Evaluation of the Nuclear Charge Form Factor with Intrinsic Hyperspherical-Coordinate Wave Functions*

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Evaluation of the nuclear charge form factor requires the evaluation of a many-body matrix element. In this paper it is shown how this many-body matrix element can be reduced to an integral over single-particle matrix elements when hyperspherical coordinates are used.

1. INTRODUCTION

The nuclear charge form factor is defined by the matrix element

$$F(q) = \langle \Psi_I | e^{i\vec{q}\cdot(\vec{r}_1 - \vec{R})} | \Psi_I \rangle , \tag{1}$$

where $\vec{r}_1 - \vec{R}$ is an intrinsic coordinate and Ψ_I is the intrinsic nuclear wave function depending on intrinsic coordinates. Evaluation of this matrix element requires integration over intrinsic coordinates which, due to the presence of the c.m. coordinate $\vec{R}$ in the exponent, can be very difficult to perform. Indeed $\exp[i\vec{q}\cdot(\vec{r}_1 - \vec{R})]$ is an A-body operator for an A-mass nucleus. Equation (1) can be easily evaluated if one does not use intrinsic coordinates, which can be done by requiring the factorization of the wave function into the intrinsic wave function and an arbitrary c.m. factor,[1] say

$$\Psi_F = \Psi_I \chi_0 . \tag{2}$$

In this case[2]

$$F(q) = \left[\int d\vec{R}\, e^{i\vec{q}\cdot\vec{R}} |\chi_0(\vec{R})|^2 \right]^{-1} \langle \Psi_F | e^{i\vec{q}\cdot\vec{r}_i} | \Psi_F \rangle , \tag{3}$$

which is the matrix element of a one-body operator. In this way, however, the difficulty has been shifted from the problem of using explicitly intrinsic coordinates, to the problem of obtaining factorized wave functions. This is an easier problem if one only demands approximate factorization, but this approximation can cause large errors at high momentum transfer.[3] In view of this fact it can be worthwhile to try to use intrinsic coordinates at the cost of a major complication.

No general method of evaluation of Eq. (1) is known when intrinsic coordinates are used. The purpose of this paper is to show that, with a particular choice of intrinsic coordinates, the intrinsic hyperspherical coordinates, $F(q)$ can be evaluated in terms of integrals of single-particle matrix elements. The integrals involved are the same occurring in the evaluation of the matrix elements of the potential.[4,5] Indeed the result we are going to present is obtained by a modification of the procedure developed in Refs. 4 and 5 for the evaluation of matrix elements of two-body operators.

2. EVALUATION OF THE CHARGE FORM FACTOR

The intrinsic hyperspherical coordinates[6] consist of a hyperradius ρ_I and a set of $(3A-4)$ intrinsic angular coordinates which we shall denote by Ω_{ρ_I}. The hyperradius is defined in terms of the single-particle coordinates $\vec{r}_s$ and the c.m. coordinate $\vec{R}$ by the equation

$$\rho_I{}^2 = \sum_{s=1}^{A} \rho_s{}^2 , \quad \vec{\rho}_s = \vec{r}_s - \vec{R} . \tag{4}$$

A set of spin-isospin coordinates which we shall denote by $\Omega_{\sigma\tau}$ complete the dynamical variables. Using these dynamical variables Ψ_I reads

$$\Psi_I = \sum_{K\nu} a_{K\nu} \Psi_{K\nu} , \tag{5}$$

where

$$\Psi_{K\nu} = \rho_I^{-(3A-4)/2} \chi_{K\nu}(\rho_I) U_{K\nu}(\Omega_{\rho_I}, \Omega_{\sigma\tau}) . \tag{6}$$

The $U_{K\nu}$ can be expressed in terms of the harmonic polynomials of degree K, $P_{K\nu}$, by the formula

$$U_{K\nu} = \frac{N_{K\nu}}{\sqrt{A!}} \rho_I^{-K} P_{K\nu} , \tag{7}$$

where $N_{K\nu}$ are normalization factors. These polynomials are constructed as a superposition of Slater determinants of the functions

$$\varphi_{nlm\mu\tau}(\vec{\rho}_j) = C_{nl} (\rho_j{}^2)^{n+l/2} Y_{lm}(\hat{\rho}_j) \alpha_{\mu\tau}(j) , \tag{8}$$

where

$$C_{nl} = (-)^n \sqrt{2} \left[\Gamma(n+1) \Gamma(n+l+\tfrac{3}{2}) \right]^{-1/2} ,$$

and $\alpha_{\mu\tau}(j)$ is a spin-isospin function. Note that $\varphi_{nlm\mu\tau}(\vec{\rho}_j)$ is not a single-particle function, because $\vec{\rho}_j = \vec{r}_j - \vec{R}$. Let us define the auxiliary quan-

tities

$$\mathfrak{F}_{KK'\nu\nu'}(q,\rho_I)=\sum_{\sigma\tau}\int d\Omega_{\rho_I}U^{+}_{K\nu}[\varphi_\beta(\vec{\rho}_j)]\,e^{i\vec{q}\cdot(\vec{r}_1-\vec{R})}U_{K'\nu'}[\varphi_\beta(\vec{\rho}_j)]\,, \tag{9}$$

where the label β stands for the quantum numbers $nlm\mu\tau$. In terms of these quantities $F(q)$ reads

$$F(q)=\sum_{KK'\nu\nu'}a^{*}_{K\nu}a_{K'\nu'}\int d\rho_I\chi^{*}_{K\nu}(\rho_I)\mathfrak{F}_{KK'\nu\nu'}(q,\rho_I)\chi_{K'\nu'}(\rho_I)\,. \tag{10}$$

In order to evaluate the $\mathfrak{F}_{KK'\nu\nu'}(q,\rho_I)$, first introduce into Eq. (9) integration over ρ_I along with a δ function and then a convenient representation of the δ function

$$(q,\rho_I)=2\frac{N_{K\nu}N_{K'\nu'}}{A!}\rho_I^{-(K+K'+3A-5)}\sum_{\sigma\tau}\int d\Omega_{\rho_I}\int d\rho_I\,\rho_I^{3A-4}\delta\Big[\sum_{s=1}^{A}\rho_s{}^2-\rho_I{}^2\Big]P^{+}_{K\nu}[\varphi_\beta(\vec{\rho}_j)]P_{K'\nu'}[\varphi_\beta(\vec{\rho}_j)]\,e^{i\vec{q}\cdot(\vec{r}_1-\vec{R})}$$

$$=\frac{N_{K\nu}N_{K'\nu'}}{\pi A!}\rho_I^{-(K+K'+3A-5)}\int dt\sum_{\sigma\tau}\int d\Omega_{\rho_I}\int d\rho_I\,\rho_I^{3A-4}P^{+}_{K\nu}[\varphi_\beta(\vec{\rho}_j)]$$

$$\times\exp\Big[i\Big(t\sum_{s=1}^{A}\rho_s{}^2-t\rho_I{}^2-\vec{q}\cdot\vec{R}\Big)\Big]P_{K'\nu'}[\varphi_\beta(\vec{\rho}_j)]\,e^{i\vec{q}\cdot\vec{r}_1}\,. \tag{11}$$

This enables us to get rid of the c.m. coordinate $\vec{R}$ in the exponent using the following identities

$$\exp\Big(\frac{i}{t}\frac{q^2}{4A}\Big)\Big(\frac{At}{i\pi}\Big)^{3/2}\int d\vec{R}\exp[i(AtR^2+\vec{q}\cdot\vec{R})]=1\,, \tag{12}$$

$$A^{3/2}\,d\vec{R}\,d\Omega_{\rho_I}\,d\rho_I\,\rho_I^{3A-4}=d\vec{r}_1\,d\vec{r}_2\cdots d\vec{r}_A\,, \tag{13}$$

which gives

$$\mathfrak{F}_{KK'\nu\nu'}(q,\rho_I)=\frac{N_{K\nu}N_{K'\nu'}}{\pi A!}\rho_I^{-(K+K'+3A-5)}\int dt\Big(\frac{t}{i\pi}\Big)^{3/2}\exp\Big(\frac{i}{t}\frac{q^2}{4A}-it\rho_I{}^2\Big)$$

$$\times\sum_{\sigma\tau}\int d\vec{r}_1\int d\vec{r}_2\cdots\int d\vec{r}_A P^{+}_{K1}[\varphi_\beta(\vec{\rho}_j)]\,\exp\Big(-\frac{t}{i}\sum_{s=1}^{A}r_s{}^2\Big)P_{K'\nu'}[\varphi_\beta(\vec{\rho}_j)]\exp(i\vec{q}\cdot\vec{r}_1)\,. \tag{14}$$

From now on Ref. 5 can be followed step by step. For the sake of completeness we report the essential points leading to a computation in terms of single-particle coordinates. First, since the $P_{K\nu}$ are harmonic polynomials

$$P_{K\nu}[\varphi_\beta(\vec{\rho}_j)]=\Big(\frac{i}{t}\Big)^{K/2}P_{K\nu}\Big\{\varphi_\beta\Big[\Big(\frac{t}{i}\Big)^{1/2}\vec{\rho}_j\Big]\Big\}\,. \tag{15}$$

Second, use the identity

$$P_{K\nu}\Big\{\varphi_\beta\Big[\Big(\frac{t}{i}\Big)^{1/2}\vec{\rho}_j\Big]\Big\}=\sum_{K_1\nu_1}a^{K\nu}_{K_1\nu_1}\Big(\frac{i}{t}\Big)^{(3/4)A}P_{K_1\nu_1}\Big\{\psi_\beta\Big[\Big(\frac{t}{i}\Big)^{1/2}\vec{\rho}_j\Big]\Big\}\,, \tag{16}$$

where

$$\psi_{nlm\mu\tau}\Big[\Big(\frac{t}{i}\Big)^{1/2}\vec{\rho}_j\Big]=\Big(\frac{t}{i}\Big)^{3/4}[2\Gamma(n+1)]^{1/2}[\Gamma(n+l+\tfrac{3}{2})]^{-1/2}L_n^{l+1/2}\Big(\frac{t}{i}\rho_j{}^2\Big)\Big(\frac{t}{i}\rho_j{}^2\Big)^{l/2}Y_{lm}(\hat{\rho}_j)\alpha_{\mu\tau}(j)\,, \tag{17}$$

the $L_n^{l+1/2}$ being Laguerre polynomials. This identity follows from the fact that the ψ are linear combinations of the φ. Note that $\psi[(t/i)^{1/2}\vec{r}_j]\exp[(-t/2i)r_j{}^2]$ is a harmonic-oscillator eigenfunction of complex length $(i/t)^{1/2}$. The evaluation of $\mathfrak{F}_{KK'\nu\nu'}(q,\rho_I)$ would have led to the standard evaluation of matrix elements with harmonic-oscillator wave functions, if it were not for the presence of the many-particle arguments $\vec{\rho}_j$.

To arrive at the final result it is only necessary to express

$$P_{K_1\nu_1}\Big\{\psi_\beta\Big[\Big(\frac{t}{i}\Big)^{1/2}\vec{\rho}_j\Big]\Big\}\exp\Big(-\frac{t}{2i}\sum_{s=1}^{A}r_s{}^2\Big)$$

in terms of Slater determinants of harmonic-oscillator single-particle wave functions. This is obtained by putting $\vec{\rho}_j = \vec{r}_j - \vec{R}$ and expanding the $P_{K_1\nu_1}$ with respect to $\vec{R}$. Only a finite (and in general small) number of terms are obtained. Explicit formulas for this job are given in Ref. 5. In order to show the type of expressions to be finally evaluated, we specialize to a "magic nucleus" and to $K = K' = K_{\min}$, the lowest K harmonic, in which case there is a unique U_K, so that the label ν can be dropped

$$\mathcal{F}_{K_{\min}K_{\min}}(q,\rho_I) = \frac{N_{K_{\min}}{}^2}{\pi A!}\rho_I^{-(2K_{\min}+3A-5)}\int dt\left(\frac{t}{i\pi}\right)^{3/2}\left(\frac{i}{t}\right)^{K_{\min}+(3/2)A}\exp\left(\frac{i}{t}\frac{q^2}{4A} - it\rho_I{}^2\right)$$
$$\times\sum_{\sigma\tau}\int d\vec{r}_1\int d\vec{r}_2\cdots\int d\vec{r}_A P^+_{K_{\min}}\left\{\psi_\beta\left[\left(\frac{t}{i}\right)^{1/2}\vec{\rho}_j\right]\right\}\exp\left(-\frac{t}{i}\sum_{s=1}^{A} r_s{}^2\right)$$
$$\times P_{K_{\min}}\left\{\psi_\beta\left[\left(\frac{t}{i}\right)^{1/2}\vec{\rho}_j\right]\right\}\exp(i\vec{q}\cdot\vec{r}_1)$$
$$= \frac{N_{K_{\min}}{}^2}{\pi}\rho_I^{-(2K_{\min}+3A-5)}\int dt\left(\frac{t}{i\pi}\right)^{3/2}\left(\frac{i}{t}\right)^{K_{\min}+(3/2)A}\exp\left(\frac{i}{t}\frac{q^2}{4A} - it\rho_I{}^2\right)$$
$$\times\sum_\beta\int d\vec{r}\psi^*_\beta\left[\left(\frac{t}{i}\right)^{1/2}\vec{r}\right]\exp\left(-\frac{t}{i}r^2\right)\psi_\beta\left[\left(\frac{t}{i}\right)^{1/2}\vec{r}\right]\exp(i\vec{q}\cdot\vec{r}).$$

The sum over β runs over the single-particle states occupied in the ground state. In the general case one has a linear combination of terms of this form.

The full evaluation of the charge form factor requires the integration over t and ρ_I of single-particle matrix elements, as the evaluation of potential matrix elements requires the integration over t and ρ_I of two-particle matrix elements. If the integration over t is performed with the saddle point method,[4] one is left in both cases with the integration over ρ_I.

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