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Truncation of Configuration Space and the Nature of Effective Two-Body Interaction*†

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Using the renormalized G matrix (the G_r) results of Kuo (for the d-s shell) and of Kuo and Brown (for the f-p shell), a re-renormalized effective G matrix (the G_{eff}) for only two active orbits is calculated numerically, which would be valid for use in $(d_{5/2}\,s_{1/2})^n$ and $(f_{7/2}\,p_{3/2})^n$ configurations. The operators G_r and G_{eff} as well as the bare G matrix (the G_b) of Kuo and Brown, which thus define the transformation chain $G_b \rightarrow G_r \rightarrow G_{\text{eff}}$, are then analyzed in the light of truncation of the shell-model space. The analysis shows that the effect of this truncation can largely be taken into account by the addition of a pairing-force term to the effective two-body interaction. The relative change in $T=1$ and $T=0$ parts of the interaction due to the truncation of configuration space in various ways is also discussed. Finally, a comparison between the available empirically fitted effective interactions and our theoretical G_{eff} interaction is carried out.

1. INTRODUCTION

Ever since the advent of the nuclear shell model, one of the practical problems in describing the various nuclear properties has been the choice of configuration space and the corresponding effective nuclear force. The problem has become more involved with the growing use of realistic nuclear forces. Usually one likes to keep the calculations simple by limiting the configuration space. However, a theoretical justification for using very simple configuration space can only be given if one knows how to reduce a respectable realistic interaction to an effective interaction, to be used only in the restricted configuration space. Generally, in most shell-model calculations a need for increasing the configuration space is felt (either in the sense of including core excitations or by including higher-energy configurations) because the theoretical predictions with effective interactions have failed to agree with experiments. In recent years several attempts have been made to investigate the effect of truncation of shell-model space on various nuclear properties in the framework of the so-called pseudonuclei.[1-5] It has been shown that for many nuclear properties the effect of truncating the configuration space can be counterbalanced or absorbed by suitably modifying the effective two-body interaction. However, although this apparently works very well where the configuration mixing is only of two-particle–two-hole type, it is not possible to conceal the configuration mixing by modifying the effective interaction when the mixing is also of one-particle–one-hole type or if one wants to study properties such as inelastic electron scattering.[3,4]

The crux of the problem of truncating configuration space, however, lies in the possibility of finding an equivalent satisfactory effective interaction in the reduced space. In this paper we make an attempt to investigate this effect of truncation of shell-model space on the nature of two-body effective interaction. For this purpose, we choose to work with the G matrix calculated with the Hamada-Johnston potential by Kuo and Brown.[6-8] We consider two examples $(1d_{5/2}2s_{1/2})^2_{JT}$ and $(1f_{7/2}2p_{3/2})^2_{JT}$ configurations, and begin with the values of the G matrix in which the effects of very high-lying configurations are implicitly included,[7,8] referred to as the bare G matrix. Tables I and II list these numbers, respectively, for the $d_{5/2}s_{1/2}$ and the $f_{7/2}p_{3/2}$ configurations (denoted as G_b). As a second step, one can reduce the configuration space by excluding excitations from the so-called core configurations [$(1s)^4(1p)^{12}$ in the case of d-s shell and $(1s)^4(1p)^{12}(1d2s)^{24}$ in the case of f-p shell] and modify the effective interaction by including the core-excitation effects in the effective two-body matrix elements. Such a G matrix, which may be used in the limited shell-model space (the complete d-s and the complete f-p shells in the present two examples) has been calculated by Kuo[7] and Kuo and Brown.[8] We refer to it as the renormalized G matrix. These numbers are also listed in Tables I and II (denoted as G_r). As a next step, one would like to truncate the shell-model space further to include only the very minimum two orbits and thus obtain the two-body matrix elements which, for the two examples under study, would be valid for use in $(1d_{5/2}2s_{1/2})^n$ and $(1f_{7/2}2p_{3/2})^n$ configurations. Such a calculation for the G matrix has not been made in the past and we refer to it as the renormalized effective G matrix (denoted as G_{eff}). Thus the purpose of this paper is twofold: firstly to calculate such a re-renormalized effective nuclear interaction which would

complete the transformation chain $G_b \rightarrow G_r \rightarrow G_{\text{eff}}$ and secondly to analyze these operators in a few different possible ways, in order to get *some simple feeling* for the nature of two-body effective interaction due to this type of renormalization of the force. This is done in Secs. 2 and 3, respectively. Finally, in Sec. 4 we carry out a brief comparison between the G_{eff} and the empirically determined effective interactions. A summary of our results is given in Sec. 5.

2. CALCULATION OF THE RENORMALIZED EFFECTIVE G MATRIX FOR ONLY TWO ACTIVE ORBITS

Using the prescription given by Kuo and Brown[6-8] in a series of papers, one should in principle be able to calculate the G matrix for any choice of the shell-model space. However, the physical operators in a small space would be different than the operators in the full space. Though the theoretical expressions for the effective operators to be used in a limited basis space are now available[9-11] (derived by using perturbation theory) there are many difficulties such as the convergence of the perturbation expansion for the operators.[12] In the following, we give a numerical procedure through which the effective G matrix for the case of only two active orbits, can be obtained in a simple manner by making an explicit truncation and using the available results (the G_r, referred above) of a large configuration space.

TABLE I. Two-body matrix elements for the basis space $(1d_{5/2}2s_{1/2})^2$. All energies are in MeV and the notation is

$$\langle j_1 j_2 : JT | H | j_1' j_2' : JT \rangle \equiv \langle 2j_1\, 2j_2\, 2j_1'\, 2j_2' : JT \rangle ,$$

where $5 \equiv 1d_{5/2}$ and $1 \equiv 2s_{1/2}$.

Configuration	J	T	Hamada-Johnston G_b	G_r	G_{eff}	Effective interactions V_{ACLM}[a]	V_{WMHG}[b]
5555	0	1	−1.24	−2.44	−3.52	−3.41	−4.04
5511	0		−0.63	−0.97	−1.18	−1.04	−1.59
1111	0		−2.05	−1.95	−2.00	−2.17	−2.55
5555	2		−1.01	−1.03	−1.25	−1.21	0.71
5551	2		−0.56	−0.85	−1.10	−0.88	−2.71
5151	2		−1.17	−1.29	−1.69	−1.17	−0.51
5151	3		−0.29	0.17	0.17	1.16	1.12
5555	4		−0.43	−0.05	−0.52	−0.08	−0.56
5555	1	0	−0.30	−1.03	−3.70	0.01	−2.94
5511	1		−0.27	−0.60	−1.95	−4.27	−0.78
1111	1		−3.01	−3.18	−4.08	−3.67	−2.18
5151	2		−0.53	−0.62	−3.24	−3.70	−6.74
5555	3		−0.79	−0.86	−1.55	0.38	−2.35
5551	3		−1.24	−1.56	−1.87	−3.53	−2.56
5151	3		−3.12	−3.69	−3.70	−2.60	−1.62
5555	5		−3.42	−3.66	−3.66	−4.26	−3.60

[a] The single-particle energies used in this calculation are given in Table III.

[b] The single-particle energies adjusted to obtain this interaction are $E_{d_{5/2}} = -4.313$ MeV and $E_{s_{1/2}} = -3.265$ MeV.

The original eigenvalue problem is

$$H\psi_j = E_j \psi_j ,$$

where

$$\psi_j = \sum_{\alpha=1}^{N} a_{j\alpha}\phi_\alpha$$

and

$$\langle \phi_\alpha | \phi_\beta \rangle = \delta_{\alpha\beta} . \tag{1}$$

Here N defines the dimension of the space (fixed by the G_r matrix, in our problem). For example, when two identical nucleons are distributed over the four orbits $(f_{7/2},\ p_{3/2},\ f_{5/2},\ p_{1/2})$ of $1f2p$ shell, $N=8$ and $j=1$ to 8 for $J=2^+$, $T=1$ states, since eight such states can be formed.

Now, we wish to transform this problem to a smaller space which we shall refer to as the model space. Our procedure is to use the results of the original eigenvalue problem such that the effect of truncating the configuration space ($sd \rightarrow d_{5/2}s_{1/2}$ and $fp \rightarrow f_{7/2}p_{3/2}$) is counterbalanced by renormalizing the effective interaction. We, therefore, write the eigenvectors of the model space as the projection of ψ_j onto the model space,

$$\psi_j' = \sum_{\alpha=1}^{d} a_{j\alpha}\phi_\alpha , \tag{2}$$

where d is the dimension of the model space. For the example cited above, if the configuration space is reduced to only $(f_{7/2},\ p_{3/2})$ orbits, $d=3$ and $j=1$ to 3. If we now assume that the interaction remains Hermitian, the eigenvectors ψ_j' have to be reorthogonalized. This can be done by using the standard Gram-Schmidt method[13] of orthogonalization. We start our Gram-Schmidt procedure from the lowest-lying level (justified, in the following) for each (J,T) and on normalization obtain the eigenvectors of the model space, say,

$$\psi_j^{(m)} = \sum_{\alpha=1}^{d} b_{j\alpha}\phi_\alpha ,$$

where

$$\langle \psi_j^{(m)} | \psi_k^{(m)} \rangle = \delta_{jk} . \tag{3}$$

The model-space eigenvalue problem can then be denoted as

$$H^{(m)}\psi_j^{(m)} = E_j \psi_j^{(m)} , \tag{4}$$

where $H^{(m)}$ is some model Hamiltonian acting in the truncated space. Equation (4) says that $H^{(m)}$ acting on the orthonormalized projected wave func-

tions $\psi_j^{(m)}$ produces the same energies E_j as the full Hamiltonian H acting on ψ_j. (Of course, all the energies E_j would not be obtained for the truncated space.) The Hermitian interaction $H^{(m)}$, thus obtained by solving Eq. (4) would give *all* the "right" eigenvalues and the "right" eigenvectors for the *lowest* state of a given (J, T), but slightly "wrong" eigenvectors for the higher state of the given (J, T) [where we have used the words "right" or "wrong" in the sense of these numbers matching or not matching exactly with the full-shell eigenvalues E_j and their truncated eigenvectors, given by Eq. (2)]. Apparently, if we start our Gram-Schmidt procedure with any other level of a given (J, T), this is equivalent to choosing different $\psi_j^{(m)}$ for the same model space. The interaction $H^{(m)}$ would consequently be different. However, our calculations show that the resulting change in $H^{(m)}$, due to a different choice of the basis state for the Gram-Schmidt orthogonalization procedure, is insignificantly small. [For example, the two-body matrix elements for $J=0$, $T=1$ states in d-s shell (refer to Table I) changes from −3.52, −1.18, and −2.00 MeV to, respectively, −3.55, −1.14, and −1.97 MeV; and similar order of differences occur in the other G_{eff} numbers.] We, however, choose to orthogonalize our eigenvectors with respect to the lowest-lying level for each (J, T) since this might have an advantage in calculations for systems of three and four active nucleons, because their low-lying states might have particularly large coefficients of fractional parentage involving the lowest-lying states of the two-active-nucleon system.[14]

Alternatively,[15] one can generate an interaction that exactly fits not only the eigenvalues E_j of the

TABLE II. Two-body matrix elements for the basis space $(1f_{7/2}\ 2p_{3/2})^2$. All energies are in MeV and the notation is same as in Table I with $7 \equiv 1f_{7/2}$ and $3 \equiv 2p_{3/2}$.

Configuration	J	T	G_b	Hamada-Johnston G_r	Hamada-Johnston G_{eff}	G'_{eff}	Effective interaction V_{eff} [a]
7777	0	1	−0.87	−1.81	−2.40	−3.01	−2.75
7733	0		−0.47	−0.78	−1.01	−1.25	−1.22
3333	0		−1.00	−1.21	−1.64	−1.76	−1.72
7777	2		−0.66	−0.78	−0.86	−1.12	−1.04
7773	2		−0.28	−0.50	−0.59	−0.68	−0.72
7733	2		−0.17	−0.27	−0.29	−0.36	−0.58
7373	2		−0.63	−0.86	−1.02	−1.06	−1.03
7333	2		−0.23	−0.32	−0.37	−0.40	−0.58
3333	2		−0.45	−0.38	−0.58	−0.60	−0.64
7373	3		−0.21	−0.03	−0.03	−0.03	−0.17
7777	4		−0.30	−0.09	−0.17	−0.27	−0.36
7773	4		−0.15	−0.31	−0.37	−0.43	−0.38
7373	4		−0.22	−0.05	−0.27	−0.30	−0.19
7373	5		−0.12	0.14	0.14	0.14	0.15
7777	6		−0.12	0.23	0.13	0.08	0.20
7777	1	0	−0.23	−0.52	−1.66	−1.88	−2.71
7733	1		−0.08	−0.28	−0.70	−0.78	b
3333	1		−0.65	−0.64	−2.88	−2.90	b
7373	2		−0.27	−0.29	−1.09	−1.09	−0.60
7777	3		−0.21	−0.21	−0.58	−0.67	−1.55
7773	3		−0.39	−0.48	−0.72	−0.77	b
7733	3		−0.24	−0.30	−0.46	−0.48	b
7373	3		−0.68	−0.60	−1.33	−1.35	b
7333	3		−0.44	−0.51	−0.77	−0.79	b
3333	3		−1.83	−1.83	−2.03	−2.04	b
7373	4		−0.10	−0.16	−0.23	−0.23	−0.70
7777	5		−0.60	−0.50	−0.61	−0.65	−1.04
7773	5		−0.75	−0.82	−0.86	−0.90	−0.92
7373	5		−2.08	−2.16	−2.19	−2.22	−2.77
7777	7		−2.18	−2.20	−2.20	−2.27	−2.47

[a] The single-particle energies used in this calculation are given in Table III.

[b] These matrix elements were taken to be equal to their corresponding G_r counterparts (Ref. 25).

TABLE III. Single-particle energies, derived from experiments, for both the d-s and f-p shells.

Configuration	Energy (MeV)	Configuration	Energy (MeV)
$1d_{5/2}$	0	$1f_{7/2}$	0
$2s_{1/2}$	0.87	$2p_{3/2}$	2.1
$1d_{3/2}$	5.08	$2p_{1/2}$	3.9
		$1g_{9/2}$	5.9
		$1f_{5/2}$	6.5

full space but also uses the untreated, nonorthogonal truncated eigenvectors given by Eq. (2). Such an interaction would, however, be non-Hermitian but for "many-body" shell-model calculations could be made Hermitian by some averaging procedure. From a private communication with Barrett and Halbert we understand that they (in collaboration with McGrory) are carrying out such calculations and refer to this method as Brandow's method. They make their interaction Hermitian simply by averaging it with its own transpose. This then gives somewhat "wrong" values for *all* the eigenvalues and somewhat "wrong" values for *all* the eigenvectors, though in many cases the "slightly wrong" answers are almost "right." We thus find that our projection method, which maintains an Hermitian interaction, seems better in that (i) it gives a greater number of "right" eigenvalues and eigenvectors for the case of two active nucleons and (ii) our choice of orthogonalizing the wave function with respect to the lowest-lying state for each (J, T) might provide an advantage for its application to low-lying states in systems of three and four active nucleons.

We have applied this method to both the $1d$-$2s$ and $1f$-$2p$ shells. The G_τ matrices for the two shells are taken from Kuo[7] and Kuo and Brown,[8] respectively. The single-particle energies used in the calculations are listed in Table III. The results of our calculation are given in Table I for the d-s shell and in Table II for the f-p shell (denoted as $G_{\rm eff}$). In Table II, $G_{\rm eff}$ refers to the case of $g_{9/2}$ orbit omitted in the original calculations of ψ_j and E_j and $G'_{\rm eff}$ to the case when this orbit is also included in the calculations of ψ_j and E_j.

As a further application of our method, we have tested the idea[16] that for the $J=0^+$, $T=1$ ground state, the effects of leaving out $d_{3/2}$ orbits in $1d$-$2s$ shell calculations are absorbed by modifying only the d^2 part of the eigenvectors, and that in the case of f-p shell (with $g_{9/2}$ omitted) the effects of limiting the configuration space (f-$p \rightarrow f_{7/2}p_{3/2}$) are shown in the components of their respective orbital angular momenta. Specifically, for the $J=0$, $T=1$ states in the complete d-s and f-p shells,

$$\psi_j = a_{1j}(d_{5/2})^2 + a_{2j}(d_{3/2})^2 + a_{3j}(s_{1/2})^2\,, \quad (j=1,2,3) \tag{5a}$$

and

$$\phi_j = b_{1j}(f_{7/2})^2 + b_{2j}(f_{5/2})^2 + b_{3j}(p_{3/2})^2 + b_{4j}(p_{1/2})^2\,, \quad (j=1,2,\ldots,4) \tag{5b}$$

the model states in the truncated space can be written as

$$\psi'_j = (a_{1j}{}^2 + a_{2j}{}^2)^{1/2}(d_{5/2})^2 + a_{3j}(s_{1/2})^2\,, \quad (j=1,2)\,, \tag{6a}$$

$$\phi'_j = (b_{1j}{}^2 + b_{2j}{}^2)^{1/2}(f_{7/2})^2 + (b_{3j}{}^2 + b_{4j}{}^2)^{1/2}(p_{3/2})^2\,, \quad (j=1,2)\,, \tag{6b}$$

where the eigenvectors now remain normalized. Using the procedure described above, the $G_{\rm eff}$ matrix elements can then be calculated. The results of this calculation are given in Table IV. We have also included in this table, the corresponding $G_{\rm eff}$ matrix elements from Tables I and II. It is interesting to observe that the two sets of numbers are very much identical.

3. NATURE OF THE EFFECTIVE TWO-BODY INTERACTION SEEN IN THE RENORMALIZATION OF G MATRIX

As already stated in the Introduction, in this section we examine the operators G_b, G_τ, and $G_{\rm eff}$, listed in Tables I and II, with a view to infer the nature of effective two-body interaction. We do this by studying the relative changes (i) in the two-body matrix elements of each (J, T) for $G_b \rightarrow G_\tau \rightarrow G_{\rm eff}$ and (ii) in the centroids of certain sets of these matrix elements. This is given in subsec-

TABLE IV. $G_{\rm eff}$ matrix elements for $J=0$, $T=1$ states in both $(d_{5/2}s_{1/2})^2$ and $(f_{7/2}p_{3/2})^2$ configurations, derived by the use of Eq. (6), and the corresponding numbers from Tables I and II, respectively.

Configuration	$G_{\rm eff}$ Table I	$G_{\rm eff}$ Eq. (6a)	Configuration	$G_{\rm eff}$ Table II	$G_{\rm eff}$ Eq. (6b)
5555	−3.52	−3.55	7777	−2.40	−2.36
5511	−1.18	−1.14	7733	−1.01	−1.12
1111	−2.00	−1.97	3333	−1.64	−1.69

tions A and B, respectively. We have also carried out our analysis by reparametrizing these operators in terms of (i) radial integrals and (ii) the parameters of Donnelly's velocity-dependent potential,[17] using least-squares fitting procedures. The parametrization of the two-body matrix elements in terms of the diagonal radial integrals has been advocated by Pandya[18] for some time and the properties of various radial integrals for the d-s shell configuration have been discussed by Pandya and Kulkarni.[19,20] In our calculations, we have assumed for simplicity only a central potential. However, this part of our analysis turns out to be rather approximate since the least-squares-fitted parameters for both the radial integrals and the Donnelly potential, given an inexact reproduction of the operators G_b, G_r, and $G_{\rm eff}$ (though most of the matrix elements are reproduced within 5–10%, some are in error by as much as 50%). Since the results of this calculation could not be conclusive, we do not give any further details here but summarize them in Sec. 5 along with the other results.

A. jj Two-Body Matrix Elements

We first proceed by comparing the matrix elements of $5555:JT$ and $7777:JT$ configurations, listed in Tables I and II (for notation, refer to the table caption). We notice that when the core configurations are removed ($G_b \to G_r$), the major changes occur in 5555 : 01 and 5555 : 10 matrix elements for the d-s shell and in 7777 : 01 and 7777 : 10 matrix elements for the f-p shell, where by the word change we mean the absolute change in MeV. In each case a large attraction is added. For $T=1$, $J=0$, 2, 4 states in the d-s shell, the spectrum goes over from an almost rotational to a vibrational type. Next, when the $d_{3/2}$ subshell is removed ($G_r \to G_{\rm eff}$), the major change is once again in $J=0$, $T=1$ and $J=1$, $T=0$ matrix elements and these matrix elements become further attractive. Furthermore, if we compare $G_{\rm eff}$ and $G'_{\rm eff}$ matrix elements in Table II, the effect of omitting the $g_{9/2}$ orbit is also seen in making only the 7777 : 01 matrix element further attractive. In the case of other diagonal matrix elements, we notice that whereas the matrix element 1111 : 01 remains essentially undisturbed, the 1111 : 10 is gradually depressed (becomes attractive). For the f-p shell, however, the 3333 : 01 matrix element is depressed gradually whereas the 3333 : 10 is depressed suddenly in going from G_r to $G_{\rm eff}$. The above noted changes in the jj matrix elements, when analyzed in terms of the centroids of the sets of matrix elements, reflect on the nature of the effective two-body interaction in an explicit manner. We might, however, mention here that the quoted changes in $G_b \to G_r$ (and $G_r \to G_{\rm eff}$) depend on the particular selection of very simple perturbation approximations, and would be different if a different selection had been made.

B. Centroid Calculations

Talmi[21] has defined the parameters

$$V_0 = \langle 2j2j2j2j : 01\rangle ,$$

$$\overline{V}_2 = \frac{1}{(j+1)(2j-1)} \sum_{J>0,\ \text{even}}^{2j-1} (2J+1)\langle 2j2j2j2j : J1\rangle , \tag{7}$$

and

$$\overline{V}_1 = \frac{1}{(j+1)(2j+1)} \sum_{J,\ \text{odd}}^{2j} (2J+1)\langle 2j2j2j2j : J0\rangle . \tag{8}$$

These parameters can be used to calculate the energies of the lowest-seniority (v_0) states for $(j)^nT$ (and these calculated energies may be considered as estimates of the ground-state energies):

$$\langle (j)^nT, v_0 | H | (j)^nT, v_0\rangle = \frac{n(n-1)}{2}\alpha + \left[\frac{n}{2}\right]\beta + [T(T+1) - (\tfrac{3}{4})n]\gamma , \tag{9}$$

where $[n/2]$ is a step function which is equal to $n/2$ for even n and $(n-1)/2$ for odd n, and the interaction parameters are

$$\alpha = \frac{(6j+5)\overline{V}_2 + (2j+1)\overline{V}_1 - 2V_0}{4(2j+1)},$$

$$\beta = \frac{2(j+1)}{2j+1}(V_0 - \overline{V}_2), \tag{10}$$

$$\gamma = \frac{(2j+3)\overline{V}_2 - (2j+1)\overline{V}_1 - 2V_0}{2(2j+1)},$$

which refer, respectively, to the quadratic term, pairing term, and the isospin-dependent symmetry term in the estimated ground-state binding energies.

We have calculated V_0, $\overline{V}_1$, and $\overline{V}_2$ as well as the parameters α, β, and γ using the matrix elements $5555:JT$ and $7777:JT$ for the operators G_b, G_r, and $G_{\rm eff}$, listed in Table I and II. The results of the calculation are given in Table V. We observe that for both the $(d_{5/2})^2$ and $(f_{7/2})^2$ configurations, the centroid V_0 changes more strongly than $\overline{V}_1$ and $\overline{V}_2$. The truncation of configuration space thus shows up in widening of the gap between V_0 and $\overline{V}_2$, and hence in the variation of β, representing an increase in the pairing component of the interaction. This result is similar to that obtained by Cohen, Lawson, and Soper[1] from their study of pseudonium isotopes. It is interesting to note that

this similarity occurs despite the fact that pseudonium study involves omission or inclusion of only core-excited states whereas our analysis of the transformation $G_b \rightarrow G_r \rightarrow G_{\text{eff}}$ involves omission or inclusion of the core-excited states as well as some orbits outside the core. On the other hand, the centroid, $\bar{V}_1$ is lowered gradually with the truncation of space. The relative magnitudes of $\bar{V}_1$ and $\bar{V}_2$ indicate more attraction in $T=0$ states, on the average in agreement with the binding energy calculations of Talmi.[21] Finally, the quadratic energy parameter α is shown to remain essentially insensitive to the truncation of shell-model space, whereas the symmetry energy term γ changes by a factor of about 2.

4. COMPARISON BETWEEN THE G_{eff} AND THE EMPIRICALLY FITTED EFFECTIVE INTERACTIONS

No doubt phenomenological effective interactions are, in general, obtained with different aims in mind. However, the final aim of all such calculations should be to get a picture of the experiments, consistent with some theory. In this section, we aim at seeing such a consistency between the available empirically fitted effective interactions and our theoretical G_{eff} interaction which is designed to reproduce some properties of G_r for one to two active nucleons.

For the d-s shell, the 16 $(d_{5/2}s_{1/2})^2$ matrix elements were obtained by Arima *et al.*[22] by fitting a number of experimentally observed levels in nuclei $A=17-20$ and by Wildenthal *et al.*[23] by fitting some of the observed levels in nuclei $A=20-28$, in terms of the configuration $(d_{5/2}s_{1/2})^n$; thus $n=1-4$ in the calculations of Arima *et al.* and 4–12 in those of Wildenthal *et al.* The two operators thus obtained are listed in Table I (denoted as V_{ACLM} and V_{WMHG}, respectively). The single-particle energies for V_{ACLM} were taken from experiments (same as given in Table III) and for V_{WMHG}, these were treated as parameters along with the 16 matrix elements. The single-particle energies thus obtained for V_{WMHG} are given at the bottom of Table I. For the f-p shell, recently McGrory and Halbert[24] have obtained 23 (of the total 30) $(f_{7/2}p_{3/2})^2 : JT$ matrix elements by reproducing some theoretical levels which were calculated by using a modified form of G_r acting in the space of one to four active nucleons ($A=41-44$). This effective interaction is listed in Table II (denoted as V_{eff}). The single-particle energies used are the same as listed in Table III. The remaining seven matrix elements, which could not be well determined (these are all for $T=0$ states) were taken to be equal to their G_r counterparts.[25] We might mention here that, based on an earlier calculation,[26] these authors did not include the $g_{9/2}$ orbit in their analysis and we have made a similar omission in our calculation of the G_{eff} operator. Thus we see that whereas the operator V_{eff} has a direct connection to a modification of G_r, the operators V_{ACLM} and V_{WMHG} do not have any direct connection with $G_r \rightarrow G_{\text{eff}}$. However, in view of our earlier remark, a comparison between the G_{eff} and these empirically fitted interactions should be of interest. We notice that whereas for the f-p shell the comparison between the G_{eff} and V_{eff} is very good (as expected, since both are obtained as modifications of G_r), it is rather poor for the d-s shell. Even the two empirically fitted interactions (V_{ACLM} and V_{WMHG}) are significantly different from each other. Apart from the obvious fact that these two operators are derived by fitting different experimental data, this situation for the d-s shell might partly be due to the fact that $d_{3/2}$ plays an important role in $T=0$ matrix elements and partly due to the poor determination of some of these matrix elements in the least-squares fitting procedure of Arima *et al.*[22] There is also a possibility[27] that this difference has its origin in the use of different single-particle energies for the two calculations.

TABLE V. The centroids and the interaction parameters, calculated by use of Eqs. (7)–(10), for the $(d_{5/2})^2 : JT$ and $(f_{7/2})^2 : JT$ configurations.

Configuration	Parameters	Hamada-Johnston G_b	G_r	G_{eff}
$(d_{5/2})^2$	V_0	−1.24	−2.44	−3.52
	$\bar{V}_2$	−0.63	−0.40	−0.78
	$\bar{V}_1$	−2.10	−2.35	−2.96
	α	−0.95	−0.72	−1.10
	β	−0.70	−2.38	−3.20
	γ	0.83	1.31	1.55
$(f_{7/2})^2$	V_0	−0.87	−1.81	−2.40
	$\bar{V}_2$	−0.28	−0.06	−0.15
	$\bar{V}_1$	−1.16	−1.15	−1.35
	α	−0.46	−0.23	−0.31
	β	−0.66	−1.96	−2.53
	γ	0.51	0.76	0.88

5. SUMMARY OF THE RESULTS

Using the renormalized G_r matrix calculations of Kuo[7] (for the d-s shell) and of Kuo and Brown[8] (for the f-p shell), a re-renormalized effective G_{eff} matrix for only two active orbits is calculated numerically, which would be valid for use in $(d_{5/2}s_{1/2})^n$ and $(f_{7/2}p_{3/2})^n$ configurations. An analysis of the transformation chain $G_b \rightarrow G_r \rightarrow G_{\text{eff}}$ shows that the truncation of the shell-model space both in terms of the core excitation and some orbits

outside the core affects mainly the $J=0$, $T=1$ and $J=1$, $T=0$ states due to the configurations 5555 and 7777. It is further shown that the effects of truncation of the space can be largely taken into account by the addition of a pairing-force term to the effective two-body interaction. This result is similar to that obtained from the study of so-called pseudonium isotopes,[1] though there exists a qualitative difference in the nature of transitions analyzed in the two studies. Next, our renormalization leads to more attraction for the $T=0$ part of the interaction and the quadratic-energy term is shown to remain essentially insensitive to the choice of shell-model space. The reparametrization of the operators G_b, G_τ, and G_{eff} in terms of the radial integrals and the parameters of Donnelly's velocity-dependent potential (using least-squares fitting procedure) suggests the following: For both the transformations $G_b \to G_\tau$ and $G_\tau \to G_{\text{eff}}$, large changes occur in the s-state radial integrals which are known to have pairing-force-like properties.[19,20] The analysis of the parameters of Donnelly's potential indicates that the truncation of configuration space affects mainly the odd-state forces—the velocity-independent part (mainly the triplet-odd strength) becoming more and more repulsive and the velocity-dependent more attractive. Renormalization for severe truncation of configuration space to include only two orbits, also leads to more attraction in velocity-dependent even-state forces. Finally, a comparison between the results of this calculation (G_{eff}) and the available phenomenological effective interactions shows that the interactions fitted empirically by using a very limited shell-model space can sometimes turn out very different from those obtained by renormalizing realistic forces.

We conclude with a remark that it will be of interest to look for the similar effects of truncation of space on the nature of effective two-body interaction in terms of noncentral forces. This is a subject of our next investigation.

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[15] Added during the revision of the manuscript as a result of correspondence with Professor B. R. Barrett and Professor E. C. Halbert, initiated after the author sent a preprint of this work to Professor J. B. McGrory.

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