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Effect of Two-Body Correlations on Sum Rules for Nuclear Charge Monopole and Dipole Modes

A. M. Lane
Theoretical Physics Division, Atomic Energy Research Establishment, Harwell, Didcot, Berkshire, England
and
A. Z. Mekjian*
Physics Department, Rutgers University, New Brunswick, New Jersey 08903
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We have evaluated energy- and non-energy-weighted sum rules (N and M) for the nuclear charge monopole and dipole modes. Sums are given for different intermediate isospins, using the method in which sums are split into isoscalar, vector, and tensor parts. This facilitates the separation of model-dependent quantities, and we particularly consider possible modifications of shell-model evaluations arising from correlations. Four important specific results are: (i) The shell-model value of M in the dipole case is carefully calculated for ^{208}Pb, and is 30% larger than the observed value; this difference must arise from correlations. (ii) The previously suggested smallness of the isotensor part of M is not general, but depends on using a shell-model wave function; it is shown theoretically and from experiment that the isotensor part may be large. (iii) The effects of correlations on M can be displayed in a form requiring no new calculations (given published Tamm-Dancoff diagonalizations). (iv) There is a close relation between the charge-exchange part of N in the dipole and monopole problems, so one can use data on the former to evaluate the latter; also, a connection with the effective $M1$ charge and effective mass is shown.

I. INTRODUCTION

The present paper arose out of a study[1] of Coulomb mixing in nuclei. For such a project, an essential preliminary is the evaluation of the parameters of the nuclear vibrational mode of the charge monopole type. We have set up a theory for this based on sum rules. No data exists on this mode, but it is closely analogous to the charge dipole mode, for which data is available. Therefore we have also applied our theoretical methods to the dipole mode, and have checked the results against experiment, thereby validating the methods. The object of the present paper is to present several new results (mostly on the dipole mode) that have emerged.

In the dipole problem, sum rules have been used for many years (for example, see Refs. 2 and 3) and with great profit to analyze photonuclear data. The energy-weighted sum rule, determines the integrated absorption cross section $\sigma_0 \equiv \int \sigma\, dE$, and the non-energy-weighted sum rule determines the bremsstrahlung-weighted integrated cross section $\sigma_{-1} \equiv \int dE(\sigma/E)$. For economy of words we will refer to the former as the N sum and the latter as the M sum, these being the symbols we use for these quantities in the text. The kinetic-energy part of the former is model-independent; this part is often called the classical sum rule. The latter and potential-energy part of the former are model-dependent, and are normally evaluated theoretically with the shell model. Formerly, it was said[4] that there was reasonable agreement with experiment. In a recent presentation[5] of the data, however, the observed value of σ_{-1} is of the order of 50% of the traditional oscillator shell-model evaluation[4] of Levinger ($0.36A^{4/3}$ mb). This fact has largely passed without comment, perhaps because it was considered that other potential well shapes might strongly reduce the shell-model value (Levinger quoted[4] a reduction of 15% when the finite square well is used in place of the oscillator). In fact, we will argue that there is little uncertainty (much less than 15%) in the shell-model value, once the rms radii of the outer orbits is prescribed, and that there is a real discrepancy of order 30% between the data and the best shell-model value. Barring large changes in experimental values, this means that *two-body correlations reduce the shell-model value of M by this amount*. This is an important result because there are remarkably few examples in nuclear physics where one can pinpoint the effect of correlations in so explicit a way. It is true that there are several other areas where the shell model needs correction. For instance, strong $E2$ and $E3$ transitions to individual low-lying states often require collective two-body correlations for their explanation. However, these situations are not in the same category as the present one in the

sense that the $E2$, $E3$ data do not yield a value for a ground-state expectation value of a two-body operator to parallel that given by σ_{-1}, viz. $\langle 0|\sum_{i,j} t_{3i} t_{3j} z_i z_j |0\rangle$. (Indeed, the only other case of such a value is that of the total energy where the operator is the Hamiltonian.) In random-phase-approximation (RPA) treatments of dipole problems, it is implicit that the calculated value of σ_{-1} is substantially smaller than the shell-model value, but this important fact is usually not brought out.

Turning to another aspect of the dipole case, there has recently been interest[5-9] in the separate sum rules for the different isospins of the states excited by the dipole operator. The separation is very conveniently made by expressing the squared dipole operator as a sum of its isoscalar, vector, and tensor parts. This has the great merit of facilitating separation of the model-independent aspects (those involving one-body operators) from the model-dependent aspects (those involving two-body operators). It turns out that the model-independent quantities are the vector part of M, and the scalar and tensor parts of the kinetic contribution to N. All others are model-dependent and can, in principle, be affected by correlations. We have already mentioned that σ_{-1} is so affected. The major part of this is the scalar part of M, so this must be affected. There is special interest in the tensor part of M, which has been shown to be small compared to the scalar part when evaluated with the shell model. (If the dipole operator is referred to the well center rather than the centroid, the shell-model value becomes zero.) It is tempting to assume that the smallness of the shell-model value of the tensor term arises from the tensor nature rather than the use of the shell model, in which case the term is also small in the presence of correlations. Leonardi[8] has recently made this assumption and then inferred properties of the neutron and proton densities in nuclei from the photonuclear data. Unfortunately there are experimental and theoretical reasons to doubt the validity of the assumption, as we will show in our second main result: *The smallness of the tensor term in M is a shell-model property, and does not apply to correlated wave functions in general.*

Little explicit attention has been given to the effect of correlations on the sum rules. Levinger[4] quotes a detailed numerical calculation by Okamoto[10] which found a reduction of σ_{-1} by a few percent arising from the first-order effect of two-body forces. We have studied the question of the first-order effects and found the important result that these effects can be expressed in terms of known quantities, viz. the N sum rule and the giant resonance energy as obtained from the diagonalization of 1p-1h states. In other words, there is no need to embark on numerical work: We find that correlations can have an effect of the order of the 30% effect deduced from the data, i.e., much larger than Okamoto's value.

Finally, turning to the monopole problem, we find great assistance from the dipole results in making estimates of the monopole sum rules. Not only does the structure of the formal analysis apply, but one can arrive at conclusions about the reliability of shell-model values and the effect of correlations. In the case of the potential part of the N sum, the relation is much closer than mere analogy. It turns out, remarkably, that the vital radial integral on the two-body potential is the same in both problems. Thus its evaluation from the dipole data can be directly used to provide an evaluation of the N sum in the monopole problem.

In Sec. II we present formal results for the N and M sums without making any separation of different isospin contributions. Shell-model evaluations are given. In Sec. III, comparison with photonuclear dipole data for ^{208}Pb shows the inadequacy of the shell model for the M sum. Section IV gives details of the method developed for including correlation corrections to the shell-model value of the M sum. Section V extends the formal definitions of N and M to separate isospins and introduces corresponding isoscalar, vector, and tensor quantities. Section VI adapts the method for correlation corrections to the case of separated isospins. Finally, Sec. VII discusses the monopole problem, and Sec. VIII describes a connection with effective $M1$ charge and effective mass.

II. STRENGTH DISTRIBUTIONS AND SUM RULES

A. Formal Expressions

The general form of a one-body isovector operator causing excitations from a given state $|\phi_0\rangle = |\phi_0(T, T_3)\rangle$ is

$$F = \sum_i t_{3i} [f(\boldsymbol{r}_i) - \hat{f}] , \tag{1}$$

where $\hat{f}$ is a constant chosen so that $\langle \phi_0|F|\phi_0\rangle = 0$. The action of F on the state vector $|\phi_0\rangle$ creates other states of the system, $|\phi_\lambda\rangle$ say:

$$F|\phi_0\rangle = \sum_\lambda |\phi_\lambda\rangle\langle\phi_\lambda|F|\phi_0\rangle . \tag{2}$$

Two crucial quantities for characterizing the effect of F on $|\phi_0\rangle$ are the non-energy-weighted **sum-rule quantity M and the energy-weighted**

sum-rule quantity N defined by the equations

$$M \equiv \sum_\lambda \langle \phi_\lambda | F | \phi_0 \rangle^2 ,$$
$$N \equiv \sum_\lambda (E_\lambda - E_0) \langle \phi_\lambda | F | \phi_0 \rangle^2 , \tag{3}$$

where $H|\phi_\lambda\rangle = E_\lambda |\phi_\lambda\rangle$, $H|\phi_0\rangle = E_0|\phi_0\rangle$, H being the Hamiltonian. The sums here are unrestricted in the isospin indices. The isovector operator F admixes states with $T' = T-1$, T, $T+1$. In this and Secs. III, IV, we consider the unrestricted sum rules; the restricted ones are discussed in Secs. V, VI.

By using closure, the sums of Eq. (3) can be reduced to expectation values

$$M = \langle \phi_0 | F^2 | \phi_0 \rangle ,$$
$$N = \tfrac{1}{2} \langle \phi_0 | [F,[H,F]] | \phi_0 \rangle . \tag{4}$$

Since $H = K + U$, where K is the kinetic energy operator and U is the potential-energy operator, the energy-weighted sum rule is a linear sum of two parts, one from K and one from U:

$$N = N(K) + N(U) . \tag{5}$$

From now on, expectation values will be understood to be taken in the state $|\phi_0\rangle$. Inserting (1) in (4), one readily finds

$$N(K) = (\hbar^2/8m) \left\langle \sum_i | \vec{\nabla}_i f(r_i) |^2 \right\rangle \tag{6}$$

and

$$N(U) = \tfrac{1}{2} \left\langle \sum_{i,j} b_{ij} \right\rangle \tag{7}$$

with

$$b_{ij} = [f(r_i) - f(r_j)]^2 v(r_{ij}) (\vec{t}_i \cdot \vec{t}_j - t_{3i} t_{3j}) (M P^\sigma_{ij} + H) . \tag{8}$$

In obtaining Eq. (7) we have taken $U = \frac{1}{2} \sum_{i \neq j} v_{ij}$, with

$$v_{ij} = (W - M P^\sigma_{ij} P^\tau_{ij} + B P^\sigma_{ij} - H P^\tau_{ij}) v(r_{ij}) . \tag{9}$$

The above results apply to a "parent" state, i.e., one with $T_3 = T$. More general expressions valid for $T \neq T_3$ will be given in Sec. V. Finally, we define a quantity β to be the ratio of the potential-energy contribution to N to the kinetic-energy contribution:

$$\beta \equiv N(U)/N(K) . \tag{10}$$

B. Specialization to the Dipole Case

The correct $E1$ operator is $\mathfrak{D} = \sum_i (\frac{1}{2} - t_{3i}) z_i'' = -\sum t_{3i} z_i''$, where z_i'' is taken relative to the centroid and therefore satisfies $\sum_i z_i'' = 0$. In terms of z_i referred to a fixed point, this becomes $\mathfrak{D} = D + D_0$ where $D = -\sum_i t_{3i} z_i$, $D_0 = (1/A)(\sum_i t_{3i})(\sum_j z_j)$. When used with internal wave functions, the part D_0 has no effect and can be dropped. In practice, one rarely uses such functions, so the D_0 term should be retained. However, in order to keep the discussion focused on essentials and to stress the parallel with the monopole case, we first present results for D and later we note the small effects of changing to $\mathfrak{D}$.

Since most of the present work is concerned with the dipole case, we will henceforth use the notation of Sec. II A for this case, with the understanding that the general operator F is now the specific operator D; $f(r) = -z$, $\hat{f} = 0$.

1. Evaluation with the Oscillator Shell Model

If the oscillator shell model is used, the quantity $\langle D^2 \rangle$ is for most parent states $\phi_0(T, T)$ simply related to the model-independent counterpart $N(K)$ of (6). Such states are those for which all components of $D\phi_0(T, T)$ have the same excitation energy, viz. $\hbar\omega$, the oscillator quantum. The action of D on a general oscillator state ϕ_0 excites components of excitation energies $\pm\hbar\omega$. However, for nearly all low-lying states ϕ_0, the Pauli principle prohibits any components with negative excitation. From the definitions of M and N, it follows that, for such a state ϕ_0,

$$N(K) = \hbar\omega M = \hbar\omega \langle D^2 \rangle . \tag{11}$$

From Eq. (6), putting $f(r) = -z$, $\hat{f} = 0$ (z being referred to the well center):

$$N(K) = (\hbar^2/8m) A . \tag{12}$$

Thus the oscillator value for the model-dependent quantity $\langle D^2 \rangle$ is

$$\langle D^2 \rangle = (\hbar/8m\omega) A = \tfrac{1}{8} A b^2 , \tag{13}$$

where b is the oscillator size parameter such that $\hbar^2/mb^2 = \hbar\omega$. Table I gives evaluations of various quantities for ^{90}Zr and ^{208}Pb using oscillator wave functions. According to Levinger,[4] the value of $\langle D^2 \rangle$ for a finite square well is about 15% less than for the oscillator. (In Sec. III we mention the value for the Woods-Saxon well.) The table also shows that the Pauli principle reduces $\langle D^2 \rangle$ by about 70%.

2. Evaluation with the Dipole Operator Referred to the Centroid

Let us now expose the effect of replacing D by $\mathfrak{D} \equiv D + D_0$, where $\mathfrak{D}$ and D_0 have been defined at the beginning of Sec. II B. We denote the sum rule quantities referring to $\mathfrak{D}$ by a double prime: M'',

N''. First, we note that

$$\langle \mathfrak{D}^2 \rangle - \langle D^2 \rangle = -\langle D_0^2 \rangle + 2\langle D_0 \mathfrak{D} \rangle \ . \tag{14}$$

Next, the oscillator gives $\langle D_0 \mathfrak{D} \rangle = 0$, since the wave function is separable in the two coordinates D_0 and $\mathfrak{D}$. Also, it gives $\langle D_0^2 \rangle = T^2 b^2/2A$, so

$$M'' \equiv \langle \mathfrak{D}^2 \rangle = \langle D^2 \rangle - \frac{T^2 b^2}{2A} = \frac{NZ}{2A} b^2 = \langle D^2 \rangle \left(1 - \frac{4T^2}{A^2}\right). \tag{15}$$

This is a special case of a well-known result. On writing $\mathfrak{D} = D_n - D_p$, where $D_n = (Z/A)\sum(\frac{1}{2} + t_{3i})z_i$, $D_p = (N/A)\sum(\frac{1}{2} - t_{3i})z_i$, the shell-model property $\langle D_n D_p \rangle = 0$ implies $\langle \mathfrak{D}^2 \rangle = \langle D_n^2 \rangle + \langle D_p^2 \rangle$. For the oscillator, the use of (13) for D_n^2, D_p^2 gives (15).

The evaluation of $\langle \mathfrak{D}^2 \rangle$ for ^{90}Zr and ^{208}Pb is given in Table I.

The effect of replacing D by $\mathfrak{D}$ on the energy-weighted sums can also be easily obtained. It is straightforward to show that

$$N''(K) = \tfrac{1}{2}\langle [\mathfrak{D}, [K, \mathfrak{D}]] \rangle = \frac{\hbar^2}{2m}\frac{NZ}{A} = N(K) - \frac{\hbar^2 T^2}{2mA} \ . \tag{16}$$

This result is also model-independent. One sees that the change equals $\hbar\omega$ times the corresponding change for the oscillator in the non-energy-weighted case above. For the potential part, we immediately obtain the result $N''(U) = N(U)$, since the isoscalar part of $\mathfrak{D}$ commutes with U.

TABLE I. Oscillator evaluation of various quantities relevant to the dipole problem occurring in Secs. II and V. Subscripts n, p on $\langle r^2 \rangle$ signify average taken over neutrons and protons, respectively; no subscript means average over all nucleons. The same oscillator parameter, $\hbar\omega = \hbar^2/\mathrm{mb}^2$, is used for neutrons and protons, thereby ensuring isospin purity of states but also giving a larger neutron mean-square radius: $\langle r^2 \rangle_n > \langle r^2 \rangle_p$. For densities with the observed property $\langle r^2 \rangle_n \approx \langle r^2 \rangle_p$, $M_1/\langle r^2 \rangle = \frac{1}{12}(N - Z)$.

	^{90}Zr	^{208}Pb
$\frac{\langle r^2 \rangle_n}{b^2}$	41/10	693/126
$\frac{\langle r^2 \rangle_p}{b^2}$	15/4	393/82
$\frac{\langle r^2 \rangle}{b^2}$	71/18	1086/208
$\frac{\langle D^2 \rangle}{\langle r^2 \rangle}$ (with Pauli principle)	2.82	5.00
$\frac{\langle D^2 \rangle}{\langle r^2 \rangle}$ (Pauli principle ignored)	7.5	17.3
$\frac{M_1}{\langle r^2 \rangle}$	1.16	4.80
$\frac{\langle B(K) \rangle}{\hbar^2/m} = \frac{2N(K)}{\hbar^2/m}$	22.5	52
$\frac{\langle N_1(K) \rangle}{\hbar^2/m}$	4.57	25.0
Two-body part of $\frac{\langle N_1(K) \rangle}{\hbar^2/m}$	3.32	19.5
$\frac{\langle \mathfrak{D}^2 \rangle}{\langle r^2 \rangle}$	2.79	4.78

C. Potential-Energy Contribution to the Energy-Weighted Sums

We now turn to an evaluation of $N(U)$. If the parent state is a Slater determinant, then

$$N(U) = \tfrac{1}{2}\left\langle \sum b'_{ij} \right\rangle_{\mathrm{nas}} , \tag{17}$$

with

$$b'_{ij} = b_{ij}(1 - P^x_{ij} P^\sigma_{ij} P^\tau_{ij}), \tag{18}$$

and where nas means evaluated with a product (i.e., nonantisymmetric) wave function. On performing this we obtain

$$N(U) = -\tfrac{1}{2} NZ(H + 2M)\, \mathfrak{F}_E(np), \tag{19}$$

where N, Z are the neutron, proton numbers of the parent and $\mathfrak{F}_E(np)$ is the average space-exchange integral for the operator $\frac{1}{2}[f(r_i) - f(r_j)]^2 v(r_{ij})$. This is explicitly defined and evaluated in Appendix A.

III. EXPERIMENTAL EVALUATIONS OF SUM RULES AND COMPARISON WITH SHELL-MODEL PREDICTIONS

If higher multipole contributions and finite wavelength modifications are ignored, the absorption cross section gives N'', M'' through the relations $\int \sigma\, dE = (4\pi^2 e^2/\hbar c)N'' = 0.288 N''$ and $\int (\sigma/E) dE = (4\pi^2 e^2/\hbar c)M'' = 0.288 M''$.

For experimental numbers we refer to ^{208}Pb, where the data[11] is especially good. The two integrals from 7.3 to 25 MeV are 3.48 (±0.23) b MeV and 0.251 (±0.020) b. The contributions outside 7.3 to 25 MeV can be estimated on the basis of a Lorentzian shape, which gives an excellent fit to the data inside this range. (A Breit-Wigner shape gives an almost identical fit, but has the properties that it does not vanish at zero energy, and $\int \sigma\, dE$ diverges. If the shape is cut off at zero energy and at a large positive energy, say 50 MeV, then it is essentially equivalent to the Lorentzian in the present context.) This fit implies that the region above 25 MeV increases the two integrals

by 13 and 4% while the region below 7.3 MeV increases them by $1\frac{1}{2}$ and 5%, respectively. However, there is data below 7.3 MeV suggesting that the latter are too large; we estimate the correct values to be $\lesssim\frac{1}{2}$ and $\lesssim 2\%$. The resulting value of $\int\sigma dE$ is 4.03 b MeV, implying $N''=1395$ MeV fm^2, or 36% larger than the kinetic energy value $2\pi^2e^2\hbar NZ/mcA$. The value of $\int(\sigma/E)dE$ is 0.267 b, implying $M''=93$ fm^2.

A. Non-Energy-Weighted Sum $M''=\langle\mathfrak{D}^2\rangle$

If we use the oscillator, and allow different oscillator parameters for neutrons and protons, then

$$\langle\mathfrak{D}^2\rangle=\frac{\hbar^2}{m}\frac{NZ}{2A^2}\left(\frac{Z}{\hbar\omega_n}+\frac{N}{\hbar\omega_p}\right). \tag{20}$$

If we choose the parameters to reproduce the observed rms proton radius $\langle r^2\rangle_p^{1/2}=5.42$ fm and assume that the neutron one is equal, then we get (using the values of $\langle r^2\rangle_n,\langle r^2\rangle_p$ in Table I): $\hbar\omega_n=7.75$ MeV, $\hbar\omega_p=6.75$ MeV. Inserting these in (20) gives

$$\langle\mathfrak{D}^2\rangle=145\text{ fm}^2. \tag{21}$$

A better value is obtained by choosing the parameters to reproduce the radii of the higher orbits that are "active" for $E1$ transitions, i.e., are allowed to contribute to $\langle\mathfrak{D}^2\rangle$ by the Pauli principle. From a detailed study[12] of overlaps of oscillator functions and Woods-Saxon functions chosen to fit observed radii and particle energies, we find $\hbar\omega_n\approx 9$ MeV, $\hbar\omega_p\approx 7$ MeV, giving

$$\langle\mathfrak{D}^2\rangle=135\text{ fm}^2. \tag{22}$$

We have results[13] for $E1$ integrals for the case of a Woods-Saxon potential $V_0[1+\exp(r-a)/(r_0A^{1/3})]^{-1}$ with parameters $a=0.60$ fm, $r_0=1.30$ fm, $V_0=43$ MeV for neutrons, 58 MeV for protons. In fact, integrals are given only for those transitions which are nonzero in the oscillator. For these, the energy-weighted sum rule is 0.935 times the classical sum rule, indicating that other transitions account for 6.5%. Since these transitions will have energy $\geq 3\hbar\omega$, the correction to $\langle\mathfrak{D}^2\rangle$ is 2%. For the oscillator-type transitions alone $\langle\mathfrak{D}^2\rangle=137$ fm^2, so we deduce

$$\langle\mathfrak{D}^2\rangle=140\ (\pm 2)\text{ fm}^2. \tag{23}$$

This particular Woods-Saxon calculation corresponds to too large $\langle r^2\rangle_n$, as one can see by comparing with a calculation[14] which is chosen to reproduce $\langle r^2\rangle_n=\langle r^2\rangle_p=(5.42)^2$ fm^2, and particle energies $a=0.75$ fm, $r_0=1.18$ fm (neutrons), 1.26 fm (protons), $V_0=51\ (\pm 1)$ MeV (neutrons), $61\ (\pm 1)$ MeV (protons). The small variation (± 1 MeV) in values of V_0 from orbit to orbit arises from fitting observed energies accurately. Unfortunately, V_0 may be different for two orbits with large $E1$ transition, so this transition does not obey the classical sum rule. However, the effect should be small since the variation is small. The oscillator-type transitions of this calculation give $\langle\mathfrak{D}^2\rangle=128$ fm^2. (Note that Ref. 15 places the $j_{15/2}$ state 2 MeV higher than Ref. 13, from evidence on spectroscopic factors. This decreases the transition strength to the $i_{13/2}$ state by 24% and decreases $\langle\mathfrak{D}^2\rangle$ by about 3%.) Allowing for non-oscillator transitions finally gives

$$\langle\mathfrak{D}^2\rangle=131\ (\pm 2)\text{ fm}^2. \tag{24}$$

Thus, we see that the shell-model evaluation of $\langle\mathfrak{D}^2\rangle$ is closely determined at ≈ 131 fm^2, once the potential is chosen to fit obvious parameters like $\langle r^2\rangle_p$, $\langle r^2\rangle_n$, and particle energies. This value needs to be reduced by 29% to fit the observed value 93 fm^2. This reduction is much larger than any uncertainty in shell model, or experimental error in the observed value. Thus we deduce that the shell-model value of $\langle\mathfrak{D}^2\rangle$ is reduced in practice by 29% by correlation effects.

A possible objection to our shell-model evaluation is that we have used different wells for neutrons and protons, so the wave function does not have pure isospin. First we note that using the same oscillator well, and imposing the condition that $\hbar\omega$ reproduce the mean of best overlaps for neutrons and protons ($\hbar\omega\approx 8$ MeV) gives essentially the same value. More important, however, is the fact that the quantity $\langle\mathfrak{D}^2\rangle$ itself involves no evident relation with isospin purity, since it is related to the sum M without isospin distinctions. It is "observed" in the sense that $\int dE(\sigma/E)=0.288\langle\mathfrak{D}^2\rangle$. The requirement of isospin purity is more critical when discussing separate isospins (Secs. V, VI).

B. Energy-Weighted Sum N''

Let us consider the ratio β'' of potential to kinetic energy contributions to the N'' sum rule. From (10), (12), and (19)

$$\beta''=-(Am/\hbar^2)(H+2M)\mathfrak{F}_E. \tag{25}$$

This ratio is an experimental quantity, viz., the fractional increase in the integrated photonuclear sum over the classical kinetic-energy part. Recent data[12,15] on ^{208}Pb, ^{197}Au, ^{181}Ta, ^{165}Ho, ^{159}Tb, and ^{139}La suggest that it lies in the range 0.25–0.4. The actual integrated cross sections to 25 MeV are 10–22% above the classical limit. Extrapolation to higher energies with a Lorentzian shape fitted to $\sigma(E)$ below 25 MeV gives an increase of

~15%. For light nuclei, $A \leq 40$, data below 30 MeV indicate no evidence for $\beta'' > 0$. Apparently then, in lighter nuclei the giant resonance has a higher, more extensive tail than in heavier nuclei. This agrees with the fact that $\sigma(3E_p/2)/\sigma(E_p)$ is of order 0.3 in light nuclei, while, for ^{208}Pb, the ratio is 0.11, E_p being the peak energy.

Wide classes of forces v_{ij} give values of $(H+2M)\mathfrak{F}_E$ in accord with the observed range of values. We defer giving sample values until the end of Sec. IV, when we present the corresponding values of the M sum rule corrected for correlation effects.

C. Mean Energy $\bar{E}''$

Although $\bar{E}''$ is trivially related to the previous quantities M'', N'' by $\bar{E}'' \equiv N''/M''$, we consider it explicitly to stress that it differs significantly from the observed peak energy E_p. The observed value of N''/M'' is 15.0 MeV, which is 1.7 MeV above the energy of the observed peak, 13.3 MeV. This difference arises from the asymmetry in the Lorentzian shape fitted to the observed cross section, especially the high-energy tail. For a symmetrical shape like the Breit-Wigner one $(\sigma/E) \propto [(E-E_p)^2 + \frac{1}{4}\Gamma^2]^{-1}$, the difference does not occur, i.e., $\bar{E}'' = E_p$. From the viewpoint of line-broadening theory, a difference between $\bar{E}''$ and E_p is expected, viz., the second-order shift associated with the spreading width of the dipole resonance, as described in Appendix B. The distinction between $\bar{E}''$ and E_p will not concern us further since we will not be considering second-order effects (see Sec. IV).

IV. EFFECT OF TWO-BODY CORRELATIONS ON SUM RULES

We have seen in Sec. III that the shell-model value of $M'' \equiv \langle \mathfrak{D}^2 \rangle$ needs to be reduced by 29% to fit experiment. Actually, this should not surprise us if we note the relation $M'' \equiv N''/\bar{E}''$, where $\bar{E}''$ is the mean energy of the giant dipole resonance. For ^{208}Pb, the observed value of $\bar{E}''$ is of the order of twice the shell-model value ($\hbar\omega \approx 7$ MeV); we have seen in Sec. III that N'' is about 36% larger than the shell-model value, so it follows directly that $N''/\bar{E}''$ is appreciably less than its shell-model value.

This reduction in $\langle \mathfrak{D}^2 \rangle$ is not contained in a theory of the Tamm-Dancoff (TD) sort where the shell-model ground state is retained. Since this theory gives a strong shift to $\bar{E}''$, it is not consistent in the sense that $\bar{E}''$ is given to higher accuracy than $\langle \mathfrak{D}^2 \rangle$. A consequence of this is that the TD evaluation of the N'' sum rule is incorrect, as is well known. In contrast, an RPA evaluation[16] of the sum rule is correct, essentially because, as Thouless[17] showed, the ground state in the RPA theory has correlations in it. As a result, the RPA evaluation of $\langle \mathfrak{D}^2 \rangle$ is smaller than the shell-model value, as required by experiment. Instead of using an RPA framework to describe correlations, we will use an alternative and essentially novel method which takes account of correlations consistently to first order in v_{ij}. To make an evaluation of $\langle \mathfrak{D}^2 \rangle$ with this method requires only the results of a TD calculation and an evaluation of β''. Since the special selection of second and higher-order corrections included in RPA theory has not been shown to be physically significant, the results of the present first-order theory are as good as those of RPA theory.

A. Qualitative Remarks

We can see by simple qualitative argument that two-body correlations reduce $\langle \mathfrak{D}^2 \rangle$. We have

$$\begin{aligned}\langle \mathfrak{D}^2 \rangle &= \left\langle \sum_{i,j} t_{3i}\, t_{3j}\, z_i''\, z_j'' \right\rangle \\ &= \tfrac{1}{4}\left\langle \sum_i z_i''^{\,2} \right\rangle + \left\langle \sum_{i \neq j} t_{3i} t_{3j} Z_{ij}^{\,2} \right\rangle \\ &\quad - \tfrac{1}{4}\left\langle \sum_{i \neq j} t_{3i} t_{3j}\, z_{ij}^{\,2} \right\rangle, \end{aligned} \tag{26}$$

where $Z_{ij} \equiv \frac{1}{2}(z_i'' + z_j'')$, $z_{ij} \equiv (z_i'' - z_j'')$. The first two terms are expected to be unaffected by correlations to a good approximation, while the third is obviously affected by them. Writing

$$\begin{aligned} 4\left\langle \sum_{i \neq j} t_{3i}\, t_{3j}\, z_{ij}^{\,2} \right\rangle &= \left\langle \sum_{nn} z_{ij}^{\,2} \right\rangle + \left\langle \sum_{pp} z_{ij}^{\,2} \right\rangle \\ &\quad - \left\langle \sum_{np} z_{ij}^{\,2} \right\rangle, \end{aligned} \tag{27}$$

where nn means summation over neutron-neutron pairs, etc., we see that a reduction in $\langle \mathfrak{D}^2 \rangle$ requires a net increase in this combination. Clearly the use of antisymmetrized wave functions acts in this direction since the chance of a nn or pp pair being in a spatial antisymmetric state is greater than the chance for a np pair. (We already know from Table I that the Pauli principle reduces $\langle \mathfrak{D}^2 \rangle$ strongly.) Furthermore, we see that a further reduction in $\langle \mathfrak{D}^2 \rangle$ occurs if np forces are more attractive than nn or pp forces. Since this is a known fact, we may say that we understand qualitatively the origin of the reduction in $\langle \mathfrak{D}^2 \rangle$ from its shell-model value. Essentially it is a symmetry energy effect, and agrees with the elementary picture that the amplitude of dipole oscillation of neutrons against protons is necessarily reduced if neutrons and protons attract

each other.

The two kinds of correlation (Pauli and dynamical) are distinguished by the quantity $\langle D_n D_p \rangle$ occurring in $\langle \mathfrak{D}^2 \rangle = \langle D_n^2 \rangle + \langle D_p^2 \rangle - 2\langle D_n D_p \rangle$ with D_n, D_p defined below (15). $\langle D_n D_p \rangle = 0$ for Pauli correlations but $\neq 0$ for dynamical ones.

B. Quantitative Analysis

Now we will make a quantitative analysis. We can do this for the dipole operator referred to well center or centroid. For economy of notation the following discussion is made for the former case, but it applies equally to the latter when D is replaced by $\mathfrak{D}$ and double primes inserted.

Let us write the shell-model wave function as Φ_0, and expand the true wave function ϕ_0 to first order in the two-body forces. If $H_0 \equiv K + \sum_i V_i$ is the shell-model Hamiltonian and $H = K + \frac{1}{2}\sum_{i \neq j} v_{ij}$ is the true one, then

$$\phi_0 = \left(1 - \frac{Q}{H_0 - E_0} H'\right) \Phi_0 , \tag{28}$$

where $H' = H - H_0 = \frac{1}{2}\sum_{i \neq j} v_{ij} - \sum_i V_i$, $Q = 1 - |\Phi_0\rangle\langle\Phi_0|$, and E_0 is the shell-model eigenvalue of Φ_0:

$$H_0 \Phi_0 = E_0 \Phi_0 . \tag{29}$$

It follows that the correction to the shell-model value of M is

$$\langle \phi_0 | D^2 | \phi_0 \rangle - \langle \Phi_0 | D^2 | \Phi_0 \rangle = -2\langle \Phi_0 | D^2 \frac{Q}{H_0 - E_0} H' | \Phi_0 \rangle . \tag{30}$$

Our problem is to evaluate this correlation effect. At this point, we could try to proceed in direct fashion by developing the expression on the right-hand side. However, there is an alternative means of evaluation. Recalling that $\overline{E}$ is, by definition, equal to N/M, we evaluate the two quantities N and $\overline{E}$, then take the quotient $M = N/\overline{E}$. The merit of this procedure is that the perturbation expansions and approximations are more readily displayed in the quantities $N, \overline{E}$ than in M. This is particularly so in the case of $N \equiv N(K) + N(U)$. As shown by (12), $N(K)$ is just a number, $\hbar^2 A/8m$, and is not subject to any perturbation corrections. Since the operator in $N(U) = \frac{1}{2}\langle \phi_0 | [D, [H', D]] | \phi_0 \rangle$ is first order in H', it follows that corrections arising from replacing ϕ_0 by Φ_0 are second order, whence

$$N = N^{(0)} + O(H'^2/2\hbar\omega), \tag{31}$$

where $N^{(0)} \equiv \frac{1}{2}\langle \Phi_0 | [D, [H, D]] | \Phi_0 \rangle$. The denominator $2\hbar\omega$ represents a typical excitation energy of corrections to Φ_0 arising from the perturbation H'.

Now we turn to $\overline{E}$, which we may write

$$\overline{E} = \sum_\lambda (E_\lambda - E_0) a_\lambda^2 , \tag{32}$$

where a_λ^2 is defined as $\langle \phi_\lambda | D | \phi_0 \rangle^2 / \langle \phi_0 | D^2 | \phi_0 \rangle$ and satisfies $\sum_\lambda a_\lambda^2 = 1$. Let us discuss separately the quantities $(E_\lambda - E_0)$ and a_λ^2. In lowest order, E_0 is $\langle \Phi_0 | H | \Phi_0 \rangle$ while E_λ are the energies obtained by diagonalizing H' amongst any near-degenerate shell-model states of spin 1^-. The lowest of these are clustered around the oscillator energy $\hbar\omega$, the next around $3\hbar\omega$, and so on. Diagonalization of H' in such a cluster is a Tamm-Dancoff calculation and the resulting values of $(E_\lambda - E_0)$ are the TD energies. Interaction between clusters produces changes in $(E_\lambda - E_0)$ of order $O(H'/2\hbar\omega)^2$, provided only that the original diagonalizations cause no levels to migrate into adjoining clusters. An example of a level moving appreciably out of its cluster is the collective dipole state, which separates from the cluster at $1\hbar\omega$ as described by Brown and Bolsterli.[18] However, it moves less than halfway to the $3\hbar\omega$ cluster, so that the proviso is met.

The quantities a_λ^2 are, in contrast to $(E_\lambda - E_0)$, subject to first-order changes when their TD values are corrected. Although this is true in principle, in practice the changes are very small, and no larger than second order. There is a simple reason for this. TD diagonalizations have the remarkable property that nearly all ($\geq$90%) of the sum $\sum_\lambda a_\lambda^2$ arises from states in the immediate vicinity of the giant dipole state. In other words, actual distributions are quite close to the extreme Brown-Bolsterli[18] distribution in which a_λ^2 is zero for all states but one, the giant dipole. In this extreme case, changes in a_λ^2 are obviously second order, since a_λ is itself first order. Thus we see that, in the approximation where we neglect any a_λ^2 outside the giant dipole region, $\overline{E}$ is given to second order by the TD value, $\overline{E}_{\mathrm{TD}} \equiv \sum (E_\lambda - E_0)_{\mathrm{TD}}\, a_\lambda^2\,(\mathrm{TD})$.

Now we check on this approximation by comparing a TD calculation for ^{208}Pb with the corresponding RPA calculation.[19] The latter includes almost all first-order and a selection of second- and higher-order effects. From the results we find that $\overline{E}_{\mathrm{TD}} = 11.15$ MeV, while $\overline{E}_{\mathrm{RPA}} = 10.85$ MeV. The difference of 3% is negligible. Upon more detailed study, we find that $1\frac{1}{2}$% arises from the change in $(E_\lambda - E_0)$, and $1\frac{1}{2}$% from the change in a_λ^2. Thus we see directly that the changes in a_λ^2, while formally first order, are no larger than second order. Only 7% of $\sum_\lambda a_\lambda^2\,(\mathrm{TD})$ arises from the 24 states below 10 MeV, implying a mean value for $a_\lambda^2(\mathrm{TD})$ of 0.003. (If this 7% were ignored, the value of $\overline{E}_{\mathrm{TD}}$ increases by only 2.5%

to 11.42 MeV.) For comparison, 9% of $\sum_\lambda a_\lambda^2$(RPA) arises from states below 10 MeV, so that the mean value of a_λ^2(RPA) is 0.004. Thus the change in distribution is only 2%. In contrast, the RPA value of $M \equiv \langle \phi_0 | D^2 | \phi_0 \rangle$ is 19% less than the TD value. Thus we see that, to an excellent approximation, first-order effects bring about a large change in the over-all normalization of dipole strengths $\langle \phi_\lambda | D | \phi_0 \rangle^2$, but cause essentially no change in their distribution. This remarkable fact means that *M may be evaluated as the quotient* $N^{(0)}/\bar{E}_{\rm TD}$, *with corrections effectively of second order in* $H'/2\hbar\omega$.

Before passing on to exploit the formula for M, we pause to note some interesting features of it and its derivation.

(i) It may seem odd that the value of M given by this formula includes first-order correlation corrections due to matrix elements $\langle {\rm pp'} | v | {\rm hh'} \rangle$, yet neither $N^{(0)}$ or $\bar{E}_{\rm TD}$ apparently involve reference to these quantities. $\bar{E}_{\rm TD}$ certainly does not, since it involves only $\langle {\rm ph'} | v | {\rm p'h} \rangle$. In fact, the missing quantities occur implicitly in N, but do not arise explicitly because it is a sum-rule quantity in which closure is performed, so that these quantities are removed in favor of the diagonal expectation value of the commutator $[D,[U,D]]$.

(ii) We have mentioned that the RPA theory includes first-order effects, along with a selection of higher-order effects. The leading corrections to it are of the phonon self-screening type (involving two-body matrix elements of types $\langle {\rm pp'} | v | {\rm p''h} \rangle$ and $\langle {\rm hh'} | v | {\rm h''p} \rangle$). These [19] change $\bar{E}$ by <0.1 MeV. It has been shown that, at least for some modes like the monopole one,[20] the omitted terms are comparable with the included ones and tend to cancel them. Thus RPA results are not to be trusted beyond first order. For our purposes of exposing first-order effects, they are sufficient. The fact that RPA theory includes first-order effects is reflected in Thouless's theorem[17] that the RPA eigenvalues and dipole matrix elements satisfy the energy-weighted sum rule to first order. In fact, with self-consistency the sum is exactly $N^{(0)}$, which, as we have seen, differs from N in second order.

(iii) We have seen that the essential reason why the first-order effects on the distribution of a_λ^2 are so small is that the TD values of a_λ^2 are very small for states outside the giant dipole region. If the correction to a_λ(TD) is δa_λ, then the correction to a_λ^2 contains the first-order part $2a_\lambda({\rm TD})\delta a_\lambda$, which is small for small a_λ(TD). This provides the basis for our result that the distribution of a_λ^2 is very little changed by first-order terms. There is a further aspect of the corrections to a_λ^2 which, although not needed for our results, is remarkable in itself, viz., RPA distribution follows the TD one in details,[19] besides showing the giant dipole dominance. This means that the first-order correction δa_λ to a_λ tends to be proportional to a_λ, which means that the first-order correction to $\langle \Phi_\lambda | D | \Phi_0 \rangle$ tends to be proportional to $\langle \Phi_\lambda | D | \Phi_0 \rangle$. As far as we know, this feature has not been exposed or discussed in the literature. We comment further in Appendix C.

(iv) It is noteworthy that the derivation of the result $M = N^{(0)}/\bar{E}_{\rm TD} + O(H'/2\hbar\omega)^2$ does not depend on any special choice of single-particle potential V_i; in particular, it need not be the self-consistent potential derived from the two-body potential v_{ij}. However the usefulness of the result requires that V_i be reasonably close to self-consistency, otherwise some of the second-order correction terms to $(E_\lambda - E_0)$ may be large, viz., those containing matrix elements of H' in which H' excites a single particle in Φ_0 to a higher principle quantum number. These vanish under self-consistency.

C. Evaluation of Sum Rules for ^{208}Pb

First let us consider a TD calculation with zero-range forces (e.g. Ref. 13) such that the $E1$ strength is concentrated near the observed peak energy $E_p = 13.3$ MeV $\approx 1.73\hbar\omega$ with $\hbar\omega \approx 7.89$ MeV. Because of the zero range, $\beta'' = N(U) = 0$. This disagrees with the ^{208}Pb data $\beta'' = 0.36$. The implied value of M'' is $N''(K)/\bar{E}''_{\rm TD}$ or 42% less than the shell-model value $N''(K)/\hbar\omega$. The observed value is 29% less. Thus, zero-range TD calculations do not constitute a satisfactory fit to the data, even when they reproduce E_p, since M'', β'' are too small.

In order to see whether the same criticism must be leveled at the finite-range calculation, we estimate β'' implicit in such a calculation.

If an RPA calculation were available then, from Thouless's result[17] that this gives the correct value of $N^{(0)''}$ (assuming self-consistency), we could obtain β''. Unfortunately, it cannot be obtained from a TD calculation so we must fall back on a direct evaluation from (25) which we can write as

$$\beta'' = -A(H+2M)\frac{m}{\hbar^2}\,\mathfrak{F}_D(0)\,\frac{\mathfrak{F}_E(0)}{\mathfrak{F}_D(0)}\,F_E(a/R), \qquad (33)$$

where $\mathfrak{F}_D(0)$ is the average direct integral of (A10) and $F_E(a/R)$ is the finite-range correction factor defined by (A14).

As an example of a finite-range TD calculation, we consider the work of Perez,[14] who uses a Yukawa force of range $a = 1.55$ fm, strength $v_0 = 40$ MeV, and exchange mixture $H = 0.025$, M

$=0.625$, $W=0.125$, $B=0.225$. Choosing the radius parameter for ^{208}Pb as described at the end of Appendix A, $R=6.5$ fm, (A10) gives $m\mathcal{F}_D(0)/\hbar^2 = 0.093$. The ratio $\mathcal{F}_E(0)/\mathcal{F}_D(0)$ is, from (A15) with Fermi momentum $k_F=1.25$ fm^{-1}, equal to 0.030, while $F_E=0.75$. Inserting these values in (33) gives $\beta''=0.54$.

It may be that this evaluation of β is somewhat inaccurate [since the method of Appendix A for $\mathcal{F}_E(0)/\mathcal{F}_D(0)$ is based on an extrapolation from light nuclei]. On using it, the implied value of M'' is 90% of the shell-model value. Thus this particular finite-range calculation overestimates β'' and M''. It seems certain that a calculation with smaller range would reproduce both to better accuracy.

Finally we mention a calculation[19] where both TD and RPA results are available for the same v_{ij}. (Another calculation[21] gives similar results.) The peak energies are (using $\hbar\omega=7.3$ MeV) 11.5 and 11.3 MeV for TD and RPA (both being about 2 MeV below the observed peak). The values of N are $1.54N''(K)$, $1.23N''(K)$, respectively. The value of M'' from the RPA results is $0.74M^{(0)}{}''$, which is quite close to the value from our first-order theory, $0.80M^{(0)}{}''$, if we assume that $N^{(0)}{}''$ is the RPA value $1.23N''(K)$. (Since the calculation has no Hartree-Fock self-consistency, there is no guarantee of this, and this probably accounts for the small discrepancy.)

V. ISOSPIN-RESTRICTED SUM RULES

A. General Discussion

Recently, there has been considerable interest[5-9] in the separate sum rules for the different isospins of the states excited by the dipole operator. In this section we set up the restricted sum rules and then, in Sec. VI, we study the effect of correlations upon them.

We first define the restricted sum rules by

$$M(T')=\sum_{\lambda(T')}\langle\phi_\lambda(T',T_3)|F|\phi_0(T,T_3)\rangle^2,$$
$$N(T')=\sum_{\lambda(T')}[E_\lambda(T')-E_0(T)]\times\langle\phi_\lambda(T',T_3)|F|\phi_0(T,T_3)\rangle^2, \tag{34}$$

where the notation $\lambda(T')$ means that the summation is over states $\phi_\lambda(T',T_3)$ of isospin T' only.

B. Decomposition into Isoscalar, Isovector, and Isotensor Parts

1. Non-Energy-Weighted Sums

Since F is an isovector, the Wigner-Ekhart method can be used to express $M(T')$ in terms of "reduced" quantities $M(\|T'\|)$:

$$M(T')=C^2(TT_3;10|\,T'T_3)M(\|\,T'\|), \tag{35}$$

where these may be written

$$M(\|\,T-1\|)=\frac{1}{2T(2T-1)}[T(2T+1)\langle F^+F^-\rangle - 2(2T+1)\langle F^2\rangle+\langle F^-F^+\rangle], \tag{36}$$
$$M(\|\,T\|)=\frac{1}{2T}[2(T+1)\langle F^2\rangle-\langle F^-F^+\rangle],$$
$$M(\|\,T+1\|)=\tfrac{1}{2}\langle F^-F^+\rangle;$$

also

$$M=\frac{1}{2T(2T-1)}[(T^2-T_3^2)(\langle F^+F^-\rangle+\langle F^-F^+\rangle) + 2(2T_3^2-T)\langle F^2\rangle]. \tag{37}$$

These results have been given previously.[22] F^-,F^+ are obtained from F by changing t_3 to t_-,t_+, respectively. The brackets signify expectation value taken in the parent state of $\phi_0(TT_3)$, viz., that with $T=T_3$, e.g.

$$\langle F^2\rangle=\langle\phi_0(TT)|F^2|\phi_0(TT)\rangle. \tag{38}$$

For reasons which will appear, it is very useful to express the operators in Eq. (36) in terms of isoscalar, isovector, and isotensor combinations as done in Ref. 6:

$$F^2=F_0^2=\sum_Y C(10;10|Y0)(\underline{F}\times\underline{F})_Y,$$
$$\langle F^2\rangle=M_0-\tfrac{2}{3}M_2,$$
$$F^+F^-=-2F_1F_{-1}=\sum_Y C(11;1\text{-}1|Y0)(\underline{F}\times\underline{F})_Y, \tag{39}$$
$$\langle F^+F^-\rangle=2(M_0+M_1+\tfrac{1}{3}M_2),$$
$$F^-F^+=-2F_{-1}F_1=\sum_Y C(1\text{-}1;11|Y0)(\underline{F}\times\underline{F})_Y,$$
$$\langle F^-F^+\rangle=2(M_0-M_1+\tfrac{1}{3}M_2),$$

where:

$$M_0\equiv-\sqrt{\tfrac{1}{3}}\langle(\underline{F}\times\underline{F})_0\rangle=\tfrac{1}{3}\langle F_0^2+\tfrac{1}{2}(F^-F^++F^+F^-)\rangle = \tfrac{1}{3}\langle\underline{F}\cdot\underline{F}\rangle,$$
$$M_1\equiv-\sqrt{\tfrac{1}{2}}\langle(\underline{F}\times\underline{F})_1\rangle=\tfrac{1}{4}\langle F^+F^--F^-F^+\rangle, \tag{40}$$
$$M_2\equiv-3\sqrt{\tfrac{1}{6}}\langle(\underline{F}\times\underline{F})_2\rangle=\langle\tfrac{1}{4}(F^-F^++F^+F^-)-F_0^2\rangle = \tfrac{1}{2}\langle\underline{F}\cdot\underline{F}-3F_0^2\rangle.$$

Substitution of (39) in $M(\|T'\|)$ gives:

$$M(\|T'\|)=\langle F^2\rangle-\frac{1}{2T}[T'(T'+1)-T(T+1)-2]\langle M_1-M_2\rangle + \delta_{T',T-1}2\left(\frac{2T+1}{2T-1}\right)M_2,$$
$$\sum M(T')=\langle F^2\rangle+\frac{2(T^2-T_3^2}{T(2T-1)}M_2. \quad (41)$$

Besides facilitating systematic analysis, the advantage of expressing $M(T')$ in terms of M_0, M_1, M_2 is that the model-dependent and -independent aspects of $M(\|T'\|)$ are made clear. M_1 involves the one-body operator

$$\tfrac{1}{4}(F^+F^- - F^-F^+)=\tfrac{1}{2}\sum_i t_{3i}[f(r_i)-\hat{f}\,]^2, \quad (42)$$

so M_1 depends only on the neutron and proton densities, ρ_n and ρ_p:

$$M_1=\tfrac{1}{4}\int[f(r)-\hat{f}\,]^2[\rho_n(r)-\rho_p(r)]\,d^3r, \quad (43)$$

and is therefore essentially model-independent. In contrast, M_0 and M_2 involve two-body operators so are model-dependent. [The relation between our quantities and those of Ref. 6 are: $\langle F^2\rangle = S(0)$, $M_1=\alpha_1 S(0)$, $M_2=\alpha_2 S(0)$.]

2. Energy-Weighted Sums

Now we turn to the energy-weighted sums where

$$N(T')=C^2(TT_3;10|\,T'T_3)N(\|T'\|). \quad (44)$$

The reduced quantities $N(\|T'\|)$ are given by expressions like those given in (36) for $M(\|T'\|)$ with the substitutions

$$\begin{cases} 2F_0^2 \to B\equiv[F,[H,F]], \\ 2F^+F^- \to A^{+-}\equiv F^+[H,F^-]+[F^+,H]F^-, \\ 2F^-F^+ \to A^{-+}\equiv F^-[H,F^+]+[F^-,H]F^+. \end{cases} \quad (45)$$

Each of these operators B, A^{+-}, A^{-+} can be expressed in terms of isoscalar, isovector, and isotensor operators, exactly as before, with the result

$$N(\|T'\|)=\tfrac{1}{2}\langle B\rangle-\frac{1}{2T}[T'(T'+1)-T(T+1)-2]\langle N_1-N_2\rangle + \delta_{T',T-1}2\left(\frac{2T+1}{2T-1}\right)N_2, \quad (46)$$
$$\sum_{T'}N(T')=\tfrac{1}{2}\langle B\rangle+\frac{2(T^2-T_3^2)}{T(2T-1)}N_2,$$

where

$$N_0=\tfrac{1}{6}\langle B+\tfrac{1}{2}(A^{+-}+A^{-+})\rangle,$$
$$N_1=\tfrac{1}{8}\langle A^{+-}-A^{-+}\rangle, \quad (47)$$
$$N_2=\langle -\tfrac{1}{2}B+\tfrac{1}{8}(A^{+-}+A^{-+})\rangle.$$

We note that the combination $(A^{+-}+A^{-+})$ is a sum of double commutators:

$$A^{+-}+A^{-+}=[F^+,[H,F^-]]+[[F^+,H],F^-] =[F^+,[H,F^-]]+[F^-,[H,F^+]]. \quad (48)$$

Since $N(\|T'\|)$, B, N_0, N_1, N_2 are linear in $H=K+U$, we can again write each of these quantities as a sum of two parts, one from the kinetic energy K and one from the potential energy U. First we consider the former.

(i) Kinetic-energy contribution. Denoting this by argument K, it is easy to see that

$$N_2(K)=0. \quad (49)$$

This comes from the vanishing of the operator, i.e., it does not depend on any model wave function. Thus:

$$N(\|T'\|,K)=\tfrac{1}{2}\langle B(K)\rangle-\frac{1}{2T}[T'(T'+1)-T(T+1)-2]N_1, \quad (50)$$

where

$$\langle B(K)\rangle=(\hbar^2/4m)\left\langle\sum_i|\vec{\nabla}_i f(r_i)|^2\right\rangle,$$
$$N_1(K)=\langle(\hbar^2/8m)\sum_{\substack{i,j\\(i\neq j)}}(t_{i+}t_{j-}-t_{i-}t_{j+})f(r_j) \times[f(r_i),\nabla_i^2]\rangle+\langle(\hbar^2/4m)\sum_i t_{3i}|\vec{\nabla}_i f(r_i)|^2\rangle. \quad (51)$$

$\langle B(K)\rangle$ is model-independent but, unfortunately, N_1 is a model-dependent quantity, because its first term is a two-body one.

(ii) Potential-energy contribution. If a shell-model wave function is assumed then, in terms of the direct and exchange integrals, $\mathfrak{F}_D$ and $\mathfrak{F}_E$ of $\frac{1}{2}[f(r_i)-f(r_j)]^2 v_{ij}$, the potential-energy contributions are (Appendix A):

$$\langle B(U)\rangle=-NZ(H+2M)\mathfrak{F}_E,$$
$$N_1(U)=\tfrac{1}{8}(N^2-Z^2)[(2H+M)\mathfrak{F}_D-(2M+H)\mathfrak{F}_E],$$
$$N_2(U)=\tfrac{1}{8}(N-Z)^2[(2H+M)\mathfrak{F}_D-(2M+H)\mathfrak{F}_E],$$
$$N(\|T'\|,U)=-\tfrac{1}{2}NZ(H+2M)\mathfrak{F}_E-[(2H+M)\mathfrak{F}_D-(2M+H)\mathfrak{F}_E] \times\left\{\tfrac{1}{4}Z[T'(T'+1)-T(T+1)-2]-\delta_{T',T-1}\left(\frac{2T+1}{2T-1}\right)T^2\right\}, \quad (52)$$
$$\sum_{T'}N(T',U)=-\tfrac{1}{2}NZ(H+2M)\mathfrak{F}_E+\frac{(T^2-T_3^2)T^2}{T(2T-1)} \times[(2H+M)\mathfrak{F}_D-(2M+H)\mathfrak{F}_E].$$

We note that, in contrast to $N_2(K)=0$, $N_2(U)$ is not only not zero, but is simply related to $N_1(U)$. For a parent state $T=T_3$ the total sum rule, $\sum_{T'} N(T', U)$, depends on the exchange term $\mathfrak{F}_E$ only.

C. Specialization to the Dipole Case

With f put equal to $-z$, Eqs. (43), (49), and (51) give the model-independent results:

$$M_1=\frac{1}{12}\int r^2[\rho_n(r)-\rho_p(r)]\,d^3r\,,$$
$$N_2(K)=0\,, \tag{53}$$
$$\langle B(K)\rangle=\hbar^2 A/4m\,.$$

1. Oscillator-Model Evaluations

For the model-dependent quantities we may use (11) and its extensions:

$$N_1(K)=\hbar\omega M_1\,;\quad N_2(K)=\hbar\omega M_2\,. \tag{54}$$

These follow from definitions if no negative-energy components occur when D acts on parent and analog states, i.e., if $D\phi_0(T, T)$ and $D\phi_0(T, T-1)$ contain only components of excitation energy $\hbar\omega$. Equivalently,

$$N(\|T'\|, K)=\hbar\omega M(\|T'\|) \tag{55}$$

for $T'=T-1$, T, $T+1$. (If negative energy components occur for the parent, but not the analog, this is valid only for $T'=T$, $T+1$.)

From (11), (53), (54), the model-dependent quantities are

$$N_1(K)=\tfrac{1}{12}\hbar\omega(N\langle r^2\rangle_n-Z\langle r^2\rangle_p),$$
$$M_2=0\,, \tag{56}$$
$$\langle D^2\rangle=\frac{A\hbar}{8\omega m}=\tfrac{1}{8}Ab^2\,.$$

From (39), $\langle D^2\rangle=M_0-\frac{2}{3}M_2$, so the oscillator value of $M_0=\langle D^2\rangle$. Table I gives evaluations of the various quantities for ^{90}Zr, ^{208}Pb.

2. Potential-Energy Contributions

From (52) we see that for large T, the ratio of the isospin-dependent part to the isospin-independent part is equal to the factor

$$\left(\frac{2H+M}{2M+H}\frac{\mathfrak{F}_D}{\mathfrak{F}_E}-1\right)$$

times a factor which is $-2T/A$, $+2/A$, $+2T/A$ for $T'=T+1$, T, $T-1$, respectively. For the Rosenfeld force of range $a=1.4$ fm $\mathfrak{F}_D\mathfrak{F}_E=12$ (see Appendix A), and $(2H+M)/(2M+H)=0.25$, so the first factor is ≈ 2. This corresponds to a relatively small splitting, decreasing $N(\|T+1\|, U)$ by about 20% for ^{208}Pb. For the Hamada-Johnston force, the long-range part has $(2H+M)=0$, so gives an effect of opposite sign. When the shorter-range part is included, the net value of $[(2H+M)/(2M+H)]$ $(\mathfrak{F}_D/\mathfrak{F}_E)$ is 0.15 so the first factor is -0.85 for the Hamada-Johnston force.

Unfortunately, there are no data against which to check these estimates. However, the values cited are sufficiently small that we may dismiss this potential-energy source of isospin-dependence as fairly negligible for conventionally used forces. It is of the order of 10% (or less) of the isospin splitting of the kinetic-energy term $N(\|T'\|, K)$, if the latter is estimated with the oscillator.

D. Evaluation with Dipole Operator Referred to Centroid

1. Model-Independent Results

Let us now expose the effect of replacing D by $\mathfrak{D}\equiv D+D_0$. We may approach this problem either by writing D_0 as $(1/A)(\sum_i t_{3i})(\sum_j z_j)$ and regarding it as a (two-body) isovector operator, or by writing it as $[(N-Z)/2A]\sum_j z_j$ and regarding it as a (one-body) isoscalar. We take the former attitude since $\mathfrak{D}$ is then an isovector and we may use the structure of Eqs. (39) for $M(\|T'\|)$ in terms of M_0, M_1, M_2 (note that the formulas apply to two-body as well as one-body operators). Let us again denote the quantities referring to $\mathfrak{D}$ by a double prime. Since D_0 cannot change isospin, $M(\|T\pm 1\|)$ are unchanged by its inclusion.

From the relations

$$M(\|T\|)=\frac{1}{2T+1}[3TM(\|T+1\|)-(T-1)M(\|T-1\|)]+2(T+1)M_1\,,$$
$$\langle D^2\rangle=\frac{1}{2T+1}[(2T^2+T+1)M(\|T+1\|)-T(2T-1)M(\|T-1\|)]+2TM_1\,, \tag{57}$$

we deduce

$$M''(\|T\|) - M(\|T\|) = 2(T+1)[M_1'' - M_1],$$
$$\langle \mathfrak{D}^2 \rangle - \langle D^2 \rangle = 2T[M_1'' - M_1]. \quad (58)$$

Also,

$$M_2'' - M_2 = -(2T-1)(M_1'' - M_1). \quad (59)$$

The merit of expressing the changes in terms of M_1 is that the operator is the simplest:

$$M_1'' - M_1 = \tfrac{1}{2}\left\langle \sum_i t_{3i}(z_i''^2 - z_i^2) \right\rangle$$
$$= -\frac{1}{2T}\langle D_0^2 \rangle - \frac{1}{T}\langle D_0 \mathfrak{D} \rangle. \quad (60)$$

The effect on the energy-weighted sums is exactly parallel. $N(\|T \pm 1\|, K)$ cannot be affected by the replacement of D by $\mathfrak{D}$. Thus the changes in $B(K)$, $N_1(K)$, $N_2(K)$ are related. From the result of Eq. (16),

$$\langle B''(K) \rangle = \frac{\hbar^2}{m}\frac{NZ}{A} = \langle B(K) \rangle - \frac{\hbar^2 T^2}{mA}, \quad (61)$$

it then follows that

$$N_2''(K) - N_2(K) = \frac{\hbar^2}{m}\frac{T(2T-1)}{4A},$$
$$N_1''(K) - N_1(K) = -\frac{\hbar^2}{m}\frac{T}{4A}, \quad (62)$$
$$N''(\|T\|, K) - N(\|T\|, K) = \frac{\hbar^2}{m}\frac{T}{A}.$$

These results are all model-independent. As noted in Sec. II, the potential-energy contribution is unaffected by referring the dipole operator to the centroid, so $N_1''(U) = N_1(U)$, $N_2''(U) = N_2(U)$, $\langle B''(U) \rangle = \langle B(U) \rangle$.

2. *Oscillator-Model Evaluations*

An important feature is that this analysis does not depend on any model so far. Now, to proceed further, a model must be introduced. The oscillator gives $\langle D_0 \mathfrak{D} \rangle = 0$, since the wave function is separable in the two coordinates D_0 and $\mathfrak{D}$. Also, it gives $\langle D_0^2 \rangle = T^2 b^2/2A$, so

$$M_1'' - M_1 = -(T/4A)b^2. \quad (63)$$

From (58) and (59) all other quantities follow. In particular, from $M_2 = 0$,

$$M_2'' = \frac{T(2T-1)}{4A} b^2. \quad (64)$$

Using Table I, (63) gives $M_1'' - M_1 \approx -T\langle r^2 \rangle/16A \approx -3M_1/8A$. Thus the fractional change in M_1 is very small ($\lesssim 0.3\%$ for all nuclei with $A \geq 90$). M_2'' rises to $0.05\langle \mathfrak{D}^2 \rangle$ in heavy nuclei.

If the oscillator potential is replaced by another shape, we expect that the results quoted for fractional changes in the non-energy-weighted quantities are not essentially changed. Some insight into this is gained by rewriting (60) as

$$M_1'' - M_1 = \frac{T}{2A^2}\left\langle \left(\sum_i z_i \right)^2 \right\rangle + \frac{1}{2A}\left[\left\langle \left(\sum_{i=p} z_i \right)^2 \right\rangle - \left\langle \left(\sum_{i=n} z_i \right)^2 \right\rangle \right], \quad (65)$$

where the three sums on i are over nucleons, protons, and neutrons. With the oscillator the three expectation values are $\frac{1}{2}Ab^2$, $\frac{1}{2}Zb^2$, $\frac{1}{2}Nb^2$: whence the second term is minus twice the first, giving the quoted result $M_1'' - M_1 = -(T/4A)b^2$. For any well shape, the first expectation value is the sum of the others, so

$$M_1'' - M_1 = \frac{1}{2A^2}\left[(A+T)\left\langle \left(\sum_{i=p} z_i \right)^2 \right\rangle - (A-T)\left\langle \left(\sum_{i=n} z_i \right)^2 \right\rangle \right]. \quad (66)$$

With the oscillator, there is quite a strong cancellation between the neutron and proton terms in this expression, the result being $\approx (2T/A)$ times a single term, i.e., 0.22 for ^{208}Pb. We have seen in Sec. III A that the expectation values are insensitive (to within 3%) to change of well shape provided that the rms radius of the "active" orbits is unchanged. Thus we conclude that the degree of cancellation is not essentially changed, and the oscillator evaluation $(M_1'' - M_1)/M_1 \approx -3/8A$ remains valid for other wells provided that the rms radii of active neutrons and protons—especially their relative value—is unchanged.

E. Experimental Lower Limit on M_2'' for ^{208}Pb

Applying (41) to the case where the operator is referred to the centroid, we have

$$\frac{M_2''}{\langle \mathfrak{D}^2 \rangle} = \left(1 + \frac{1}{T}\frac{M_1''}{\langle \mathfrak{D}^2 \rangle}\right)\frac{M(\|T+1\|)}{M''(\|T\|)} + \frac{M_1''}{\langle \mathfrak{D}^2 \rangle} - 1. \quad (67)$$

Using the experimental value $\langle \mathfrak{D}^2 \rangle = 93$ fm^2 quoted in Sec. III and estimating $M_1'' = 108$ fm^2 on the basis of $\langle r^2 \rangle_n = \langle r^2 \rangle_p = (5.42 \text{ fm})^2$ gives

$$\frac{M_2''}{\langle \mathfrak{D}^2 \rangle} = 1.05\left(\frac{M(\|T+1\|)}{M''(\|T\|)}\right) + 0.16. \quad (68)$$

We see that, since the first term is positive, we have the lower limit

$$M_2''/\langle \mathfrak{D}^2 \rangle > 0.16, \quad (69)$$

which is considerably larger than the shell-model value from (15) and (64), viz., $T(2T-1)/2NZ = 0.05$. In fact, there is an experimental observation[23] of a cross section ~3.3 MeVmb to $T+1$ states in ^{208}Pb at energy 25 MeV. This indicates $M(\|T+1\|)/M''(\|T\|) \geq 0.02$, whence

$$\frac{M_2''}{\langle \mathfrak{D}^2 \rangle} \geq 0.18 . \tag{70}$$

Of course, since cancellation of large numbers is involved, and numbers given are subject to experimental error, this result must be taken with some caution. Further, the physical states involved slightly violate the condition of isospin purity. If M_2'' is assigned its shell-model value ($0.05\langle \mathfrak{D}^2 \rangle$), then (67) implies $M_1'' = 95.5$ fm^2, which gives $\langle r^2 \rangle_n^{1/2} = 0.98 \langle r^2 \rangle_p^{1/2}$. To the extent that it is unlikely that $\langle r^2 \rangle_n^{1/2} < \langle r^2 \rangle_p^{1/2}$, so it is likely that M_2'' exceeds the shell-model value.

VI. CORRELATIONS AND THE SUM RULES WITH RESTRICTED ISOSPINS

A. Sum Rules for Separate Isospins

In practice, the diagonalization of TD or RPA matrices for separate isospins has rarely been attempted. If j_0, j_u, j_e are wholly occupied, wholly unoccupied, and neutron excess orbits, separation of isospins requires that the basis include states of type $(j_0^{-1} j_e)_p (j_e^{-1} j_u)_n$ in addition to $(j_0^{-1} j_u)_p$ and $(j_0^{-1} j_u)_n$. When isospins are not distinguished, the former may be omitted since they have no dipole strength. (They are in the same energy region as the other, so can cause some splitting and redistribution of strength.) They must be included to separate isospins however, and this increases the size and complication of the matrices to be diagonalized. Only ^{48}Ca, ^{88}Sr, ^{90}Zr have been studied[24,25] for separate T. One could also include ^{208}Pb since only 2 of the 35 basis states of energy $1\hbar\omega$ are of type $j_0^{-1} j_u$, i.e., almost all basis states are of pure isospin ($T=22$) so that the usual TD results are relevant to this isospin.

In view of the absence of calculations for separate isospins, it is very useful to have a method for estimating by other means. In particular, we may calculate $M(\|T'\|)$ as $N(\|T'\|)/\bar{E}_{\mathrm{TD}}(T')$, i.e., apply the result of Sec. IV for M to separate isospins. (The equations and analysis in the present section refer to the dipole operator referred to well center. They also hold for the operator referred to centroid if D is replaced by $\mathfrak{D}$ and double-primed quantities are used.) The derivation is the same, and the correction is second order if the distribution of a_λ^2 for the separate isospins is strongly peaked. [For the few cases mentioned where isospins have been separated,[24,25] this is true.] It is important to realize that, while $N = N^{(0)}$ to second order, it is not true that $N(\|T'\|) = N^{(0)}(\|T'\|)$ to second order since $N_1(K)$ has first-order corrections. Thus, for given $\bar{E}(T')$ the relations $N(\|T'\|) = M(\|T'\|)\bar{E}(T')$ do not provide values of $M(\|T'\|)$ from values of $N(\|T'\|)$, but rather constitute three relations between the six sum-rule quantities $N(\|T'\|)$, $M(\|T'\|)$. To separate the quantities which are readily evaluated from those that are not, we replace these quantities by the six quantities N_Y, M_Y ($Y=0, 1, 2$). Let us write $\bar{E}(T) = \epsilon_0$, $\bar{E}(T+1) = \epsilon_0 + (T+1)\delta_+$, $\bar{E}(T-1) = \epsilon_0 - T\delta_-$. Then the three relations give, using (41) and (46):

$$\begin{aligned}
&\tfrac{1}{2}\langle B \rangle = \langle D^2 \rangle (\epsilon_0 + \delta_+) - (M_1 - M_2)\delta_+ , \\
&(N_1 - N_2) = -\langle D^2 \rangle T\delta_+ + (M_1 - M_2)(\epsilon_0 + T\delta_+) , \\
&(N_1 - N_2) + 2\left(\frac{2T+1}{2T-1}\right) N_2 = -T\delta_- \langle D^2 \rangle \\
&\qquad - (M_1 - M_1)[T+1)\delta_- - \epsilon_0] \\
&\qquad + \left(\frac{2T+1}{2T-1}\right) 2(\epsilon_0 - T\delta_-) M_2 .
\end{aligned} \tag{71}$$

These may be supplemented by the inequalities resulting from the positive-definite nature of $M(\|T+1\|)$, $N(\|T+1\|)$, viz., $M_1 - M_2 \leq \langle D^2 \rangle$, $N_1 - N_2 < \frac{1}{2}\langle B \rangle$. (Actually, given one of these two inequalities, the above relations imply the other.)

As a special case, we have $\delta_+ = \delta_-$ ($=\delta$, say); this corresponds to a simple symmetry-type splitting of $\bar{E}(T')$, and leads to

$$\begin{aligned}
&\tfrac{1}{2}\langle B \rangle = \langle D^2 \rangle (\epsilon_0 + \delta) - (M_1 - M_2)\delta , \\
&N_1 = -\langle D^2 \rangle T\delta + M_1(\epsilon_0 + \tfrac{1}{2}\delta) - M_2 \delta \left(\frac{2T+1}{2}\right) , \\
&N_2 = -\left(\frac{2T-1}{2}\right)\delta M_1 + (\epsilon_0 - \tfrac{1}{2}\delta) M_2 .
\end{aligned} \tag{72}$$

The three relations may be applied to observed quantities to try to give observed values for the six quantities, or they may be applied to theoretical estimates. The input and output are different in the two cases:

(i) Observed values. Given $\bar{E}(T')$ and the observed values of $\langle B \rangle$, $\langle D^2 \rangle$, M_1, the three relations are sufficient in principle to determine observed values of the quantities, N_1, N_2, M_2. Unfortunately, this cannot be done in practice since the first equation gives M_1-M_2 as the difference between two large numbers, both with experimental errors, i.e., the first equation essentially confirms that $\frac{1}{2}\langle B \rangle = \langle D^2 \rangle \epsilon_0$, and gives no new relation. The last two equations provide relations between the three

quantities N_1, N_2, M_2 but do not determine them. If we accept the theoretical estimate of N_2, indicating that it is negligible in the last equation (Sec. VII), then N_1, M_2 can be evaluated.

(ii) Theoretical values. First we note that, in the case of the shell model without two-body forces, the three relations reduce to (11) and (54). Putting $\delta = 0$, $\epsilon_0 = \hbar\omega$:

$$\frac{\frac{1}{2}\langle B\rangle}{\langle D^2\rangle} = \frac{N_1}{M_1} = \frac{N_2}{M_2} = \hbar\omega\,. \tag{73}$$

When first-order correlation effects are present, we assume that $\bar{E}(T')$ are given by the Tamm-Dancoff values $\bar{E}_{\rm TD}(T')$. The quantities $\langle B\rangle$, N_2 are readily specified to second order (Sec. IV), so the three equations give us three relations between the four quantities $\langle D^2\rangle$, M_1, M_2, N_1. As we have seen above, the first equation essentially gives $\langle D^2\rangle = \langle B\rangle/2\bar{E}_{\rm TD}(T)$, so the last two equations give two relations between the three quantities M_1, M_2, N_1. If we insert the observed value of M_1, then M_2, N_1 may be evaluated. This last step requires that the two-body forces used in the evaluation of N_2 are consistent with the observed one-body densities in M_1.

In this discussion of the evaluation of sum-rule quantities for separate isospins, we have assumed that the mean energies $\bar{E}(T')$ can be specified and used as input. The important subject of the isospin splitting of the dipole peak has been previously discussed qualitatively by Bohr and Mottelson,[26] and more quantitatively by Akyüz and Fallieros.[7] The next section is devoted to this topic.

At this stage, some remarks on the isospin purity are relevant. We have noted (Sec. IV) that it is acceptable to use isospin-impure wave functions (like the self-consistent shell-model wave function) for the evaluation of $\langle D^2\rangle$, since this does not depend on its definition or evaluation on isospin purity. For M_0, M_1, M_2, however, the situation is different because these functions are set up for isospin-pure states. Further, if one tries to evaluate them with isospin-impure states, one can see that drastic undesirable effects may appear. Thus one should evaluate these quantities with isospin-pure states (except for the combination $M_0 - \frac{2}{3}M_2$, since this is $\langle D^2\rangle$).

In zeroth order (i.e., when working with pure shell-model states), this means that one must not use the self-consistent potential, but rather a nonphysical potential that is the same for neutrons and protons. However, in first order the unpleasant feature disappears since we can assume that the first-order wave function has isospin purity restored to high accuracy, whatever potential is used. In particular, it is quite in order to evaluate M_0, M_1, M_2 from the self-consistent potential, if first-order effects are included.

B. Specification of $\bar{E}_{\rm TD}$ (T')

We have seen in Sec. VI A that detailed calculations of the TD or RPA types have rarely been done for separate isospins, also that the mean energies $\bar{E}(T')$ are essential input parameters for evaluating sum-rule quantities.

In the absence of calculations, we try to set up a systematic prescription for $\bar{E}_{\rm TD}(T')$ as a function of A and T'. In fact, this has already been done,[7] and we merely justify and slightly modify the previous result. The basic starting point is the Brown-Bolsterli result for the collective displacement of a state. If matrix elements $\langle {\rm ph}^{-1}|v|{\rm p'h'}^{-1}\rangle$ are approximately separable into a product of the matrix elements $g d_{\rm ph} d_{\rm p'h'}$ of the dipole operator, then the collective shift is $g\sum d_{\rm ph}{}^2$, which is $gM^{(0)}$, where $M^{(0)} = \langle\Phi_0|D^2|\Phi_0\rangle$, the shell-model value of M. In order to apply this result to separate T' one must check that the separate Tamm-Dancoff problems have the essential character of 1p-1h diagonalizations. We now do this.

First let us consider $T' = T$ and $T+1$. As an example, consider ^{208}Pb. Almost all of the 1p-1h states of oscillator energy $1\hbar\omega$ involve the excess orbits so need no supplementary 2p-2h states. They have pure isospin T (= ground-state isospin). Exceptions are $1h_{1/2}{}^{-1}\,1i_{11/2}$ and $1h_{11/2}{}^{-1}\,2g_{9/2}$. For each of these, two 1p-1h and six 2p-2h states combine to give one state of isospin $T+1$ and seven of isospin T. In the isobaric nucleus ^{208}Tl, in which states $T+1$ occur but not states T, one sees that the counterparts of the $T+1$ states are of pure 1p-1h type, viz., $(1h_{11/2}{}^{-1})_p(1i_{11/2})_n, (1h_{11/2})_p(2g_{9/2})_n$. Thus the structure of the $T+1$ states may be approached as a 1p-1h diagonalization by considering the isobaric nucleus. The occurrence of the twelve 2p-2h states partially disturbs the structure of the isospin-T states in ^{208}Pb which cannot be described as pure 1p-1h superpositions. However, the effect of these 2p-2h states is very small. Both of the cited configurations have very small dipole matrix elements (because of the spin flip for $1h_{11/2}{}^{-1}\,1i_{11/2}$ and because of poor radial overlap for $1h_{11/2}{}^{-1}\,2g_{9/2}$). For the same reasons, and because of their higher energy, these configurations couple weakly to the others and have little effect on the diagonalization of isospin-T states; in particular, little effect on the giant dipole state.

Now let us consider states of $T-1$ occurring as dipole excitations of the analog state in ^{208}Bi. Generally these are mixtures of $(1p)_p(1h)_n$ and

those 2p-2h states of the same energy. The latter have mostly isospin $T-1$, and there are more of them, so there are more $T-1$ than T states in the basis (by about a factor of 6). Nevertheless, the T states have nearly all the dipole strength as evidenced by the result $M(T) \approx TM(T-1)$. This is explained by the fact that the $T-1$ states have strength through their 1p-1h components. The 2p-2h states contribute nothing to $M(T-1)$, as can be seen by writing:

$$D(T_-\phi_0) = T_-(D\phi_0) + [D, T_-]\phi_0 \ . \tag{74}$$

The two-body operator T_-D gives states T, $T+1$, while $[D, T_-]$ is a one-body operator exciting mostly $T-1$ states. We know that diagonalization of the 1p-1h and 2p-2h states gives rise to a strong upward collective shift of at least one state, viz., the analog of the dipole state (isospin T) of the parent, $T_-(D\phi_0)$; this is the first of the above two terms. (Since such dipole states are a systematic feature of all low-lying states of nuclei, we also expect five more states to separate, viz., those of $T-1$ corresponding to the dipole states of the five "configuration" states in ^{208}Bi.) We also expect that the $T-1$ part of $D(T_-\phi_0)$, which is mostly the 1p-1h state $D_-\phi_0$, also has the same upward shift. We expect that this state would also have this shift if 2p-2h states were dropped from the diagonalization, i.e., the shift results from the interaction of $(1p)_p(1h)_n$ states, just as for the T states of the parent system.

Now we return to the problem of setting up a systematic formula for $\bar{E}_{TD}(T')$. Akyüz and Fallieros[7] give

$$\bar{E}_{TD}(T') = E_{ph} + \tfrac{1}{2}[T'(T'+1) - T(T+1) - 2]\, V_1/A + gM^{(0)}(\|T'\|); \tag{75}$$

E_{ph} is the mean particle-hole energy ($\approx \hbar\omega$). The second term is the symmetry splitting that is always present to split states of the same configuration (or kinetic energy) and different T'. In the diagonalization of two-body forces amongst the 1p-1h and 2p-2h states, it emerges automatically as a collective effect arising from coherent off-diagonal terms between any given 1p-1h state and those 2p-2h states of the same nucleonic configuration. The last term in the formula represents the collective shift and splitting of the dipole states. This would also automatically emerge in a numerical diagonalization. The difference with the symmetry term is that the latter exists for each nucleonic 1p-1h state, while the dipole shift emerges from coherent effects between different 1p-1h states. Note that the collective dipole shift contains the reduced sum rule, not $M^{(0)}(T', T_3)$; this must be so since we are dealing with a charge-independent property of H, which does not change with T_3.

The essential feature of the formula is the extension to separate isospins T' of the Brown-Bolsterli result that the shift is proportional to sum-rule strength $M^{(0)}$. The above discussion, showing that the separate diagonalizations are essentially between 1p-1h states, justifies this extension. The only remaining issue is whether the constant of proportionality, g, might depend on T'. At first sight this seems likely. The diagonalization problem for $T' = T$ states involves interaction of 1p-1h states, both between like nucleons ($T=1$) and unlike nucleons (mixture of $T=0$, 1). In contrast, the problems for $T' = T \pm 1$ involves interaction between unlike nucleons only, e.g. for $T+1$ the problem is essentially a diagonalization between (neutron-particle, proton-hole) states. However, for zero-range forces, one can show that the combination of W, M, B, H occurring in the collective shift in the case $T' = T$ is the same as that occurring in the matrix elements of the $T' = T \pm 1$ cases, viz., $(W+M) + 2(B+M)$. (Note that this is also the combination occurring in V_1.) Thus there is no general reason from this viewpoint why g should depend on T'.

While accepting the general spirit of the above formula, there are two modifications for $T' = T+1$ that are expected and required in practice:

(1) It is clear that the form of $\bar{E}_{TD}(T')$ needs modification in heavy nuclei. In the form given, $\bar{E}_{TD}(T+1)$ for ^{208}Pb is several MeV below the lowest 1^- state of $T' = T+1$ at 25 MeV. The origin of this anomaly is easy to find. The value of E_{ph} relevant for $T' = T+1$ is much higher than for $T' = T$. The energies of the $1h_{11/2}^{-1}1i_{11/2}$ and $1h_{11/2}^{-1}2g_{9/2}$ states in ^{208}Pb are about 13 MeV instead of the average of 7 MeV for other transitions. No such effect occurs in ^{90}Zr, since it depends on a large neutron excess; 1p-1h states needed for $T' = T+1$ involve excitation across the neutron excess. For a systematic treatment one can add a term $\delta_{T',T+1}\epsilon_{ph}$ with $\epsilon_{ph} \approx \frac{1}{4}T$ MeV to a sufficient approximation.

(2) Another modification is suggested by the calculation[25] on ^{90}Zr. The collective shifts implied by the results suggest that g is smaller for $T' = T+1$ than for $T' = T$. E_{ph} is about 10.5 MeV and V_1 is taken as 120 MeV, whence

$$\begin{aligned} \bar{E}_{TD}(T) &= 9.2 + gM^{(0)}(\|T\|), \\ \bar{E}_{TD}(T+1) &= 16.0 + gM^{(0)}(\|T+1\|). \end{aligned} \tag{76}$$

Taking the value of $\bar{E}_{TD}(T')$ from the main peaks in the dipole strength, we get 15.5 and 19 for $T' = T$, $T+1$ implying shifts of 6.0 and 2.5. The ratio of these is less than the value $M^{(0)}(\|T+1\|)/M^{(0)}(\|T\|)$

$\simeq 0.60$ from Table I. Thus the effective value of g for $T' = T+1$ is apparently less by a factor $\approx \frac{2}{3}$. This is not surprising since the basis of the above formula is the schematic model in which the interaction matrix elements are separable: $\langle \mathrm{ph}|v|\mathrm{p'h'}\rangle \propto z_{\mathrm{ph}} z_{\mathrm{p'h'}}$. Inspection of calculated matrix elements shows this to apply well to states ph with large values of z_{ph}, but less well to states of small z_{ph}. Thus, for the latter states, matrix elements are more random and less likely to give a large shift. To go to the extreme case of ^{208}Pb, there are only two 1p-1h states giving $T+1$, both with very small z_{ph}. The collective shift is essentially zero. We may take $\frac{2}{3}g$ for $T' = T+1$ for all nuclei as a working guide. For ^{208}Pb, the value of $M^{(0)}(\|T+1\|)$ is so small that the value of g is of little significance.

C. Use of Modified Akyüz-Fallieros Formula and Evaluation of Sum Rules

With modification (1) and using (41) along with (56), we may write (75) thus:

$$\overline{E}_{\mathrm{TD}}(T') = E_{\mathrm{ph}} + gM_0^{(0)} + \tfrac{1}{2}[T'(T'+1) - T(T+1) - 2]\left(\frac{V_1}{A} - g\frac{M_1^{(0)}}{T}\right) + \delta_{T',T+1}[\epsilon_{\mathrm{ph}} - (g - g_{T+1})(M_0^{(0)} - M_1^{(0)})]. \tag{77}$$

The quantity g_{T+1} is the effective value of g for $T' = T+1$; as discussed above, we will take $g_{T+1} = \frac{2}{3}g$. g itself is fixed by requiring that the collective shift for $T' = T$ equals the observed value, which is about 6 MeV for all nuclei:

$$gM_0^{(0)} \approx 6 \text{ MeV}. \tag{78}$$

E_{ph} is $\hbar\omega \approx 41A^{-1/3}$ MeV. V_1 is usually taken to be ≈ 120 MeV, while ϵ_{ph} is (see above) $\approx \frac{1}{4}T$ MeV. With these values, and Table I, we find that gM_1/T is about $0.37V_1/A$, $0.45V_1/A$ for ^{90}Zr, ^{208}Pb, respectively. Thus the net T splitting is reduced below that from the symmetry effect by a factor ≈ 0.41 (Akyüz and Fallieros[7] quote ≈ 0.6). The last term is negligible for ^{90}Zr (due to cancellation of the two parts, each ≈ 1.2 MeV); for ^{208}Pb, it is dominated by the term $\epsilon_{\mathrm{ph}} \approx 5.5$ MeV, if Table I is used for $M_0^{(0)} - M_1^{(0)}$. The best values of $\overline{E}_{\mathrm{TD}}(T')$ are given in Table II for these two nuclei.

TABLE II. Values of $\overline{E}_{\mathrm{TD}}(T')$ from the modified Akyuz-Fallieros formula. All values in MeV. V_1 is taken as 120 MeV.

	Input			Output $\overline{E}_{\mathrm{TD}}(T')$		
	E_{TD}	ϵ_{ph}	$gM_0^{(0)}$	$T-1$	T	$T+1$
^{90}Zr	10.5	~1.2	6.0	12.8	15.9	19.6
^{208}Pb	7.0	~5.5	6.5	8.5	13.3	24.5

VII. EVALUATION OF ISOSPIN-RESTRICTED SUM-RULE QUANTITIES FROM EXPERIMENTAL DATA AND FROM PUBLISHED TD CALCULATIONS

We have seen in Sec. V that, once $\overline{E}''(T')$ are given, the observed values of $\langle \mathfrak{D}^2\rangle$, $\langle B''\rangle$, M_1'' can be used, in principle, to deduce the remaining sum-rule quantities M_2'', N_1'', N_2''. In practice, because of errors on experimental values, one cannot do this; rather one estimates a value of $N_2''(U)$, then deduces corresponding values of M_2'' and N_1''.

Again, we will concentrate on ^{208}Pb. From Sec. III, we have the experimental values

$$N'' = 1396 \text{ MeV fm}^2, \qquad M'' \equiv \langle \mathfrak{D}^2\rangle = 93 \text{ fm}^2. \tag{79}$$

To give an experimental value to $M_1'' = \frac{1}{12}(N\langle r^2\rangle_n - Z\langle r^2\rangle_p)$, we will set $\langle r^2\rangle_n = \langle r^2\rangle_p$, and use the reported observed value $\langle r^2\rangle_p = (5.42 \text{ fm})^2$; this gives $M_1'' = 108$ fm^2.

A. Evaluation of M_2''

From (71) and (77) with the dipole operator referred to the centroid and ignoring the $\delta_{T',T+1}$ term

$$M_2'' = (\epsilon_0 - \tfrac{1}{2}\delta)^{-1}[\tfrac{1}{2}(2T-1)M_1''\delta + N_2''], \tag{80}$$

where δ is $[(V_1/A) - (gM_1^{(0)}/T)]$. We evaluate $N_2(U)$ thus, using (52), (16), and (19):

$$\frac{N_2(U)}{N(U)}\frac{N(U)}{N''(K)}N''(K) = \frac{T^2}{2A}\beta''\frac{\hbar^2}{m}\left[\frac{(2H+M)\mathfrak{F}_D}{(2M+H)\mathfrak{F}_E} - 1\right]. \tag{81}$$

If we then estimate the last factor as of order unity (as concluded in Sec. V C 2) and use (62) for $N_2''(K)$, we see that $N_2(U)$ is of order $\beta''N_2''(K)$. Since all estimates of β'' indicate that it is <1, we have, with $M_1 \approx \frac{1}{6}T\langle r^2\rangle$,

$$\frac{N_2''}{\frac{1}{2}(2T-1)M_1''\delta} \approx \frac{42}{A\langle r^2\rangle\delta}. \tag{82}$$

For ^{90}Zr, δ is 1.1 MeV (Sec. VI) and this ratio is 0.023; for ^{208}Pb, δ is 0.21 and the ratio is 0.032. Although the estimate that the factor in square brackets in Eq. (81) is of order unity or less is based on Rosenfeld and Hamada-Johnston forces, it seems very unlikely that other forces could increase this by a factor of 30. [Note that such an increase would imply $N_2(U) \approx N(U)$.] Thus we neglect the N_2'' term and get

$$\frac{M_2''}{\langle \mathfrak{D}^2\rangle} = \tfrac{1}{2}(2T-1)\frac{\delta}{\epsilon_0 - \frac{1}{2}\delta}\frac{M_1''}{\langle \mathfrak{D}^2\rangle}. \tag{83}$$

For ^{208}Pb, using the observed values of $\langle \mathfrak{D}^2 \rangle$, M_1'' quoted above, this ratio is 0.37; for ^{90}Zr, estimating $\langle \mathfrak{D}^2 \rangle$ by its shell-model value $2.79\langle r^2 \rangle/12$, the ratio is 0.10. In each case, the ratio is greater than the shell-model value by about a factor of 10. In the case of ^{208}Pb the large value of M_2'' is clearly very important, while for ^{90}Zr the effects of M_2'' are relatively small.

Now we consider the effect of the term $\delta_{T',T+1}$ in (77). We have, from (71) instead of (80),

$$\frac{M_2''}{2T-1}[(2T+1)\,2\epsilon_0+(2T^2-T)\delta_+-(2T^2+T+1)\delta_-]$$

$$=M_1''\,[(T+1)\delta_-+T\delta_+]-\langle \mathfrak{D}^2\rangle\, T(\delta_+-\delta_-)$$

$$+2\left(\frac{2T+1}{2T-1}\right)N_2'' \quad (84)$$

We drop the term in N_2'' as small for the same reasons as above; upon inserting values of $\delta_+ = 0.49$, $\delta_- = 0.22$ MeV taken from Table II, we find, for ^{208}Pb,

$$M_2''/\langle \mathfrak{D}^2\rangle = 0.47(M_1''/\langle \mathfrak{D}^2\rangle) - 0.17 = 0.37\,. \quad (85)$$

Thus we see that the evaluation of M_2'' is essentially unaffected by the increase in δ_+ over δ_-.

Referring back to (83) and dropping small terms, we have

$$M_2''/M_1'' \approx T\delta/\epsilon_0\,. \quad (86)$$

Thus we see that M_2''/M_1'' is equal to the ratio of the isospin splitting of $\bar{E}(T')$ to the mean value. This comes mainly from the familiar symmetry splitting of states of a given configuration [corresponding to the splitting by the term $2V_1(\vec{t}\cdot\vec{T})/A$ in the optical model]. Thus we see that the isotensor term (which is zero for the shell model) acquires a large value when the most elementary correction to the model is taken consistently into account, viz., the $(\vec{t}\cdot\vec{T})$ splitting effect. Of course, it is quite reasonable that any isotensor effect, which is zero in the absence of isospin splitting, should be proportional to such splitting. We note that the present result is consistent with the previous lower limit (69) for M_2'', viz., $M_2'' \geq 0.16\langle \mathfrak{D}^2\rangle$. Further, the result implies $M(\|T+1\|)/M''(\|T\|) \approx 0.2$, or $M(T+1)/M''(T) \approx 0.01$. The observed $T+1$ states at 25.2, 25.8 MeV have an observed integrated absorption cross section[23] of about 3.3 mb MeV, or about 0.008 times the total amount observed (4070 mb MeV). This implies a contribution to $M(T+1)$ of about $0.0004\,M''(T)$. Thus the two reported states account for only 4% of the value of $M(\|T+1\|)$ derived from our estimate $M_2'' = 0.37\langle \mathfrak{D}^2\rangle$. The calculated dipole strength[13] of the states (based on the shell-model configurations $1h_{11/2}^{-1}1i_{11/2}^{-1}$, $1h_{11/2}^{-1}2g_{9/2}$) is about 3 times the observed value, and about 12% of the estimated $M(\|T+1\|)$ for a Woods-Saxon shape. {For each state, the transition matrix element to the $T+1$ component is $[2(T+1)]^{-1/2}(d_n - d_p)$, where d_n, d_p are the single-particle values for neutron, proton normalized such that the contribution to M is $d_n^2 + d_p^2$.} (For the oscillator, the factor is 5, equivalent to 20%.) Since there is a strong cancellation between $\langle \mathfrak{D}^2\rangle$ and M_1'' involved in the shell-model evaluation (with pure isospin) of $M(\|T+1\|)$, viz., $0.04M(\|T\|)$ (see Table I), it is not surprising that correlation corrections drastically increase the value; in fact, shell models without isospin purity[15] chosen to make $\langle r^2\rangle_n = \langle r^2\rangle_p$, give a value of $(\langle \mathfrak{D}^2\rangle - M_1'')$, which is $\approx 0.20\langle \mathfrak{D}^2\rangle$, i.e., 5 times that for pure isospin. [The value of $M(\|T+1\|) = \langle \mathfrak{D}^2\rangle - M_1'' + M_2''$ cannot be given, since it requires M_2'', which is not known to us for the mixed-isospin shell model.]

Our conclusion that the isotensor term may be large is in conflict with other authors. Some[8,9] have been tempted to hypothesize that the shell-model result[6] $M_2 = 0$ of (56) is more general than the shell model, and can be assumed to hold for actual nuclei. This is a matter of considerable importance since, if it were true, one could infer a value of M_1'' (and thence the crucial nuclear parameter $\langle r^2\rangle_n$) from photonuclear data. We will now discuss in more detail the smallness of the shell-model evaluation of M_2, and see how this smallness may be lost when shell-model wave functions are superimposed. We note that, even with the oscillator shell model, M_2 does not vanish unless the state is of pure isospin (requiring equal neutron and proton potentials). In Sec. V, we showed that $M_2 = 0$ for the case of the oscillator if no downward dipole excitations from the analog are allowed by the Pauli principle. This is true more generally for any shell model under the same conditions. To show this, and to generally examine the magnitude of M_2, it is best to start from[7]

$$M_2 = \tfrac{1}{2}T[\langle \phi_0(TT-1)|\,D^2\,|\phi_0(TT-1)\rangle$$

$$-\langle \phi_0(TT)|\,D^2\,|\phi_0(TT)\rangle\,]\,, \quad (87)$$

which follows from the equality of the scalar part of D^2 in the two terms, plus the fact that the average of a tensor operator in $\phi_0(TT-1)$ is $(T-3)/T$ times that in $\phi_0(TT)$. From this, the shell-model result $M_2 = 0$ follows without any reference to the oscillator. The condition of no downward dipole excitations from the analog means that the excess neutron orbits in the parent are all of one parity, or the j values of opposite parity orbits differ by more than one unit (as in ^{208}Pb). When this is violated, M_2 is nonzero and positive. For shell-model states M_2 will generally be positive (when not zero) since the action of the Pauli principle in removing certain excitations in $D|\,\phi_0(TT_3)\rangle$ can be

expected to be most pronounced when there are the largest number of particles of one sort, i.e., when $T_3 = T$. Thus one expects

$$\langle \phi_0(TT-1)|\, D^2 \,|\phi_0(TT-1)\rangle$$

to be larger than $\langle \phi_0(TT)|\, D^2 \,|\phi_0(TT)\rangle$, in which case $M_2 > 0$. (Note, however, that if the spatial factor is removed from D, the resulting tensor quantity $\langle T^2 - 3T_3^2\rangle$ is strongly negative, so the result $M_2 \geq 0$ depends on the spatial nature of the operator.) Let us now estimate M_2 for an oscillator state for which it is not zero. Consider the ground state of ^{209}Pb.

The difference between the average of D^2 in the parent and analog arises from the downward transitions $2g_{9/2} \to 2f_{7/2}$, $1h_{9/2}$. Since these are two transitions out of roughly $8T$, and since the analog proton is in the $2g_{9/2}$ orbit in only fraction $1/T$ of the analog state, M_2 is $\approx \langle D^2\rangle/2T$. Thus, even when not zero, M_2 is small for shell-model states, at least those of low excitation in actual nuclei. (It may be larger for highly excited states, or those containing mostly one type of nucleon.)

Of course, the fact that M_2 is very small for all low-lying states of pure configuration for physical N, Z does not mean that one can infer M_2 to be generally small for two reasons:

(1) Although the diagonal shell-model elements in M_2 are small, the off-diagonal ones are not. Thus, small admixtures of certain excited states into the shell-model state can increase M_2 drastically. As a schematic situation, suppose that the shell-model state Φ_0 has an admixture of the form $\alpha(Q\Phi_0)$; $\phi_0 = (1 + Q\alpha)\Phi_0$, where Q is the tensor operator such that $M_2 = \langle Q\rangle$. M_2 is increased by $2\alpha\langle Q^2\rangle$; for positive α, this can be much larger than the original M_2.

(2) Superpositions of shell-model states may be such that there is constructive interference of their contributions to M_2 with the result that M_2 is much larger than M_2 of the separate states.

B. Evaluation of $(N_1 - N_2)$

From the second equation (71) for the dipole operator referred to the centroid, we get, using values quoted above of $\langle D^2\rangle$, M_1'', M_2'', δ_+ for ^{208}Pb: $N_1'' - N_2'' \approx -1000 + 1850 = 850$ MeV fm^2. For comparison, $N''(K)$ is 1030 MeV fm^2 and the shell-model value of $N_1''(K) - N_2''(K)$ is 990 MeV fm^2 [Table I and (62)]. The above equations do not permit one to decide whether the reduction from 1050 to 850 is due to correlation corrections to $N_1''(K)$, or to the presence of $[N_1(U) - N_2(U)]$. Estimating the latter quantity in the way that $N_2(U)$ was estimated above we find $[N_1(U) - N_2(U)]$ is of order $0.06N''(K) \approx 60$ MeV fm^2. Thus we conclude that the reduction of $N_1'' - N_2'' \approx 850$ below the shell-model value of $[N_1''(K) - N_2''(K)] \approx 990$ may be due *either* to correlation corrections to $N_1(K)$ *or* the presence of the potential term $[N_1(U) - N_2(U)]$, or to both.

VIII. CHARGE MONOPOLE MODE

The appropriate form of f in this case is $f = r^2$, while $\hat{f}$ is

$$\hat{f} = \left\langle \sum_i t_{3i} r_i^2 \right\rangle \Big/ T_3 . \tag{88}$$

The model-independent quantities are from (43), (49), (50), (51):

$$\begin{aligned} M_{M1} &= \frac{1}{4}\left[N\langle r^4\rangle_n - Z\langle r^4\rangle_p - \frac{(N\langle r^2\rangle_n - Z\langle r^2\rangle_p)^2}{N-Z}\right] \\ &= \frac{1}{4}\left[N(\langle r^4\rangle_n - \langle r^2\rangle_n^2) - Z(\langle r^4\rangle_p - \langle r^2\rangle_p^2)\right. \\ &\quad \left. - \frac{NZ}{N-Z}(\langle r^2\rangle_n - \langle r^2\rangle_p)^2\right], \end{aligned} \tag{89}$$

$$N_{M0}(K) = (\hbar^2/2m)A\langle r^2\rangle ,$$

$$N_{M2}(K) = 0 ,$$

where the subscript M denotes the monopole case.

A. Oscillator Evaluations

The model-dependent quantities M_{M0}, M_{M2}, N_{M1} can be evaluated if the oscillator shell model is used. Exactly as in the dipole case (Sec. V C 1), we argue that, if $F|\Phi_0\rangle$ contains no components of negative excitation energy, both for parent and analog states Φ_0, then all excitations are at the same energy, $2\hbar\omega$, so

$$N_M(\|T'\|) = 2\hbar\omega M_M(\|T'\|) \tag{90}$$

for $T' = T-1$, T, $T+1$, whence:

$$\begin{aligned} M_{M0} &= (2\hbar\omega)^{-1}N_{M0}(K) = \tfrac{1}{4}Ab^2\langle r^2\rangle , \\ N_{M1}(K) &= 2\hbar\omega M_{M1} , \\ M_{M2} &= (2\hbar\omega)^{-1}N_{M2}(K) = 0 , \end{aligned} \tag{91}$$

where $\hbar\omega = \hbar^2/mb^2$. In Table III, we give evaluations for these quantities and others for ^{90}Zr and ^{208}Pb, using the oscillator model. The model-independent quantity M_{M1} may be evaluated more correctly with realistic densities. If we take ρ_n and ρ_p to have the same shape, $\{1 + \exp[(r-a)/c]\}^{-1}$ with $a = 0.6$ fm, $c = 4.8$, 6.5 fm for ^{90}Zr, ^{208}Pb, we find that M_{M1} is reduced to about $\frac{2}{3}$ of the oscillator value based on the same $\langle r^2\rangle$.

From Table III, we see that in heavy nuclei the two-body part of $N_{M1}(K)$ is comparable with the one-body part, that the Pauli principle reduces M_{M0} by about 30%, and that the isovector term in $M_M(\|T'\|)$ or $N_M(\|T'\|, K)$ is comparable with the leading term for $T' = T \pm 1$.

B. Potential-Energy Contributions

These contributions to $N(\|T'\|)$ are given by (46), with $B(U)$, $N_1(U)$, $N_2(U)$ given by (52). The appropriate forms for the direct and exchange integrals $\mathcal{F}_{MD}$, $\mathcal{F}_{ME}$ for the monopole case are given in Appendix A. The remarkable similarity of the integrals to the dipole case means that we can infer values for the monopole case from those for the dipole case. In particular, since $\mathcal{F}_E$ can be assigned an experimental value from the potential contribution to the energy-weighted sum rule, we can infer a value for the monopole parameter $\mathcal{F}_{ME}$. We have, from Appendix A,

$$\langle B_M(U)\rangle = (\mathcal{F}_{ME}/\mathcal{F}_E)\langle B(U)\rangle = \tfrac{12}{5}R^2\langle B(U)\rangle\,, \tag{92}$$

whence:

$$\beta_M \equiv \frac{\langle B_M(U)\rangle}{\langle B_M(K)\rangle} = \frac{3R^2}{5\langle r^2\rangle}\,\frac{\langle B(U)\rangle}{\langle B(K)\rangle}\,\frac{F_{ME}(a/R)}{F_E(a/R)}\,. \tag{93}$$

With $\langle r^2\rangle = \frac{3}{5}R_F{}^2$ and $R = R_F - 0.5$ fm and $F_{ME}(a/R)$ typically 20% less than $F_E(a/R)$—see Tables IV and V—we have

$$\beta_M \approx 0.8\beta\left(\frac{R_F - 1\text{ fm}}{R_F}\right). \tag{94}$$

As seen in Sec. III, experimental values[1,5] of β for $A > 130$ lie between 0.25 and 0.4. For ^{208}Pb the value is 0.34, corresponding to $\langle B_M(U)\rangle/\langle B_M(K)\rangle$ of 0.23.

TABLE III. Oscillator evaluation of various quantities relevant to the charge monopole problem for ^{90}Zr and ^{208}Pb. The same oscillator parameter $\hbar\omega = \hbar^2/\text{mb}^2$ is used for neutrons and protons, in order to be consistent with the fact that the states are assumed to have pure isospin. Note (i) for densities which have the observed property $\langle r^2\rangle_n \approx \langle r^2\rangle_p$, $\hat{f} = \langle r^2\rangle$ and (ii) for realistic densities (see text), the model-independent quantity M_{M1} is such that $M_{M1}/\langle r^2\rangle^2 = 0.60$, 2.75 for ^{90}Zr, ^{208}Pb, respectively.

	^{90}Zr	^{208}Pb
$\frac{\langle r^4\rangle_n}{b^4}$	91/4	10 185/252
$\frac{\langle r^4\rangle_p}{b^4}$	39/2	5055/164
$\frac{N\langle r^2\rangle_n - Z\langle r^2\rangle_p}{(N-Z)\langle r^2\rangle} = \frac{\hat{f}}{\langle r^2\rangle}$	1.40	1.31
$\frac{M_{M0}}{\langle r^2\rangle^2}$ (with Pauli principle)	5.70	10.0
$\frac{M_{M0}}{\langle r^2\rangle^2}$ (Pauli principle ignored)	6.85	14.3
$\frac{M_{M1}}{\langle r^2\rangle^2}$	0.88	4.75
$\frac{N_{M0}(K)}{\hbar^2/m\langle r^2\rangle}$ $(=\frac{1}{2}A)$	45	104
$\frac{N_{M1}(K)}{\hbar^2/m\langle r^2\rangle}$	7.0	50
Two-body part of $\frac{N_{M1}(K)}{\hbar^2/m\langle r^2\rangle}$	0	21

TABLE IV. Finite-size reduction factors for the dipole case as a function of the range a of Yukawa force to nuclear radius R. The values of a/R correspond to heavy, medium, and light nuclei. The exchange factors F_E are calculated with $ak_F = 1.75$. (For $a = 1.41$ fm, this implies a Fermi energy 32 MeV.) For this value, the ratio $\mathcal{F}_E(0)/\mathcal{F}_D(0)$ is 0.037.

a/R	0	0.20	0.26	0.31
$F_D(a/R)$	1	0.44	0.35	0.30
$F_E(a/R, ak_F)$	1	0.78	0.72	0.68

IX. EFFECTIVE CHARGE AND EFFECTIVE MASS

A. Effective $M1$ Charge

The exchange forces responsible for the increase in the N sum rule from its classical value for the dipole excitation are also responsible for a renormalization of other electromagnetic properties of nucleons. It is interesting to note that the effective $M1$ charge involves the same radial integral $\int r^4C^2(k_Fr)v(r)dr$ which appears in the N sum rule of the dipole and monopole modes [in the integral $\mathcal{F}_E(0)$ of (A14)]. This result is obtained by considering the exchange current defined by

$$\vec{J}_{\text{ex}} = \left[U, \sum_{i=1}^{A} ie\vec{r}_i(\tfrac{1}{2} - t_{3i})\right]. \tag{95}$$

When evaluated using a Slater determinant of plane waves, the exchange current of an excess

TABLE V. Finite-size reduction factors for the monopole case a function of the range a of Yukawa force to nuclear radius R. The three values of a/R correspond to heavy, medium, and light nuclei. The exchange factors F_E are calculated with $ak_F = 1.75$. For this value, the ratio $\mathcal{F}_{ME}(0)/\mathcal{F}_{MD}(0)$ is 0.037.

a/r	0	0.20	0.26	0.30
$F_{MD}(a/R)$	1	0.22	0.14	0.10
$F_{ME}(a/R, ak_F)$	1	0.65	0.57	0.53

neutron with momentum $\vec{k}_F$ is

$$\langle \vec{J}_{ex}(\vec{k}_F)\rangle = -Z(e/2)\vec{k}_F \times \int (r^4/R^3)C^2(k_F r)v(r)dr(H+2M). \quad (96)$$

Defining the effective $M1$ charge by equation

$$\langle \vec{J}_{ex}(\vec{k}_F)\rangle = (e/m)e^*_{M1}\vec{k}_F, \quad (97)$$

the quantity e^*_{M1} is simply related to $B(U)/B(K) \equiv \beta$ of the dipole N sum rule. This relationship is, using (A4), (A14), and (53),

$$2e^*_{M1} = \beta, \quad (98)$$

and is valid for a force of arbitrary range and shape. In arriving at this result we have assumed an $N=Z$ core. From it we can also obtain a connection between the change in orbital g factor δg_l due to exchange currents and the β, which reads

$$2\delta g_l = \beta. \quad (99)$$

The last result has also been obtained by Fujita and Hirata.[27]

Using the experimentally determined values[11,15] of β, which range from 0.25 to 0.4, we find that exchange currents change g_l by 12 to 20%. The exchange currents arising from a one-pion-exchange potential result in $\beta=0.2$ and $\delta g_l = 0.1$. For a Rosenfeld force of range $a=1.4$ fm and depth $v_0 = 48$ MeV, $\beta=0.65$ and $\delta g_l = 0.32$, while a Serber mixture gives $\beta=0.4$ and $\delta g_l = 0.2$. The difference between the phenomenological potentials and the one-pion-exchange potential arises from higher-order processes. For example, the charge-exchange part of the Hamada-Johnston potential results in $\beta=0.4$ and $\delta g_l = 0.2$, an increase of 100% above the one-pion-exchange part of this force.

B. Effective Mass

The radial integral $\int r^4 C^2(k_F r)v(r)dr$ also appears in an evaluation of the effective mass at the Fermi energy. However, in this case, Wigner and Bartlett forces will contribute. An evaluation of the one-body isoscalar velocity-dependent potential energy V_α of a nucleon in orbit α interacting with a Fermi-gas core results in the following expression:

$$V_\alpha = A(W - \tfrac{1}{4}M + \tfrac{1}{2}B - \tfrac{1}{2}H)d - A(\tfrac{1}{4}W - M + \tfrac{1}{2}B - \tfrac{1}{2}H)e_\alpha(k_F). \quad (100)$$

The direct and exchange integrals d and $e_\alpha(k_F)$, respectively, are

$$d = \langle k_\alpha k_i | v | k_\alpha k_i\rangle,$$

$$e_\alpha(k_F) = \sum_{k_i} \langle k_i k_\alpha | v | k_\alpha k_i\rangle / \sum_{k_i} 1, \quad (101)$$

where we ignore the difference between k_F for neutrons and protons that exists when $N \neq Z$. The direct integral is independent of the velocity, while the velocity-dependent exchange integral can be reduced to Van Vleck's form[28]

$$e_\alpha(k_F) = \int \frac{3r^2}{R^3} C(k_F r)v(r)\frac{\sin k_\alpha r}{k_\alpha r}dr. \quad (102)$$

For a Yukawa shape, the $e_\alpha(k_F)$ can be evaluated analytically and is

$$e_\alpha(k_F) = \frac{9v_0}{4R^3 k_F{}^3}\left\{\frac{1+k_F{}^2a^2 - k_\alpha{}^2a^2}{2k_\alpha a}\ln\left[\frac{1+(k_\alpha a + k_F a)^2}{1-(k_\alpha a - k_F a)^2}\right] - 2\tan^{-1}\left(\frac{2k_F a}{1+k_F{}^2a^2 - k_\alpha{}^2a^2}\right) + 2k_F a\right\}. \quad (103)$$

Comparing the resulting expression of V_α with a quadratic approximation in k_α, we find that V_α follows the quadratic law very well for $k_\alpha \lesssim k_F$. If we next define the isoscalar effective mass m^* at the Fermi energy through the relation

$$\frac{m}{m^*} - 1 = \frac{m}{k_F{}^2}\frac{\partial V_\alpha}{\partial k_\alpha}\bigg|_{k_\alpha = k_F}, \quad (104)$$

we obtain the following equation:

$$m/m^* - 1 = \tfrac{1}{2}Am(W - 4M + 2B - 2H) \times \int r^4 C^2(k_F r)v(r)dr/2R^3. \quad (105)$$

Comparing this equation with that for $\beta \equiv B(U)/B(K)$ of the dipole N-sum rule, we arrive at the result

$$\frac{m/m^* - 1}{\beta} = 1 - \frac{W+2B}{2H+4M}. \quad (106)$$

For both a Rosenfeld and a Serber mixture and for the one-pion-exchange potential the factor $(W+2B)/(2H+4M) = \frac{1}{4}$, so that

$$\frac{m^*}{m} = \frac{1}{1+\frac{3}{4}\beta} \quad (107)$$

for these forces. With β taken as 0.33, the median of the range of experimental values (Sec. III), this gives $m^*/m = 0.8$. Values cited for typical occupied states are typically 0.6 or 0.7. For orbits near the Fermi surface, there is no clear cut value from experiment or theory although there is a suggestion[29] that m^*/m increases here.

X. CONCLUSIONS

In the present work, we have considered those aspects of dipole and monopole sum rules where previous studies were incomplete or need revision, especially the following:

(i) *Lack of precision in the literature about the extent to which the shell model fits, or fails to fit, the value of* $M'' \equiv \langle \mathfrak{D}^2 \rangle$ *derived from* o_{-1}. We have evaluated this quantity for ^{208}Pb with a shell model, chosen to reproduce the rms radius. The conclusion is that the observed value is 30% less than the shell-model value.

(ii) *Lack of systematic discussion of the effect of two-body correlations on sum-rule quantities, especially M.* Mostly, correlations have been taken into account with the RPA method, which includes a restricted class of correlation diagrams to infinite order. In such studies, the effect of correlations is usually left implicit. Neither the shell-model nor RPA value of $\langle \mathfrak{D}^2 \rangle$ are quoted in many publications. Various incidental questions are unanswered, e.g. to what extent are the effects of correlations taken into account by the lowest- (first-) order terms alone? We have chosen to make a systematic first-order study, rather than use RPA. One reason is that it turns out that the first-order effects can be evaluated simply in terms of a quantity ($\bar{E}_{\mathrm{TD}}$) known from a Tamm-Dancoff calculation, and an evaluation of $N^{(0)}$, the shell-model value of N: $M = N^{(0)}/\bar{E}_{\mathrm{TD}}$. (Other reasons are cited in Sec. V.) It seems quite clear nowadays that the higher-order terms included in an RPA calculation are without significance, and can be dismissed as an unnecessary refinement (at least for high-energy modes). It is more instructive to work with an explicit first-order treatment of correlations.

(iii) *Lack of attempt to evaluate the effect of correlations on the isotensor term* M_2. It has been pointed out[6] some time ago (and subsequently quoted[5,8,9] without extension) that the shell model implies $M_2 = 0$ for most low-lying states of nuclei. This leaves the crucial question: Does the smallness of M_2 depend on the use of the shell model, or is it a property of a wider (correlated) class of wave function? If the latter is true, this is of great importance, since one can hope[8] to evaluate M_1 from data and thereby determine an experimental value of $\langle r^2 \rangle_n$, the neutron rms radius. In Sec. VI, we have used our first-order correlation theory to estimate the effect of correlations on M_2. We find that a large value of M_2 is induced by the simplest correlation effect, the splitting of centroids of different isospin states of the same configuration. This means that the result $M_2 = 0$ is a shell-model pathology, and cannot be used for other models.

(iv) *Lack of attempt to relate the dipole and monopole sum rules.* The monopole sum rules are crucial for the calculation of Coulomb mixing effects, and they have often[30,31] been discussed. However, no attempt has been made to obtain values of parameters from a parallel study of the dipole problem where experimental data exists. It turns out (Sec. VIII) that the two essential unknown parameters of the N-sum rules (the direct and exchange integrals over the two body force) are identical for the two modes for large systems. Although less directly related, the M-sum rules for the monopole problem are illuminated by the study of their dipole counterparts, e.g. the fact that correlations reduce the dipole M by 30% means that one must allow for a similar kind of effect in the monopole case. The evaluation of the monopole sum rules will be given in a study[1] of Coulomb mixing in nuclei.

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APPENDIX A

Potential-Energy Contributions to N Sums

In the text, we see from (46) that the evaluation of the potential-energy contributions $N(\|T'\|, U)$ requires the three quantities $N_0(U)$, $N_1(U)$, $N_2(U)$ or, on using (47), $B(U)$, $A^{+-}(U)$, $A^{-+}(U)$. The quantity $B(U)$ was already implicitly met in Sec. II. From (44), (45), and (7)

$$B(U) = [F,[U,F]] = \sum_{ij} b_{ij}\,, \tag{A1}$$

where b_{ij} is given by (8). If the parent wave function is a Slater determinant, then:

$$\langle B(U) \rangle = \left\langle \sum b'_{ij} \right\rangle_{\mathrm{nas}}\,, \tag{A2}$$

with

$$b'_{ij} \equiv b_{ij}(1 - P^x_{ij}P^\sigma_{ij}P^\tau_{ij})\,, \tag{A3}$$

where nas means evaluated with a product (i.e., nonantisymmetric) wave function. On performing this

$$\langle B(U) \rangle = -NZ(H + 2M)\mathfrak{F}_E(NP)\,, \tag{A4}$$

where N, Z are the neutron, proton numbers of the parent ($N = \frac{1}{2}A + T$, $Z = \frac{1}{2}A - T$) and $\mathfrak{F}_E(np)$ is

the average space-exchange integral

$$\mathcal{F}_E(np) = \sum_{\alpha,\beta} \langle u_\alpha(r_i)\mu_\beta(r_j)\,|\,\tfrac{1}{2}[f(r_i)-f(r_j)]^2 v(r_{ij})\,|\,u_\alpha(r_j)u_\beta(r_i)\rangle \Big/ \sum_{\alpha,\beta} 1\,, \tag{A5}$$

where α, β label proton, neutron spatial orbits.

For A^{-+} and A^{+-}, we find, from (45),

$$\begin{aligned} A^{-+}(U) &= \sum_{i,j} (b_{ij} + x_{ij}^{-+})\,, \\ A^{+-}(U) &= \sum_{i,j} (b_{ij} + x_{ij}^{+-})\,, \end{aligned} \tag{A6}$$

where x_{ij}^{-+}, x_{ij}^{+-} are obtained from b_{ij} by replacing the isospin factor by $2t_{i3}t_{j3} - \frac{1}{2}(t_{i3}+t_{j3})$, $2t_{i3}t_{j3} + \frac{1}{2}\ (t_{i3}+t_{j3})$. The expectation values emerge as

$$\begin{aligned} \left\langle \sum_{i,j} x_{ij}^{-+} \right\rangle &= (2H+M)[Z^2\mathcal{F}_D(pp) - NZ\,\mathcal{F}_D(np)] - (2M+H)Z^2\mathcal{F}_E(pp)\,, \\ \left\langle \sum_{i,j} x_{ij}^{+-} \right\rangle &= (2H+M)[N^2\mathcal{F}_D(nn) - NZ\,\mathcal{F}_D(np)] - (2M+H)N^2\mathcal{F}_E(nn)\,, \end{aligned} \tag{A7}$$

where $\mathcal{F}_E(pp)$ is an exchange integral like $\mathcal{F}_E(np)$ above except that both orbits are proton orbits; similarly, $\mathcal{F}_E(nn)$ contains neutron orbits. $\mathcal{F}_D$ are the average direct integrals

$$\mathcal{F}_D = \sum_{\alpha,\beta} \langle u_\alpha(r_i)u_\beta(r_j)\,|\,\tfrac{1}{2}[f(r_i)-f(r_j)]^2 v(r_{ij})\,|\,u_\alpha(r_i)u_\beta(r_j)\rangle \Big/ \sum_{\alpha,\beta} 1\,. \tag{A8}$$

The terms $\alpha = \beta$ are included in the summations. If we assume that the averages in $\mathcal{F}_D$, $\mathcal{F}_E$ are the same irrespective of whether neutron or proton orbits are involved, then we get expressions (52) in the text.

Evaluation of $\mathcal{F}_D$, $\mathcal{F}_E$ in the Dipole Case

From the short-range nature of the two-body force, we may evaluate these with the Wigner-Seitz procedure. This means replacing spatial states u_α by plane waves within a sphere equal to the nuclear radius R. (The precise value to be assigned to R is discussed below.) The direct integral becomes

$$\mathcal{F}_D = (\tfrac{4}{3}\pi R^3)^{-2}\tfrac{1}{2} \int d^3r_i \int d^3r_j (z_i - z_j)^2 v(r_{ij})\,, \tag{A9}$$

where

$$|r_i| \le R\,, \quad |r_j| \le R\,.$$

We may write this as the product of the leading term for large R times a finite-size correction factor F_D. If a is the range of $v(r)$,

$$\begin{aligned} \mathcal{F}_D(a/R) &= \mathcal{F}_D(0)F_D(a/R)\,, \\ \mathcal{F}_D(0) &= \tfrac{1}{2}R^{-3}\int_0^\infty dr\, r^4 v(r)\,, \\ F_D(a/R) &= \int_0^2 g(x)v(xR)dx \Big/ \int_0^\infty x^4 v(xR)dx\,, \end{aligned} \tag{A10}$$

where

$$\begin{aligned} g(x) &= x^4(1 - \tfrac{3}{4}x + \tfrac{1}{16}x^3) \\ &= x^4(1-\tfrac{1}{2}x)^2(1+\tfrac{1}{4}x)\,. \end{aligned} \tag{A11}$$

For the Yukawa potential

$$v(r) = v_0(a/r)\exp[-(r/a)]\,, \tag{A12}$$

$$\begin{aligned} \mathcal{F}_D(0) &= 3(a/R)^3 a^2 v_0\,, \\ F_D(y) &= (1 - 3y + \tfrac{15}{2}y^3) \\ &\quad - e^{-2/y}(6y^3 + 30y^4 + 72y^5 + 90y^6 + 45y^7)\,, \end{aligned} \tag{A13}$$

where $y = a/r$. Values in Table IV show that the finite size corrections are appreciable.

The exchange integral $\mathcal{F}_E$ is the same as $\mathcal{F}_D$ except that the integrand includes the Wigner-Seitz exchange function $C^2(k_F|r_{ij}|)$ where k_F is the Fermi wave number, and the function $C(x)$ is $(3/x)j_1(x) = (3/x^3)(\sin x - x\cos x)$. We have

$$\mathcal{F}_E(a/R) = F_E(a/R)\mathcal{F}_E(0)\,, \tag{A14}$$

where all of these quantities are obtained from the corresponding direct quantities above by multiplying $v(r)$ by $C^2(k_F|r|)$. This factor causes the integrand to decrease much more rapidly with increasing r_{ij}. This means that the finite-size corrections are less important (although still appreciable). If we take k_F corresponding to

Fermi energy 32 MeV (i.e., $k_F = 1.24$ fm^{-1}), then we get the values given in Table IV. As expected, the finite-size correction is less crucial (but still important) for $\mathfrak{F}_E$.

It is evidently true that the estimates depend quite sensitively on the value of the range a, and therefore put a premium on our choice $a = 1.4$ fm. It is noteworthy however, that any components of $v(r)$ of shorter range will be strongly suppressed by the fourth-power radial weighting in the integrals.

For a Yukawa force, the ratio of exchange-to-direct integrals for an infinite system ($a/R = 0$) can be evaluated explicitly:

$$\mathfrak{F}_E(0)/\mathfrak{F}_D(0) = \tfrac{3}{4}\alpha^4\left[\frac{L}{2}\ln\left(\frac{L+1}{L-1}\right) - 1\right], \qquad \text{(A15)}$$

where $L \equiv 1 + \frac{1}{2}\alpha^2$ and $\alpha \equiv (ak_F)^{-1}$. For $\alpha = 0.4$, 0.5, 0.6, the ratio is 0.014, 0.027, 0.040, respectively.

Check Against Levinger's Shell-Model Results for ^{16}O, ^{40}Ca

Levinger[2] has made detailed shell-model calculations of $B(U)$ for ^{16}O, ^{40}Ca. For a certain recipe for the relative values of parameters k_F and R of the Fermi-gas approach and of $\langle r^2\rangle$ of the shell model, the former *without finite-size correction* reproduces closely the value of the latter. The recipe is that $\langle r^2\rangle = \frac{3}{5}R_F{}^2$ and $R_F = (9\pi A)^{1/3}/2k_F$, where we add the subscript F to R to denote that it satisfies these two relations. With this recipe, the Fermi-gas approach gives values that are 4, 8% too high for ^{16}O, ^{40}Ca when $R_F = 1.2A^{1/3}$. For $R_F = 1.5A^{1/3}$, the values are 9 and 13%. Thus the Fermi-gas model without the factor F_E and with the Levinger recipe for relative values of $\langle r^2\rangle$, k_F, R is accurate to within about 10% for a wide range of choices of R_F. Let us note that the only occurrence of R in $B(U)$, when F_E is removed, is the factor R^{-3}. Thus, comparing our expression (including F_E) with Levinger's (without F_E) for the same value of $k_F a$, we find that they are the same if

$$F_E(a/R, ak_F)R^{-3} = (R_F)^{-3}. \qquad \text{(A16)}$$

In Table IV we have used $ak_F = 1.75$. We see that $F_E(x, y)$ is not very sensitive to $x \approx 0.25$ for fixed $y = 1.75$. We expect a similar lack of sensitivity to y for fixed x. [Since the factors C^2 and v in $\mathfrak{F}_E$ act similarly as radial cutoffs of ranges $k_F{}^{-1}$, a, respectively, we expect them to enter $\mathfrak{F}_E$ roughly in the combination $(1/a + k_F)$ so that F_E will be fixed by $R(1/a + k_F) \equiv (1 + y)/x$.] Taking the value $F_E(0.36, 1.75) = 0.64$, we find that the condition is satisfied if $R = 0.86R_F$. For a shell-model system with $\langle r^2\rangle$ equal to its observed value, the rule $R_F{}^2 = \frac{5}{3}\langle r^2\rangle$ gives $R_F = 1.32A^{1/3}$ fm for $A \approx 40$ (corresponding to $k_F = 1.15$ fm^{-1}). Thus $R = 1.13A^{1/3}$ fm.

Let us now choose R so that our formula reproduces the shell-model value in the case of ^{40}Ca. As mentioned above, this is smaller than the Levinger value by 8–13% depending on R_F. Let us take 10% for $R_F = 1.32A^{1/3}$ fm; then we find R should be 3% larger, viz. $R = 0.89R_F = 1.17A^{1/3}$ fm for ^{40}Ca. We note that this value gives $Rk_F = 4.55$, which is close to the value implied by the arguments used in $F_E(0.36, 1.75)$, viz. 4.85, so we have consistency.

Having established the parametric values that ensure that our formula gives the correct value of $B(U)$ for ^{40}Ca, it remains to extrapolate to larger nuclei. Since, for very large R, $F_E \to 1$ and we know that the Fermi-gas approach becomes exact, we must not extrapolate on the basis of a fixed ratio R/R_F. More appropriate is a fixed difference: $R = R_F - 0.50$ fm. For ^{208}Pb, this prescription gives a value 5% less than that from a simple application of the Levinger prescription to these nuclei. [This follows since $F_E \approx 0.76$, $(R/R_F)^3 = 0.80$.]

Thus our conclusion is that our formulas are accurate (for F_E at least) if R is chosen as $R_F - 0.50$ fm, where $R_F = (9\pi A)^{1/3}/2k_F$) and the shell-model density is such that $\langle r^2\rangle = \frac{3}{5}R_F{}^2$.

Monopole Case

We evaluate the average spatial integrals $\mathfrak{F}_{MD}$, $\mathfrak{F}_{ME}$ as in the dipole case by considering a wave function that is a Slater determinant of plane waves distributed over a sphere of nuclear radius R; this gives

$$\mathfrak{F}_{MD}(a/R) = (\tfrac{4}{3}\pi R^3)^{-2}\tfrac{1}{2}\int\int d^3r_i d^3r_j (r_i{}^2 - r_j{}^2)^2 v(r_{ij})$$
$$= F_{MD}(a/R)\mathfrak{F}_{MD}(0), \qquad \text{(A17)}$$

where

$$\mathfrak{F}_{MD}(0) = \frac{6}{5R}\int_0^\infty r^4 v(r)dr. \qquad \text{(A18)}$$

F_{MD} is the correction factor for finite nuclear size, and a is the range of $v(r)$. For infinite size, $R \to \infty$, $F_D \to 1$. For finite R:

$$F_{MD}(a/R) = \int_0^2 g_M(x)v(xR)dx \Big/ \int_0^x x^4 v(xR)dx \qquad \text{(A19)}$$

where

$$x \equiv r/R,$$

and

$$g_M(x) = x^4(1-\tfrac{1}{2}x)^4(1+\tfrac{1}{6}x)$$
$$= x^4(1-\tfrac{15}{8}x+\tfrac{5}{4}x^2-\tfrac{5}{16}x^3\tfrac{1}{128}x^5)\,. \quad \text{(A20)}$$

This is simply related to the function I_C of earlier work[22]:

$$g_M(x) = \tfrac{10}{3}R^3 I_C(r)\,. \quad \text{(A21)}$$

$g_M(x)$ has a strong maximum at $x \approx 1$. Unless $v(r)$ cuts off very sharply at range $r=a$, the factor x^4 in the integrands means that the major part of the integrals arises from well beyond $r=a$. For the Yukawa form

$$v(r) = v_0 \frac{a}{r} e^{-r/a} \quad \text{(A22)}$$

the maximum of $r^4 v(r)$ is at $r=3a$. Taking $a \approx 1.4$ fm, and $R=3a$, $6a$ for typical light, heavy nuclei, the maximum of $g_M(x)v(xR)$ is at $r \approx 1.5a = 0.5R$ for the light nucleus, at $r \approx 2.1a = 0.35R$ for the heavy nucleus. The explicit form of F_{MD} for the Yukawa form is

$$F_{MD}(y) = 1-\tfrac{15}{2}y+25y^2-\tfrac{75}{2}y^3+\tfrac{105}{2}y^5$$
$$-e^{-2/y}(\tfrac{5}{2}y+20y^2+\tfrac{135}{2}y^3+105y^4+\tfrac{105}{2}y^5)\,, \quad \text{(A23)}$$

where $y=a/R$. For the above parameters, the last term with the exponential factor is negligible for light and heavy nuclei. In Table V, we give values of F_{MD}. Even for large nuclei this is $\ll 1$, showing that the finite size correction to $\mathcal{F}_{MD}$ is always of crucial importance.

The exchange integral $\mathcal{F}_{ME}(a/R)$ is the same as the above $\mathcal{F}_{MD}$ except that the extra function $C^2(k_F r_{ij})$ occurs in the integrand. This cuts off with increasing r_{ij}, so that $\mathcal{F}_{ME}$ is less subject to finite-size modification than $\mathcal{F}_{MD}$. The entries in Table V show this.

An important feature of the above results is that the integrals in $\mathcal{F}_{MD}(0)$, $\mathcal{F}_{ME}(0)$ have exactly the same integrands as in the dipole case. The relations between corresponding quantities are

$$\mathcal{F}_{MD}(0) = \tfrac{12}{5}R^2\mathcal{F}_D(0)\,,$$
$$\mathcal{F}_{ME}(0) = \tfrac{12}{5}R^2\mathcal{F}_E(0)\,. \quad \text{(A24)}$$

Evidently the values given for the ratio $\mathcal{F}_E(0)/\mathcal{F}_D(0)$ apply to the monopole case also.

APPENDIX B: RELATION BETWEEN THE MEAN ENERGY $\bar{E}$ AND THE PEAK ENERGY E_p

The Tamm-Dancoff energy E_{TD} may be regarded as an approximate evaluation of the mean energy $\bar{E} \equiv \langle\phi_0|DHD|\phi_0\rangle / \langle\phi_0|D^2|\phi_0\rangle$, the approximation consisting in the replacement of the true ground state ϕ_0 by the shell-model state Φ_0. It is important to realize that even the exact value of the mean energy $\bar{E}$ is not equal to the energy E_p of the physical giant dipole state; we now consider the relation between these two quantities.

First we define the strength function $s(E)$ describing the distribution of the state $D|\phi_0\rangle / \langle\phi_0|D^2|\phi_0\rangle^{1/2}$ amongst eigenstates ϕ_λ of H:

$$s(E) = \left[\frac{\langle\phi_\lambda|D|\phi_0\rangle^2}{\langle\phi_0|D^2|\phi_0\rangle}\right]_E \rho_\lambda(E)\,, \quad \text{(B1)}$$

where the square brackets indicate average over states ϕ_λ near energy E. A standard result of line-broadening theory is

$$s(E) = \frac{(1/2\pi)\Gamma}{(\bar{E}+\Delta-E)^2+\frac{1}{4}\Gamma^2}\,, \quad \text{(B2)}$$

where

$$\Gamma(E) = 2\pi[\langle n|[H,D]|\phi_0\rangle^2]_E \rho_n(E)\,,$$
$$\Delta(E) = -P\int \frac{(1/2\pi)\Gamma(E')}{E'-E}\,dE'\,. \quad \text{(B3)}$$

Here the states n are eigenstates of QHQ where Q projects out the state $D|\phi_0\rangle / \langle\phi_0|D^2|\phi_0\rangle^{1/2}$. In general, Δ and Γ vary with energy E. However, from normalization and the definition of $\bar{E}$, we have

$$\int s(E)\,dE = 1\,,$$
$$\int Es(E)\,dE = \bar{E}\,. \quad \text{(B4)}$$

These are evidently satisfied in the special case when the set of states n have uniform character for all energies; then Γ = constant, $\Delta = 0$.

As a more realistic case, we may consider that Δ, Γ have a mild long-range energy variation than $s(E)$ has a peak at energy E_p, given by

$$E_p = \bar{E} + \Delta(E_p)\,. \quad \text{(B5)}$$

Since Γ cuts off at $E=0$, this gives a tendency for Δ to be negative. The fit to the data in terms of a Lorentzian shape described in Sec. III implies an experimental value $\Delta \approx -1.7$ MeV. (A similar value results from using a Breit-Wigner shape cutoff at $E=0$ and 50 MeV.)

A positive contribution to Δ comes from the kinetic-energy part of H and those 1p-1h states with energies E_{ph} scattered around an energy δ below E_p. We find

$$\Delta = (2\pi\delta)^{-1}[\langle E_{\mathrm{ph}}{}^2\rangle - \langle E_{\mathrm{ph}}\rangle^2]\,, \quad \text{(B6)}$$

where the angular brackets denote mean value over 1p-1h states with weight factors $z_{\mathrm{ph}}{}^2$. For the oscillator, where all E_{ph} are equal ($=\hbar\omega$), this Δ

is zero. For a more realistic set of levels we find $\langle E_{\rm ph}{}^2\rangle - (\bar{E}_{\rm ph})^2$ is of order 1 MeV, so, with $\delta \approx 6$ MeV, Δ is $\approx$30 keV, i.e., negligible. Thus we infer that Δ is probably dominated by the potential-energy part of H. This is likely to correspond to intermediate states n of energy above E_p, i.e., to a negative value of Δ. However, we have not pursued this numerically to see if the experimental value (≈ -1.7 MeV) can be reproduced by the theory.

APPENDIX C: INSENSITIVITY OF DIPOLE STRENGTH DISTRIBUTION TO FIRST-ORDER CORRECTIONS

In Sec. V, we mention that comparison of TD and RPA results show that the first- and higher-order corrections incorporated in the RPA have very little effect on the distribution of dipole strength, although the summed strength is reduced by 20%. We refer to calculations[20] on ^{208}Pb for illustration. For all states, the amplitude corrections are of the order of 10% of the TD values. Since the TD values are 10 times less for states below the peak than at the peak, this means that the corrections tend to be proportional to the original values, thereby preserving the distribution. Even fine-structure details in the below-peak region are preserved. The only exception is that there is destructive interference in the peak region (reducing strength by 20%), while it tends to be constructive below the peak, increasing strength by about 10%.

Formally the first-order change in $\langle\Phi_\lambda|D|\Phi_0\rangle$ arises from changes in Φ_λ and Φ_0. Since the former change contributes zero in the oscillator approximation (no transitions of energy other than $\hbar\omega$), the latter is expected to dominate, so

$$\langle\phi_\lambda|D|\phi_0\rangle \approx \langle\Phi_\lambda|D|\Phi_0\rangle - \langle\Phi_\lambda|D\frac{Q}{H_0-E_0}H'|\Phi_0\rangle\,. \tag{C1}$$

The above results cited for ^{208}Pb suggest that the last term tends to be 10% of the leading term for states near the peak, and -10% for other states. This simple proportionality is unexpected, and has not been brought out in the literature. One might have expected that states excited by H' were sufficiently complicated that their matrix elements with states Φ_λ were not affected by the extent to which Φ_λ overlaps $D|\Phi_0\rangle$. Apparently this is not so. To develop some feeling for this situation, let us consider the commonly used model in which matrix elements of the two-body force are separable into a product of single-particle dipole matrix elements:

$$\langle {\rm ph}|H'|{\rm p'h'}\rangle = \langle({\rm ph})({\rm p'h'})|H'|\Phi_0\rangle = \xi D_{\rm ph}D_{\rm p'h'}\,. \tag{C2}$$

This model implies that $QH'|\Phi_0\rangle$ is $\xi QD^2|\Phi_0\rangle$. From the viewpoint of the collective model, this state contains two dipole phonons. When inserted in the above correction term the operator D destroys one, leaving the one-phonon state $D|\Phi_0\rangle$. Thus we see that there is a suggestion that the correction term for general state λ depends on the extent to which Φ_λ overlaps $D|\Phi_0\rangle$. Let us now see how this suggestion is borne out by actual results.

Let us label particle-hole states by κ, and their energies and dipole matrix elements by $\epsilon_\kappa, D_\kappa$. If we define the response function for the TD solution

$$S_{\rm TD}(e) = \xi\sum_\lambda \frac{\langle\Phi_\lambda|D|\Phi_0\rangle^2}{e-e_\lambda}\,, \tag{C3}$$

where Φ_λ, E_λ are the TD states and energies, then it is straightforward to solve the dynamical problem with the solution

$$S_{\rm TD}(e) = [1 - S_{\rm ph}(e)]^{-1} - 1\,, \tag{C4}$$

where

$$S_{\rm ph}(e) \equiv \xi\sum_\kappa \frac{D_\kappa{}^2}{e-\epsilon_\kappa}$$

is the particle-hole response function.

The corresponding RPA result is

$$S_{\rm RPA}(e) \equiv \sum_\lambda \frac{\langle\Phi_\lambda|D|\phi_0\rangle^2}{e-E_\lambda} = [1 - S_{\rm ph}(e) - S_{\rm ph}(-e)]^{-1} - 1 + \frac{\xi}{e}\sum_\kappa D_\kappa{}^2\,. \tag{C5}$$

Let us first consider the case where all ϵ_κ are degenerate ($=\epsilon$). It is well known that this corresponds to all dipole strength being in one (collective) state D with properties:

$$e_D = \epsilon + \Delta\,, \qquad \langle\Phi_D|D|\Phi_0\rangle^2 = \sum_\kappa D_\kappa{}^2 \tag{C6}$$

for the TD case where $\Delta \equiv \xi\sum_\kappa D_\kappa{}^2$, and

$$E_D = +(\epsilon^2 + 2\epsilon\Delta)^{1/2}\,, \qquad \langle\phi_D|D|\phi_0\rangle^2 = (\epsilon/E_D)\left(\sum_\kappa D_\kappa{}^2\right) \tag{C7}$$

for the RPA case. If $\Delta \leq \epsilon$, E_D can be approximated as $e_D[1 - \frac{1}{2}(\Delta/e_D)^2]$. For our purpose, the important feature is that the inclusion of first-order effects by RPA causes no redistribution of dipole strength, which is 100% in one state.

Now let us consider that the ϵ_κ are not degenerate. If their density does not cut off outside a confined region, but extends at a low level on

either side (e.g. as with a Gaussian form), then we can conveniently describe the distributions with strength functions defined with Lorentzian average, e.g. in the RPA case

$$s_{\mathrm{RPA}}(e) \equiv \xi\,\delta \sum_{\lambda} \frac{\langle \phi_\lambda | D | \phi_0 \rangle^2}{(E_\lambda - e)^2 + \delta^2} \ . \tag{C8}$$

δ is an energy a few times larger than the spacing of E_λ. We find

$$s_{\mathrm{TD}}(e) = \frac{s_{\mathrm{ph}}}{[1-d(e)]^2 + (\pi s_{\mathrm{ph}})^2} \ ,$$
$$s_{\mathrm{RPA}}(e) = \frac{s_{\mathrm{ph}}}{[1-d(e)-d(-e)]^2 + (\pi s_{\mathrm{ph}})^2} \tag{C9}$$

where $s_{\mathrm{ph}}(e)$ is the strength function of the original strength $D_\kappa{}^2$ and

$$d(e) \equiv P \int \frac{s_{\mathrm{ph}}(e')}{e - e'}\, de' \ . \tag{C10}$$

If the collective shift $\Delta \equiv \xi \sum_\kappa D_\kappa{}^2 = \int s_{\mathrm{ph}}(e) de$ is greater than the width of the distribution of s_{ph}, then one can see that $d(e)$, which vanishes at the centroid (ϵ, say) of s_{ph}, rises above unity on the upper side of the distribution. It passes unity inside the half-width point, and again near $e = \epsilon + \Delta$, the latter corresponding to the collective state.

The first-order difference between the TD and RPA approaches is represented by $d(-e)$. This is typically of order $\Delta/2\epsilon$. Although of first order, it is increased by the coherent collective effect represented by Δ; nevertheless we can see that its effect on the distribution is small. For e inside the distribution of s_{ph}, $s_{\mathrm{ph}}(e)$ is typically of order (Δ/ω) where ω is the width of the distribution of s_{ph}. Observed values of the collective shift Δ are $\gtrsim \omega$; in ^{208}Pb, $\Delta \approx 6$ MeV, while $\omega \approx 2$ or 3 MeV. Thus $(\pi s_{\mathrm{ph}})^2$ is a large number of order 50, implying that $s_{\mathrm{TD}}/s_{\mathrm{ph}}$ is small. In contrast $d(-e)$ is of order $-\Delta/2\epsilon \approx -0.4$, while $d(e)$ is zero near ϵ with maximum and minimum values of about $\pm 3\Delta/2\omega$ at $\epsilon \pm \frac{1}{2}\omega$. With $d \approx 2\omega$, the effect of the term $d(-e)$ on $s(e)$ is $\lesssim 10\%$ for e inside the distribution of s_{ph}. Because of the dominance of the term $(\pi s_{\mathrm{ph}})^2$ over $(1-d)^2$, any fluctuation of s_{ph} from its general over-all peaked shape will tend to produce an opposite effect on s_{TD}, i.e., a peak will produce a dip, and vice versa. The inclusion of $d(-e)$ will not affect this conclusion. Thus, to this extent, we see how s_{TD} and s_{RPA} will show parallel fluctuations.

We can see why the inclusion of $d(-e)$ causes opposite effects on the strength in the peak and below-peak regions. In the latter, the effect is to cause the zero of $(1-d)^2$ to move upwards to the region where s_{ph} is smaller, thereby giving an increase in s_{RPA} over s_{TD}. In the peak region, the effect causes the zero to move downwards to where s_{ph} is larger, giving a decrease in s_{RPA} relative to s_{TD}.

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