PHYSICAL REVIEW C VOLUME 8, NUMBER 6 DECEMBER 1973

Pauli-Principle Effects in Pion-Nucleus Scattering*

Rubin H. Landau and Malcolm McMillan
Department of Physics, University of British Columbia, Vancouver 8, Canada
(Received 14 June 1973)

A quantitative estimate is made of the changes in pion-nucleus scattering caused by incorporating the Pauli principle in a realistic optical potential. A modification is hypothesized in which the theoretical momentum-space potential is simply multiplied by a Pauli suppression factor. Several forms for this factor are derived, combined with the optical potential, and the Schrödinger equation including these potentials is solved in momentum space. It is found that including the Pauli principle in a realistic optical potential can have significant albeit not drastic effects on calculated low- and intermediate-energy elastic scattering. The effects on differential cross sections are not readily noticeable, except at large angles, whereas the approximately 15% changes in total cross sections can significantly improve the agreement with data.

I. INTRODUCTION

Experimental data on the scattering of pions from light nuclei is reproduced well by an optical potential constructed completely from first principles by means of multiple-scattering theory.[1-4] In fact, the properly constructed potential, even in lowest order, is so successful at intermediate energies that one hesitates to attempt significant refinements. Nevertheless, as more precise data become available, especially at lower energies, the first-order optical potential will need modification.[1,2]

In this paper we estimate quantitatively the changes in predicted pion-nucleus scattering brought about by incorporating the Pauli principle in the optical potential. By no means do we claim our work represents the last word on this subject; rather we feel that it indicates the size and nature of the expected changes and the desirability of more refined calculations.

Some consequences of the Pauli principle in pion-nucleus scattering are included in recent investigations, notably those of Dover and Lemmer,[5] Schmit,[6] and Bethe.[7] These investigations describe the πN interaction by a Chew-Low theory modified to include the Pauli principle *during* the scattering process within infinite nuclear matter. In contrast, we neither assume a microscopic form for the πN interaction in terms of a field theory nor include restrictions during the πN scattering process. Instead we take the off-energy-shell collision matrix (the blob in Fig. 1) as our elementary two-body input and without further dissection of this amplitude, include the Pauli principle restriction on the *final* nucleon momentum. This effectively modifies the optical potential (a more complete theory includes restrictions on all states). Since the elementary πN scattering process is ultimately iterated to obtain pion-nucleus scattering, our restriction on the final nucleon state is actually a restriction on intermediate multiple scatterings.

Furthermore, the optical potential and nuclear model used in our investigation are very different and possibly more realistic than those used in Refs. 5–7; we do not always treat the nucleus as an infinite slab of nuclear matter, nor describe the wave nature of the propagation by means of a local index of refraction, nor restrict the πN collision to either forward scattering or just one partial wave. Instead, we have solved an actual relativistic wave equation for pion-nucleus scattering with a modified form of the momentum-space optical potential derived from multiple-scattering theory.[1,3] In this way we account for the surface and the nonlocal nature of the interaction and obtain a clear estimate of the importance of the Pauli principle in a model which reproduces actual pion-nucleus scattering.

In the next section we indicate how the Pauli principle enters into the theoretical optical potential and then describe our hypothesized modification of the lowest-order potential to include the Pauli principle, Eq. (4). In Sec. III we derive several forms for a Pauli suppression factor Q_P to modify the optical potential. By using a Fermi-gas model to describe the nucleus we obtain simple analytic forms for Q_P which can be combined easily with the optical potential; these are given by Eqs. (10) and (13)–(17). In Sec. IV we combine Q_P with the optical potential, solve the Schrödinger equation in momentum space, and hence estimate the influence of the Pauli principle on pion-nucleus scattering.

II. MODIFIED OPTICAL POTENTIAL

We begin by describing the present form of the optical potential. The first-order potential in momentum space is[1-3]

$$\langle \vec{k}'|U|\vec{k}\rangle = A\int (\vec{k}', \vec{p}-\vec{q}|t^{\pi N}|\vec{k}, \vec{p})F(\vec{p}-\vec{q}, \vec{p})d^3p\,, \tag{1}$$

where for simplicity we consider only spin-zero nuclei with equal proton and neutron distributions. The nuclear overlap F can be expressed in terms of the momentum-space nuclear ground-state wave function $\psi(\vec{p}, \vec{p}_2 \ldots \vec{p}_A)$

$$F(\vec{p}-\vec{q}, \vec{p}) = \int \psi^*(\vec{p}-\vec{q}, \vec{p}_2, \ldots \vec{p}_A)\psi(\vec{p}, \vec{p}_2, \ldots \vec{p}_A) \times \delta\left(\vec{p} + \sum_i \vec{p}_i\right) d^3p_2 \cdots d^3p_A\,. \tag{2}$$

A is the nucleon number, $(|t|)$ is an average πN collision matrix for the scattering of a pion with momentum $\vec{k}$ and a free nucleon with momentum $\vec{p}$ into $\vec{k}'$ and $\vec{p}-\vec{q}$, respectively, (Fig. 1) and $\vec{q}=\vec{k}'-\vec{k}$ is the three-momentum transfer. Since energy is not conserved in this collision, the t matrix is "off the energy shell."

Most calculations using the optical potential assume in addition that $(|t|)$ can be taken outside of the integral in (1) and evaluated at some average value of $\vec{p}$. In this "factored approximation" the optical potential has the simple form

$$\langle \vec{k}'|U|\vec{k}\rangle = A(\vec{k}'|t^{\pi N}|\vec{k})\rho(q)\,, \tag{3}$$

where $\rho(q)$ is the average nuclear form factor for neutrons and protons. The familiar Kisslinger[8] and Laplacian[4] forms of the pion-nucleus optical potential are obtained from (3) by assuming specific behavior for $t^{\pi N}$ off the energy shell and then Fourier transforming to coordinate space.

The Pauli principle enters the optical potential in several places. First of all, it influences the optical potential through the antisymmetric nature of the wave functions in (2). Most of this influence, however, is removed by the integration over the $A-1$ nucleon momenta, especially for large A. Furthermore, when the factored approximation is made, any Pauli principle effects contained in the nuclear wave function ψ are effectively lost since they are not explicitly in the nuclear form factor $\rho(q)$. Yet, independent of the wave-function symmetry, the factored approximation should be valid if $k \gg p$ and the nucleus is large since it is based only on the observation that compared to $(\vec{k}', \vec{p}-\vec{q}|t|\vec{k}, \vec{p})$ the nuclear overlap $F(\vec{p}-\vec{q}, \vec{p})$ is a sharply peaked function of the momentum $\vec{p}$. (The rapid fall off of F is a consequence of the nuclear size being large compared to the range of the πN interaction.)

Secondly, the Pauli principle should enter the potential via the collision matrix. In writing (1) in terms of $t^{\pi N}$ we have implicitly made the "impulse approximation" by assuming that the elementary scattering occurs too quickly for the target nucleons to have dynamical influence on the πN collision matrix. The scattered nucleon, however, is both initially and finally bound within the nucleus and hence must come from and be placed back into an antisymmetric state with respect to all other nucleons. This effect of the Pauli principle is not explicitly accounted for when the free $t^{\pi N}$ is used in (1).[9]

In addition, the second-order potential, the correction to U second-order in $t^{\pi N}$, explicitly involves the two-nucleon correlation function, which in turn primarily arises from the antisymmetric nature of the nuclear wave function—the Pauli principle. There are many calculations of this term,[1,10] primarily for nucleon scattering and predominately using a Fermi-gas model to calculate the correlation function within the nucleus. The second-order term in the pion optical potential should be included when making a detailed comparison of theory and experiment.

We now hypothesize that an estimate of the significance of the Pauli principle in pion-nucleus scattering is obtained by a simple modification of the factored potential (3), a modification correcting primarily the impulse approximation, although in some sense also the factored approximation. We hypothesize the modified factored approximation

$$\langle \vec{k}'|U|\vec{k}\rangle = A(\vec{k}'|t|\vec{k})Q_P(\vec{k}', \vec{k})\rho(q)\,. \tag{4}$$

Thus the collision matrix determines the extent to which the πN collision can transfer momentum $\vec{q}$ to a nucleon at a particular energy, the "Pauli suppression factor" Q_P measures the average probability that a nucleon within the nucleus can absorb a momentum $\vec{k}-\vec{k}'$ and recoil into an *unoccupied* state, and the form factor $\rho(q)$ measures

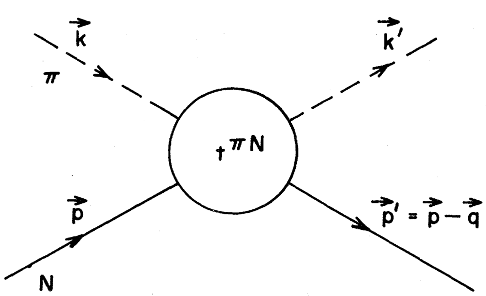

FIG. 1. The elementary scattering of a pion from a bound nucleon as described by an off-energy-shell collision matrix $t^{\pi N}$.

the ability of the nucleus to absorb momentum $\vec{q}$ and remain in the ground state. We note that since the dependence on nucleon momenta is integrated out of the optical potential (4), Q_P represents some average over all nucleons within the nucleus.

As is well known,[11] an expression of the form (4) using a particular form of Q_P has been rather successful when used in the nucleon-nucleus optical potential. In the next section we derive several forms for this Pauli suppression factor which are applicable to pion-nucleus scattering.

III. FORMS FOR THE PAULI SUPPRESSION FACTOR

To calculate Q_P we describe the neutrons and protons within the nucleus as two noninteracting zero-temperature Fermi gases. Although we use a realistic model[3] to calculate the form factor, we use the Fermi-gas model to calculate Q_P since it yields analytic forms which permit a convenient estimate of the Pauli principle effect on pion-nucleus scattering. In the numerical estimates which follow we choose a numerical value for the Fermi momentum which crudely accounts for the surface nature of pion-nucleus scattering.[12] We now derive three possible forms for the suppression factor which permit us to determine the model dependence of our estimate.

A. Goldberger-Clementel-Villi Form

We first derive a Pauli suppression factor for pion-nucleus scattering analogous to the one derived by Goldberger[13] and by Clementel and Villi[14] for nucleon-nucleus scattering. We consider the scattering process shown in Fig. 1 as occurring within a Fermi gas of nucleons. The effective cross section for this process in the lab system is[13-15]

$$\bar{\sigma}(k) = \int d^3p N(\vec{p}) \frac{|\vec{k} - \mu\vec{p}/m|}{k} \frac{d\sigma}{d\Omega_{\text{c.m.}}}(\kappa, \hat{\kappa}'\cdot\hat{\kappa}) d\Omega_{\text{c.m.}}, \tag{5}$$

where $d\sigma/d\Omega_{\text{c.m.}}$ is the differential cross section in the center-of-mass system. The first factor, $d^3pN(\vec{p})$, is the probability of finding a nucleon with momentum $\vec{p}$ within the nucleus, where

$$N(\vec{p}) = (4\pi k_f^3/3)^{-1}\theta(k_f - p). \tag{6}$$

(θ is the unit step function.) The next (flux) factor is the relative πN velocity for a moving nucleon, divided by that for a stationary nucleon. $\vec{\kappa}$ and $\vec{\kappa}'$ are the initial and final πN center-of-mass momenta, e.g.,

$$\vec{\kappa} = (m\vec{k} - \mu\vec{p})/(m+\mu), \tag{7}$$

where m and μ are the nucleon and pion masses, respectively.

There are basically three restrictions on the integration region in (5). First, the initial nucleon momentum must lie within the Fermi sphere, i.e., $p \le k_f$; second, the final nucleon momentum must lie outside the Fermi sphere, i.e., $|\vec{p} - \vec{q}| \ge k_f$; finally, energy and momentum must be conserved:

$$\vec{k} + \vec{p} = \vec{k}' + \vec{p}', \tag{8}$$

$$k^2/2\mu + p^2/2m = k'^2/2\mu + p'^2/2m. \tag{9}$$

Note that for nucleon-nucleon scattering both $\vec{k}$ and $\vec{k}'$ must also lie outside the Fermi sphere.

The evaluation of this integral subject to these three restrictions is particularly tedious for the unequal mass case ($\mu \ne m$); we give details in an Appendix. For $k > k_f$ the final result is very simple:

$$\begin{aligned} \bar{Q}_P(k) &\equiv \bar{\sigma}/\sigma \\ &= 1 - (1 + 2\mu/m)k_f^2/5k^2 \\ &= 1 - 0.26k_f^2/k^2, \end{aligned} \tag{10}$$

where σ is the average value of $d\sigma/d\Omega_{\text{c.m.}}$ and $\bar{Q}_P$ is our notation for this form of the Pauli factor. For $k < k_f$, the result is more complicated; we show this "historical" Pauli factor over the complete range of k in Fig. 2. We notice first, that as the pion momentum is lowered this factor monotonically suppresses a greater fraction of the pion-nucleon scatterings, and second, that as a consequence of averaging, the factor is independent of the angle between $\vec{k}$ and $\vec{k}'$.

If we compare the suppression factor (10) for pion-nucleus scattering with the suppression factor for nucleon-nucleus scattering,[13,14]

$$\begin{aligned} \bar{Q}_P(k) &= 1 - 7k_f^2/5k^2 \quad (k^2 > 2k_f^2) \\ &= 1 - 1.4k_f^2/k^2, \end{aligned} \tag{11}$$

we see that there is considerably more suppression for nucleons than for pions. This is caused by the smaller mass of the pion ($\mu/m \simeq \frac{1}{7}$) and by the Pauli principle restricting only one particle (the nucleon) in pion-nucleon scattering but both in nucleon-nucleon scattering.

B. q-Dependent Form

The Pauli factor of the last section was derived with the restriction that the collisions conserve energy. Since the off-diagonal ($k' \ne k$) part of the optical potential (4) explicitly describes a collision in which energy is not conserved we wish now to find a Pauli factor applicable to the *off*-diagonal part of the optical potential. Again we consider the πN collision pictured in Fig. 1. We now take Q_P as the fraction, for fixed momentum transfer

$-\vec{q}$, of all initial states for which the final state is outside the Fermi sea:

$$Q_P(q) = \int d^3p N(\vec{p})\theta[(\vec{p}-\vec{q})^2 - k_f^2], \qquad (12)$$

where $N(p)$ is given by (6).

Note, this Pauli factor sets no requirement that energy be conserved and includes no average over scattering angle. The integration region dictated by the θ function can be found geometrically by using the familiar construction of two intersecting spheres,[7, 13] or the integral can be done using the Legendre series expansion of the θ function.[16] The result is

$$Q_P(q) = \begin{cases} \frac{3}{4}(q/k_f - q^3/12k_f^3) & 0 < q \leq 2k_f \\ 1 & q > 2k_f \end{cases}, \qquad (13)$$

where

$$q = (k^2 + k'^2 - 2kk'\cos\theta_{kk'})^{1/2}. \qquad (14)$$

We note that small momentum-transfer collisions are strongly suppressed; finite momentum must be transferred to the average nucleon to remove it from the Fermi sphere.

We thus hypothesize a second ("mixed") form for the Pauli factor; use the Goldberger-Clementel Villi form $\bar{Q}_P$ [(10)] for the diagonal terms in the optical potential (4) and the q-dependent form $Q_P(q)$ [(13)] for the off-diagonal terms:

$$Q_P(\vec{k}', \vec{k}) = \begin{cases} \bar{Q}_P(k) & k' = k \\ Q_P(q) & k' \neq k. \end{cases} \qquad (15)$$

C. Angle-Averaged Form

We note here that it is unacceptable to use the q-dependent factor $Q_P(q)$ for both the energy-conserving and -nonconserving T matrices in the optical potential (4), since this factor greatly suppresses small-angle πN scattering and thus causes a forward-angle dip in pion-nucleus scattering. This unrealistic behavior is caused by the unphysical infinite-nuclear-matter model used to calculate $Q_P(q)$.

Since we aim only to estimate the Pauli principle suppression of a realistic optical potential, we will be somewhat phenomenological in removing the strong, unphysical forward suppression in $Q_P(q)$. Accordingly, we obtain another form for the Pauli factor to be used both when k' equals and does not equal k by taking the angle average of $Q_P(q)$:

$$\langle Q_P \rangle = \frac{1}{4\pi}\int d\Omega Q_P(q)$$
$$= (k_f^2/8kk')\left[\frac{k_<(k+k')^2 - |k-k'|^3}{k_f^3} - \frac{k_<^5 - |k-k'|^5}{20k_f^5}\right], \qquad (16)$$

where $k_<$ is the lesser of $k+k'$ or $2k_f$. We call (16) the "angle-averaged" Pauli factor.

The Goldberger-Clementel-Villi form (10) can be related to the other forms we have discussed by simply evaluating the angle-averaged Pauli factor $\langle Q_P \rangle$ when energy is conserved. Setting $k' = k$ in (16) we obtain

$$\langle Q_P \rangle |_{k'=k} = (1 - k_f^2/5k^2), \qquad (17)$$

when k is greater than k_f. Aside from the relatively small $2\mu/m$, arising from the velocity ratios in (5), this is the historical Pauli factor (10).

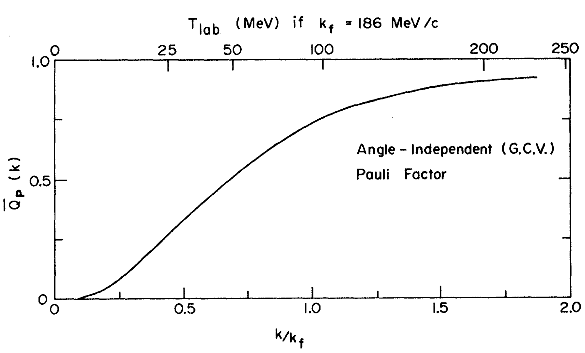

FIG. 2. The Goldberger-Clementel-Villi (G.C.V.) form for the Pauli suppression factor $\bar{Q}_P(k)$ as a function of the pion laboratory momentum (k_f is the Fermi momentum). In the simplest case the Pauli principle is included in pion-nucleus scattering by multiplying the optical potential with this factor.

IV. MODIFIED POTENTIAL AND RESULTS

We have solved the Lippman-Schwinger equation

$$\langle \vec{k}' | T | \vec{k}_0 \rangle = \langle \vec{k}' | U | \vec{k}_0 \rangle + \int \frac{d^3k \langle \vec{k}' | U | \vec{k} \rangle \langle \vec{k} | T | \vec{k}_0 \rangle (A-1)/A}{E_0 - (\mu^2 + k^2)^{1/2} - (A^2m^2 + k^2)^{1/2} + i\epsilon} \qquad (18)$$

for the pion-nucleus collision matrix T with potentials of the general form (4). Details of our technique are described elsewhere.[3, 4] By working in momentum space we are able to examine various possible behaviors for Q_P without performing difficult and laborious Fourier transformations of the optical potential to coordinate space. In order to include the 3-3 resonance directly into our potential and to generate a reasonable off-energy-shell behavior of the πN transition matrix, a separable model is used to construct the πN transition matrix.[17] The optical potentials we use have no adjustable parameters (the inputs are the measured πN phase shifts and nuclear densities) and in all cases we include a proper transformation of the πN transition matrix from the pion-nucleon to pion-nucleus c.m. frames.

The only new technical detail needed to solve (18) with a partial expansion of T is the Legendre series expansion of $Q_P(q)$ which must be combined with those of $(|t|)$ and $\rho(q)$ to obtain the partial wave decomposition of the potential. We do this by rewriting (13) as

$$Q_P(q) = [\tfrac{3}{4}(q/k_f - q^3/12k_f^3) - 1]\theta(4k_f^2 - q^2) + 1 \quad (19)$$

and then combining the Legendre expansion of the θ function[16] with those of q and q^3. 10 to 20 terms in the expansion usually represent Q_P accurately except at a discontinuity; the calculated pion-nucleus cross sections are stable if 10 or more terms are included.

In Figs. 3–5 we show the calculated differential and total cross sections for the pure nuclear scattering of pions from carbon. The solid curves are calculated with an optical potential having no Pauli suppression built in, the dashed curves are calculated with the energy-conserving angle-independent Goldberger-Clementel-Villi form (10), and the dot-dashed curves are calculated with the off-shell q-dependent Pauli factor (15).

The predicted *differential* cross sections are not changed grossly by these Pauli factors particularly at smaller angles where the simple model, (3), is most valid. In contrast, the calculated peaks in the total cross sections, Figs. 4 and 5, are generally decreased in magnitude and shifted slightly upwards in energy; they now agree better with the data than those calculated without a Pauli factor.[18]

The angle-averaged off-energy-shell Pauli factor

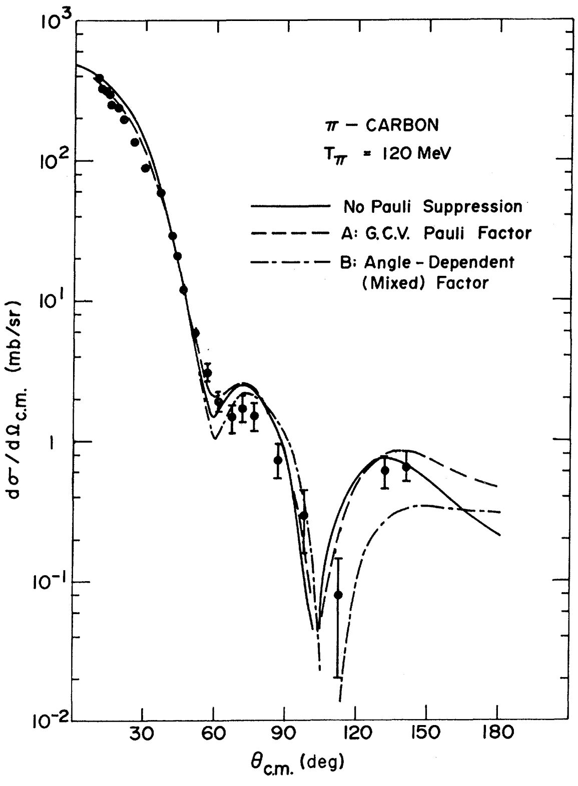

FIG. 3. The calculated pion-carbon differential cross section for pure nuclear scattering based on a separable potential model for the πN interaction. The solid curve has no Pauli suppression included, the dashed curve has Pauli suppression included via the energy-conserving angle-independent G.C.V. form, and the dot-dashed curve has been calculated with an off-shell q-dependent Pauli factor. The pion kinetic energy is 120 MeV and the data are from F. Binon *et al.* Nucl. Phys. B17, 168 (1970).

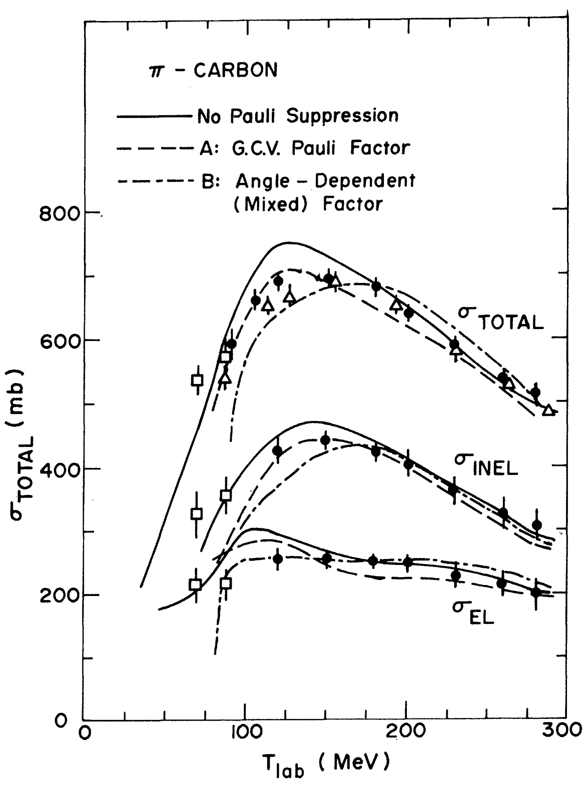

FIG. 4. Calculated pion-carbon total cross sections with the separable πN-based optical potential. The labeling of the different curves are the same as in Fig. 3. The Δ's are the charge-averaged data from A. S. Clough *et al.* [Phys. Lett. 43B, 476 (1973)], ●'s are from F. Binon *et al.* [Nucl. Phys. B17, 168 (1970)], and the □'s are from M. Crozon *et al.* [Nucl. Phys. 64, 567 (1965)].

(16) affects the cross sections in a manner essentially identical to the on-energy-shell Goldberger-Clementel-Villi form. This indicates that pion-nucleus scattering is most sensitive to the diagonal ($k=k'$) terms of the potential and not particularly sensitive to the precise value of the Pauli factor. Actually, it may seem somewhat surprising that the calculated pion-nucleus scattering is not affected *more* by these Pauli factors since at 120 MeV the Pauli factor reduces the potential by 20% (Fig. 2) but reduces the *total* cross section by only 7% (Figs. 4 and 5). This indeed is a reflection of the relative insensitivity of elastic pion-nucleus scattering to the precise depth of the potential—an observation made possible only by using a realistic model for the nuclear scattering.

The total cross sections calculated with the mixed q-dependent Pauli factor (15) have their peak shifted (back) up in energy by too great an amount and their shapes considerably modified. The origin of this shift is simply a suppression of multiple scattering [the integral in Eq. (18)]. The multiple-scattering contribution to the total cross section usually changes from constructive to destructive interference as the energy passes upwards through resonance.[2] Suppressing multiple scattering thereby shifts the peaks upwards. We feel the two curves A and B indicate the model dependence of our estimate.

A method of improving the behavior of the Fermi-gas-modified πN collision matrix is to impose the requirement of unitarity on it. Bethe,[7] in fact, emphasizes that this should be an important part of a pion-nucleus optical model. The unmodified optical potential we use without the Pauli factor is based on a potential model for the πN interaction and so naturally has unitary πN collision matrices. For the angle-independent Pauli factors the off-shell unitarity conditions of Watson and Nuttall[19] can be employed to make our modified amplitudes unitary. The effect is slight. If the Pauli principle were included consistently throughout the entire theory, these unitarization procedures would probably be unnecessary.

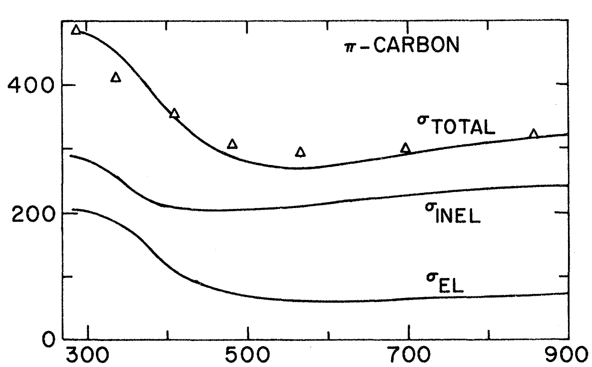

FIG. 5. Calculated pion-carbon total cross sections with the separable πN-based optical potential. In this higher-energy region the Pauli suppression has an insignificant effect on the calculated cross sections. The labeling is the same as Figs. 3 and 4.

V. CONCLUSIONS

We have found that including the Pauli principle in a realistic pion-nucleus optical potential can have significant albeit not drastic effects on calculated low- and intermediate-energy elastic scattering. Specifically, the effects on differential cross section are not readily noticeable although the ~15% changes in total cross sections can significantly improve the agreement with data. Even though the Pauli principle strongly suppresses the elementary πN amplitude within nuclear matter, its effect on the pion-nucleus amplitude is much less because of the insensitivity of elastic scattering to the precise depth of the optical potential.

In order to reach these conclusions it is necessary to use a realistic model for nuclear scattering, such as the optical potential in momentum space. Similar results can also be obtained in coordinate space with the Pauli factor given by Eq. (17). The use of an unmodified Fermi-gas model to include the effects of the Pauli principle in the elementary πN collision matrix is not very acceptable and for this reason we have removed some of the unphysical properties arising from the Fermi-gas model when performing our calculations.

Our calculation is not meant to be final, but rather just an estimate of the size and nature of the expected changes and an indication of the desirability of more complete calculations. In this regard we feel there are several corrections to the lowest-order optical potential, probably of similar magnitude to the one we have calculated, which should be examined if detailed predictions of the theory at lower energies are required. Specifically, the validity of the impulse and factored approximations and the modifications introduced by the second-order potential might well be examined, in addition to the contribution to pion-nucleus scattering arising from true pion absorption.

ACKNOWLEDGMENTS

It is a pleasure to thank many of our colleagues for their advice and helpful comments. In particular we wish to acknowledge stimulating and illuminating conversations with Dr. F. Tabakin, Dr. L. Scherk, Dr. J. Grabowski, Dr. D. Beder, and Dr. H. Bethe. We also wish to thank Dr. W. J. McDonald for a tabulation of his data prior to publication.

APPENDIX

We wish to evaluate the integral in Eq. (5) subject to the constraints $p < k_f$, $p' > k_f$, and energy-momentum conservation, (8) and (9). To insure energy conservation and simplify the geometrical considerations, we replace the differential element of solid angle $d\Omega_{\text{c.m.}}$ by a three-dimensional volume element in momentum space

$$d\Omega_{\text{c.m.}} \frac{\delta(\kappa - \kappa')}{\kappa^2} d^3\kappa . \tag{A1}$$

We now follow Refs. 13 and 14 and neglect the energy and angular dependence of the cross section, i.e., remove $d\sigma/d\Omega$ from under the integral in (5), and replace it by an angle-independent $\sigma/4\pi$:

$$\begin{aligned} \overline{Q_P} &= \bar{\sigma}/\sigma \\ &= (3/16\pi^2 k_f^3) \int d^3p \, \frac{|\vec{k} - \mu\vec{p}/m|}{k} \\ &\qquad \times \theta(k_f - p) \int d^3\kappa' \, \frac{\delta(\kappa' - \kappa)}{\kappa^2} \, \theta(p' - k_f) . \end{aligned} \tag{A2}$$

After a convenient change of variables, the inner integral takes the form

$$\begin{aligned} \int d^3\kappa' \, \frac{\delta(\kappa' - \kappa)}{\kappa^2} \, \theta(p' - k_f) &= \frac{2\pi(\mu + m)^2}{2m^2 |\vec{k} - \mu\vec{p}/m| \, |\vec{p} + \vec{k}|} \\ &\qquad \times \int_{p'^2_{\min}}^{p'^2_{\max}} d(p'^2) \theta(p' - k_f) . \end{aligned} \tag{A3}$$

$p'_{\max}$ and $p'_{\min}$ are the maximum and minimum values of p', for a given $\vec{p}$ and $\vec{k}$, consistent with the constraints. $p'_{\min} = k_f$ and $p'_{\max}$ is the greater of k_f and P, where

$$P = (m/m + \mu)(|\vec{k} + \vec{p}| + |\vec{k} - \mu\vec{p}/m|) . \tag{A4}$$

For $p > k_f$, P is always greater than k_f so $p'_{\max} = P$. We obtained (A4) by first noting that for fixed energy, $(p^2/2m + k^2/2\mu)$, p' will be a maximum when k' is minimum. For k' to be a minimum, momentum conservation requires $\vec{p}'$ to be parallel to $\vec{p} + \vec{k}$. In this case

$$\begin{aligned} p^2/2m + k^2/2\mu &= p'^2/2m + (\vec{p} + \vec{k})^2/2\mu \\ &\qquad + p'^2/2\mu - 2p'|\vec{p} + \vec{k}|/2\mu . \end{aligned} \tag{A5}$$

(A4) is a solution of (A5).

With these values for $p'_{\min}$ and $p'_{\max}$ we obtain

$$\begin{aligned} \overline{Q_P} &= [3(\mu + m)^2/16\pi k_f^3 m^2] \\ &\qquad \times \int d^3p \, \frac{(p'^2_{\max} - k_f^2)}{k|\vec{k} + \vec{p}|} \, \theta(k_f - p) . \end{aligned} \tag{A6}$$

If $k > k_f$ this integral gives the very simple result

$$\overline{Q_P} = 1 - (1 + 2\mu/m) k_f^2/5k^2 . \tag{A7}$$

For $k < k_f$ the result is more complicated and we show it graphically in Fig. 2.

---

*Work supported in part by National Research Council of Canada.

[1] A. K. Kerman, H. McManus, and R. M. Thaler, Ann. Phys. (N.Y.) 8, 551 (1959).

[2] D. S. Koltun, Adv. Nucl. Phys. 3, 71 (1969).

[3] R. H. Landau, S. C. Phatak, and F. Tabakin, Ann. Phys. (N.Y.) 78, 299 (1973).

[4] M. M. Sternheim and E. H. Auerbach, Phys. Rev. Lett. 25, 1550 (1970); E. H. Auerbach, D. M. Fleming, and M. M. Sternheim, Phys. Rev. 162, 1683 (1967); 171, 1781 (1968); S. C. Phatak, F. Tabakin, and R. H. Landau, Phys. Rev. C 7, 1803 (1973).

[5] C. B. Dover and R. H. Lemmer, Phys. Rev. C 7, 2312 (1973).

[6] C. Schmit, Nucl. Phys. A197, 449 (1972).

[7] H. A. Bethe, Phys. Rev. Lett. 30, 105 (1973).

[8] L. S. Kisslinger, Phys. Rev. 98, 761 (1955).

[9] The impulse approximation can be corrected in the nucleon-nucleus optical potential by using a Brueckner G matrix in place of the collision matrix. See, e.g., C. Shakin and R. M. Thaler, Phys. Rev. C 7, 494 (1973).

[10] R. R. Johnston, Nucl. Phys. 36, 368 (1962); R. C. Johnson and D. C. Martin, Nucl. Phys. A192, 496 (1972); E. Kujawski, Phys. Rev. C 1, 1651 (1970); J. S. Chalmers, Phys. Rev. C 3, 968 (1971); J. S. Chalmers and A. Saperstein, Phys. Rev. 168, 1145 (1968).

[11] A. Bohr and B. R. Mottelson, *Nuclear Structure* (Benjamin, New York, 1969), p. 261.

[12] As displayed in Fig. 3A of Ref. 3, the pion-carbon interaction peaks at $r \simeq 2.5$ fm which determines a Fermi momentum $k_f = (3\pi^2\rho/2)^{1/3} \simeq 186$ MeV/c.

[13] M. L. Goldberger, Phys. Rev. 74, 1269 (1948); G. L. Shaw, Ann. Phys. (N.Y.) 8, 509 (1959); S. Hayakawa, M. Kawai, K. Kikuchi, Prog. Theoret. Phys. 13, 415 (1955).

[14] E. Clementel and C. Villi, Nuovo Cimento 2, 176 (1955).

[15] M. L. Goldberger and K. M. Watson, *Collision Theory* (Wiley, New York, 1964).

[16] E. L. Lomon and M. McMillan, Ann. Phys. (N.Y.) 23, 439 (1963).

[17] The popular Laplacian or local potential was also examined. It gave similar numerical results to the separable πN-based potential.

[18] The corresponding peak in the πN total cross section is ~40 MeV higher in energy than in pion-nucleus scattering. The unmodified optical potential predicts the nuclear peaks to be shifted down by ~ 50 MeV, slightly too much. As indicated in Ref. 5, the Pauli principle shifts these peaks back up somewhat.

[19] K. M. Watson and J. Nuttall, *Topics in Several Particle Dynamics* (Holden-Day, San Francisco, 1967), p. 17.