Hauser-Feshbach Theory and Ericson Fluctuations in the Presence of Direct Reactions

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Neglecting level-level correlations, we show that consistency conditions derived from general properties of the S matrix determine the distribution of resonance parameters. This is also true in the presence of direct reactions. Formulas are given for average cross sections, autocorrelation, and cross-correlation functions.

I. RESULTS

With the aid of a unitary transformation we reduce the problem of calculating fluctuation cross sections (often called compound-nucleus cross sections) in the presence of direct reactions to diagonal form, i.e., to the well-known Hauser-Feshbach problem[1] without direct reactions. In doing so, we use Moldauer's sum rule[2] for resonance reactions, which plays a central role in the theory. We neglect level-level correlations throughout. It follows from the unitarity, analyticity, and time-reversal invariance of the S matrix that this can consistently be done only in the case of weak absorption in all channels. Moreover, the resulting consistency conditions, together with the assumption that the partial-width amplitudes follow a Gaussian distribution, completely determine the distribution of resonance parameters.

Explicit formulas are given from which fluctuation cross sections, polarizations, etc., can be found easily. We also calculate cross-correlation functions between cross sections relating to different channels. These do not vanish in the presence of direct reactions. We show that the "direct" cross section defined in Ericson theory is given in terms of the average S matrix element. Until recently, the relation between these two quantities has been the subject of debate.[3]

II. HAUSER-FESHBACH THEORY

A. Definitions and Previous Results

For fixed values J^π of spin and parity of the compound nucleus, the unitary and symmetrical S matrix has the form

$$S_{ab}=S^{(0)}_{ab}-i\sum_{\mu} g_{\mu a}g_{\mu b}/(E-\xi_\mu)\,, \tag{1}$$

where the background matrix $S^{(0)}$, the partial-width amplitudes $g_{\mu c}$ and the complex resonance energies $\xi_\mu=E_\mu-\frac{1}{2}i\Gamma_\mu$ are assumed constant, and where E is the energy of the system. The average over energy is indicated by a bracket and given by

$$\langle S_{ab}\rangle=S^{(0)}_{ab}-(\pi/D)\langle g_{\mu a}g_{\mu b}\rangle_\mu\,, \tag{2}$$

where D is the average level spacing. The index μ on the bracket denotes an average over levels. We introduce Satchler's transmission matrix[4]

$$P_{ab}=\delta_{ab}-\sum_c\langle S_{ac}\rangle\langle S^*_{bc}\rangle\,. \tag{3}$$

Moldauer's sum rule[2] for resonance reactions takes the form

$$(2\pi/D)\langle g_{\mu a}g_{\mu b}\rangle_\mu=\sum_c P_{ac}(\langle S^*\rangle^{-1})_{cb}\,. \tag{4}$$

The fluctuating part of S is given by $S^{\text{fl}}=S-\langle S\rangle$. The unitarity of S implies

$$P_{ab}=\sum_c\langle S^{\text{fl}}_{ac}S^{\text{fl}*}_{bc}\rangle\,. \tag{5}$$

Following Moldauer,[5] we define $N_\mu=\Gamma_\mu{}^{-1}\sum_a|g_{\mu a}|^2$. A calculation quite similar to the one performed in Ref. 5 shows that under neglect of level-level correlations[6] one gets

$$\langle S^{\text{fl}}_{ab}S^{\text{fl}*}_{cd}\rangle=(2\pi/D)\langle N_\mu g_{\mu a}g_{\mu b}g^*_{\mu c}g^*_{\mu d}(\sum_k|g_{\mu k}|^2)^{-1}\rangle_\mu\,. \tag{6}$$

For $c=a$, $d=b$, this agrees with Eq. (59) of Ref. 5.

B. Transformation to Diagonal Form

In the presence of direct reactions, $\langle S\rangle$ and $\langle S^*\rangle$ are both not diagonal. Since these two matrices do not commute, they cannot be simultaneously diagonalized by the *same* transformation. Let U be the unitary matrix which diagonalizes the Hermitian matrix P of Eq. (3),

$$(UPU^\dagger)_{ab}=\delta_{ab}p_a\,,\quad 0\le p_a\le 1\,. \tag{7}$$

Defining the symmetric matrix $A=U\langle S\rangle U^T$ and noting that $U^\dagger U=1$ implies $U^TU^*=1$, we can write Eq. (7) in the form $(AA^*)_{ab}=\delta_{ab}(1-p_a)=(A^*A)_{ab}$. This and the symmetry of A show that A is normal[7] and, hence, can be diagonalized. The eigenvectors of A, of $A^\dagger$, and of $AA^\dagger$ coincide so that these three matrices can be diagonalized simultaneously. Since $AA^\dagger$ is diagonal already, we have

$$(U\langle S\rangle U^T)_{ab}=\delta_{ab}(1-p_a)^{1/2}\exp(2i\phi_a)\,, \tag{8}$$

where the ϕ_a are real. If some eigenvalues of P

coincide, U can be chosen such that Eqs. (7) and (8) both hold. Note that $U^*\langle S^*\rangle U^\dagger$ is also diagonal. We introduce the quantities $v_{\mu a}=\sum_b U_{ab}\, g_{\mu b}$. Multiplying Eq. (4) from the left by U, from the right by U^T, we find

$$(2\pi/D)\langle v_{\mu a}v_{\mu b}\rangle_\mu=\delta_{ab}\,p_a(1-p_a)^{-1/2}\exp(2i\phi_a)\,. \qquad (9)$$

This and Eqs. (2) and (8) imply that $US^{(0)}U^T$ is also diagonal. Using Eqs. (5), (6), and (9) in Eq. (7), we obtain

$$(2\pi/D)\langle N_\mu v_{\mu a}v_{\mu b}^*\rangle_\mu=\delta_{ab}\,p_a\,. \qquad (10)$$

Transformation with U thus reduces the problem to diagonal form, i.e., to the form which is characteristic of Hauser-Feshbach theory without direct reactions.

C. Distribution of Resonance Parameters

Several authors[5,8] have shown that the N_μ defined below Eq. (5) obey the condition $N_\mu \geq 1$. It then follows from Eqs. (9) and (10) that

$$p_a \geq p_a(1-p_a)^{-1/2}\,. \qquad (11)$$

This relationship is fulfilled (with the equality sign) to lowest order in p_a, i.e., in the case of small absorption in all channels, while it obviously fails to hold if higher-order terms in p_a are considered. This shows that the neglect of level-level correlations is not justified[9] unless $p_a \ll 1$ for all a. For the case without direct reactions, a similar conclusion has been reached by Moldauer.[10] For $\Gamma \gg D$, the effect of level repulsion is negligible.[5] Our result thus implies the existence of correlations between $g_{\mu a}$ and $g_{\nu b}$ for $\mu \neq \nu$ unless all $p_a \ll 1$. Since the nature of such correlations is not known, one has little choice but to neglect them. We accordingly keep only lowest-order terms in p_a. The relation (11) then becomes an equality which implies $N_\mu = 1$ and $\langle |v_{\mu a}|^2\rangle_\mu = |\langle v_{\mu a}{}^2\rangle_\mu|$. The last equality shows that the $v_{\mu a}$ are, for fixed a, distributed along a straight line in the complex plane. This fact and Eq. (9) together show that we have $v_{\mu a}=\exp(i\phi_a)x_{\mu a}$, where the $x_{\mu a}$ are real and obey the conditions

$$(2\pi/D)\langle x_{\mu a}x_{\mu b}\rangle_\mu=\delta_{ab}\,p_a\,. \qquad (12)$$

It is thus consistent both with our results obtained so far and with the fact that for thermal neutrons the $|g_{\mu a}|^2$ obey a Porter-Thomas distribution to assume that the variables $x_{\mu a}$ and $x_{\mu b}$ are uncorrelated for $a \neq b$, and that each has a Gaussian distribution centered at zero with variance (12). The distribution of $g_{\mu a}$ with

$$g_{\mu a}=\sum_b \exp(i\phi_b)x_{\mu b}U_{ba}^* \qquad (13)$$

is then specified completely. We see that the basic quantities of the theory are the uncorrelated real random variables $x_{\mu a}$. The existence of direct reactions implies correlations between the complex random variables $g_{\mu a}$. However, the nature of these correlations is under neglect of level-level correlations completely specified by the unitary transformation U and by the ϕ_a, i.e., by the matrix $\langle S\rangle$. It follows, in particular, that when a group of channels is connected through direct reactions the corresponding partial width amplitudes *must* be correlated, and conversely. Note that the case without direct reactions is contained in the treatment given above in a trivial way, since then $U\equiv 1$. Summing Eq. (12) over all a, we find

$$2\pi\Gamma/D=\sum_a p_a=\mathrm{Tr}(P)\,. \qquad (14)$$

This consistency condition is implied by the neglect of level-level correlations.

D. Cross-Section Formulas

Equation (6) contains the most general expression of interest in Hauser-Feshbach theory. From this expression, one can calculate the fluctuation part of cross sections, polarizations, polarization transfer experiments, etc. (None of these quantities need vanish in the presence of direct reactions. However, fluctuation cross sections remain symmetric about 90° c.m. since different J^π values do not interfere.) Using Eqs. (12) and (13), we find

$$\langle g_{\mu a}g_{\mu b}g_{\mu c}^*g_{\mu d}^*(\sum_k |g_{\mu k}|^2)^{-1}\rangle_\mu=\sum_{jk}\langle x_{\mu j}{}^2x_{\mu k}{}^2(\sum_m x_{\mu m}{}^2)^{-1}\rangle_\mu \times[(1-\delta_{jk})U_{ja}^*U_{kb}^*(U_{jc}U_{kd}+U_{jd}U_{kc}) +\exp(2i\phi_j-2i\phi_k)U_{ja}^*U_{jb}^*U_{kc}U_{kd}]\,. \qquad (15)$$

The calculation of the expectation value on the right-hand side involves only integrations and can be carried out in the way described by Moldauer.[11] Thus, $\langle S_{ab}^{\mathrm{fl}}S_{cd}^{\mathrm{fl}*}\rangle$ is known. If the number Λ of channels contributing to the sum $\sum_m x_{\mu m}{}^2$ is large, and if the $\langle x_{\mu m}{}^2\rangle_\mu$ of all contributing channels are comparable in magnitude (these two assumptions are jointly referred to as $\Lambda \gg 1$), we may replace the sum $\sum_m x_{\mu m}{}^2$ by its expectation value. From Eqs. (6), (12), (15), and the Gaussian distribution of the $x_{\mu a}$ one then finds the result

$$\langle S^{\rm fl}_{ab} S^{\rm fl*}_{cd}\rangle = (D/2\pi\Gamma)(P_{ac}P_{bd} + P_{ad}P_{bc} + G_{ab}G^*_{cd}) , \tag{16}$$

where $G_{ab} = \sum_j U^*_{ja}U^*_{jb}p_j \exp(2i\phi_j)$. This result is consistent with Eq. (14) since $\Lambda \gg 1$. In the absence of direct reactions ($U \equiv 1$), Eq. (16) gives a Hauser-Feshbach cross section which coincides with the standard value[1] for inelastic scattering, while the elastic cross section[1] is enhanced by a factor of 3, which arises because the $g_{\mu a}$ are distributed along a straight line.[11] For $U \neq 1$, the values of the G_{ab}'s depend on the phases ϕ_j. When comparing Eq. (16) with the approach of Ref. 12, where we put $\alpha = 1 = \beta$, we notice that the assumptions made in Ref. 12 (no level-level correlations, Gaussian distribution for the g_{qa}, and $\Gamma_q = \Gamma$) are implied by those made here. In particular, $\Lambda \gg 1$ implies $\Gamma_q = \Gamma$. Solving Eq. (10) of Ref. 12 in the limit $\Lambda \gg 1$ for $\hat{X}_{ab}$, we find from Eq. (18) of Ref. 12 our present Eq. (16) if $\alpha = 1 = \beta$. The approach of Ref. 12 and our Eq. (16) are thus consistent within their common domain of validity $\Lambda \gg 1$. A similar remark applies to the comparison of our Eq. (16) and the results of Ref. 13, except that there the additional assumption $\langle g_{qa} g_{qb}\rangle_q \approx 0$ has been made for $\Gamma \gg D$. (Note, however, that the g_{qa} of Ref. 13 differ from our $g_{\mu a}$.) For our $g_{\mu a}$, the validity of such an assumption depends upon the values of the ϕ_j. This additional assumption has also tacitly been made in Ref. 14. We have shown that this assumption fails to hold in the case $U \equiv 1$.

All our results (and those of standard Hauser-Feshbach theory) can be derived only if $p_a \ll 1$ for all a. Standard Hauser-Feshbach theory has, of course, been applied with success outside this range.

III. ERICSON FLUCTUATIONS

Since the distribution of the $g_{\mu a}$ is known completely, the calculation of correlation functions for various cross sections under neglect of level-level correlations is straightforward. We confine ourselves to the Ericson limit $\Gamma \gg D$, although it is not difficult to calculate correction terms of order D/Γ. Using $\Lambda \gg 1$, we find

$$\langle \sigma_{ab}(E)\sigma_{cd}(E+\epsilon)\rangle - \langle\sigma_{ab}\rangle\langle\sigma_{cd}\rangle = [\Gamma^2/(\Gamma^2+\epsilon^2)] \times \{ |\langle S^{\rm fl}_{ab}S^{\rm fl*}_{cd}\rangle|^2 + [(1-i\epsilon/\Gamma)\langle S^{\rm fl}_{ab}S^{\rm fl*}_{cd}\rangle(\langle S^*_{ab}\rangle - \delta_{ab})(\langle S_{cd}\rangle - \delta_{cd}) + \text{c.c.}]\} . \tag{17}$$

Equation (17) shows that in the presence of direct reactions cross correlations involving different channels do not vanish, and that they are asymmetric about $\epsilon = 0$. Indications for nonvanishing cross correlations have, e.g., been found in Ref. 15. It would be of interest to analyze these and to check our Eq. (17). In the autocorrelation function $F_{ab}(\epsilon)$ [put $c = a$, $d = b$ in Eq. (17)] the term asymmetric in ϵ vanishes, and $F_{ab}(0)$ has the value $\sigma^{\rm fl}_{ab}(2\sigma^{\rm dir}_{ab} + \sigma^{\rm fl}_{ab})$, where under omission of kinematical and geometrical factors we have $\sigma^{\rm dir}_{ab} = |\langle S_{ab}\rangle - \delta_{ab}|^2$ and $\sigma^{\rm fl}_{ab} = \langle |S^{\rm fl}_{ab}|^2\rangle$ for direct and fluctuation cross section, respectively.

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[9]Alternatively, the sum rule (4) may not be valid. It can be shown, however, that the inclusion of level-level correlations in Eq. (10) removes the inconsistency (11) without affecting the validity of the sum rule (4). Moreover, Eq. (4). can be derived without using the statistical assumptions employed originally (Ref. 2). It applies provided only that the Γ_μ are bounded from above, so that one can choose the averaging interval $I \gg \Gamma_\mu$ for all μ. This holds to a very good approximation if the $g_{\mu a}$ have a Gaussian distribution. The inconsistency (11) is thus definitely due to the neglect of level-level correlations. (These arguments will be presented in detail elsewhere by one of us.)

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