Transition Operator for Matrix Potentials: An Application to the Pion-Nucleon Interaction

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The transition operator corresponding to a separable nonlocal and of a matrix-type potential is studied theoretically. It is shown that the transition operator possesses the same $\vec{r}$ (or $\vec{k}$) dependence and properties of separability and nonlocality as the potential and that it is also of a matrix type. This transition operator is then explicitly calculated. A numerical application is made to the simple but physically important case of the low-energy pion-nucleon interaction. The behavior of the transition operator coefficients versus the incident energy are discussed in detail for S, P, and D waves.

1. INTRODUCTION

Several nonlocal interactions between quantum particles have been proposed in the past few years[1-3] and have been shown to be very useful.[3] The main justifications for using such a type of interaction are the following: (i) The "true" physical interactions have to be considered between particles having some space extension rather than between point (well-localized) particles and it is known that the interaction between nonpoint particles is nonlocal; (ii) in many cases, the problem of two interacting particles which are more or less elementary is in fact a many-body problem so that the resulting potential is of an optical and nonlocal type; (iii) it is well known that if relativistic corrections are taken into account or if retarded effects or source recoil are correctly treated (in mesonic theories for example), nonlocality effects appear. Finally, one needs to emphasize the simplicity of the calculations. In many cases a nonlocal separable potential allows us to get through the whole analytical calculation without any approximation.[3]

In Sec. 2, we propose a nonlocal separable potential which is characterized by being of a matrix type and so is a generalisation of the existing nonlocal potentials (the utility of this type of potential has been pointed out elsewhere in the particular case of the nucleon-nucleon interaction and its applications to nuclear matter and finite nuclei[4]). The transition operator being often more useful than the potential itself in practical applications, we then present a study of the remarkable properties of this transition operator corresponding to a matrix potential. A general, explicit, and simple method for calculating its coefficients will be given.

In Sec. 3 we make an application of these results to the case of the pion-nucleon interaction. This example has been firstly chosen for its importance in the interactions of pions with nuclei which has now drawn attention due to the imminent availability of intense pion beams at meson factories. As a matter of fact, it seems useful and justified to start from realistic pion-nucleon interactions to obtain good pion-nucleus potentials. At present, very simple pion-nucleon potentials (as for example those of Refs. 5 and 6) have been extensively used. In particular, the velocity-dependent Kisslinger optical model[6] allowed a better understanding of (i) the sensitivity of elastic $\pi^{\pm}$-nucleus scattering to the pion form factors[7]; (ii) the pion elastic scattering from carbon[8,9] and oxygen,[8] the relative neutron-proton distribution in light nuclei[10,11]; (iii) the sensitivity of the pion-nucleus total cross sections to the assumed off-shell behavior of the pion-nucleon scattering amplitude[12]; and (iv) the energy variation of the effective nuclear radius arising from P-wave pion-nucleon interaction.[12] However, the nonlocal optical potentials of the Kisslinger type cannot be used in all cases[12,13] and a lot of problems are still to be solved.[14] This is one reason why more sophisticated pion-nucleon potentials have been proposed.[15,2] In Ref. 15, Landau *et al.* propose a general method to construct absorptive separable potentials directly from the phase shifts and absorption parameters. The result for the pion-nucleon system is an S- and P-wave complex interaction satisfying the relativistic Schrödinger equation and fitting phase shifts and absorption parameters up to 2.5-GeV/c pion laboratory momentum. From another point of view, the restrictive case of the low energies seems to be of interest because in nuclear physics applications, such as the pion-nucleus scattering, energies greater than 200 MeV (pion laboratory energy) are rarely involved. In addition, the availability of simple analytical expressions for both the potential and transition operators is certainly of importance for these further applications. Thus, we here use the "realistic" potential of Ref. 2, which fits the experimental nucleon phase shifts up to the first energy resonance. This potential is nonlocal separable and a particular case of the matrix-

type potential proposed in Ref. 4 and leads to analytical calculations very easy to handle. The interest of considering the transition operator (t) of a pion-nucleon potential is that, in most nonrelativistic theories of the interaction between π mesons and nuclear matter or finite nuclei, the knowledge of the matrix elements of t on and off the energy shell is needed and that the result depends appreciably on the off-shell behavior of t. The numerical results for the coefficients defining t are presented for the relative S, P, and D waves of the pion-nucleon system; their physical meaning is discussed, particularly the relative importance of the real and imaginary parts of t versus the energy.

2. TRANSITION OPERATOR FOR A NONLOCAL SEPARABLE MATRIX POTENTIAL

A. Definition of the Potential

A general form for a two-body potential operator conserving the spin S^2, the isospin T^2, T_z, and the total angular momentum J^2, J_z can be written

$$W=\frac{\hbar^2}{2\mu}\sum_{LL'\nu M}\sum_{ij}\lambda_{ij}^{LL'\nu}|v_{iL\nu}\rangle|LS;JM\rangle\langle L'S;JM|\langle v_{jL'\nu}|P_T, \tag{1}$$

where μ is the reduced mass of the relative particle, P_T is the projection operator onto the isotopic state T, ν stands for the quantum numbers SJT, $\vec{L}$ is the relative orbital angular momentum coupled to the spin $\vec{S}$ ($\vec{J}=\vec{L}+\vec{S}$) and the kets $|LS;JM\rangle$ are eigenkets of L^2, S^2, J^2, and J_z:

$$|LS;JM\rangle=\sum_{m_L m_S}|Lm_L\rangle|Sm_S\rangle\langle Lm_L\,Sm_S|JM\rangle. \tag{2}$$

To have a real and Hermitian potential, the λ coefficients (in $\mathrm{fm}^{-(L+L'+3)}$ must satisfy

$$(\lambda_{ij}^{LL'\nu})^*=\lambda_{ij}^{LL'\nu}=\lambda_{ji}^{L'L\nu}. \tag{3}$$

If $L=L'$ these coefficients correspond to the "depth of the potential well" in the $(L\nu)$ states; if $L\neq L'$ they describe the intensity of the tensor force in the $(LL'\nu)$ coupled states (e.g., ${}^3S_1+{}^3D_1$ for the two-nucleon system). The number of terms in the potential (i and $j=1, n$) and the choice of the kets $|v_{iL\nu}\rangle$ leading to the r (or k) dependence of the potential will be defined in the practical applications. In relative coordinates ($\vec{r}$) the potential of Eq. (1) can be expressed in a matrix form

$$\langle\vec{r}|W|\vec{r}'\rangle=\sum_{LL'\nu M}[\langle r|v_{1L\nu}\rangle\langle r|v_{2L\nu}\rangle\cdots\langle r|v_{nL\nu}\rangle]$$
$$\times\begin{bmatrix}\lambda_{11}^{LL'\nu} & \lambda_{12}^{LL'\nu} & \cdots & \lambda_{1n}^{LL'\nu}\\ \lambda_{21}^{LL'\nu} & \lambda_{22}^{LL'\nu} & \cdots & \lambda_{2n}^{LL'\nu}\\ \vdots & \vdots & & \vdots\\ \lambda_{n1}^{LL'\nu} & \lambda_{n2}^{LL'\nu} & \cdots & \lambda_{nn}^{LL'\nu}\end{bmatrix}\begin{bmatrix}\langle v_{1L'\nu}|r'\rangle\\ \langle v_{2L'\nu}|r'\rangle\\ \vdots\\ \langle v_{nL'\nu}|r'\rangle\end{bmatrix}\langle\hat{r}|LS;JM\rangle\langle L'S;JM|\hat{r}'\rangle P_T. \tag{4}$$

We remark that the usual nonlocal potentials are particular cases of the matrix-type potentials defined here. If we cancel all the matrix elements $\lambda_{ij}^{LL'\nu}$ except the first two diagonal (i.e., if we set $\lambda_{ij}^{LL'\nu}=B_i^{LL'}\delta_{ij}$ and $i=1,2$ only in the whole formulation) we find the potentials of Refs. 1–3. Of course, for the systems where L is conserved $L=L'$ everywhere.

B. Properties of the Transition Operator

We now propose a demonstration of the following important result: If a potential operator W is defined by Eqs. (1)–(3), i.e., is nonlocal, separable, and of the matrix type, then the transition operator $t^{\pm}$ has the same properties and the corresponding $\vec{r}$ (or $\vec{k}$) dependence are the same for W and $t^{\pm}$. From the mathematical point of view, this means that we can write similar expressions for both v and $t^{\pm}$ operators, namely,

$$t^{\pm}=\frac{\hbar^2}{2\mu}\sum_{LL'\nu M}\sum_{ij}\mu_{ij}^{\pm LL'\nu}(k_0)|v_{iL\nu}\rangle|LS;JM\rangle\langle L'S;JM|\langle v_{jL'\nu}|P_T, \tag{5}$$

where the μ coefficients will be shown to be complex functions of the incident energy $E_0 = \hbar^2 k_0^2/2\mu$ and implicit functions of the "form factors" of the potential ($\langle k|v_{iLv}\rangle$).

The transition operator satisfying the Lippman-Schwinger equation may be defined by

$$t^{\pm} = W + WG^{\pm}t^{\pm} = W\sum_{n=0}^{\infty}(G^{\pm}W)^n = \sum_{n=0}^{\infty} t_n^{\pm} \tag{6a}$$

with

$$G^{\pm} = \frac{1}{E_0 - H_0 \pm i\epsilon} = \frac{2\mu}{\hbar^2}\int \frac{|\vec{q}\rangle d^3q\langle\vec{q}|}{k_0^2 - q^2 \pm i\epsilon} \tag{6b}$$

and

$$H_0|\vec{q}\rangle = \frac{\hbar^2}{2\mu}q^2|\vec{q}\rangle, \tag{6c}$$

where G^+ and G^- are the Green's operators corresponding to a divergent and a convergent wave, respectively, ($\epsilon > 0$). It is sufficient to show that each term $t_n^{\pm}$ defined by Eqs. (6) has an expression of the same type as the potential. For $n=0$, the assertion is evident and it can be verified directly for $n=1$. Let us suppose (induction hypothesis) that the assertion is true for $n-1$, i.e.,

$$t_{n-1}^{\pm} = \frac{\hbar^2}{2\mu}\sum_{LL'\nu M}\sum_{ij}\rho_{ij}^{\pm LL'\nu}(k_0)|v_{iL\nu}\rangle|LS;JM\rangle\langle L'S;JM|\langle v_{jL'\nu}|P_T. \tag{7}$$

Then,

$$t_n^{\pm} = t_{n-1}G^{\pm}W = \int t_{n-1}|\vec{q}\rangle d^3q\langle\vec{q}|G^{\pm}|\vec{q}'\rangle d^3q'\langle\vec{q}'|W$$

$$= \frac{\hbar^2}{2\mu}\sum_{LL'\nu M}\sum_{ij}\sigma_{ij}^{\pm LL'\nu}(k_0)|v_{iL\nu}\rangle|LS;JM\rangle\langle L'S;JM|\langle v_{jL'\nu}|P_T, \tag{8}$$

where we set

$$\sigma_{ij}^{\pm LL'\nu}(k_0) = \sum_{i'j'}\sum_{l}\rho_{ii'}^{\pm Ll}(k_0)q_{i'j'}^{\pm l\nu}(k_0)\lambda_{j'j}^{lL'\nu} \tag{9}$$

and

$$q_{ij}^{\pm L\nu}(k_0) = \int_0^{\infty}\langle v_{iL\nu}|q\rangle\frac{q^2dq}{k_0^2 - q^2 \pm i\epsilon}\langle q|v_{jL\nu}\rangle$$

$$= q_{ji}^{\pm L\nu}(k_0) = [q_{ij}^{\mp L\nu}(k_0)]^*. \tag{10}$$

We have used the relation

$$\langle\vec{q}|G^{\pm}|\vec{q}'\rangle = \frac{2\mu}{\hbar^2}\frac{\delta(\vec{q}-\vec{q}')}{k_0^2 - q^2 \pm i\epsilon}. \tag{11}$$

Equation (8) shows that if the operator $t_{n-1}^{\pm}$ is nonlocal separable and of a matrix type then t_n has the same properties and the same $\vec{r}$ (or $\vec{k}$) dependence. It follows that the transition operator $t^{\pm}$ must have the form given in Eq. (5). The μ coefficients are combinations of the σ coefficients of Eq. (9) and consequently are functions of the incident energy and of the form factors of the potential through the complex coefficients $q_{ij}^{\pm L\nu}(k_0)$.

C. Explicit Derivation of the μ Coefficients of the Transition Operator

We indicate here a general but simple method to calculate the coefficients $\mu_{ij}^{\pm LL'\nu}$ of the transition operator as written in Eq. (5).

The expressions of W, $G^{\pm}$, $t^{\pm}$ as written in Eqs. (1), (6b), and (5), respectively, with the above property and definition [Eqs. (10) and (11)], lead to the following development of the operator $WG^{\pm}t^{\pm}$:

$$WG^{\pm}t^{\pm} = \frac{\hbar^2}{2\mu}\sum_{LL'\nu M}\sum_{ij}\Big[\sum_{l}\sum_{i'j'}\lambda_{ii'}^{Ll\nu}q_{i'j'}^{\pm l\nu}(k_0)\mu_{j'j}^{\pm lL'\nu}(k_0)\Big]|v_{iL\nu}\rangle|LS;JM\rangle\langle L'S;JM|\langle v_{jL'\nu}|P_T. \tag{12}$$

We substitute Eqs. (1) and (12) in Eq. (6a) and, identifying this expression of $t^{\pm}$ with that of Eq. (5), we obtain

$$\mu_{ij}^{\pm LL'\nu}(k_0)=\lambda_{ij}^{LL'\nu}+\sum_{l}\sum_{i'j'}\lambda_{ii'}^{Ll\nu}q_{i'j'}^{\pm l\nu}\mu_{j'j}^{\pm lL'\nu}(k_0). \tag{13}$$

The practical utilization of the preceding equations is very much simplified if, for a given set of quantum numbers $\nu=SJT$, we consider the $\lambda_{ij}^{LL'\nu}$ coefficients as elements of a matrix (λ) constructed with submatrices $(\lambda)^{LL'\nu}$. In the usual case when two values at most are possible for the orbital momentum, then

$$(\lambda)=\begin{bmatrix}(\lambda)^{LL\nu} & (\lambda)^{LL'\nu}\\ (\lambda)^{L'L\nu} & (\lambda)^{L'L'\nu}\end{bmatrix}, \tag{14}$$

where the $(\lambda)^{LL'\nu}$ submatrices are explicitly written in Eq. (4). In the same manner for each ν, we define the $(\mu^{\pm})$ matrix with the coefficients $\mu_{ij}^{\pm LL'\nu}(k_0)$ as matrix elements and the $(q^{\pm})$ matrix with matrix elements $q_{ij}^{\pm LL'\nu}(k_0)$ defined by

$$q_{ij}^{\pm LL'\nu}(k_0)=q_{ij}^{\pm L\nu}(k_0)\delta_{LL'}. \tag{15}$$

Thus, Eq. (13) and the preceding definitions lead to the simple matrix equation

$$(\mu^{\pm})=[1-(\lambda)(q^{\pm})]^{-1}(\lambda). \tag{16}$$

It is easy to show from this equation and from the symmetry relations of the λ and q coefficients [Eqs. (3) and (10)] that

$$\mu_{ij}^{\pm LL'\nu}(k_0)=\mu_{ji}^{\pm L'L\nu}(k_0). \tag{17}$$

3. APPLICATION TO THE PION-NUCLEON INTERACTION

A. Details of the Calculation

We use the pion-nucleon potential of Ref. 2 which, as mentioned above, is a particular case of the general matrix potential of Sec. 2. The corresponding potential operator reads [see the remark under Eq. (4)]

$$W=\frac{\hbar^2}{2\mu}\sum_{i=1}^{2}\sum_{LJMT}B_i^{LJT}|v_{iLJT}\rangle|L\tfrac{1}{2};JM\rangle\langle L\tfrac{1}{2};JM|\langle v_{iLJT}|P_T, \tag{18}$$

where μ is the reduced mass of the pion-nucleon system and LJT correspond to the relative orbital angular momentum, the total angular momentum, and the isospin ($T=\frac{1}{2},\frac{3}{2}$), respectively. The kets $|LS;JM\rangle$ are defined from Eq. (2) and the single value for the spin is $S=\frac{1}{2}$. The r dependance of the potential has been chosen

$$\langle r|v_{iLJT}\rangle=r^{L-1}e^{-r\alpha_{iLJT}}. \tag{19a}$$

So, if we adopt the following normalization conventions

$$\langle\vec{r}|\vec{k}\rangle=(2\pi)^{-3/2}e^{i\vec{k}\cdot\vec{r}}$$

and

$$\int|\vec{r}\rangle d^3r\langle\vec{r}|=\int|\vec{k}\rangle d^3k\langle\vec{k}|=1, \tag{19b}$$

the k dependence of the potential is[16]

$$\begin{aligned}\langle k|v_{iLJT}\rangle&=\frac{1}{i^L}\left(\frac{2}{\pi}\right)^{1/2}\int_0^{\infty}j_L(kr)\langle r|v_{iLJT}\rangle r^2dr\\&=\frac{1}{i^L}\left(\frac{2}{\pi}\right)^{1/2}\frac{L!(2k)^L}{(k^2+\alpha_{iLJT}{}^2)^{L+1}}=\frac{1}{i^L}\left(\frac{2}{\pi}\right)^{1/2}p_{iLJT}(k).\end{aligned} \tag{19c}$$

We see in Eq. (19c) that the k dependence of the potential $\langle k|v_{iLJT}\rangle$ satisfies all the analytic conditions required for a separable interaction[17]: (i) It is a positive-definite function of k^2 vanishing as k^{2l} when k^2 goes to zero; (ii) it has a branch cut from $k^2=-\alpha_{iLJT}{}^2$ to $-\infty$; (iii) it goes to zero when $k^2\to\infty$. For S waves, it corresponds to a Yukawa-type potential and, for $L\geq 1$, to a modulated Yukawa-type potential. This allows

us to obtain a good fit of S-P-D waves to experimental phase shifts without changing the values of the inverse range parameters α_{iLJT}.

The transition operator is, from Eq. (5),

$$t^{\pm}=\frac{\hbar^2}{2\mu}\sum_{i,j=1}^{2}\sum_{LJMT}\mu_{ij}^{\pm LJT}(k_0)|v_{iLJT}\rangle|L\tfrac{1}{2};JM\rangle\langle L\tfrac{1}{2};JM|\langle v_{jLJT}|P_T\,, \tag{20}$$

where we use $\mu_{ij}^{\pm LJT}$ instead of $\mu_{ij}^{\pm LL'v}$, since $L=L'$, $S=\frac{1}{2}$. In this particular case, Eq. (17) gives

$$\mu_{ij}^{\pm LJT}(k_0)=\mu_{ji}^{\pm LJT}(k_0)\,. \tag{21}$$

The μ coefficients are complex and solutions of the system of algebric equations, obtained from Eq. (13):

$$\mu_{ij}^{\pm L}(k_0)=B_i^L\delta_{ij}+B_i^L\sum_{i'}q_{ii'}^{\pm L}(k_0)\mu_{i'j}^{\pm L}(k_0)\,, \tag{22}$$

where the JT superscripts are implicit from now on. The real and imaginary parts of the q functions are easily separated:

$$q_{ij}^{\pm L}(k_0)=q_{ij}^{L}(k_0)\mp ik_0p_{iL}(k_0)p_{jL}(k_0)\,, \tag{23}$$

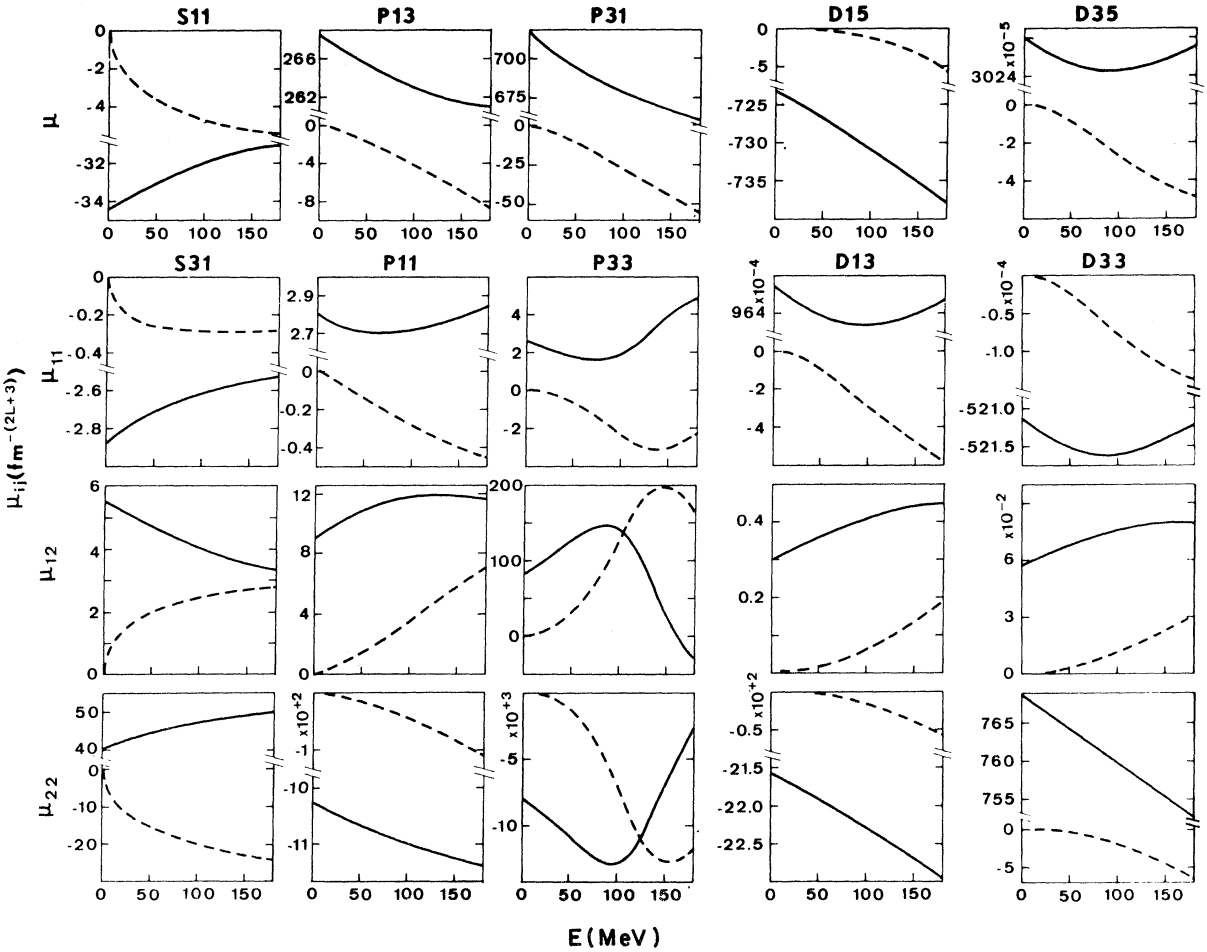

FIG. 1. Transition operator coefficients (in $\mathrm{fm}^{-(2L+3)}$) versus energy of the pion in the laboratory (in MeV). They are calculated from the potential of Ref. 2 determined by the fitting of the pion-nucleon phase shifts (see parameters in Table I). Each of the upper five figures shows the real and imaginary parts (respectively, in solid and dotted line) of the unique μ coefficient which is different from zero when a one-term potential is used: μ_{22} for the $S11$, $P13$, $P31$, and $D15$ waves and $\mu=\mu_{11}$ for the $D35$ wave. The three other groups of five figures compare the real and imaginary parts of each of the three complex coefficients μ_{11}, μ_{12}, and μ_{22} of the five waves $S31$, $P11$, $P33$, $D13$, and $D33$ for which a two-term potential has to be used. These curves are discussed in Sec. 3.

where

$$q_{ij}^{L}(k_0)=P\int_0^\infty \frac{\langle v_{iL}|q\rangle q^2 dq\langle q|v_{jL}\rangle}{k_0^2-q^2}. \tag{24}$$

The evaluation of these integrals is not obvious; it is possible to show after a lengthy and complicated calculation that the $q_{ij}^{L}(k_0)$ may be related to the functions $H_{iL}(r,k)$ defined in Ref. 2 [see Eqs. (13) and (14) of this reference]:

$$H_{iL}(r,k)=j_L(kr)\int_r^\infty n_L(kr')\langle r'|v_{iL}\rangle r'^2 dr'+n_L(kr)\int_0^r j_L(kr')\langle r'|v_{iL}\rangle r'^2 dr',$$

where j_L and n_L are the spherical Bessel and Neumann functions with the phase conventions of Schiff.[18] We find

$$q_{ij}^{L}(k_0)=k_0\int_0^\infty \langle r|v_{iL}\rangle H_{jL}(r,k_0)r^2 dr. \tag{25}$$

This means that the $q_{ij}^{L}(k_0)$ defined in Eq. (24) are the same as those used in Ref. 2 where they are explicitly given for $L=0,1,2$. In the Appendix, we shall give the detailed solution of Eq. (22).

B. Numerical Results and Discussion

The real and imaginary parts of the $\mu_{ij}^{+L}(k_0)$ coefficients of the transition operator t^+ (denoted $\mu_{ij}^{(r)}$ and $\mu_{ij}^{(i)}$, respectively) have been numerically calculated for the 10 (L-2T-2J) waves considered in Ref. 2 ($L=0,1,2$). They are plotted versus the energy of the pion in the laboratory on Fig. 1. We consider this energy up to about 180 MeV which corresponds to the first resonance ($P33$ wave) and consequently to the validity range of the potential parameters.[2]

Several general observations may be made on the behavior of the transition operator in this energy range. (i) First, we note that the real and imaginary parts of the off-diagonal terms $\mu_{i\neq j}^{+L}(k_0)$ are of the same order of magnitude, while the imaginary part is small compared to the real part for the diagonal term $\mu_{i=j}^{+L}(k_0)$:

$$|\mu_{i=j}^{(i)}|\ll|\mu_{i=j}^{(r)}|$$

and

$$|\mu_{i\neq j}^{(i)}|\lesssim|\mu_{i\neq j}^{(r)}|.$$

The only exception to this rule is the $P33$ wave. (ii) Second, we note that the absolute variations of the real and imaginary parts of the μ functions have always the same order of magnitude for a given wave. The imaginary parts go to zero at the origin and their absolute values increase smoothly with increasing energy (except, once more, for the $P33$ wave). On the contrary the real parts differ from zero at low energy and present a more irregular behavior versus the energy particularly for $P33$, $D35$ waves (all the $\mu_{ij}^{(r)}$) and $P11$, $D13$, $D33$ waves ($\mu_{11}^{(r)}$ only). It is clear from the above remarks the relative variations of the imaginary parts are far more important than those of the real parts in the case of the off-diagonal ($i\neq j$) coefficients and are of the same order of magnitude in the case of the diagonal ($i\neq j$) coefficients. (iii) For a given wave, the numerical value of $\mu_{11}^{(r)}$ (or $\mu_{22}^{(r)}$) is of the same order of magnitude as the numerical value of the potential parameters B_1 (or B_2) except for $P33$, while the off-diagonal

TABLE I. Comparison between the pion-nucleon potential parameters (Ref. 2) (B_i) and the real parts of the corresponding coefficients of the transition operator t^+ ($\mu_{ij}^{(r)}$). The unit is fm$^{-(2L+3)}$ for both B_i and $\mu_{ij}^{(r)}$, for this last quantity the minimum and maximum values are given. We recall that B_1 and B_2 correspond to the inverse ranges $\alpha_1=1.5$ fm^{-1} and $\alpha_2=3.5$ fm^{-1}, respectively.

Wave	B_1	$\mu_{11}^{(r)}$	$\mu_{12}^{(r)}$	$\mu_{22}^{(r)}$	B_2
$S11$	0	0	0	−34 −31	−24.54
$S31$	−2.37	−2.9 −2.5	5 13	40 50	127
$P11$	3.19	2.80 2.70	8 12	−1100 −1200	−702
$P13$	0	0	0	269 261	307.9
$P31$	0	0	0	720 660	1088
$P33$	3.79	2 5	150 0	−12 000 −3000	−1755
$D13$	0.097	0.0963 0.0966	0.3 0.4	−2100 −2300	−1725
$D15$	0	0	0	−724 −737	−667
$D33$	−0.052	−0.0521 −0.0522	0.006 0.008	770 750	844
$D35$	−0.0303	0.030 24 0.030 26	0	0	0

$|\mu_{12}^{(r)}|$ is between B_1 and B_2 (see Table I).

We see that the $P33$ wave is an exception to the rules above and recall that in Ref. 2 it was pointed out the poor agreement between experimental and calculated values of the $P33$ scattering volume. The low-energy behavior of this particular wave seems to be perturbed by the existence of a resonance. The imaginary part of the μ coefficients of the transition operator are important and have a sharp minimum ($\mu_{11}^{(i)}$ and $\mu_{22}^{(i)}$) or a maximum ($\mu_{12}^{(i)}$), near the resonance energy. It is the only wave presenting extrema for the imaginary parts of $\mu_{ij}^{+L}(k_0)$ between 0 and 180 MeV. This seems to indicate a correlation between the behavior of the imaginary part of the transition operator and the existence (or the absence) of resonance in a given energy range. Finally, in spite of the tormented aspect of the $P33$ plots of the μ coefficients we note that the three real terms on the one hand, and the three imaginary terms on the other hand have their extrema at the same energies (90 and 160 MeV, respectively).

4. CONCLUSION

It has been demonstrated that the transition operator for a nonlocal separable matrix potential takes the same separable and matrix form as the potential and can be calculated analytically without any approximation. This result has been used in the numerical calculation of the transition operator of a pion-nucleon interaction for the S, P, and D relative waves. It has been shown that the imaginary parts are monotonic functions of the pion energy except for the $P33$ wave for which they present extrema at an energy a little lower than the resonance energy. At low energy, these imaginary parts are negligible. The real parts may present extrema in the considered (0–180 MeV) energy range and, for a given wave, they are placed at about the same energy. Because of its simple analytical form, this pion-nucleon transition operator may be convenient in low-energy nuclear physics calculations. It would be interesting to know the corresponding pion-nucleus potential.

APPENDIX

To facilitate the use of our results, we here give several supplementary relations concerning the $\mu_{ij}^{\pm L}(k_0)$ coefficients of the transition operator in the case of the pion-nucleon potential. These coefficients are the complex solutions of the set of linear algebric equations.

$$\sum_{i'=1}^{2}[B_i^{LJT}q_{ii'}^{\pm LJT}(k_0)-\delta_{ii'}]\mu_{i'j}^{\pm LJT}(k_0)=-B_i^{LJT}\delta_{ij} \tag{A1}$$

or, equivalently,

$$(\mu^{\pm})=[1-(B)(q^{\pm})]^{-1}(B), \tag{A2}$$

where the matrices are defined as in Eq. (14).

For each L-$2T$-$2J$ wave the solutions of this equation are

$$\mu_{11}=\frac{B_1}{DD^*}\{D^{(r)}(1-B_2q_{22}^{(r)})-D^{(i)}B_2q_{22}^{(i)}-i[D^{(i)}(1-B_2q_{22}^{(r)})+D^{(r)}B_2q_{22}^{(i)}]\}, \tag{A3}$$

$$\mu_{12}=\mu_{21}=\frac{B_1B_2}{DD^*}[D^{(r)}q_{12}^{(r)}+D^{(i)}q_{12}^{(i)}+i(D^{(r)}q_{12}^{(i)}-D^{(i)}q_{12}^{(r)})]. \tag{A4}$$

One obtains μ_{22} from Eq. (A3) by permuting the subscripts 1 and 2. We have defined

$$D=1-B_1q_{11}^{(r)}-B_2q_{22}^{(r)}+B_1B_2(q_{11}^{(r)}q_{22}^{(r)}-q_{11}^{(i)}q_{22}^{(i)}-q_{12}^{(r)2}+q_{12}^{(i)2})$$
$$+i[-B_1q_{11}^{(i)}-B_2q_{22}^{(i)}+B_1B_2(q_{11}^{(r)}q_{22}^{(i)}+q_{11}^{(i)}q_{22}^{(r)}-2q_{12}^{(r)}q_{12}^{(i)})]. \tag{A5}$$

The L, J, T superscripts and the dependence of μ_{ij}, q_{ij}, and D on k_0 are implicit. The labels (r) and (i) correspond to real and imaginary parts; for D they are given from the above equation and for q_{ij} they are given from Eq. (24) and Ref. 2. Equations (A3) and (A4) are, of course, valid for both $\mu_{ij}^{+LJT}(k_0)$ and $\mu_{ij}^{-LJT}(k_0)$. One needs only to use $q_{ij}^{+LJT}(k_0)$ or $q_{ij}^{-LJT}(k_0)$.

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[1]Y. Yamaguchi, Phys. Rev. **95**, 1628 (1954); J. H. Naqvi, Nucl. Phys. **58**, 289 (1964); F. Tabakin, Ann. Phys. (N.Y.) **30**, 51 (1964); T. F. Hammann, G. Oberlechner, G. Trapp, and J. Yoccoz, J. Phys. (Paris) **28**, 755 (1967); T. R. Mongan, Phys. Rev. **175**, 1260 (1968); Phys. Rev. **178**, 1597 (1969); Phys. Rev. **180**, 1514 (1969); A. P. S. Sirohi and M. K. Srivastava, Nucl. Phys. **A179**, 524 (1972); Nucl. Phys. **A201**, 66 (1973).

[2]P. Desgrolard and T. F. Hammann, Phys. Rev. C **6,** 482 (1972).
[3]T. F. Hammann, Ph. D. thesis, University of Strasbourg, France, 1970 (unpublished).
[4]T. F. Hammann, L. Chetouani, and P. Desgrolard, unpublished.
[5]M. L. Goldberger and K. M. Watson, *Collision Theory* (Wiley, New York, 1964); N. C. Francis and K. M. Watson, Phys. Rev. **92,** 291 (1953).
[6]L. S. Kisslinger, Phys. Rev. **98,** 761 (1955); R. M. Frank, J. L. Gammel, and K. M. Watson, Phys. Rev. **101,** 891 (1956).
[7]E. H. Auerbach, D. M. Fleming, and M. M. Sternheim, Phys. Rev. **171,** 1781 (1968).
[8]J. F. Marshall, M. E. Nordberg, Jr., and R. L. Burman, Phys. Rev. C **1,** 1685 (1970); F. Binon, P. Duteil, J. P. Garron, J. Gorres, L. Hugon, J. P. Peigneux, C. Schmit, M. Spighel, and J. P. Stroot, Nucl. Phys. **B17,** 168 (1970); R. M. Edelstein, W. F. Baker, and J. Rainwater, Phys. Rev. **122,** 252 (1961).
[9]M. M. Sternheim and E. H. Auerbach, Phys. Rev. Lett. **25,** 1500 (1970); M. Krell and S. Barmo, Nucl. Phys. **B20,** 461 (1970).
[10]D. K. Anderson, D. A. Jenkins, and R. J. Powers, Phys. Rev. Lett. **24,** 71 (1970).
[11]L. S. Kisslinger, R. L. Burman, J. H. Koch, and M. M. Sternheim, Phys. Rev. C **6,** 469 (1972).
[12]R. R. Silbar and M. M. Sternheim, Phys. Rev. C **6,** 764 (1972).
[13]G. Fäldt, Phys. Rev. C **5,** 400 (1971).
[14]T. E. O. Ericson, in *Interaction of High-Energy Particles with Nuclei, International School of Physics "Enrico Fermi," course 38* (Academic, New York, 1967); D. S. Koltun, *Interaction of Pions with Nuclei, Advances in Nuclear Physics* (Academic, New York, 1970), Vol. 3.
[15]R. H. Landau and F. Tabakin, Phys. Rev. D **5,** 2746 (1972); R. H. Landau, S. C. Phatak, and F. Tabakin, Ann. Phys. (N.Y.) (to be published).
[16]The phase convention we adopt here differs from the convention of Ref. 2, but this choice has no consequence in the following calculations.
[17]A. N. Mitra, Phys. Rev. **123,** 1892 (1961); V. S. Varma, Phys. Rev. **163,** 1682 (1967).
[18]L. I. Schiff, *Quantum Mechanics* (McGraw-Hill, New York, 1968).