Two-potential formula and its application to proton-proton scattering

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Within the framework of potential scattering theory we derive an analytical two-potential formula for the on-shell partial wave scattering amplitude. This formula embodies a large number of possible applications, including long range Coulomb forces as well as short distance singular potentials. As an example illustrating the use of the formula we analyze the determination of the strong proton-proton scattering s-wave phase shift from the experimentally determined Coulomb phase when the one-pion exchange and two-pion exchange chiral potentials are taken into account and analyze the relevant scales of the problem.

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I. INTRODUCTION

The two-potential formalism, developed in the fifties by Watson [1] and Gell-Mann and Goldberger [2], relates the scattering due to the sum of two different potentials and has a widespread use in scattering theory. The usual example is the treatment of Coulomb distortion for strongly interacting particles. The problem is to determine the total scattering amplitude T from a potential constructed as the sum of two potentials $V = V_S + V_L$ in terms of the scattering amplitude T_S due only to the potential V_S. The result can be found in a straightforward manner using the Lippmann-Schwinger equation

$$T = V + V G_0 T, \tag{1}$$

with V being the potential operator and $G_0 = (E - H_0)^{-1}$ the resolvent of the free Hamiltonian. The outgoing boundary condition corresponds to $E \to E + \mathrm{i}0^+$. The T matrix can then be expressed as[1]

$$T = T_S + (1 + T_S G_0)\, \hat{T}_L\, (G_0 T_S + 1), \tag{2}$$

where T_S is the short distance T matrix, and $\hat{T}_L$ is the long distance one distorted by short range effects,

$$T_S = V_S + V_S G_0 T_S, \tag{3}$$

$$\hat{T}_L = V_L + V_L G_S \hat{T}_L, \tag{4}$$

with $G_S = G_0 + G_0 V_S G_S$ being the full propagator for V_S.

While the result above solves the problem, it does not explicitly relate the on-shell scattering amplitudes of the full and short distance potentials. The reason is that the Lippmann-Schwinger equation involves the off-shell behavior of the potentials, which allows one to treat nonlocal potentials quite straightforwardly, but precisely because of this it is hard to profit specifically from the simplifying features that arise in the interesting and quite frequent case of local potentials arising, e.g., in a particle exchange picture. For the local case, a coordinate space formulation of the scattering problem is more convenient (an effective field theory example is provided by Ref. [3]).

In this article we derive a two-potential formula, which relates the phase shifts (i.e. the on-shell scattering matrix) of the full and short range potentials V and V_S, and which is based on two assumptions: (i) the potentials are local and (ii) the short range potential dominates at short distances. Our result will be amenable to rather detailed analytical study, hence enlarging the class of situations one may cover. The connection to momentum space renormalization with counterterms of the Lippmann-Schwinger equation is also analyzed. This is particularly enlightening in the case of singular potentials and their renormalization, a subject of recent interest (see, e.g., Ref. [3]). As an illustrative application we show how our two-potential formula may be used to deduce the strong proton-proton phase shifts from the experimentally measured ones when long distance corrections coming from one- and two-pion exchange contributions are taken into account.

II. TWO-POTENTIAL FORMULA

We consider the nonrelativistic scattering of two particles by a spherically symmetric potential V that can be decomposed into the following two pieces:

$$V(r) = V_L(r) + V_S(r), \tag{5}$$

where V_L and V_S, respectively, represent the long and short distance components of the interaction. We assume that the short range potential V_S dominates at short distances $r = r_c$, i.e.,

$$V_S(r_c) \gg V_L(r_c), \tag{6}$$

for r_c small enough. The system can be described by solving the reduced Schrödinger equation (for simplicity, we only consider here the s-wave case)

$$-u_k'' + 2\mu\, [V_L(r) + V_S(r)]\, u_k(r) = k^2 u_k(r), \tag{7}$$

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[1]We have chosen this particular formulation of the two-potential trick to have a more consistent notation throughout the article. Normally it is written in terms of the distorted short range T matrix and the undistorted long range T matrix, but in either case the resulting T matrix is the same.

with $u_k(r)$ being the reduced wave function, μ the reduced mass of the two-body system, and k the center-of-mass momentum. The phase shift can then be obtained by matching the reduced wave function u_k to the usual asymptotic boundary condition for $r \to \infty$,

$$u_k(r) \to \cot\delta \, \sin kr + \cos kr, \tag{8}$$

where we have assumed that the long range potential V_L decays faster than $1/r^2$ at large distances; therefore, phase shifts are well defined. We also consider the corresponding scattering problem for which only the short range potential V_S is present. In such a case, the reduced Schrödinger equations reads

$$-u_k^{S\prime\prime} + 2\mu \, V_S(r) \, u_k^S(r) = k^2 u_k^S(r), \tag{9}$$

with u_k^S being the *short* reduced wave function. The phase shift can be extracted from the asymptotic behavior of u_k^S,

$$u_k^S(r) \to \cot\delta^S \, \sin kr + \cos kr, \tag{10}$$

for $r \to \infty$.

The problem is to relate the full phase shift $\delta(k)$ with the short phase shift $\delta^S(k)$. For that purpose, we assume that at short enough distances $r = r_c$ the full and short reduced wave function u_k and u_k^S are approximately equal, $u_k(r_c) \simeq u_k^S(r_c)$. The previous approximation can be restated in terms of the logarithmic derivatives of the reduced wave functions,[2]

$$\frac{u_k'(r_c)}{u_k(r_c)} = \frac{u_k^{S\prime}(r_c)}{u_k^S(r_c)}. \tag{11}$$

This expression will hold true when the condition expressed in Eq. (6) is fulfilled. We now make use of the superposition principle to represent the full solution as the following linear combination:

$$u_k(r) = \cot\delta \, J_k(r) - Y_k(r), \tag{12}$$

where J_k and Y_k are solutions of Eq. (7), subjected to the asymptotic conditions

$$J_k(r) \to \sin kr, \tag{13}$$

$$Y_k(r) \to -\cos kr. \tag{14}$$

Analogously, we write the short range solution as

$$u_k^S(r) = \cot\delta^S \, J_k^S(r) - Y_k^S(r), \tag{15}$$

with J_k^S and Y_k^S being solutions of Eq. (9), such that

$$J_k^S(r) \to \sin kr, \tag{16}$$

$$Y_k^S(r) \to -\cos kr, \tag{17}$$

for $r \to \infty$. By matching the logarithmic derivatives we arrive at our final expression

$$\cot\delta(k) = \frac{\mathcal{A}(k, r_c)\cot\delta^S(k) - \mathcal{B}(k, r_c)}{\mathcal{C}(k, r_c)\cot\delta^S(k) - \mathcal{D}(k, r_c)}, \tag{18}$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ are defined as

$$\mathcal{A}(k, r_c) = W\left(J_k^S, Y_k\right)\Big|_{r=r_c} \tag{19}$$

$$\mathcal{B}(k, r_c) = W\left(Y_k^S, Y_k\right)\Big|_{r=r_c} \tag{20}$$

$$\mathcal{C}(k, r_c) = W\left(J_k^S, J_k\right)\Big|_{r=r_c} \tag{21}$$

$$\mathcal{D}(k, r_c) = W\left(Y_k^S, J_k\right)\Big|_{r=r_c}, \tag{22}$$

with $W(f, g)|_{r=r_c} = f(r_c)g'(r_c) - f'(r_c)g(r_c)$ being the Wronskian between different wave functions evaluated at the cutoff radius $r = r_c$. The bilinear structure is reminiscent of the Moebius transformation invariance discussed at length in Ref. [4] in the context of the renormalization group analysis with boundary conditions. It should be noted that although the matching of log-derivatives to obtain long range correlations is not new, one nice example being the Landau-Smorodinsky derivation of the effective range expansion [5], or the treatment of hadronic atoms in Ref. [6], its use in combination with the superposition principle to derive direct relations between phase shifts is less common, and it has only been partially exploited in some effective field theory r-space computations [7–9].

In passing we also note that Eq. (18) cannot be derived from the Lippmann-Schwinger equation. The reason is that the two-potential formula depends on the *explicit* formulation of the following: (i) the superposition principle via Eqs. (12) and (15), and (ii) the short distance boundary condition for the Schrödinger equation, Eq. (11). These two conditions are included in the Lippmann-Schwinger equation, but *implicitly*, in a way that they cannot be directly handled, impeding the derivation of the previous formula [but allowing the derivations of formulas relating the full off-shell scattering amplitudes, like the two-potential trick, Eq. (2)].

The two-potential formula also holds in certain cases for nonlocal potentials. The necessary condition for its application is that the nonlocal potential does not involve derivatives of order higher than two, e.g., potentials of the type

$$V_S^{NL} = \{\nabla^2, f_S(r)\}, \tag{23}$$

where $\{,\}$ represents the anticommutator. In such a case the short distance boundary condition for the Schrödinger equation can be expressed as the log-derivative of the wave function, Eq. (11). For nonlocal potentials involving higher derivatives, the two-potential formula can still be applied when the cutoff radius r_c is larger than the range at which the nonlocalities appear.

III. COULOMB SCATTERING

The case where the long range potential is of Coulomb type requires a special treatment because the usual asymptotic behavior described in Eq. (8) does not apply. For definiteness, we analyze here the Coulomb repulsion between two unit charge particles in the s wave. The full system is now described by the following equation,

$$-u_k^{C\prime\prime} + 2\mu\left[V_S(r) + \frac{\alpha}{r}\right] u_k^C(r) = k^2 u_k^C(r), \tag{24}$$

where we have added the ^{C} superscript for labeling the Coulomb solution and α represents the fine structure constant.

[2]We are not necessarily assuming a regular solution of the Schrödinger equation; i.e., V_S can contain zero-range pieces.

The correct asymptotics for u_k^C is given by

$$u_k^C(r) \to \cot\delta^C\, F_0(\eta,\rho) + G_0(\eta,\rho), \tag{25}$$

with δ^C being the Coulomb modified phase shift and $F_0(\eta,\rho)$ and $G_0(\eta,\rho)$ the usual s-wave Coulomb wave functions (see, for example, Ref. [10]), which depend on the parameters $\eta = 1/(ka_B)$ and $\rho = kr$, $a_B = 1/(\mu\alpha)$ is the Bohr radius of the two-particle system. F_0 and G_0 are solutions of the reduced Schrödinger equation for the Coulomb potential $V_C(r) = \alpha/r$, with the asymptotic behavior

$$F_0(\eta,\rho) \to \sin(kr - \eta\log 2kr + \sigma_0), \tag{26}$$

$$G_0(\eta,\rho) \to \cos(kr - \eta\log 2kr + \sigma_0), \tag{27}$$

where σ_0 is a phase shift defined as $\sigma_0 = \arg\Gamma(1+i\eta)$. As in the previous case, we can use the superposition principle to rewrite the full solution

$$u_k^C(r) = \cot\delta^C\, F_k^C(r) - G_k^C(r), \tag{28}$$

with $F_k(r)$ and $G_k(r)$ being solutions of Eq. (24) subjected to the asymptotic boundary conditions

$$F_k^C(r) \to F_0(\eta,\rho), \tag{29}$$

$$G_k^C(r) \to -G_0(\eta,\rho). \tag{30}$$

The short range system is described by Eq. (9), and the short range wave function u_k^S is again parametrized by Eq. (15).

After matching logarithmic derivatives we find

$$\cot\delta^C(k) = \frac{\mathcal{A}(k,r_c)\cot\delta^S(k) - \mathcal{B}(k,r_c)}{\mathcal{C}(k,r_c)\cot\delta^S(k) - \mathcal{D}(k,r_c)}, \tag{31}$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ are now defined as

$$\mathcal{A}(k,r_c) = W\left(J_k^S, G_k^C\right)\big|_{r=r_c} \tag{32}$$

$$\mathcal{B}(k,r_c) = W\left(Y_k^S, G_k^C\right)\big|_{r=r_c} \tag{33}$$

$$\mathcal{C}(k,r_c) = W\left(J_k^S, F_k^C\right)\big|_{r=r_c} \tag{34}$$

$$\mathcal{D}(k,r_c) = W\left(Y_k^S, F_k^C\right)\big|_{r=r_c}, \tag{35}$$

in complete analogy with Eq. (18). Previous relationships between Coulomb and short distance scattering can be found for some specific cases in Refs. [11–13].

A. Contact short range potential

A simple application of the previous formula corresponds to a situation where the short range potential is zero for distances greater than the cutoff radius r_c,

$$V_S(r) = 0 \quad \text{for} \quad r > r_c, \tag{36}$$

while, for distances shorter than r_c, the potential is very strong. The previous potential corresponds to a δ-type contact interaction regularized at the length scale r_c. In such a case, the $F_k^C(r)$ and $G_k^C(r)$ wave functions are equal to their asymptotic behavior for $r \geqslant r_c$, and by taking into account their behavior at small radii

$$F_k^C(r) \to k\,C(\eta)\left[r + \frac{r^2}{a_B} + O(r^3)\right], \tag{37}$$

$$G_k^C(r) \to -\frac{1}{C(\eta)}\left[1 + \frac{2r}{a_B}\left(\log\frac{2r}{a_B} + 2\gamma_E - 1 + h(\eta)\right) + O(r^2)\right], \tag{38}$$

with γ_E the Euler-Mascheroni constant, and $C(\eta)$ and $h(\eta)$ defined as

$$C^2(\eta) = \frac{2\pi\eta}{e^{2\pi\eta}-1}, \tag{39}$$

$$h(\eta) = \eta^2\sum_{n=1}^{\infty}\frac{1}{n(n^2+\eta^2)} - \log\eta - \gamma_E, \tag{40}$$

the relationship given by Eq. (31) can be evaluated explicitly, yielding

$$k\,\cot\delta^S(k) = C^2(\eta)k\,\cot\delta^C(k) + 2\frac{h(\eta)}{a_B} - \frac{2}{a_B}\left[\log\frac{a_B}{2r_c} - 2\gamma_E\right] + \mathcal{O}(r_c), \tag{41}$$

where terms linear in the cutoff radius and higher powers of r_c have been ignored. As can be seen the previous expression is logarithmic divergent with respect to r_c, but can be regularized if we take into account the corresponding expression for $k = 0$, which is similar to the well-known relationship between strong and Coulomb scattering length from Blatt and Jackson [14,15],

$$-\frac{1}{\alpha_S} = -\frac{1}{\alpha_C} - \frac{2}{a_B}\left[\log\frac{a_B}{2r_c} - 2\gamma_E\right] + \mathcal{O}(r_c). \tag{42}$$

The expression above diverges in exactly the same way as Eq. (41). Subtracting the $k = 0$ expression to Eq. (41), and taking the $r_c \to 0$ limit, we arrive at the following expression,

$$k\,\cot\delta^S(k) + \frac{1}{\alpha_S} = C^2(\eta)k\,\cot\delta^C(k) + 2\frac{h(\eta)}{a_B} + \frac{1}{\alpha_C}. \tag{43}$$

The expected error of this formula can be estimated by reintroducing the cutoff r_c and interpreting it as the neglected range R_S of the short range potential V_S, yielding a relative error of $\mathcal{O}(r_c/a_B) = \mathcal{O}(R_S/a_B)$.

The corresponding formula for attractive Coulomb interaction may be of interest for the treatment of pionic atoms and can be obtained by taking $\eta = -1/(ka_B)$ negative and making the following substitution,

$$h(\eta) \to \mathrm{Re}[\psi(i\eta) - \log(-i\eta)], \tag{44}$$

with ψ being the digamma function.

Finally, the corresponding formula for p-wave repulsive Coulomb interaction can be worked out analogously to the s-wave case. The treatment of the divergences is nonetheless more involved: there is an additional logarithmic divergence proportional to k^2, due to the interplay between the Coulomb potential and the centrifugal barrier. The outcome is that two subtractions are needed to have finite results in the $r_c \to 0$

limit. The final formula is rather simple to summarize:

$$k^3 \cot\delta_1^S(k) + \frac{1}{\alpha_{1,S}} - \frac{1}{2} r_{1,S} k^2 = C_1^2(\eta)\, k^3 \cot\delta_1^C(k) + k^3 (1+\eta^2)\, 2\,\eta\, h(\eta) + \frac{1}{\alpha_{1,C}} - \frac{1}{2} r_{1,C} k^2, \tag{45}$$

where $C_1^2(\eta) = (1+\eta^2) C^2(\eta)$, $\alpha_{1,S}$ and $\alpha_{1,C}$ are the p-wave short and Coulomb scattering volumes, and $r_{1,S}$ and $r_{1,C}$ are the p-wave short and Coulomb effective ranges. The previous formula has less predictive power than the corresponding one for s waves as a consequence of the extra subtraction needed to regularize it. A possible application is nucleon-α scattering [16].

IV. APPLICATION TO PROTON-PROTON SCATTERING

Now we apply the previous results for Coulomb scattering to the specific case of proton-proton (pp) scattering in s-waves. We consider the strong pp interaction as the short range potential V_S, while the Coulomb repulsion between the protons plays the role of the long range potential V_L.

A. Pionless theory

We first consider the simplifying case in which the pion exchange interactions between the protons are neglected and the pp potential consists of contact interactions only, i.e., the pionless theory, characterized by a short distance boundary condition. In such a case the two-potential formula given by Eq. (43) applies. The previously mentioned relationship can be better understood by noticing the relationship with the strong and Coulomb effective range expansions [17], i.e.,

$$k \cot\delta^S = -\frac{1}{a_S} + \frac{1}{2} r_S k^2 + \sum_{n=2}^{\infty} v_{n,S} k^{2n}, \tag{46}$$

$$k \cot\delta^C C^2(\eta) + 2\frac{h(\eta)}{a_B} = -\frac{1}{a_C} + \frac{1}{2} r_C k^2 + \sum_{n=2}^{\infty} v_{n,C} k^{2n}, \tag{47}$$

meaning that, with the exception of the scattering length, which explicitly depends on the regularization scale r_c [see Eq. (42)], the strong and Coulomb effective range parameters for pp scattering are equal in the present approximation,

$$r_S = r_C \quad \text{and} \quad v_{n,S} = v_{n,C} \quad \text{for} \quad n \geqslant 2, \tag{48}$$

where $r_{S(C)}$ is the effective range, $v_{2,S(C)}$ the shape parameter, etc. If we compare the previous results with the parameters obtained with the Nijmegen II potential [18], we observe a small discrepancy,[3]

$$r_S = 2.84\,\text{fm}, \quad r_C = 2.76\,\text{fm}, \tag{49}$$

giving a 3% relative difference between the strong and Coulomb parameters. According to the error estimation of the previous section, we should expect a relative error of R_S/a_B, with R_S being the range of the strong pp interaction, given by one-pion exchange, $R_S = R_{\pi^0} = 1/m_{\pi^0}$, where m_{π^0} is the neutral pion mass, yielding the result $R_{\pi^0}/a_B = M_p\,\alpha/2m_{\pi^0} \sim 2.5\%$ (M_p is the proton mass) in agreement with the previous discrepancy.[4]

The corresponding results for the strong pp phase shifts, obtained from the Coulomb pp ones for the Nijmegen II potential [18] by means of Eq. (43), are shown in Fig. 1. The agreement between the strong pp Nijmegen II phase

[3] Instead of the Nijmegen II values, it is also possible to use the well-established experimental value for the Coulomb pp effective range $r_C = 2.794(14)$ fm and the model-dependent strong pp one $r_S = 2.84(4)$ fm (see Ref. [19]), although the conclusions do not change appreciably.

[4] The contributions to the strong-Coulomb effective range difference from vacuum polarization [20], or from the modified Coulomb potential of Ref. [21], are expected to be much smaller than the strong (pionic) corrections. The magnetic moment interaction [22] does not contribute to s-wave proton-proton scattering.

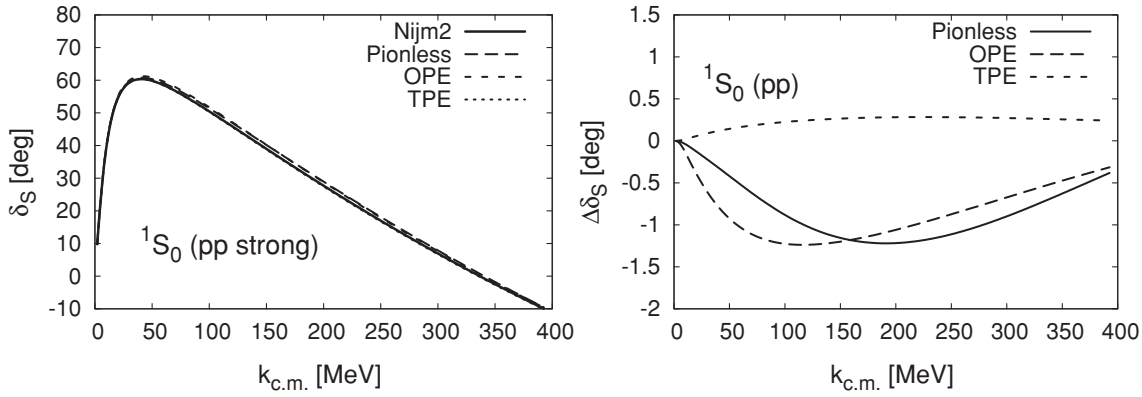

FIG. 1. (Left panel) Strong pp phase shifts computed from the Coulomb pp phase shifts (Nijmegen II potential [18]) by using the zero-range strong-Coulomb correlation of Eq. (43) and its corresponding extensions when including the one-pion exchange (OPE) and chiral two-pion exchange (TPE) (N^{2}LO) potentials with a cutoff radius of $r_c = 0.1$ fm. (Right panel) Difference between the Nijmegen phase shifts and those obtained with the strong-Coulomb correlations.

shifts and the expected ones computed from the contact theory correlation (43) is quite good, never exceeding a 1.5° difference, as expected from the relative error estimation.

B. Comparison with other approaches

The above result may be relevant to the effective field theory (EFT) formulation of low-energy pp scattering done by Kong and Ravndal [23,24] based on the power divergence subtraction (PDS) regularization scheme of Refs. [25,26]. The admitted intricacy of the momentum space formalism in those works contrasts with the much shorter and transparent discussion of the coordinate space renormalization presented above. In particular, Eq. (43) implies that there are no Coulomb corrections to the effective range once the cutoff is removed, which is in agreement with the next-to-leading order calculation of Kong and Ravndal [24], but disagrees with the next-to-next-to-leading order results of Ref. [27]. It is also implied in our treatment that the pionless treatment of pp scattering can be made almost scale independent if, apart from the usual counterterms $C_0 + C_2(p^2 + p'^2) + \mathcal{O}(p^4)$, a counterterm contribution proportional to α is included in the computations, i.e., $\alpha\, D_{e^2}$. This observation is closely related with the results of Ref. [28], where the necessity of a strong and Coulomb version of C_0 is discussed. The previous D_{e^2} counterterm fixes the difference between the strong and Coulomb scattering length, so the price to pay to remove the log scale dependence in Eq. (42) is the impossibility to relate the two, because both scattering lengths become input parameters. This seems to be in contradiction with Kong and Ravndal [23,24], who argue that the C_2 counterterm stabilizes the scattering length (see also related discussions in Refs. [27,29,30]). This counterterm is analogous to the $m_\pi^2\, D_2$ counterterm needed to renormalize Weinberg power counting at leading order [31,32]. They are both due to the similar behavior of the Coulomb and Yukawa potential at short distances.

Of course, these conclusions are based on our coordinate space analysis with cutoff regularization. Dimensional regularization with PDS yields different results [27], as Coulomb corrections to the effective range appear at next-to-next-to-leading order. These corrections depend on the off-shell behavior of the $\mathcal{O}(p^4)$ counterterms, which in the cutoff approach seems to be under control as long as nonlocalities and off-shell ambiguities happen below r_c. A more pessimistic view is presented by Gegelia in Ref. [28], where it is argued from the renormalization group behavior of the counterterms in PDS that it needs to be a strong and Coulomb version of each counterterm (to absorb the log-divergences), meaning that in the end it is impossible to relate strong and Coulomb observables. On the contrary, the renormalization group analysis with cutoff regularization of Barford and Birse [33] seems to support the idea that the Coulomb log-divergences can be absorbed in just one counterterm.[5] The observations of Gegelia [28] can be considered as an extension to any scattering observable of the results of Refs. [34,35] about the difficulty of obtaining model-independent strong scattering lengths from Coulomb ones because of short range ambiguities of the wave function. The previous seem to be in contradiction with the usual requirement of short distance independence of physical results in effective field theory. In fact, as was shown in Ref. [36], further constraints on the short range ambiguities not considered in Refs. [34,35] can noticeably reduce the model dependence of strong parameters, in better agreement with EFT expectations. Finally, we should also stress that we are only trying to separate strong from *Coulomb* corrections in nonrelativistic quantum mechanics. A complete formulation on the separation of strong and *electromagnetic* effects is only possible in the context of quantum field theory, see Ref. [37] for a modern discussion on the subject.

We should nonetheless remember that cutoff regularization is a physical regularization, in the sense that the cutoff r_c can be interpreted as a physical scale. From this point of view, the meaning of the log-divergence in the relationship between the strong and Coulomb scattering length is straightforward: it represents the expected error of the strong scattering length in the pionless approximation, which scales as $\log(R_S/a_B)$ (instead of R_S/a_B, as in the other parameters), yielding a very large, ∼350%, expected relative error (to be compared with the one for the effective range, ∼2.5%). One can also argue that the extra counterterm D_{e^2} is not needed, because the inclusion of the higher order components of the potential will reduce the scale dependence.

C. Strong-Coulomb correlations with chiral TPE potentials

The two-potential formula makes it possible to obtain the (experimentally inaccessible) strong pp phase shifts from the (experimentally accessible) Coulomb ones. While a complete analysis should of course include vacuum polarization [20], modifications to the Coulomb potential [21], and even $2\pi\gamma$ exchange [38–40], the interesting issue is that one can obtain model-independent strong phase shifts, provided that we employ a model-independent strong pp potential V_S and model-independent Coulomb phase shifts. Here, we do so with an eye to the relevant scales in the problem, an aspect that our two-potential treatment can address in a rather clean way.

For the previous purpose we use the potentials of chiral perturbation theory [41] as the short distance potential V_S of Eq. (31). These potentials include TPE effects and can be expressed as an expansion in powers of Q,

$$V_S(r) = V_\chi^{(0)}(r) + V_\chi^{(2)}(r) + V_\chi^{(3)}(r) + \mathcal{O}(Q^4), \qquad (50)$$

where Q represents either the pion mass or the momentum of the protons. We also use the Nijmegen PWA [42], which is a model-independent extraction of the pp s-wave *Coulomb*[6]

[5]The results of Ref. [33] do not exclude the existence of Coulomb corrections to all counterterms, and neither do our results if supplemented by additional subtractions. It is just that they are not needed to have scale independent results.

[6]It is important to notice that the pp phase shifts in the Nijmegen PWA are not Coulomb phase shifts, but *electromagnetic* phase shifts. By that it is meant that the pp phase shifts are defined with respect to the asymptotic solutions of the full electromagnetic potential used

phase shifts from a large proton-proton scattering database. With that information, we can obtain the strong pp phase shift and its error from the PWA by

$$\delta^C_{\rm PWA}(k) \pm \Delta\delta^C_{\rm PWA} \to (V_\chi, r_c) \to \delta^S(k) \pm \Delta\delta^S \qquad (51)$$

and analyze the resulting cutoff dependence, which is an important issue, because for large coordinate space cutoffs the higher order pieces of the chiral potential are not resolved.

It should be noted here that a complete model-independent separation between strong and electromagnetic contributions is not always possible, specially if short distance electromagnetic effects are included. One example is the inclusion of nucleon form factor corrections to the magnetic moment interaction in the proton-proton PWA of Ref. [43]. Another example is proton finite size corrections to the Coulomb potential. The formalism presented here clearly separates between *what we define* as the strong and the electromagnetic *potential*s. That does not necessarily mean that exact model independence has been achieved, especially if corrections like the ones mentioned above are added, or that strong and electromagnetic effects have been actually separated, especially as electromagnetic corrections to the proton mass or to the coupling constants have not been included.

The specific procedure we apply is analogous to the one followed in the pionless case; i.e., we do not directly use the strong-Coulomb two-potential formula, Eq. (31), but rather perform a subtraction of the equivalent two-potential formula for the scattering lengths and then check for cutoff independence of the results. This choice also allows for a better comparison between the pionless correlation given in Eq. (43) and the corresponding improvements when the strong physics are included explicitly.

in their analysis, which consists of improved Coulomb, vacuum polarization, and the magnetic moment interaction (see Ref. [42] for details). Because our current analysis is not intended to be *complete*, we ignore most of these details and simply assume that the long range potential is the usual Coulomb potential and that the full electromagnetic phase shifts roughly coincide with the Coulomb ones, $\delta^{\rm EM}_{\rm PWA} \simeq \delta^C_{\rm PWA}$.

In the present calculation we are only going to consider the chiral potentials up to the Q^3 order, i.e., next-to-next-to-leading-order (N^2LO). At this order the finite range piece of the chiral potential consists of one-pion exchange and chiral two-pion exchange. An interesting feature of the chiral two-pion exchange potentials is that they are highly singular, diverging as $\sim 1/r^6$ at N^2LO.[7] In harmony with previous findings [7–9], this divergence will become rather unimportant: the two-potential formula shows a smooth cutoff dependence for singular chiral potentials.[8] In any case, we stress that our main concern is to analyze the minimal short distance cutoff r_c for which higher order effects can be distinguished from lower order ones, rather than the specific cutoff dependence of the results.

The results for $E_{\rm lab} = 50$ MeV and $E_{\rm lab} = 200$ MeV can be seen in Fig. 2. By TPE we refer to the N^2LO chiral potential, and for compactness we skip the NLO results, which lie between the LO (OPE) and the N^2LO results. The bands represent the error coming from the original pp Coulomb phase shift in the PWA of Ref. [42]. For the strong and Coulomb scattering lengths, which are needed for the subtractions, we take the values corresponding to the Nijmegen II potential [18], i.e., $\alpha_C = -7.81$ fm and $\alpha_S = -17.25$ fm. The actual expressions for the chiral pp potential are taken from Ref. [41]. Only the long range piece of the chiral potentials is considered, and the corresponding counterterms are ignored: they are equivalent to a boundary condition for

[7]The most singular (noncontact) piece of $V^{(\nu)}_\chi$ will behave as $1/r^{3+\nu}$ in coordinate space and as $|\vec{q}|^\nu f(|\vec{q}|/m_\pi)$ in momentum space, $\vec{q}$ being the momentum exchanged between the nucleons and f a non-polynomial function.

[8]In fact, the singular chiral two-pion exchange potentials yield smoother results than the OPE potential. In the current regularization scheme, OPE shows a mild log-divergence at distances of 10^{-3} fm. This divergence can be eliminated by using a different, and more complex, subtraction prescription, but for the purposes of the present discussion it is not particularly important what happens at such small scales.

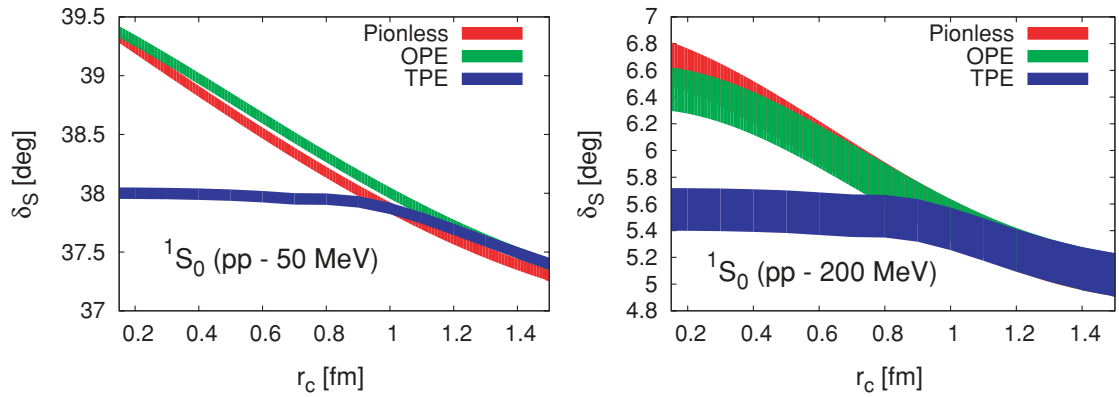

FIG. 2. (Color online) Strong pp s-wave phase shifts at $E_{\rm lab} = 50$ MeV and $E_{\rm lab} = 200$ MeV as a function of the cutoff r_c, computed from the Coulomb pp phase shifts (Nijmegen PWA [42]) by using the chiral potential truncated at different orders in the chiral expansion (pionless means no potential, OPE is the leading order potential, and TPE the full N^2LO chiral potential). The strong and Coulomb phase shifts have been related by making use of Eq. (31) with one subtraction at $k = 0$, to have finite results when the cutoff is removed.

the Schrödinger equation [44,45] and are therefore already implicitly included in the two-potential formula. We take $g_A = 1.29$, $m_{\pi^0} = 134.98$ MeV, and $f_\pi = 92.4$ MeV, which according to the definitions of Ref. [42] gives an $f^2_{pp\pi^0} = 0.0755$ for the scaling mass $m_s = m_{\pi^\pm} = 139.57$ MeV. The previous chiral pp potential explicitly depends on three chiral couplings, c_1, c_3, and c_4, which appear at $\mathcal{O}(Q^3)$ in the expansion of the potential, and which relate nucleon-nucleon and nucleon-pion scattering. We take for these chiral couplings the values obtained in Ref. [46] from analyzing the pp data alone, i.e., $c_1 = -0.76(7)$ GeV^{-1}, $c_3 = -4.78(11)$ GeV^{-1}, and $c_4 = 3.92(52)$ GeV^{-1}. As can be seen, for a cutoff above $r_c = 1.2$ fm, one cannot distinguish, within uncertainties, between lower and higher order computations; i.e., it does not matter whether pions are included in V_S or not. Actually cutoffs below $r_c = 0.8$ fm are needed to fully distinguish the chiral two-pion exchange contributions within the accuracy of the phase shifts. This result is not entirely surprising as it could be anticipated from considering the two-pion exchange related Compton wavelength scale $\lambda_{2\pi} = 1/2m_\pi \sim 0.7$ fm. A more complete analysis should include vacuum polarization, $2\pi\gamma$ and $\gamma\gamma$ exchange effects, which will affect the the precise values of the strong phase shifts but will hardly change the observation on the relevant scales. The same remarks also apply to the error analysis, which should include the error in the subtracted strong and Coulomb scattering lengths and the theoretical uncertainties in the chiral potential itself, like, for example, the error in the determination of the chiral couplings.

We can also compare the extracted strong effective ranges for the different cases considered. In this case, the Coulomb pp phase shift from the Nijmegen II potential is used as input for the two-potential formula and the resulting strong phase shifts are shown in Fig. 1. For the regularization scale $r_c = 0.1$ fm, we obtain

$$r_{S,\text{contact}} = 2.78\,\text{fm}, \tag{52}$$

$$r_{S,\text{OPE}} = 2.63\,\text{fm}, \tag{53}$$

$$r_{S,\text{TPE}} = 2.87\,\text{fm}, \tag{54}$$

to be compared with the Nijmegen II result, $r_S = 2.84$ fm. The pionless value differs from the one given in Sec. IV A due to finite cutoff effects, while OPE surprisingly contributes in the wrong direction. The TPE result reproduces the Nijmegen II one within a 1% accuracy level and agrees within error estimations with the extraction of Ref. [19], $r_S = 2.84(4)$ fm, where the error accounts for different sources of model dependence.

Finally, we note that *despite* that the TPE potential becomes highly singular at short distances, diverging as $\sim 1/r^6$, nothing dramatic happens, making the limit $r_c \to 0$ innocuous *precisely* when the TPE effects become visible, i.e., for $r_c \leqslant 0.8$ fm.[9] This particular feature is a specific merit of our two-potential formula that provides a clean separation between scales and implements in an extended distorted wave fashion the renormalization program carried out in previous works (see, e.g., Refs. [3,4]).

[9]Actually, despite Eq. (6) being fulfilled to the extreme, the TPE correction is small because the typical minimal wavelength is still not comparable to the range where the TPE correction takes over.

V. CONCLUSIONS

The two-potential formalism provides a framework where forces of different origin and ranges may be disentangled rather explicitly. We have proposed a coordinate space formulation that restates the result in a rather transparent way and fully exploits the boundary value character as well as the superposition principle of the scattering problem. Our result allows for a detailed investigation of the relevant scales built into the problem. This is particularly enlightening in the case of singular potentials and their renormalization, a subject of recent interest. We have exemplified our approach by discussing its consequences for the proton-proton system, where the electromagnetic and strong forces contribute to the scattering process, as a method to extract the strong phase shifts in a model-independent fashion. We have only discussed s-waves [with the exception of Eq. (45)] and single channel scattering. The extension to higher partial waves, as well as coupled channels, is straightforward but cumbersome (see the Appendix). Such an extended formalism might allow a discussion of further interesting applications of the present ideas to similar problems where a scale separation of different forces would be necessary.

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APPENDIX: EXTENSION TO HIGHER PARTIAL WAVES AND COUPLED CHANNELS

A. Higher partial waves

The extension of our two-potential formula to higher partial waves is straightforward. The full two-body system is described by the corresponding reduced Schrödinger equation for the l wave,

$$-u''_{k,l} + \left[2\mu\,(V_L(r) + V_S(r)) + \frac{l(l+1)}{r^2}\right] u_{k,l}(r) = k^2 u_{k,l}(r), \tag{A1}$$

where $u_{k,l}(r)$ is the l-wave reduced wave function, μ the reduced mass, and k the center-of-mass momentum. The asymptotics of $u_{k,l}$ for $r \to \infty$ is given by

$$u_{k,l}(r) \to \cot\delta_l \, \hat{j}_l(kr) - \hat{y}_l(kr), \tag{A2}$$

where $\hat{j}_l$ and $y_l(kr)$ are the reduced spherical Bessel functions, defined as $\hat{j}_l(x) = x\, j_l(x)$ and $\hat{y}_l(x) = x\, y_l(x)$. We only consider here the case of a long range potential V_L decaying faster than $1/r^2$ at large distances. By making use of the superposition principle we rewrite the full solution as

$$u_{k,l}(r) = \cot\delta_l \, J_{k,l}(r) - Y_{k,l}(r), \tag{A3}$$

where $J_{k,l}$ and $Y_{k,l}$ are solutions of Eq. (A1) subjected to the asymptotic boundary conditions $J_{k,l}(r) \to \hat{j}_l(kr)$ and $Y_{k,l}(r) \to \hat{y}_l(kr)$ for $r \to \infty$. The corresponding scattering problem for which only the short range potential V_S is present is described by the reduced Schrödinger equation

$$-u^{S''}_{k,l} + \left[2\mu\, V_S(r) + \frac{l(l+1)}{r^2}\right] u^S_{k,l}(r) = k^2 u^S_{k,l}(r), \tag{A4}$$

with $u^S_{k,l}$ the short l-wave reduced wave function. The short distance phase shift is obtained from the asymptotic behavior of $u^S_{k,l}$,

$$u^S_{k,l}(r) \to \cot\delta^S_l \, \hat{j}_l(kr) - \hat{y}_l(kr), \tag{A5}$$

for $r \to \infty$. We rewrite $u^S_{k,l}$ as

$$u^S_{k,l}(r) = \cot\delta_l \, J^S_{k,l}(r) - Y^S_{k,l}(r), \tag{A6}$$

with $J^S_{k,l}$ and $Y^S_{k,l}$ being solutions of Eq. (A4) obeying the asymptotic boundary conditions $J^S_{k,l}(r) \to \hat{j}_l(kr)$ and $Y^S_{k,l}(r) \to \hat{y}_l(kr)$.

As usual we match the logarithmic derivatives of $u_{k,l}(r)$ and $u^S_{k,l}(r)$ at the cutoff radius $r = r_c$, yielding

$$\cot\delta_l(k) = \frac{\mathcal{A}_l(k, r_c)\cot\delta^S_l(k) - \mathcal{B}_l(k, r_c)}{\mathcal{C}_l(k, r_c)\cot\delta^S_l(k) - \mathcal{D}_l(k, r_c)}, \tag{A7}$$

where $\mathcal{A}_l$, $\mathcal{B}_l$, $\mathcal{C}_l$, and $\mathcal{D}_l$ are defined as

$$\mathcal{A}_l(k, r_c) = W\left(J^S_{k,l}, Y_{k,l}\right)\Big|_{r=r_c}, \tag{A8}$$

$$\mathcal{B}_l(k, r_c) = W\left(Y^S_{k,l}, Y_{k,l}\right)\Big|_{r=r_c}, \tag{A9}$$

$$\mathcal{C}_l(k, r_c) = W\left(J^S_{k,l}, J_{k,l}\right)\Big|_{r=r_c}, \tag{A10}$$

$$\mathcal{D}_l(k, r_c) = W\left(Y^S_{k,l}, J_{k,l}\right)\Big|_{r=r_c}, \tag{A11}$$

in analogy with the s-wave case. In principle the use of the previous formula is straightforward as long as a finite cutoff is employed in the computation. On the contrary, if one tries to remove the cutoff, some divergences may appear, mostly related to the centrifugal barrier. Therefore a detailed analytical or numerical study of the divergences will be necessary to obtain a finite result in the $r_c \to 0$ limit.

B. Coupled channels

The extension to the coupled channel case is direct to obtain if an adequate notation is used. We consider the general case of N coupled channels. They can be described by the following Schrödinger equation, which in compact notation reads

$$-\mathbf{u}''_k + \left[2\mu(\mathbf{V}_L(r) + \mathbf{V}_S(r)) + \frac{\mathbf{L}^2}{r^2}\right] \mathbf{u}_k(r) = k^2 \mathbf{u}_k(r), \tag{A12}$$

where the wave function $\mathbf{u}_k$ is now an N × N matrix, each column representing a linearly independent solution. The long and short range potentials $\mathbf{V}_L$ and $\mathbf{V}_S$ are also $N \times N$ matrices (the nondiagonal terms relating the different channels), and $\mathbf{L}^2$ is the angular momentum matrix, which is diagonal and given by

$$\mathbf{L}^2 = \mathrm{diag}(l_1(l_1+1), l_2(l_2+1), \ldots, l_N(l_N+1)), \tag{A13}$$

$l_1, l_2, \ldots, l_N$ being the orbital angular momentum of each channel. In principle there are $2N$ linearly independent solutions (two per channel), but regularity conditions at the origin reduce this number to N. This is why the wave function can be represented by an $N \times N$ matrix. We have also added the simplifying assumption that there are no inelasticities, meaning that in Eq. (A12) the source of the coupling is either tensor forces or dipole-dipole interactions.

The asymptotic behavior of the wave function matrix $\mathbf{u}_k$ is given by the expression

$$\mathbf{u}_k(r) \to \mathbf{j}(kr)\,\mathbf{M}(k) - \mathbf{y}(kr), \tag{A14}$$

for $r \to \infty$, where $\mathbf{j}$ and $\mathbf{y}$ are diagonal matrices given by

$$\mathbf{j}(kr) = \mathrm{diag}(\hat{j}_{l_1}(kr), \hat{j}_{l_2}(kr), \ldots, \hat{j}_{l_N}(kr)), \tag{A15}$$

$$\mathbf{y}(kr) = \mathrm{diag}(\hat{y}_{l_1}(kr), \hat{y}_{l_2}(kr), \ldots, \hat{y}_{l_N}(kr)), \tag{A16}$$

with $\hat{j}_l$ and $\hat{y}_l$ being the reduced spherical Bessel functions as defined in the previous section. The matrix $\mathbf{M}(k)$ is analogous to $\cot\delta$ for coupled channels and is related to the S matrix by $\mathbf{M}(k) = i\,(\mathbf{S}(k) + \mathbf{1})/(\mathbf{S}(k) - \mathbf{1})$ with $\mathbf{1}$ being the identity matrix. It is a symmetric matrix and contains $N(N+1)/2$ independent scattering parameters or *phase shifts*. By making use of the superposition principle, we can rewrite the wave function matrix as[10]

$$\mathbf{u}_k(r) = \mathbf{J}_k(r)\,\mathbf{M}(k) - \mathbf{Y}_k(r), \tag{A17}$$

where $\mathbf{J}_k$ and $\mathbf{Y}_k$ are solutions of Eq. (A12) that asymptotically behave as $\mathbf{J}_k(r) \to \mathbf{j}(kr)$ and $\mathbf{Y}_k(r) \to \mathbf{y}(kr)$.

The corresponding Schrödinger equation for the short wave function is

$$-\mathbf{u}^{S''}_k + \left[2\mu\,\mathbf{V}_S(r) + \frac{\mathbf{L}^2}{r^2}\right] \mathbf{u}^S_k(r) = k^2 \mathbf{u}^S_k(r), \tag{A18}$$

where, in analogy to the full case, the short wave function matrix can be written as

$$\mathbf{u}^S_k(r) = \mathbf{J}^S_k(r)\,\mathbf{M}^S(k) - \mathbf{Y}^S_k(r), \tag{A19}$$

with $\mathbf{J}^S_k$ and $\mathbf{Y}^S_k$ being solutions of Eq. (A18) subjected to the asymptotic boundary conditions $\mathbf{J}^S_k(r) \to \mathbf{j}(kr)$ and $\mathbf{Y}^S_k(r) \to \mathbf{y}(kr)$ for $r \to \infty$.

[10]The reason why we write $\mathbf{J}_k(r)\,\mathbf{M}(k)$ instead of $\mathbf{M}(k)\,\mathbf{J}_k(r)$ in Eq. (A17) is because if $\mathbf{u}_k$ is a solution of the Schrödinger equation (A12) and $\mathbf{A}$ a constant $N \times N$ matrix, then $\mathbf{u}_k\,\mathbf{A}$ is also a solution (but this is not the case for $\mathbf{A}\,\mathbf{u}_k$).

To obtain the corresponding two-potential formula one needs to match the logarithmic derivatives of the wave functions, which for the coupled channel case means

$$\mathbf{u}_k'(r_c)(\mathbf{u}_k(r_c))^{-1} = \mathbf{u}_k^{S'}(r_c)\left(\mathbf{u}_k^S(r_c)\right)^{-1}. \tag{A20}$$

Using the Wronskian relation

$$\mathbf{u}_k^T(r_c)\mathbf{u}_k'(r_c) = \mathbf{u}_k^{T'}(r_c)\mathbf{u}_k(r_c), \tag{A21}$$

where the ^{T} superscript denotes the transpose, the boundary condition given by Eq. (A20) can be rewritten as

$$\mathbf{u}_k^T(r_c)\,\mathbf{u}_k^{S'}(r_c) = \mathbf{u}_k^{T'}(r_c)\,\mathbf{u}_k^S(r_c), \tag{A22}$$

an expression which does not involve the inverse of the wave functions. If we rewrite $\mathbf{u}_k$ and $\mathbf{u}_k^S$ in terms of Eqs. (A17) and (A19), we arrive at our final expression

$$\mathbf{M}(k) = [\mathcal{A}(k,r_c)\,\mathbf{M}^S(k) - \mathcal{B}(k,r_c)] \times [\mathcal{C}(k,r_c)\,\mathbf{M}^S(k) - \mathcal{D}(k,r_c)]^{-1}, \tag{A23}$$

with $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ defined as

$$\mathcal{A}(k,r_c) = -W\left(\mathbf{Y}_k^T, \mathbf{J}_k^S\right)\Big|_{r=r_c}, \tag{A24}$$

$$\mathcal{B}(k,r_c) = -W\left(\mathbf{Y}_k^T, \mathbf{Y}_k^S\right)\Big|_{r=r_c}, \tag{A25}$$

$$\mathcal{C}(k,r_c) = -W\left(\mathbf{J}_k^T, \mathbf{J}_k^S\right)\Big|_{r=r_c}, \tag{A26}$$

$$\mathcal{D}(k,r_c) = -W\left(\mathbf{J}_k^T, \mathbf{Y}_k^S\right)\Big|_{r=r_c}, \tag{A27}$$

where the Wronskian is given by $W(\mathbf{F},\mathbf{G})|_{r=r_c} = \mathbf{F}(r_c)\mathbf{G}'(r_c) - \mathbf{F}'(r_c)\mathbf{G}(r_c)$.

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