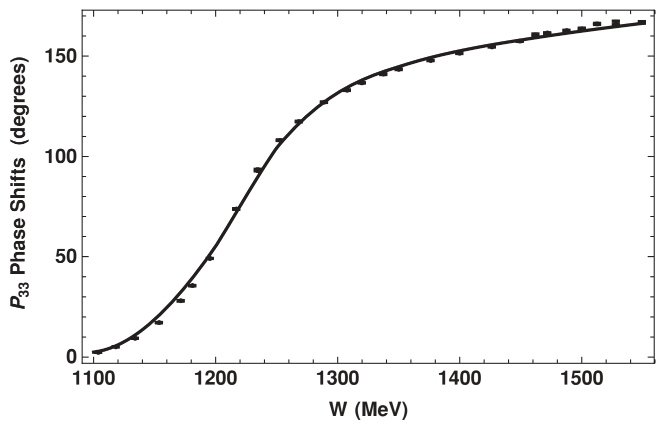

FIG. 14. P_{33} phase shifts. The curves and points have the same meaning as in Fig. 4.

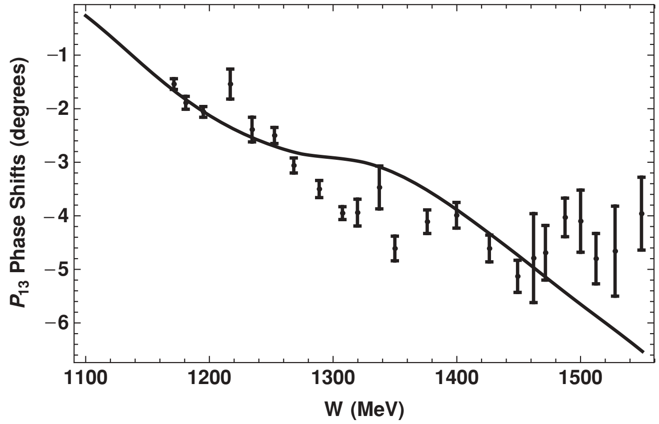

FIG. 12. P_{13} phase shifts. The curves and points have the same meaning as in Fig. 4.

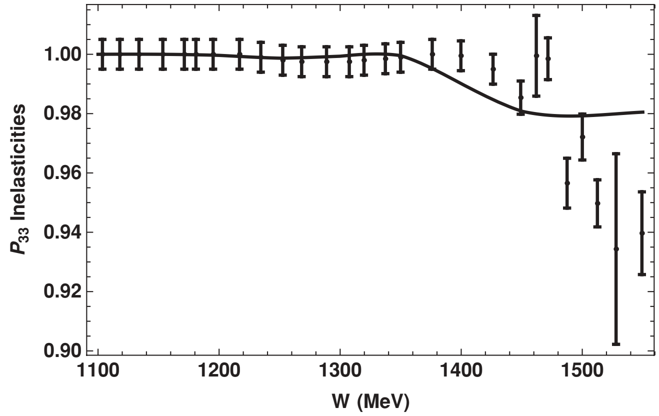

FIG. 15. P_{33} inelasticities. The curves and points have the same meaning as in Fig. 4.

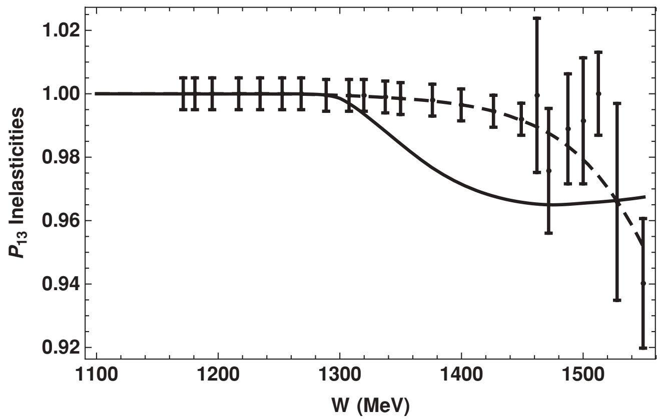

FIG. 13. P_{13} inelasticities. The curves and points have the same meaning as in Fig. 4.

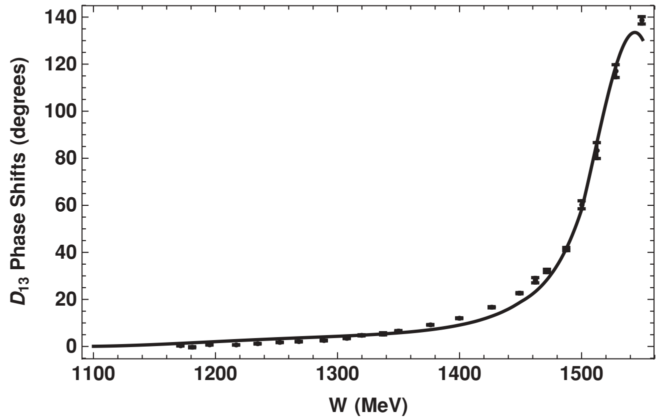

FIG. 16. D_{13} phase shifts. The curves and points have the same meaning as in Fig. 4.

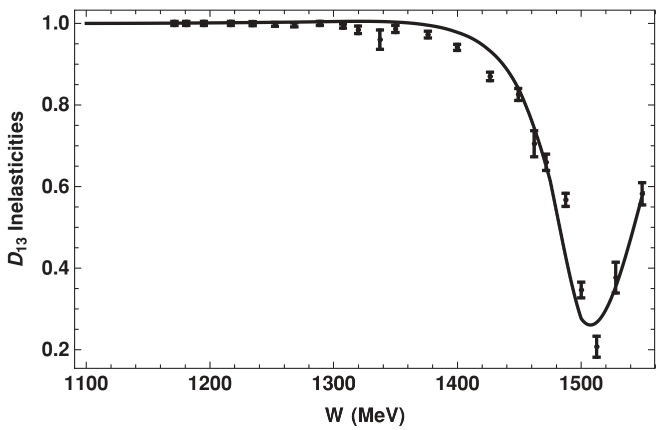

FIG. 17. D_{13} inelasticities. The curves and points have the same meaning as in Fig. 4.

A shortcoming of our model is that it does not satisfy the clustering requirement exactly [4,8–14]. If it did the renormalization parameter $Z_{\pi N}(k)$ would not depend on k. According to Table VII the dependence on k is very weak, which suggests that the neglect of the clustering requirement is a reasonable approximation. At present we are investigating an approach for satisfy the clustering requirement that deals with the square roots of operators. For example, the interaction between particles 1 and 2 in the presence of a spectator particle is taken to be

$$V_{12,3} = \sqrt{\mathbf{K}_3^2 + (M_{12})^2} - \sqrt{\mathbf{K}_3^2 + \left(M_{12}^0\right)^2}. \qquad (14.1)$$

Here M_{12}^0 and M_{12} are the noninteracting and interacting mass operators, respectively, for the isolated (12)-system, and $\mathbf{K}_3$ is the three-momentum operator for particle 3, the spectator. Our preliminary results indicate that this approach is tractable.

Even within the framework of the present model there are other things that need to be done. In particular the cross sections for the inelastic processes, $\pi N \Rightarrow \eta N$ and $\pi N \Rightarrow \pi\pi N$, should be calculated. The cross section for the process $\pi N \Rightarrow \eta N$ can be calculated from the amplitude [see Eq. (10.6)] $X^{1/2}_{\eta N,\pi N}[\mathbf{k},\mathbf{k}'; W_{\pi N}(k') + i\varepsilon]$, and pion production can be calculated from the amplitudes [see Eqs. (10.4) and (10.5)] $X^{T}_{\pi N,\pi N}[\mathbf{k},\mathbf{k}'; W_{\pi N}(k') + i\varepsilon]$ and $X^{T}_{\pi\Delta,\pi N}[\mathbf{k},\mathbf{k}'; W_{\pi N}(k') + i\varepsilon]$. In our framework the reaction $\pi N \Rightarrow \pi\pi N$ proceeds through the processes $\pi N \Rightarrow \pi N \Rightarrow \pi\pi N$ and $\pi N \Rightarrow \pi\Delta \Rightarrow \pi\pi N$. Here the processes $\pi N \Rightarrow \pi N$ and $\pi N \Rightarrow \pi\Delta$ are off-shell, that is, non-energy-conserving processes, followed by the propagation of a πN or $\pi\Delta$ state, respectively. These states then transition to the $\pi\pi N$ final state through the processes $N \Rightarrow \pi N$ and $\Delta \Rightarrow \pi N$.

One of the shortcomings of our model is the use of interactions derived from a combination of effective hadronic Lagrangians and purely phenomenological cutoff functions. The cutoff functions are needed to account for the nonelementary nature of the hadrons involved and to provide convergence of the integrals and integral equations. Ideally the interactions should be derived from QCD, but at the present time this is a challenging problem. An alternative approach is to attempt to derive the interactions from the constituent quark model. In the usual constituent quark model calculations of baryon structure the spectrum is obtained from a three-body calculation with some assumed quark-quark interaction, and the coupling of a baryon β to a meson-baryon channel $\mu\beta'$ is calculated by assuming a strong interaction transition operator T_s, and evaluating the matrix element $\langle\mu\beta'|T_s|\beta\rangle$. In general the baryon state vectors are taken from the three-quark calculation, whereas the treatment of the meson depends on the nature of the assumed transition operator. In one class of models, the so-called elementary emission models, the mesons, μ, are treated as elementary particles, and the transition operator describes a $q \Leftrightarrow q' + \mu$ coupling, where the q's are quarks [38–40]. This approach can be thought of as originating from an effective theory of hadrons discussed by Manohar and Georgi [41] in which a Lagrangian is constructed that describes the coupling of constituent quarks, gluons, and Goldstone bosons. Another type of model is the so-called pair creation model [42–51]. In such a model both the baryons and the mesons are described by constituent quark models, and the transition $\beta \Leftrightarrow \mu\beta'$ is brought about by the creation of a quark-antiquark pair from vacuum. A comparison of the various approaches is given in a review article by Capstick and Roberts [52].

Clearly, the quark wave functions should provide the shapes of our cutoff functions, and the transition operators should determine the strengths of the $\beta \Leftrightarrow \mu\beta'$ vertices. Also the baryon mass eigenvalues obtained by solving the quark model without coupling to the strong decay channels should provide the bare baryon masses, $m_\beta^{(0)}$, that appear in our equations. Hopefully, this program is tractable and will at least move the phenomenology down to the quark level.

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