

Comparison of Sub-Coulomb Stripping and Analog Resonance Results near Mass 90*

J. F. Morgan, R. G. Seyler, and J. J. Kent[†]

Department of Physics, The Ohio State University, Columbus, Ohio 43210

(Received 8 August 1973)

The (d, p) reaction has been studied, for incident energies below the Coulomb barrier, on ^{89}Y and ^{92}Mo . These data have been analyzed, together with previously measured sub-Coulomb data on ^{90}Zr and ^{92}Zr , to extract reduced normalizations for the parent states excited in the various residual nuclei. These reduced normalizations were compared with those calculated for the analogs of these parent states using three analog resonance theories. The R -matrix theory gives the best agreement with the (d, p) results.

$$\left[\begin{array}{l} \text{NUCLEAR REACTIONS } ^{89}\text{Y}, ^{92}\text{Mo}(d, p), E_d = 4.0 \text{ MeV; measured } \sigma(\theta). ^{90}\text{Y}, \\ ^{91}\text{Zr}, ^{93}\text{Zr}, ^{93}\text{Mo} \text{ levels deduced } \Lambda_{Ij}. \text{ Compared to IAR } (p, p_0) \Lambda_{Ij}. \end{array} \right]$$

I. INTRODUCTION

The experimentally determined fact that isospin is a good quantum number for medium and heavy nuclei has many consequences. In particular the spectroscopic factors determined from (d, p) reactions leading to single-particle states of these nuclei should agree with those obtained from elastic proton scattering over the isobaric-analog resonances (IAR) of the same states. Many experiments have been done to check this fact, with rather mixed results. Part of the explanation of the discrepancies lies in the well-known dependence of both the stripping and resonance analysis on the optical parameters used to generate the entrance, exit, and bound-state wave functions. Even if one could overcome this problem, there would still exist another difficulty. Whereas the distorted-wave Born approximation (DWBA) gives a well developed method for the extraction of (d, p) spectroscopic factors no such single accepted theory exists in the case of resonance analysis. Theoretical descriptions based on the R -matrix theory, the projection operator method, and the shell model have been put forth. It would be of interest to determine which of these theories gives spectroscopic factors in best agreement with the (d, p) spectroscopic factors.

Rapaport and Kerman¹ were the first to show how the difficulty of the optical-parameter dependence could be eliminated for the (d, p) reactions. They showed that having the entrance and exit channel energies below the Coulomb barrier insured that the calculated DWBA cross section (and hence the spectroscopic factor) was independent of the entrance and exit channel parameters; however a strong dependence upon the parameters used to describe the neutron bound state remained. They

defined a parameter, called the reduced normalization Λ_{Ij} which is nearly independent of the bound-state parameters. This parameter is essentially the ratio of the bound-state neutron wave function to a Hankel function evaluated at a radius far outside the nucleus, cf. Eq. (1). Recently Clarkson, von Brentano, and Harney² have shown that it is possible to define this same parameter for the analog resonance case, in the context of each of the existing theories. As for the (d, p) case, the parameter Λ_{Ij} is independent of the optical parameters used in the calculation but in this case is dependent upon the theory used for the calculation. Thus for the first time it is possible to use experimental methods to attempt to test the various analog resonance theories.

In a previous paper³ we reported on a comparison of sub-Coulomb stripping and analog resonance results for ^{93}Zr . While it was possible to obtain reduced normalizations from both the (d, p) and (p, p) data, there was not enough data to make a meaningful statement concerning which resonance theory gave the best results. In the present paper we report on sub-Coulomb (d, p) measurements on the $N=50$ nuclei ^{89}Y and ^{92}Mo . These have been used, together with the previously published ^{90}Zr - (d, p) ⁴ and $^{92}\text{Zr}(d, p)$ ³ data, to extract reduced normalizations for all the resolved parent states. These in turn have been compared with the reduced normalizations calculated for the analog resonances of these parent states, using the three theories mentioned above.

II. EXPERIMENTAL METHOD

A deuteron beam from the Ohio State University CN Van de Graaff was incident on self-supporting targets at the center of a 61-cm-diam scattering

chamber. The target thickness was measured by scattering 2.5-MeV protons from the target and assuming the scattering at forward angles to be given by the Rutherford formula. In this manner the target thicknesses were found to be 0.15 mg/cm² for ⁸⁹Y and 0.37 mg/cm² for ⁹²Mo, with an accuracy of 10%. Charged particles from the reactions were detected in three silicon surface-barrier detectors. Since all of the (*d*, *p*) reactions have large positive *Q* values (4.0–5.5 MeV) the peaks of interest were well above the elastic

deuteron peak and thus no particle identification was necessary.

Angular distributions were measured in 10° intervals from 55 to 165°. Our previous work³ had shown that in this region of the Periodic Table 4.0-MeV bombarding energy was the best com-

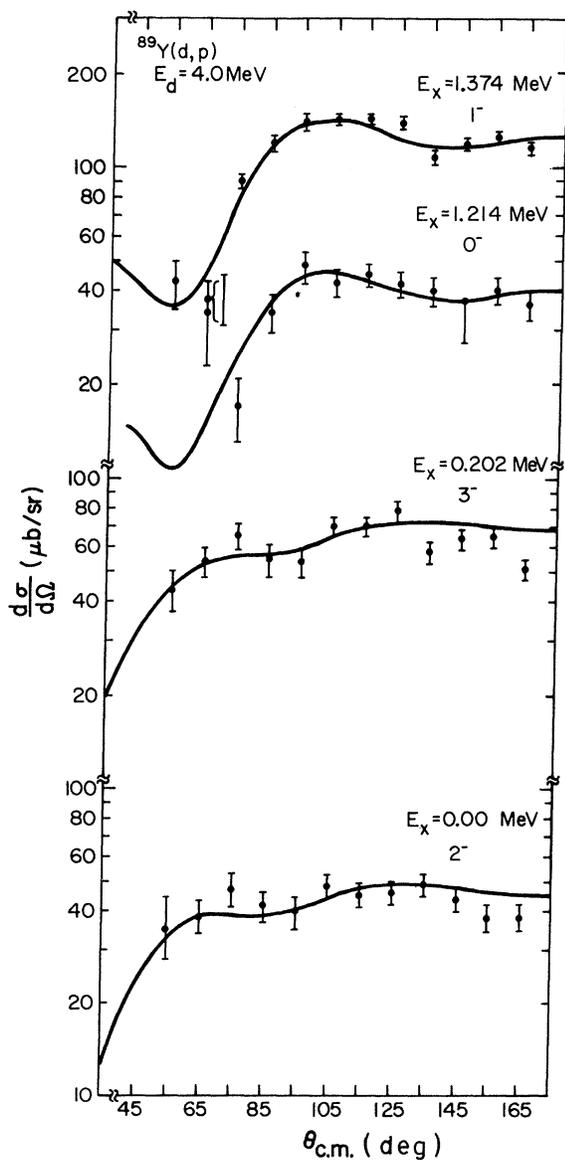


FIG. 1. ⁸⁹Y(*d*, *p*) differential cross sections at $E_d = 4.0$ MeV. The solid curves are DWBA calculations normalized to the data.

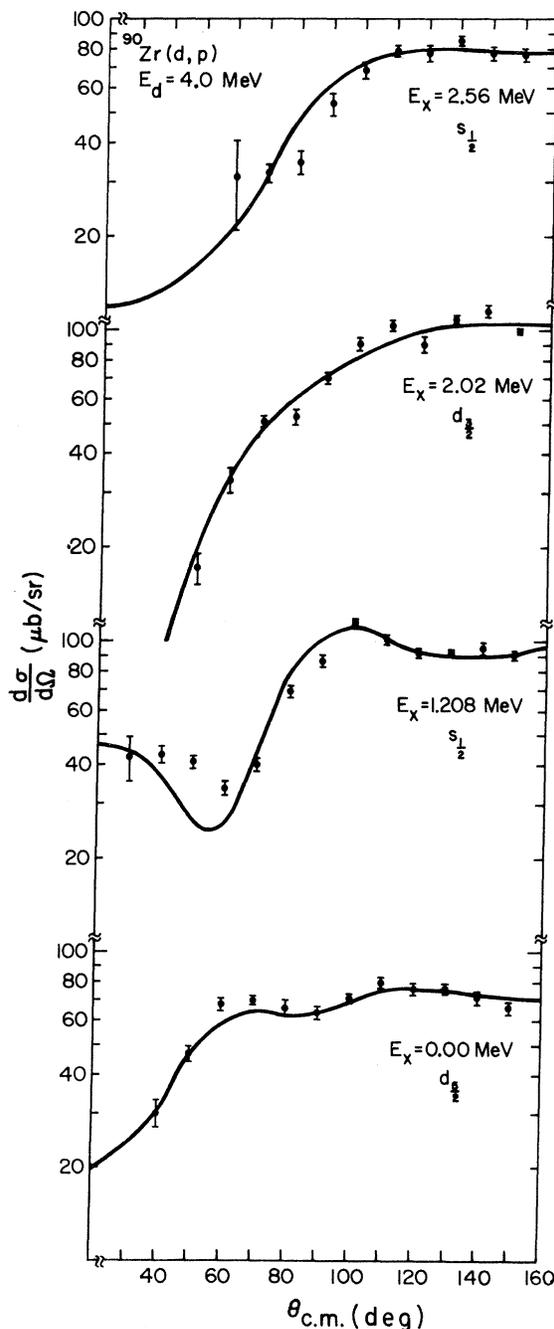


FIG. 2. ⁹⁰Zr(*d*, *p*) differential cross sections at $E_d = 4.0$ MeV. The data were taken from Ref. 4. The solid curves are DWBA calculations normalized to the data.

promise between keeping the ground-state protons below the Coulomb barrier and having a large enough cross section to separate the peaks of interest from proton peaks produced by the presence of light contaminants, mostly ^{13}C and ^{28}Si , in the target. The differential cross sections measured at 4.0 MeV are shown in Figs. 1-4 for the various targets. The error bars in the figures represent both statistical uncertainties and the errors introduced by background subtraction. The absolute cross section has an additional 10% uncertainty introduced by the target thickness. The agreement between our $^{89}\text{Y}(d, p)$ measurement and a previous one⁵ at 4.0 MeV is excellent for the states we have studied.

Also shown in Figs. 1-4 are the results of the DWBA calculations, using the code JULIE, normalized to the data. These calculations will be described below.

III. ANALYSIS

A. (d, p)

For a sub-Coulomb (d, p) reaction the radial part of the wave function of the captured neutron

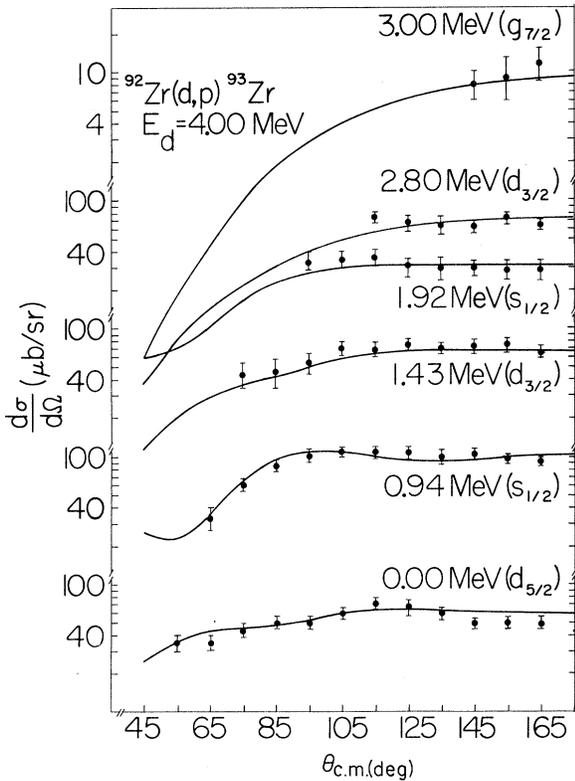


FIG. 3. $^{92}\text{Zr}(d, p)$ differential cross sections. The data were taken from Ref. 3. The solid curves are DWBA calculations normalized to the data.

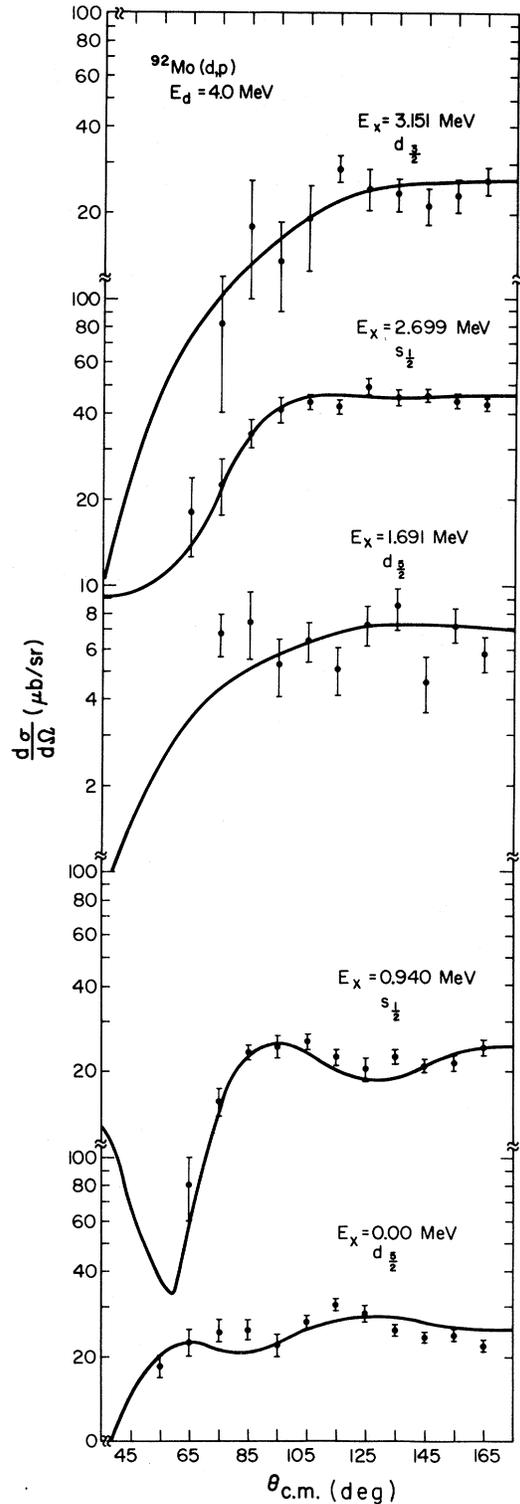


FIG. 4. $^{92}\text{Mo}(d, p)$ differential cross sections. The solid curves are DWBA calculations normalized to the data.

$\langle B|A\rangle$ outside the nuclear radius R_n may be written as

$$\psi(r) = [(2J_B + 1)\Lambda_{lj}]^{1/2} k^{3/2} h_l^{(1)}(ikr), \quad r > R_n, \quad (1)$$

where $h_l^{(1)}$ is a spherical Hankel function of the first kind, J_B the spin of the final state, and k the wave number defined as

$$k = (2\mu |B_n|)^{1/2} / \hbar,$$

where μ is the reduced mass, and B_n the neutron binding energy of the final state. Equation (1) serves as a definition of the reduced normalization Λ_{lj} .

Rapaport and Kerman have shown that the above-defined reduced normalization is related to the DWBA spectroscopic factor S_{lj} by

$$k^3 \Lambda_{lj} = S_{lj} N_{lj}^2,$$

where $N_{lj} h_l^{(1)}(ikr)$ is the asymptotic neutron radial wave function used in the DWBA code. By fitting the (d, p) data with a standard DWBA code (JULIE in our case) we can extract S_{lj} , and by comparing the code's bound-state neutron wave function with a Hankel function at a radius far outside the nucleus we can obtain N_{lj} and thus extract Λ_{lj} .

The optical parameters used for the (d, p) analysis are given in Table I and are the same as those used in our previous work.³ We review briefly the question of the sensitivity of the (d, p) results for Λ (for brevity we delete the lj subscripts) to the values chosen for the parameters of the deuteron, proton, and transferred neutron potential wells.

In the case of the deuteron well the most sensitive parameter is r_w , the radius of the surface absorption potential. An increase of 15% in r_w produces a 30% or so increase in Λ , i.e., a magnification factor of 2. On the other hand a decrease in r_w results in a magnification factor of less than 1. This sensitivity to an increase in r_w is understandable when one recalls from Table I that the absorptive potential is outside the real

TABLE I. Optical parameters for the (d, p) analysis where $V(r) = V(e^x + 1)^{-1} + i 4(d/dx')W(e^x + 1)^{-1} + V_c$ and $x = (r - r_v)/a_v$, $x' = (r - r_w)/a_w$. The neutron well depth was varied to fit the neutron binding energy of each state. The neutron spin-orbit potential strength was taken to be 25 times the Thomas form. The charged particle spin-orbit strengths were found to have negligible effect and were therefore set equal to zero.

Light particle	V (MeV)	r_v (fm)	a_v (fm)	W_{surface} (MeV)	r_w (fm)	a_w (fm)
d	100	1.15	0.81	15	1.34	0.68
p	59	1.17	0.75	12	1.32	0.60

well (r_w is 17% larger than r_v). Increasing the absorption radius increases the probability that an incoming deuteron will be absorbed and lost before it has a chance to participate in a (d, p) reaction, thus decreasing the predicted (d, p) cross section and thereby increasing Λ . The diffusivity of the absorptive well is the next most sensitive parameter with a magnification factor of almost 1. The explanation of its sensitivity is the same as that given for r_w . Other parameters have even smaller magnification factors with the well depths having the smallest. Changing the imaginary well depth by $\pm 20\%$ changes Λ by about $\pm 3\%$. The same change in the real well depth changes Λ by 1% or less.

In the case of the proton well the most sensitive parameter is r_v , the radius of the real volume potential. Increasing r_v by 15% produces a 30% increase in Λ . On the other hand decreasing r_v by 15% leads to only a 6% decrease in Λ . This sensitivity is unique to this mass region and reflects the well-known optical-model shape resonance near mass 90. An examination of the partial wave scattering amplitudes reveals that an increase in r_v permits the p -wave amplitudes to resonate, thereby decreasing the calculated (d, p) cross section and increasing Λ . For the same reason the diffusivity of the real well is also a sensitive parameter but to a smaller degree, its

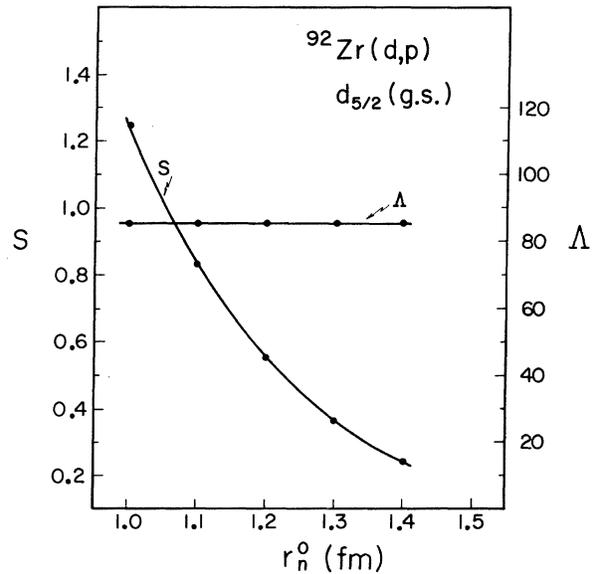


FIG. 5. Dependence of the spectroscopic factor and the reduced normalization on the radius of the potential for the transferred neutron. The calculation shown is for the $d_{5/2}$ ground state of ^{92}Zr but the result was the same for all of the states considered. Similar results apply for the diffusivity of the neutron potential.

magnification factor being less than 1. Other parameters are less sensitive with the absorptive well depth being the least sensitive.

In the case of the neutron well for given geometry, the depth is adjusted to give the right binding energy. Varying the geometry by $\pm 20\%$ and adding or subtracting a standard spin-orbit potential changes Λ by 1% or less. The fact that Λ is insensitive to such changes whereas S is so highly sensitive, especially to r_n where a change in S of nearly an order of magnitude is possible (cf. Fig. 5), is the primary motivation for introducing and using Λ .

The results of the (d, p) analysis are given in Table II where we list the Λ 's extracted for each state. With the caveat given by Fig. 5, we also give spectroscopic factors extracted from the data using the optical parameters given in Table I. It should be noted that it now seems certain that there exists a doublet in ^{93}Zr with the $d_{3/2}$ state at 1.43 MeV and a $g_{7/2}$ state at 1.48 MeV which we would not resolve at back angles (Refs. 6, 7). These states were not resolved in the (p, p_0) data either. In contradiction to Ref. 6, both this work and Ref. 7 see no evidence for a state at 1.02 MeV in ^{93}Zr which might obscure the 0.94-MeV state. Also note that, based on the works of Ellis⁸ and Ball⁹,

we take the spin of the 1.691-MeV state in ^{93}Mo to be $\frac{5}{2}$ instead of the previous¹⁰ $\frac{3}{2}$ assignment.

B. Analog Resonances

In order to compare the (d, p) results with the various IAR theories, we have calculated reduced normalizations for the analogs of the states populated in the (d, p) reaction based on the measured elastic partial widths, using the formalism of Clarkson, von Brentano, and Harney.² The optical-parameter dependence of the resonance calculations has been thoroughly studied in Ref. 2, and therefore we chose parameters for the calculations in accordance with the criteria established by Clarkson, von Brentano, and Harney. The proton well depth was first chosen to bind the $g_{7/2}$ anti-analog state in ^{90}Zr . This gave a depth of 61.83 MeV. The potential (in MeV) was then extrapolated to the energy of each resonance with the relation $V_p = 61.83 - 0.6E_{\text{res}}$ where E_{res} is the laboratory resonance energy in MeV and the coefficient 0.6 is taken from Ref. 2.

The potential parameters for the transferred particle are again chosen by the separation energy method with the radius r_n fixed at 1.20 fm and the well depth allowed to vary in order to reproduce the correct Coulomb displacement energies. The experimental parameters which were used in the analysis of each resonance were taken from the literature. For ^{89}Y the binding energies were taken from Ref. 11 and the proton widths from Refs. 12 and 13. For ^{90}Zr and ^{92}Zr the binding energies were taken from Ref. 14 and the proton widths from Refs. 15, 16, and 17 and from Refs. 18 and 19, respectively. For ^{92}Mo the binding energies were taken from Ref. 10 and the proton widths from Refs. 18 and 20.

The reduced normalizations for each state were calculated using the analog resonance theories of Thompson, Adams, and Robson¹⁵ (TAR), Mekjian and McDonald²¹ (MM), and Zaidi, Darmodjo, and Harney^{22, 23} (ZDH). These calculations were performed using the code BETTINA.²⁴

IV. RESULTS AND DISCUSSION

Before comparing the results of the (d, p) and the analog resonance experiments we must determine the precision with which the reduced normalizations can be obtained. Ignoring the dependence of the (d, p) reduced normalization on the optical parameters (see above), the main contributions to the uncertainty in its determination comes from the target-thickness determination (10%) and the counting statistics. The flat backward peaked shape of the sub-Coulomb angular distributions makes uncertainties due to normal-

TABLE II. Spectroscopic factors and reduced normalizations for the (d, p) data.

E_x	n_{lj}	S_{lj}	Λ_{lj}
^{90}Y			
0.000	$2d_{5/2}$	0.78	137.
0.202	$2d_{5/2}$	0.72	114.
1.214	$3s_{1/2}$	0.71	351.
1.374	$3s_{1/2}$	0.88	403.
^{91}Zr			
0.000	$2d_{5/2}$	0.80	166.
1.208	$3s_{1/2}$	0.67	388.
2.02	$2d_{3/2}$	0.54	31.6
2.56	$3s_{1/2}$	0.19	57.6
^{93}Zr			
0.00	$2d_{5/2}$	0.48	81
0.94	$3s_{1/2}$	0.58	305
1.43	$2d_{3/2}$	0.38	25
1.92	$3s_{1/2}$	0.19	63
2.80	$2d_{3/2}$	0.36	10
3.00	$1g_{7/2}$	0.44	0.09
^{93}Mo			
0.00	$2d_{5/2}$	0.68	219
0.94	$3s_{1/2}$	0.52	503
1.691	$2d_{5/2}$	0.07	9.88
2.699	$3s_{1/2}$	0.29	127
3.151	$2d_{5/2}$	0.17	899

izing the DWBA calculation to the experimental data negligible. Thus it seems reasonable to assign an over-all uncertainty of 15% to the (d, p) reduced normalization.

One might ask at these low bombarding energies about the contribution of a compound nuclear mechanism to the (d, p) cross sections. For all the nuclei studied the (d, n) Q value is positive and comparable to the (d, p) Q value. Thus compound nuclear decay will be dominated by the neutron channels. Experimentally this can be seen from the data. For example, the angular distribution of the 2.02-MeV $d_{3/2}$ state in ^{91}Zr follows the DWBA prediction at forward angles almost over an order of magnitude, indicating that the contribution of a compound-nuclear mechanism, which would be symmetric about 90° can be no more than about 10% of the direct reaction contribution. The possible effects of compound-nuclear formation have been ignored in extracting the spectroscopic factors.

Estimating the error in the (p, p_0) reduced normalizations is more difficult since they are based upon the proton partial widths obtained from many places in the literature. Statistical accuracy of all the elastic scattering experiments is very good so once again the main contribution to the uncertainty comes from the target thickness determination, as well as the fact that, in fitting the elastic excitation functions, the relevant parameter is Γ_p/Γ , not simply Γ_p , so an additional uncertainty is introduced. With these considerations a reasonable estimate of the uncertainty in the Λ_{pp} 's would seem to be 20%.

With these estimates of the errors the results

have been plotted in Fig. 6. In this figure the shaded areas for each state represent the (d, p) reduced normalizations with their errors, while the points give the analog resonance Λ 's for each of the three theories mentioned above. The resonance results have been offset horizontally for the sake of clarity.

To obtain a quantitative measure of the success of each of the resonance theories we define a goodness of the fit parameter I , in analogy to χ^2 , in the following manner:

$$I = \sum (\Lambda_{dp} - \Lambda_{pp})^2 [(\Delta\Lambda)_{dp}^2 + (\Delta\Lambda)_{pp}^2]^{-1},$$

where $\Delta\Lambda_{dp} = 0.15\Lambda_{dp}$ and $\Delta\Lambda_{pp} = 0.20\Lambda_{pp}$ as discussed above. The values of I obtained by summing over levels of a given spin and parity as well as over all the levels are listed in Table III. (The results for the 1.43-MeV state in ^{93}Zr , which is now known to be a doublet have not been used in Table III.) We see that for all the states excited by the sub-Coulomb stripping, with the possible exception of the $d_{5/2}$ states, the R -matrix method of calculating the analog resonance reduced normalizations (Ref. 15) gives results that are in better agreement with the (d, p) results than the other methods studied. Adapting the usual criteria that a significantly worse fit is obtained when χ^2 is increased by 50% we see that the superiority of the R -matrix method is statistically significant in nearly all the individual cases, and certainly for the aggregate of states studied.

The three resonance theories used in our analysis have been extensively reviewed by Harney and Weidenmüller,²⁵ and their conclusions can be summarized as follows:

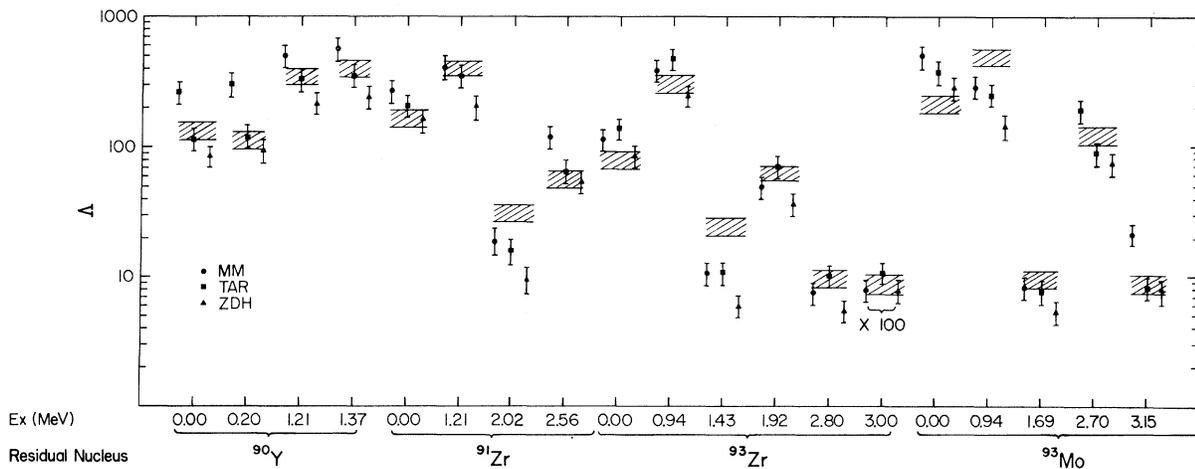


FIG. 6. Comparison of the (d, p) and analog resonance reduced normalizations. The shaded area represents the (d, p) results with a 15% error. Note that the 1.43-MeV state of ^{93}Zr is now known to be a doublet which was not resolved in this work.

TABLE III. Goodness of fit parameter (I) for the various angular momentum states.

States summed l_j	No.	Method		
		MM	TAR	ZDH
$d_{5/2}$	6	26.5	9.9	10.0
$d_{3/2}$	3	13.9	7.6	24.3
$s_{1/2}$	8	18.4	12.8	38.7
$g_{7/2}$	1	0.2	0.6	0.2
Total	18	59.0	30.9	73.2
Total/State ^a		3.3	1.7	4.1

^a Due to the definition of I a total per state of 1.0 implies that on the average the Λ 's agree to within 1.4 standard deviations.

(1) For cases where the absorptive part of the optical potential can be ignored ($W=0$) the methods of MM and ZDH are equivalent. Both methods differ, however, from the method of TAR. Harney and Weidenmüller define a "radius of equivalence" whose existence is a necessary and sufficient condition for the method of TAR to be equivalent to that of ZDH and MM, and show that for all realistic potentials such a radius does not exist.

(2) For the more realistic case where one cannot ignore the imaginary potential all three methods differ from each other. The methods of TAR and ZDH differ because they already differ at $W=0$, the imaginary potential being handled in an essentially equivalent manner in each theory. The methods of ZDH and MM differ in their fundamental assumptions. Both ZDH and MM treat explicitly only part of the full problem of the construction of the analog state. Statistical assumptions are then introduced. Since the construction of the analog state proceeds differently in each theory these assumptions have different effects on the results.

More importantly Harney and Weidenmüller show that the assumptions under which the TAR method is derived, namely no internal mixing and no external polarizing potential, seem to be violated when the method is applied to experimental data. In view of this it seems surprising that this theory

gives significantly better agreement with the (d, p) data than the other two theories. Preliminary results²⁶ indicate that this theory also gives better agreement with the (d, p) data for the isobaric-analog resonances near the $N=82$ closed shell.

A possible explanation of this anomaly may be the following. Both the MM and ZDH methods are based upon statistical assumptions which are valid only to first order in the imaginary potential W . Harney and Weidenmüller show that both theories however yield important second-order effects even for very small values of W . The TAR method avoids this problem by equating the background R matrix to an optical potential thus eliminating the need for statistical assumptions concerning the background. The experimental results therefore seem to imply that difficulties introduced by these statistical assumptions are more serious when fitting experimental data than those introduced by the violation of the assumptions of the R matrix. This conclusion, in slightly different form, was also reached by TAR in their original paper¹⁵ where they note that the advantage in choosing the matching radius small enough that the internal region displays charge independence outweighs the problem introduced by the fact that the nuclear potential then extends beyond the matching radius and the channel wave functions are no longer orthogonal.

V. CONCLUSION

Reduced normalizations have been calculated from sub-Coulomb (d, p) data for parent states in several nuclei near $A=90$. These have been compared to reduced normalizations calculated for the analogs of these parent states using three analog resonance theories. Despite doubts concerning its theoretical basis the R -matrix method gives significantly better agreement with the (d, p) data.

The authors are indebted to Professor R. G. Clarkson for many helpful discussions and for providing the code BETTINA and to Dr. G. A. Norton for aid in collecting some of the data.

*Work supported in part by the National Science Foundation.

†Present address: University of Virginia, Charlottesville, Virginia 22901.

¹J. Rapaport and A. K. Kerman, Nucl. Phys. **A119**, 641 (1968).

²R. G. Clarkson, P. von Brentano, and H. L. Harney, Nucl. Phys. **A161**, 49 (1971).

³J. J. Kent, J. F. Morgan, and R. G. Seyler, Nucl. Phys. **A197**, 177 (1972).

⁴E. B. Dally, J. B. Nelson, and W. R. Smith, Phys.

Rev. **152**, 1072 (1966).

⁵T. Yamaya, Y. Nakagome, Y. Hiratate, S. M. Lee, S. Iwasaki, T. Tohei, and S. Morita, Nucl. Phys. **A126**, 449 (1969).

⁶N. Baron, C. L. Fink, P. R. Christensen, J. Nickles, and T. Torsteinsen, Bull. Am. Phys. Soc. **17**, 50 (1972).

⁷H. Fann and U. Strohmusch, Phys. Rev. C **6**, 2299 (1972).

⁸J. L. Ellis, Ph. D. thesis, University of Wisconsin, 1970 (unpublished).

⁹J. B. Ball, Phys. Lett. **41B**, 55 (1972).

- ¹⁰J. B. Moorhead and R. A. Moyer, *Phys. Rev.* **184**, 1205 (1969).
- ¹¹N. Mishra, C. F. Moore, and C. E. Watson, *Nucl. Phys.* **A110**, 353 (1968).
- ¹²J. E. Spencer and A. K. Kerman, *Phys. Lett.* **38B**, 289 (1972).
- ¹³J. E. Spencer, M. Hasinoff, B. Small, and S. S. Hanna, Stanford University Progress Report, 1971 (unpublished); and J. E. Spencer, private communication.
- ¹⁴B. L. Cohen and O. V. Chubinsky, *Phys. Rev.* **131**, 2184 (1963).
- ¹⁵W. J. Thompson, J. L. Adams, and D. Robson, *Phys. Rev.* **173**, 975 (1968).
- ¹⁶H. L. Scott, C. P. Swann, and F. Rauch, *Nucl. Phys.* **A134**, 667 (1969).
- ¹⁷K. P. Lieb, J. J. Kent, and C. F. Moore, *Phys. Rev.* **175**, 1482 (1968).
- ¹⁸J. L. Ellis and W. Haeberli, in *Nuclear Isospin*, edited by J. D. Anderson, S. D. Bloom, J. Cerny, and W. W. True (Academic, New York, 1969), p. 651.
- ¹⁹J. J. Kent, K. P. Lieb, and C. F. Moore, *Phys. Rev. C* **1**, 336 (1970).
- ²⁰C. F. Moore, P. Richard, C. E. Watson, D. Robson, and J. D. Fox, *Phys. Rev.* **141**, 1166 (1966).
- ²¹A. Mekjian and W. M. McDonald, *Nucl. Phys.* **A121**, 385 (1968).
- ²²S. A. A. Zaidi and S. Darmodjo, *Phys. Rev. Lett.* **19**, 446 (1967).
- ²³H. L. Harney, *Nucl. Phys.* **A114**, 591 (1968).
- ²⁴R. G. Clarkson and H. L. Harney, Univ. of Oregon Report No. RLO-1925-48, 1971 (unpublished).
- ²⁵H. L. Harney and H. Weidenmüller, *Nucl. Phys.* **A139**, 241 (1969).
- ²⁶G. A. Norton, H. J. Hausman, J. J. Kent, J. Morgan, and R. G. Seyler, *Phys. Rev. Lett.* **31**, 769 (1973); and G. A. Norton, H. J. Hausman, and J. F. Morgan, to be published.