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Structure-Invariant Perturbation Theory in Three-Particle Scattering*

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A perturbation theory for three-particle scattering is formulated which in each order of approximation leaves invariant the essential structural features of the scattering amplitudes. The correct primary and secondary singularity structure is reproduced along with unitarity (both off shell and on shell), the correct threshold behaviors, and the correct residues of the double-scattering poles in the 3-to-3 amplitude. The only integral equations to be solved are of the sort which appear in the case of finite-rank two-particle interactions. The formalism, which is related to some recent work by Noyes, depends crucially on the proof of the existence of a decomposition of the two-particle transition operators into so-called essential and residual parts. The special case when the residual part is ignored provides a practical exploitation of the separable expansions found by Fuda and Osborn.

I. INTRODUCTION

An interesting and practical alternative to the full complexity of exact three-particle scattering calculations exists in the form of perturbation theories.[1] In Ref. 1 the general formulation and relative features of two distinct types of perturbation theory, one due to Alt, Grassberger, and Sandhas (AGS)[2] and the other due to Sloan,[3] were discussed. Both theories treat part of the two-particle transition operators t_α as "weak" and handle that part perturbatively while the "dominant" portion is, in effect, dealt with in an exact fashion.

The usefulness of the perturbative approach is indicated by Pieper's[4] calculations of elastic nucleon-deuteron (N-d) scattering. With the use of the Sloan method (in first order) and regarding the N-N tensor force and P-wave components as perturbations Pieper achieved the first reasonable predictions of all of the measured spin observables in N-d scattering.[5,6] The mode of calculation and the results obtained provided strong support for earlier conjectures[7] as to the physical origin of the behavior of these observables for low-energy N-d elastic scattering; further and decisive support was obtained through Doleschall's monumental exact calculation.[6]

Bencze and Doleschall[8] using the AGS method (in first order) and treating only the P-wave parts of the t_α as perturbations have recently shown that this technique also reproduces the results of the exact calculations[6] rather well. With identical N-N input it appears that the results of these authors are in somewhat closer agreement, particularly for the nucleon polarization, with the exact calculations than are those of Ref. 4.[9] This would seem to indicate that (in lowest order) the AGS method is superior to the Sloan theory at least with the treatment of the tensor force used in Ref. 4. A definitive comparison would involve using Sloan's procedure but with only the P-wave parts of the t_α treated perturbatively.

The AGS technique has several advantages not possessed by the Sloan method. For example, the former has relatively simple properties with regard to estimating the validity of the perturbation approach, carrying out higher-order approximations, and in the perturbative determination of the breakup amplitude. In addition, the Sloan perturbation theory violates unitarity in any finite order. However, we show that with a suitable choice of the dominant and weak parts of the t_α, which we shall term *essential* and *residual*, respectively, the AGS perturbation theory yields fully unitary (off-shell as well as on-shell) three-particle amplitudes in any order of the perturbation.

However, unitarity is only one aspect of the general singularity structure which a proper set of three-particle scattering amplitudes possesses. Because this singularity structure is determined only by the "essential" part of the t_α operators the preceding perturbation theory generates a succession of approximate but structure-invariant solutions each of which preserves the basic physical characteristics of the scattering. In particular, the threshold behaviors are necessarily the same as for the exact amplitudes as well as the location and residues of the physical-region double-scattering poles in the 3-to-3 amplitude.

Our development has much in common with an investigation by Noyes,[10] particularly in the exploitation of the essential-residual decomposition of t_α to obtain forms of the three-particle equations which provide a vehicle for generating a class of unitary solutions. However, our emphasis and final equations differ and the latter provide a practical means for calculating the scattering generated by the residual parts of the t_α. Moreover, we find that in order to carry out our development (or, in fact, that of Noyes[10]) *in general,* we have to exercise considerable care in the definition of the essential-residual decomposition. This, in addition to our main result of providing us with a structure-invariant form of the AGS perturbation theory, yields the realization of a practical means for exploiting the separable expansions of Fuda[11] and Osborn.[12]

II. ESSENTIAL AND RESIDUAL PARTS OF THE TWO-PARTICLE TRANSITION OPERATORS

In this section we establish the existence of a decomposition of the two-particle transition operator $t(z)$. (The latter is defined on the relative two-particle space and so we omit the particle-index subscript.) Only a few new results are established beyond what is already contained in the voluminous literature on two-particle off-shell scattering. However, these are crucial for establishing the general validity of our later investigation of three-particle scattering, and they also provide a means for actually exploiting the separable, although possibly term-by-term singular, expansions found by Fuda[11] and Osborn.[12]

$t(z)$ has the following properties[13]:

(i) $t(z)$ is operator analytic in the entire z plane cut from 0 to $+\infty$ except for possible bound-state poles for negative z. It is bounded on the cut and, except within arbitrarily small domains about the bound-state poles, for negative z as well.

(ii) Reflection property:

$$t(z^*)=t(z)^\dagger .$$

(iii) Off-shell unitarity:

$$\begin{aligned}\Delta t &\equiv t(+)-t(-)\\ &=-2\pi i t(\pm)\delta(E-K)t(\mp)\end{aligned}$$

across the cut, where K is the (relative) two-particle kinetic-energy operator. For z negative, $\Delta t=0$ except at the bound-state poles.

(iv) The residues of the bound-state poles of $t(z)$ are rank unity operators.

Next let us write

$$t(z)=t^e(z)+t^r(z), \tag{2.1}$$

where e and r refer to *essential* and *residual,* respectively. The principal result of this section is the proof that a decomposition of the form (2.1) exists with $t^e(z)$ and $t^r(z)$ satisfying the following conditions:

(i′) $t^e(z)$ contains the (possible) bound-state poles of $t(z)$.

(ii′) $t^e(z)$ and $t^r(z)$ are bounded on the cut and, except for z within some arbitrarily small domains about the bound-state poles in the case of $t^e(z)$, for negative z as well.

(iii′) $t^{e,r}(\pm)=t^{e,r}(\mp)^\dagger$ for all real z.

(iv′) $\Delta t^e=\Delta t$ for all real z.

(v′) For real z the projections of both $\theta(z)t^e(z)$ and $\theta(-z)t^e(z)$ onto states of well defined angular momentum are finite-rank operators although not necessarily of the same rank.

(vi′) For z positive and for either $|\vec{p}\,'|$ or $|\vec{p}|$ equal to $|z|^{1/2}$

$$\langle\vec{p}\,'|t^r(z)|\vec{p}\rangle=0 .$$

It follows from (iii′), (iv′), and (vi′) that

$$\Delta t^r=0 \quad \text{for all real } z ,$$

and across the cut

$$\begin{aligned}\Delta t^e &= t^e(+)-t^e(-)\\ &=-2\pi i t^e(\pm)\delta(E-K)t^e(\mp) .\end{aligned}$$

It is also easy to show as a consequence of the preceding properties that the resonance content of $t(z)$ (if any) resides in $t^e(z)$.[14]

Let us now prove the existence of a decomposition (2.1) with properties (i′)–(vi′).

A. Positive Energy

For the sake of simplicity we consider only the single-channel scattering of spinless particles. The off-shell amplitude in a state of definite orbital angular momentum, the index for which we suppress, will be denoted by $t(p,q;z)$. The extension to the case of where there exists a coupling between spin and orbital angular momenta is entirely straightforward. For positive $z=k^2\pm i\epsilon \equiv k^2\pm$, it is well known that off-shell unitarity and time-reversal invariance imply that

$$t(p,q;k^2\pm)=f(p,k)\hat{t}(k^2\pm)f(q,k)+R(p,q;k^2), \tag{2.2}$$

where f and R are real and continuous across the cut and satisfy

$$f(k,k)=1 ,$$

$$R(p,k;k^2)=R(k,q;k^2)=0 ,$$

and

$$\hat{t}(k^2\pm)\equiv t(k,k;k^2\pm) .$$

The functions f and R are well behaved except for those k for which $\hat{t}(k^2\pm)=0$.[11,12,14,15] For such k the singularities in R and in $f(p,k)\hat{t}(k^2\pm)f(q,k)$ must cancel to yield a bounded $t(p,q;k^2\pm)$. Also the half-off-shell amplitudes $f(p,k)\hat{t}(k^2\pm)$ and $\hat{t}(k^2\pm)f(q,k)$ must be bounded. Therefore, the singularities in R must have residues which factorize in p and q and separately vanish when p or q is equal to k. In addition, the residues of the singularities in $f(p,k)$ must vanish when $p=k$.[16]

In an interval of k for which $\hat{t}(k^2\pm)\neq 0$ the partial-wave amplitudes of $t^e(z)$ and $t^r(z)$ can be identified with $f(p,k)\hat{t}(k^2\pm)f(q,k)$ and $R(p,q;k^2)$, respectively.[10,17]

On the other hand, if for some partial wave we have a range of k^2 within which $\hat{t}(k^2\pm)=0$ at least once then the preceding identification is invalid because the separate boundedness conditions on t^e and t^r are violated. Our problem, then, is to find a suitable identification under these circumstances.

Let us write

$$f(p,k)=B(p,k)+S(p,k)\,, \tag{2.3}$$

where $B(p,k)$ is nonsingular and satisfies

$$B(k,k)=1\,,$$

and $S(p,k)$ contains all the possible singular structure associated with the occurrence of zeros of the on-shell amplitude. We note that $S(p,k)\hat{t}(k^2\pm)$ is nonsingular for all k and vanishes for $p=k$. Specifically, we learn from the work of Fuda[11] and Osborn[12] that $S(p,k)$ has the structure

$$S(p,k)=\sum_{i=1}^{N}\frac{\phi_i(p,k)}{D_i(k)}\,, \tag{2.4}$$

where $\phi_i(p,k)$ is nonsingular and satisfies

$$\phi_i(k,k)=0\,,$$

and

$$D_i(k_0{}^i)=0$$

corresponds to a zero of $\hat{t}(k^2\pm)$ at $k=k_0{}^i$, $i=1,\ldots,N$, such that the ratios $\hat{t}(k^2\pm)/D_i(k)$ are finite for all $k^2>0$.

If we use the expression (2.3) for f in (2.2) we obtain a decomposition of the form (2.1) with the identifications

$$\begin{aligned}t^e(p,q;k^2\pm)&=B(p,k)\hat{t}(k^2\pm)f(q,k)+S(p,k)\hat{t}(k^2\pm)B(q,k)\\&\quad+iS(p,k)\,\mathrm{Im}[\hat{t}(k^2\pm)]\,S(q,k)\\&=f(p,k)\hat{t}(k^2\pm)B(q,k)+B(p,k)\hat{t}(k^2\pm)S(q,k)\\&\quad+iS(p,k)\,\mathrm{Im}[\hat{t}(k^2\pm)]\,S(q,k)\,,\end{aligned} \tag{2.5a}$$

$$t^r(p,q;k^2\pm)=R(p,q;k^2)+S(p,k)\,\mathrm{Re}[\hat{t}(k^2\pm)]\,S(q,k)\,. \tag{2.5b}$$

It is easily seen that *each term* on the left-hand side of (2.5a) is nonsingular.[18] The residual part defined by (2.5b) is nonsingular, although *not* term by term, due to the cancellation of the singular terms at the appropriate energies.[12] It should now be clear that the identifications (2.5) satisfy conditions (i′)–(vi′) for positive z.

We have seen from (2.5a) that in any particular angular momentum state t^e will be at most of rank 3, independently of the number of on-shell zeros, or equivalently, the number of terms in the expansion (2.4). Moreover, in contrast to the expansions of Fuda[11] and Osborn,[12] (2.5a) is term-by-term nonsingular. Nonetheless, (2.5a) still is not entirely satisfactory for practical use because of the singular separable-term form factors. However, it is quite easy to rewrite (2.5a) in a form in which the latter quantities are well defined. For example, let

$$\mathfrak{D}(k)\equiv\prod_{i=1}^{N}D_i(k)\,.$$

Then we can rewrite (2.5a) as

$$\begin{aligned}t^e(p,q;k^2\pm)&=B(p,k)\bar{t}(k^2\pm)\bar{f}(q,k)+\bar{S}(p,k)\bar{t}(k^2\pm)B(q,k)\\&\quad+i\bar{S}(p,k)\left(\frac{\mathrm{Im}[\bar{t}(k^2\pm)]}{\mathfrak{D}(k)}\right)\bar{S}(q,k)\,,\end{aligned} \tag{2.6}$$

where

$$\bar{t}(k^2\pm)\equiv\hat{t}(k^2\pm)/\mathfrak{D}(k)\,,$$

$$\bar{f}(q,k)=f(q,k)\mathfrak{D}(k)\,,$$

and

$$\bar{S}(p,k)=S(p,k)\mathfrak{D}(k)$$

are each nonsingular. (2.6) is in a form suitable for use in a standard, finite-rank interactions, three-particle calculation with the approximation $t\simeq t^e$.

The implication of the work of Fuda[11] and Osborn[12] is that if $\hat{t}(k^2\pm)$ has N zeros, then one is led to a $(N+1)$-rank lowest-order approximation to $t(z)$. However, we have shown that by regrouping the relevant singularities the rank of the lowest-order approximation, obtained by neglecting t^r, need not be greater than 3 but, if there are zeros, it cannot be less than 3 if one is to make practical use of the approximation.

B. Negative Energy

For negative z the decomposition (2.1) can be realized in a variety of well-known ways since the constraints (iv′) and (vi′) are relaxed considerably. Indeed, (vi′) is irrelevant and (iv′) is quite easily satisfied in conjunction with (i′) since for $z<0$ $t(z)$ is continuous across the real axis except at the bound-state poles. So in this case the question of the existence of the decomposition (2.1) is trivial.

The real problem, of course, is to find an optimal form for the decomposition. If one appends an additional requirement that $t^e(z)$ should approximate $t(z)$ to some desired accuracy then we have the very familiar problem of representing $t(z)$ for $z<0$ by a finite-rank form.

It is possible to obtain a representation for $t(p,q;z)$ for $z=-k^2$ very similar to (2.2) by making use of the symmetry and reality properties

$$t(p,q;z)=t(q,p;z)\,, \tag{2.7a}$$

$$t(p,q;-k^2)=t(p,q;-k^2)^*\,, \tag{2.7b}$$

where (2.7b) holds except at the bound-state poles. Let us define the real function ($k>0$)

$$f^{(-)}(p,k)=t(p,k;-k^2)/\hat{t}(-k^2)\,,$$

where

$$\hat{t}(-k^2)=t(k,k;-k^2)\,,$$

and trivially,

$$f^{(-)}(k,k)=1\,.$$

We then infer from (2.7) that we can write

$$t(p,q;-k^2)=f^{(-)}(p,k)\hat{t}(-k^2)f^{(-)}(q,k)+R^{(-)}(p,q;-k^2)\,, \tag{2.8}$$

where $R^{(-)}$ is a real symmetric function such that

$$R^{(-)}(k,q;-k^2)=R^{(-)}(p,k;-k^2)=0\,.$$

The entire contribution from the bound-state poles resides in the factorizable term $f^{(-)}\hat{t}(-k^2)f^{(-)}$ in (2.8).[19]

The functions $f^{(-)}$ and $R^{(-)}$ are nonsingular except for those k for which $\hat{t}(-k^2)=0$. Unfortunately, it is not possible to relate this last property to any discernible property of the physical on-shell amplitudes. The Fuda-Osborn[11,12] analysis applies equally well to negative energies and so does the formal argument leading to (2.6). Therefore, a possible choice for t^e and t^r for $z<0$ is

$$\begin{aligned}t^e(p,q;-k^2)&=B^{(-)}(p,k)\bar{t}(-k^2)\bar{f}^{(-)}(q,k)+\bar{S}^{(-)}(p,k)\bar{t}(-k^2)B^{(-)}(q,k)\\&=\bar{f}^{(-)}(p,k)\bar{t}(-k^2)B^{(-)}(q,k)+B^{(-)}(p,k)\bar{t}(-k^2)\bar{S}^{(-)}(q,k)\end{aligned} \tag{2.9a}$$

and

$$t^r(p,q;-k^2)=R^{(-)}(p,q;-k^2)+S^{(-)}(p,k)\hat{t}(-k^2)S^{(-)}(q,k)\,, \tag{2.9b}$$

where

$$\bar{t}(-k^2)\equiv\hat{t}(-k^2)/\mathfrak{D}^{(-)}(k)\,,$$

$$\bar{f}^{(-)}(q,k)=f^{(-)}(q,k)\mathfrak{D}^{(-)}(k)\,,$$

and

$$\bar{S}^{(-)}(q,k)=S^{(-)}(q,k)\mathfrak{D}^{(-)}(k)\,.$$

We have decomposed $f^{(-)}$ in a manner similar to that depicted in (2.3) and (2.4) with obvious changes in notation.

We note that t^e as given by (2.9a) is now only of rank 2 and, in addition,

$$\langle\vec{p}'|t^r(-k^2)|\vec{p}\rangle=0$$

for $|\vec{p}'|$ or $|\vec{p}|$ equal to k. This last property does not appear as interesting as in the positive-energy case.

The preceding completes our proof of the existence of the essential-residual decomposition of $t(z)$. In the remainder of this section we exploit the two-body integral equations to infer one further aspect of the structure of the residual transition operator $t^r(z)$.

It follows from the Lippmann-Schwinger integral equation for $t(p,q;z)$ that[20]

$$\begin{aligned}R(p,q;k^2)&=\tfrac{1}{2}\pi\left(\frac{k^2-q^2}{q^2}\right)\mathfrak{R}(p,q;k^2)\\&=\tfrac{1}{2}\pi\left(\frac{k^2-p^2}{p^2}\right)\mathfrak{R}(q,p;k^2)\,,\end{aligned} \tag{2.10}$$

where the resolvent kernel[20] $\mathfrak{R}$ satisfies

$$\begin{aligned}\mathfrak{R}(p,q;k^2)&=\Lambda(p,q;k^2)+\int_0^\infty dq'\,\Lambda(p,q';k^2)\mathfrak{R}(q',q;k^2)\\&=\Lambda(p,q;k^2)+\int_0^\infty dq'\,\mathfrak{R}(p,q';k^2)\Lambda(q',q;k^2)\,,\end{aligned} \tag{2.11}$$

and

$$\Lambda(p,q;k^2)=\left(\frac{q^2}{k^2-q^2}\right)\mathfrak{V}(p,q;k^2)\,,$$

$$\mathfrak{V}(p,q;k^2)=\left(\frac{2}{\pi}\right)\left(V(p,q)-\frac{V(p,k)}{V(k,k)}V(k,q)\right)\,.$$

We are confining ourselves to positive energies. We also ignore, for the moment, the complications related to the vanishing of the on-shell amplitude.

(2.10) suggests that instead of expressing R in terms of $\mathfrak{R}$ we look for a related quantity with another inverse Green's function pulled out. Then we will have an explicit and symmetrical representation of the half-on-shell vanishing of R. To this end let us write

$$\mathfrak{R}(p,q;k^2)=\frac{(k^2-p^2)}{p^2}\bar{\mathfrak{R}}(p,q;k^2)\,. \tag{2.12}$$

We find from (2.11) and (2.12) that $\bar{\mathfrak{R}}$ satisfies the integral equations

$$\bar{\mathcal{R}}(p,q;k^2)=\Lambda_s(p,q;k^2)+\int_0^\infty dq'\,\Lambda^\dagger(p,q';k^2)\bar{\mathcal{R}}(q',q;k^2)=\Lambda_s(p,q;k^2)+\int_0^\infty dq'\,\bar{\mathcal{R}}(p,q';k^2)\Lambda(q',q;k^2)\,, \tag{2.13}$$

where

$$\Lambda^\dagger(p,q;k^2)=\left(\frac{p^2}{k^2-p^2}\right)\mathcal{V}(p,q;k^2)=\Lambda(q,p;k^2)$$

is the adjoint[12] of Λ and

$$\Lambda_s(p,q;k^2)=\left(\frac{p^2}{k^2-p^2}\right)\mathcal{V}(p,q;k^2)\left(\frac{q^2}{k^2-q^2}\right)$$

is the symmetrical inhomogeneous term. We note that

$$\mathcal{R}^\dagger(p,q;k^2)=\mathcal{R}(q,p;k^2)\,.$$

Given very weak conditions on the derivatives of $V(p,q)$ it is easily shown that Λ_s is nonsingular even in the double limit $p,q\to k$. The fact that the kernels of (2.13) are just Λ and $\Lambda^\dagger$ implies that the properties of $\bar{\mathcal{R}}$ with respect to singularity structure, in particular, are very similar to those of $\mathcal{R}$. The Fuda-Osborn[11,12] analysis can be carried through with straightforward modifications for (2.13), a fact we shall exploit below.

We infer from the preceding analysis that for positive z the residual operator has the structure

$$t^r(z)=G_0(z)^{-1}\tau(z)G_0(z)^{-1}\,, \tag{2.14}$$

where the partial-wave amplitudes of the reduced residual operator $\tau(z)$ are essentially $\bar{\mathcal{R}}$ and

$$G_0(z)=(z-K)^{-1}\,.$$

The crucial property of $\tau(z)$ is that it is continuous across the unitary cut. Since $G_0(z)$ is nonsingular for negative z, (2.14) is trivially valid in this case. (2.14) constitutes an amplification of condition (vi′) in that it is an explicit and symmetrical representation of the half-on-shell vanishing of $t^r(z)$ for positive z.

Except for the last remark similar statements apply to the representations

$$t^r(z)=G_0(z)^{-1}\tau_R(z) \tag{2.15a}$$

$$=\tau_L(z)G_0(z)^{-1} \tag{2.15b}$$

which we infer from (2.10). In particular, the partial-wave amplitudes of τ_L and τ_R are related to $\mathcal{R}$ and $\mathcal{R}^\dagger$, respectively.

(2.14) and (2.15) are valid in the general case covered by (2.6). However, in this instance τ, τ_L, and τ_R are related to the nonsingular parts of $\bar{\mathcal{R}}$, $\mathcal{R}$, and $\mathcal{R}^\dagger$, respectively. This follows from the Fuda-Osborn[11,12] eigenfunction expansions for $\mathcal{R}$ and $\mathcal{R}^\dagger$, since then each term in the expansion for R [cf. (2.10)], singular or not, can be placed in the forms (2.14) and (2.15).

III. STRUCTURE-INVARIANT PERTURBATION THEORY

We use the form of the three-body scattering integral equations introduced in Ref. 2:

$$U(z)=\bar{\delta}G_0(z)^{-1}+\bar{\delta}t(z)G_0(z)U(z)=\bar{\delta}G_0(z)^{-1}+U(z)G_0(z)t(z)\bar{\delta}\,, \tag{3.1}$$

where we have employed the usual matrix notation[21] with respect to the channel indices. That is, $U(z)$ represents the 4×4 matrix whose elements are the three-particle scattering operators $U_{\beta\alpha}(z)$, $\beta,\alpha=0,1,2,3$. $t(z)$ is a diagonal matrix whose elements are the two-particle transition operators $t_\alpha(z)$ on the three-particle space for $\alpha\neq 0$ and $t_0(z)=0$; the index α on $t_\alpha(z)$ refers to that channel α in which particle α ($=1,2,3$) is asymptotically free. $\bar{\delta}$ is the matrix with elements $1-\delta_{\beta\alpha}$, and finally

$$G_0(z)=(z-H_0)^{-1}\,,$$

where H_0 is the total three-particle kinetic energy operator.

The results of Sec. II imply that on the three-particle space $t(z)$ admits of the essential-residual decomposition

$$t(z)=t^e(z)+t^r(z)\,, \tag{3.2}$$

where for z positive $t^e(z)$ satisfies off-shell unitarity and for negative z it contains the two-body bound-state poles. $t^r(z)$ is continuous across the real z axis and has the representation

$$t^r(z)=G_0(z)^{-1}\tau(z)G_0(z)^{-1}\,, \tag{3.3}$$

where the reduced residual transition operator $\tau(z)$ is also nonsingular for real z and continuous across the real z axis. We employ also the notations

$$\tau_R(z)=\tau(z)G_0(z)^{-1}\,,$$

$$\tau_L(z)=G_0(z)^{-1}\tau(z)\,.$$

With the decomposition (3.2), (3.1) can be rewritten as

$$U(z)=\bar{U}(z)+\bar{U}(z)G_0(z)t^e(z)G_0(z)U(z)=\bar{U}(z)+U(z)G_0(z)t^e(z)G_0(z)\bar{U}(z)\,, \tag{3.4}$$

where $\overline{U}(z)$ satisfies

$$\overline{U}(z)=\bar{\delta}G_0(z)^{-1}+\bar{\delta}t^r(z)G_0(z)\overline{U}(z)$$
$$=\bar{\delta}G_0(z)^{-1}+\overline{U}(z)G_0(z)t^r(z)\bar{\delta}\,. \tag{3.5}$$

We next study the discontinuity relations satisfied by $U(z)$ and $\overline{U}(z)$ using standard results.[21] One finds from (3.5) that for any residual operator $t^r(z)$, including some approximation to some "exact" $t^r(z)$ provided the properties we have associated with a residual operator still obtain,

$$\Delta\overline{U}=\overline{U}(+)-\overline{U}(-)=\overline{U}(+)-\overline{U}(+)^\dagger$$
$$=\Delta\zeta\,, \tag{3.6}$$

where

$$\zeta=\bar{\delta}G_0(z)^{-1}$$

and we have again employed the notation $\pm$ referring to $E\pm i\epsilon$. Since

$$\Delta\zeta=-\bar{\delta}G_0(\pm)^{-1}(\Delta G_0)G_0(\mp)^{-1}\,,$$

(3.6) is a singular type of off-shell discontinuity condition. The appearance of such conditions and their necessity was first pointed out in Ref. 21.

For any $\overline{U}(z)$ satisfying (3.6) we find from (3.4) that $U(z)$ satisfies[21]

$$\Delta U_{\beta\alpha}=-2i\sum_\lambda U_{\beta\lambda}(\pm)D_\lambda U_{\lambda\alpha}(\mp)$$
$$-2i\Big[\sum_\lambda U_{\beta\lambda}(\pm)G_0(\pm)t^e_\lambda(\pm)\Big]$$
$$\times D_0\Big[\sum_\sigma t^e_\sigma(\mp)G_0(\mp)U_{\sigma\alpha}(\mp)\Big]+\Delta\zeta_{\beta\alpha}\,, \tag{3.7}$$

where

$$D_\lambda=\pi\sum_{E'_\lambda,\,\eta_\lambda}|\phi_\lambda(\eta_\lambda,E'_\lambda)\rangle$$
$$\times\delta(E-E'_\lambda)\langle\phi_\lambda(\eta_\lambda,E'_\lambda)|\,. \tag{3.8}$$

The channel states $|\phi_\alpha(E,\eta_\alpha)\rangle$ for $\alpha\neq 0$ refer to noninteracting two-particle states comprised of a particle α moving freely and a bound state of the other pair; $|\phi_0(E,\eta_0)\rangle$ corresponds to a three-particle plane-wave state. The η_λ are any other labels which are needed to specify the asymptotic configurations including an index covering the possibility of more than a single bound state in a given channel. In deriving (3.7) we have made no use of the properties of $\overline{U}(z)$ except (3.6). We have, however, employed the well-known structure of the two-body bound-state pole contributions to $t^e(z)$.

In order to obtain the correct discontinuity relations for $U(z)$ from (3.7) we have to specify a little more concerning $\overline{U}(z)$. The problem is to bring the zero-index operators $U_{\beta 0}(\pm)$ and $U_{0\alpha}(\pm)$ into (3.7).

We have

$$U_{0\alpha}(z)=\overline{U}_{0\alpha}(z)+\sum_\lambda\overline{U}_{0\lambda}(z)G_0(z)t^e_\lambda(z)G_0(z)U_{\lambda\alpha}(z) \tag{3.9a}$$

and

$$U_{\beta 0}(z)=\overline{U}_{\beta 0}(z)+\sum_\lambda U_{\beta\lambda}(z)G_0(z)t^e_\lambda(z)G_0(z)\overline{U}_{\lambda 0}(z) \tag{3.9b}$$

for all α,β. On the other hand, (3.5) implies that

$$\overline{U}_{0\alpha}(z)=G_0(z)^{-1}\Big[\bar{\delta}_{0\alpha}+\sum_\lambda\tau_\lambda(z)\overline{U}_{\lambda\alpha}(z)\Big] \tag{3.10a}$$

and

$$\overline{U}_{\beta 0}(z)=\Big[\bar{\delta}_{\beta 0}+\sum_\lambda\overline{U}_{\beta\lambda}(z)\tau_\lambda(z)\Big]G_0(z)^{-1} \tag{3.10b}$$

for all α,β.

Now, if we assume the validity of (3.10), which was not done in deriving (3.7), then

$$D_0\overline{U}_{0\alpha}(\pm)=\bar{\delta}_{0\alpha}D_0G_0(\pm)^{-1} \tag{3.11a}$$

and

$$\overline{U}_{\beta 0}(\pm)D_0=\bar{\delta}_{\beta 0}G_0(\pm)^{-1}D_0\,. \tag{3.11b}$$

The deduction of (3.11) from (3.10) can be rendered somewhat less mysterious if we note that the right-hand side of (3.11a), for example, is nonvanishing only if it operates on singular vectors proportional to $G_0(\pm)$.[21] Given the validity of (3.11) it then follows from (3.9) that

$$D_0\Big[\sum_\lambda t^e_\lambda(\mp)G_0(\mp)U_{\lambda\alpha}(\mp)\Big]=D_0[U_{0\alpha}(\mp)-\bar{\delta}_{0\alpha}G_0(\mp)^{-1}]\,, \tag{3.12a}$$

$$\Big[\sum_\lambda U_{\beta\lambda}(\pm)G_0(\pm)t^e_\lambda(\pm)\Big]D_0=[U_{\beta 0}(\pm)-\bar{\delta}_{\beta 0}G_0(\pm)^{-1}]D_0\,, \tag{3.12b}$$

and these equations, in turn, imply that

$$\Delta U_{\beta\alpha}=-2i\sum_{\lambda=0}^{3}U_{\beta\lambda}(\pm)D_\lambda U_{\lambda\alpha}(\mp)$$
$$+2i[U_{\beta 0}(\pm)D_0G_0(\pm)^{-1}\bar{\delta}_{0\alpha}+\bar{\delta}_{\beta 0}G_0(\pm)^{-1}D_0U_{0\alpha}(\mp)$$
$$+(\bar{\delta}_{\beta\alpha}-\bar{\delta}_{\beta 0}\bar{\delta}_{0\alpha})G_0(\pm)^{-1}D_0G_0(\mp)^{-1}]\,, \tag{3.13}$$

which are the correct discontinuity relations for $U(z)$.[21] As has been pointed out in Ref. 21, on-shell Eqs. (3.13) reduce to the usual statement of unitarity for three-particle scattering; the general form of (3.13), which includes the terms in square brackets which vanish on shell, is necessary to ensure the consistency of applications of the formalism which involve the operator

$$F(z)=G_0(z)U(z)G_0(z)\,.$$

The preceding argument shows that it is not enough to state that any $\overline{U}(z)$ satisfying (3.6) when used in (3.4) will yield (3.13). One must assume, in addition, that (3.11) are valid. This can be done by supposing that the $\overline{U}_{\beta\alpha}(z)$ for $\beta, \alpha \neq 0$ satisfy (3.6) and that the zero-index operators $\overline{U}_{0\alpha}(z)$ and $\overline{U}_{\beta 0}(z)$ for all β, α are *defined* by (3.10). We note that the latter definitions can be expressed entirely in terms of the $\overline{U}_{\beta\alpha}(z)$ for $\beta, \alpha \neq 0$.

However, the last point is of interest mainly in connection with the over-all consistency of the formalism and for the specification of off-shell extensions. Neither for the primary computational problem, which is to solve (3.4) for $U_{\beta\alpha}$, $\beta, \alpha \neq 0$ and which requires only those $\overline{U}_{\beta\alpha}$ for $\beta, \alpha \neq 0$, nor in the computation of the on-shell physical amplitudes (3 to 3, 3 to 2, 2 to 3) corresponding to $U_{\beta 0}$ and $U_{0\alpha}$ for $\alpha, \beta = 0, 1, 2, 3$, do we need to know anything about the operators $\overline{U}_{\beta 0}$, $\overline{U}_{0\alpha}$, $\beta, \alpha = 0, 1, 2, 3$, other than that they satisfy (3.11).

Obviously, we obtain a class of unitary approximations to a model with specified $t(z)$ by constructing various approximations to $t^r(z)$, with $t^e(z)$ fixed, which satisfy the residual constraints and using these approximate $t^r(z)$ as input in (3.5). The latter because of (2.14) will be integral equations with nonsingular kernels.

The iteration series derived from (3.5) is

$$\overline{U}(z) = \overline{\delta} G_0(z)^{-1} + \overline{\delta} t^r(z) \overline{\delta} + \overline{\delta} t^r(z) \overline{\delta} \tau_R(z) \overline{\delta} + \cdots . \tag{3.14}$$

We see that any truncation of this series which includes the term $\overline{\delta} G_0(z)^{-1}$ yields an approximate $\overline{U}(z)$ which satisfies (3.6) and (3.11) and which in turn generates a $U(z)$, via the solution of (3.5), which satisfies (3.13).

(3.14) forms the basis for not only a unitary perturbation theory but also one which leaves the principal structural properties of the scattering amplitudes invariant. This is to be expected since, as Noyes[10] has pointed out, the special case $t^r(z) \equiv 0$ constitutes a perfectly valid model for three-particle scattering. The reason for this is that the essential singularity structure of the three-particle amplitudes is independent of $t^r(z)$.

Thus, for any of the approximations discussed we obtain amplitudes which in addition to satisfying unitarity also possess the correct primary and secondary singularity structure.[22–24] As consequences of this, the correct threshold behaviors are preserved and also the existence, location, and residues of the double-scattering poles in the 3-to-3 amplitude are the same as for the exact amplitudes.

Because of the finite-rank character[25] of $t^e(z)$, (3.4) have the canonical form of the three-particle equations under the assumption of finite-rank interactions. In this instance it is much more convenient to work with the operators $F(z)$ which satisfy

$$\begin{aligned} F(z) &= \overline{F}(z) + \overline{F}(z) t^e(z) F(z) \\ &= \overline{F}(z) + F(z) t^e(z) \overline{F}(z) , \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} \overline{F}(z) &= \overline{\delta} G_0(z) + \overline{\delta} \tau_R(z) \overline{F}(z) \\ &= \overline{\delta} G_0(z) + \overline{F}(z) \tau_L(z) \overline{\delta} . \end{aligned} \tag{3.16}$$

Let us separate off the discontinuous portion of $\overline{F}(z)$ by writing

$$\overline{F}(z) = \overline{\delta} G_0(z) + \mathfrak{F}(z) ,$$

where

$$\begin{aligned} \mathfrak{F}(z) &= \overline{\delta} \tau(z) \overline{\delta} + \overline{\delta} \tau_R(z) \mathfrak{F}(z) \\ &= \overline{\delta} \tau(z) \overline{\delta} + \mathfrak{F}(z) \tau_L(z) \overline{\delta} . \end{aligned} \tag{3.17}$$

Evidently

$$\Delta \mathfrak{F} = 0 . \tag{3.18}$$

A structure-invariant perturbation theory can now be defined via the iteration solution of (3.17):

$$\mathfrak{F}(z) = \overline{\delta} \tau(z) \overline{\delta} + \overline{\delta} \tau_R(z) \overline{\delta} \tau(z) \overline{\delta} + \cdots . \tag{3.19}$$

The truncation of (3.19) in any order yields an approximate $\mathfrak{F}(z)$ which satisfies (3.18) and generates a structure-invariant $F(z)$ via (3.15).

It is unnecessary to repeat our previous discussion in connection with (3.15)–(3.19). One should keep in mind, however, that the various technical points associated with the zero-index operators [cf. (3.10), (3.11)] apply equally well to the $F(z)$ reformulation with obvious modifications.

The utility of the preceding formalism depends crucially upon the rapid convergence of the series (3.14) or (3.19) as well as on the closely related constraint of maintaining the rank of $t^e(z)$ as small as possible. Clearly, with a sufficiently accurate choice for $t^e(z)$ the effect of $t^r(z)$ may be made as small as one desires. In this connection we should point out that despite the examples of $t^e(z)$ and $t^r(z)$ given in Sec. II in the course of proving the existence of the essential-residual decomposition the latter is by no means unique for either positive or negative z.

Some general criteria for and partial estimates of the convergence in perturbation theories of the general type (AGS) considered here are contained in Refs. 1 and 8. We will expand upon these discussions very briefly.

$t^r(z)$ is the source term for the auxiliary three-body operators $\overline{U}(z)$ which satisfy integral equations [(3.5)] formally identical to the ordinary three-body equations. However, $t^r(z)$ contains no direct contributions from two-body bound or virtual states, is continuous across the entire real z axis, and for $z>0$ vanishes half on shell thus cancelling the singular behavior of the three-particle propagators. From previous studies[26] of the three-body iteration series it would appear that none of the mechanisms for generating the three-body singularities (three-body bound and virtual states) which cause the usual convergence problems in the iteration solution of the ordinary three-body equations should be present in the $t^r(z)$-generated series. Therefore, there is some reason to expect rapid convergence of the series (3.14) or (3.19). Some indirect support of this conjecture is provided by the estimates and calculations of Bencze and Doleschall,[8] but further numerical investigations are clearly needed.

Finally, we remark on the fact that the use of the essential-residual decomposition with the effects of $t^r(z)$ taken into account approximately will in general result in a loss of the global analyticity properties of the three-particle scattering amplitudes. This is a consequence of the fact that $t^e(z)$ will in general not be analytic in the entire z plane but only in the half planes $z>0$ and $z<0$, separately.[19,27] Thus, the complete *analytic* structure of the scattering amplitudes is in general not left invariant in any finite order of perturbation. However, the question of the numerical importance of the loss of global analyticity would appear to depend entirely upon the previous question of the convergence of the perturbation series. In this connection, it is interesting to note that Brayshaw[27] has proposed a decomposition of $t(z)$ (suitable for use in a perturbation theory) into a finite-rank part $t^s(z)$ and a remainder which has the property that $t^s(z)$ preserves the analytic behavior of $t(z)$. However, $t^s(z)$ does not, in general, satisfy off-shell unitarity and so that with an approximate treatment of the remainder in the three-particle equations one will violate three-particle unitarity while maintaining global analyticity.

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