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Scattering Amplitudes at High Energies. III. An Approach with Green's Functions

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The scattering amplitude at high energy is calculated in various approximations, explicitly incorporating both the single and double hard collisions. For the purpose of improving the large-angle behavior of the amplitude, approximations are introduced directly to the full Green's function rather than to the scattering function, thus avoiding the usual assumption of a straight-line trajectory. The previous study using the semiclassical Green's function gave very accurate amplitude up to a moderately large angle, and we have further examined its applicability to more general forms of potentials. Some of the practical difficulties in carrying out the amplitude integrals are pointed out, and the problem is simplified by introducing an angle-averaging procedure and also by an eikonal approximation to the semiclassical Green's function. These procedures are then tested using Gaussian and Yukawa forms of the potential; they are relatively simple to apply and the over-all accuracy is improved.

I. INTRODUCTION

This is the third in a series of reports[1] on the study of the eikonal representation of high-energy scattering amplitudes, with the emphasis on improving the behavior of the amplitude at large angles where the dynamical correlation effect of the target system plays a more prominent role. Extensive numerical calculations were performed earlier[1] to determine the effectiveness of the various forms of impact-parameter amplitudes proposed by many people. Although the need for an improved amplitude is common to both atomic and nuclear high-energy collisions, we examined the approximations in the framework of potential scattering using the Schrödinger equation, always with the understanding that much of the result obtained here may be generalized to the relativistic case by the usual kinematic adjustments.[2,3] Various forms of the potential, the Yukawa and Gaussian types, were studied at different scattering energies and scattering angles in order to make the study as model-independent as possible. Generally, cross sections with sharper diffraction structure are harder to reproduce, and we concluded in the two previous reports that none of the known approximations on the amplitude were sufficiently accurate at large angles to be used in the extraction of physical informations about the target system from experimental data.

More recently, however, we have presented[4] a new approach to this problem by explicitly constructing a simple Green's function, with full interactions, in the semiclassical approximation, $G_{sc}^{(+)}$, and its effectiveness was briefly tested for high-energy scattering. Although the actual form of the Green's function was much simpler than the corresponding WKB form,[2] the resulting amplitude was extremely accurate over wider angular ranges. In the case of a Gaussian potential, the amplitude with the Green's function $G_{sc}^{(+)}$ can be reduced immediately to a double integral. Moreover, the usual assumption of a straight-line trajectory is not required. The semiclassical formulation of the two-particle Green's function can be extended[4] to systems involving three particles, and we indicated how it can be simply generalized to many-particle cases.

However, except for the simple case with a Gaussian potential, or a Gaussian times a polynomial, the application of $G_{sc}^{(+)}$ becomes more difficult, because the amplitude integrals do not simplify so easily. In view of the effectiveness of $G_{sc}^{(+)}$ demonstrated previously,[4] it is of interest to examine in more detail some additional approximations on $G_{sc}^{(+)}$ which could reduce the amplitude integrals to simpler forms and still retain some of the desirable features of the original $G_{sc}^{(+)}$. We judge the effectiveness of a given approximation by comparing it with the Glauber amplitude, which requires two numerical integrations, one over the trajectory for the eikonal phase and the other over the impact parameter.

The necessary kinematics and notations are defined in Sec. II. We emphasize the distinction between the two ways of eikonalization of the amplitude, in which either the total wave function or the full Green's function is directly involved. A particularly useful set of variables are introduced for the evaluation of amplitudes involving Green's functions. Section III contains various approximations on the Green's function and the resulting expressions for the amplitude. The numerical test is presented in Sec. IV.

II. SCATTERING AMPLITUDES AND THE GREEN'S FUNCTION

In this section, we define notations and kinematics, review some of the conventional impact-parameter amplitudes, and point out the difficulties brought about by the assumption of straight-line trajectory. The scattering amplitude for a potential $V(\vec{r})$ is given by

$$f(\vec{K}_f, \vec{K}_i) = -\frac{1}{2\pi} \int e^{-i\vec{K}_f \cdot \vec{r}} V(\vec{r}) \Psi_i(\vec{r})\, d^3r\,, \tag{2.1}$$

where $\vec{K}_i$ and $\vec{K}_f$ are the initial and final momenta and we have set $m = \hbar = c = 1$. The total wave function Ψ_i, with the initial incoming wave boundary condition, may be conveniently written in two different forms; for small-angle scatterings, we may set[5]

$$\Psi_i \equiv e^{i\vec{K}_i \cdot \vec{r}} \Phi_i(\vec{r})\,, \tag{2.2}$$

where Φ_i is to be estimated directly from the scattering equation

$$(T + V - E)\Psi_i = 0\,, \tag{2.3}$$

where

$$T = -\tfrac{1}{2} \nabla_{\vec{r}}^{2}\,, \quad E = \frac{\hbar^2 K^2}{2m} = \frac{K^2}{2}\,, \quad K \equiv |\vec{K}_i| = |\vec{K}_f|\,. \tag{2.4}$$

That is, Φ_i satisfies

$$(T - i\vec{K}_i \cdot \vec{\nabla}_{\vec{r}} + V)\Phi_i = 0\,. \tag{2.5}$$

The form (2.2) is useful for small-angle scatterings but not for large angles. The assumption of a straight-line trajectory is equivalent to the $T\Phi_i$ term in (2.5) being small compared with the two other terms in (2.5). With such an approximation, we then obtain amplitudes of the Glauber type,[5] in which all the collisions suffered by the projectile are taken to be soft.[6] The previous study[1] was essentially based on the ansatz (2.2) for Ψ_i, and practically all the amplitudes tested there, including the $T\Phi_i$ correction, failed to provide reliable values at large angles. The result strongly suggested that the effect of both single and double hard collisions had to be incorporated into the amplitude explicitly in order to improve its behavior at larger angles.

In the high-energy scattering of pions and nucleons by complex nuclei, the multiple hard collisions can be associated with the two- and three-particle correlation functions of the target, and their effect shows up more prominently at large angles. To improve the reliability of the impact-parameter amplitude at large angles, therefore, we may try a slightly different form for Ψ_i

$$\Psi_i = e^{i\vec{K}_i \cdot \vec{r}} + \Psi_i^{\text{out}}\,, \tag{2.6}$$

where of course Ψ_i^{out} is the scattered part of Ψ_i with the outgoing wave boundary condition, and is given by

$$\Psi_i^{\text{out}} = G^{(+)}(\vec{r}, \vec{r}\,') V(\vec{r}\,') e^{i\vec{K}_i \cdot \vec{r}\,'}. \tag{2.7}$$

In (2.7), we used the full Green's function

$$G^{(+)} = (E + i\epsilon - T - V)^{-1}\,, \tag{2.8}$$

which is in general not known. However, the form (2.6) allows the particle trajectory to deviate from the original $\hat{K}_i$ direction in a very natural way. Thus, Ψ_i^{out} may be evaluated either by introducing an approximation to $G^{(+)}$, or by making an ansatz, as, e.g.,

$$\Psi_i^{\text{out}} \equiv e^{i\vec{K} \cdot \vec{r}} \varphi(\vec{r})\,, \tag{2.9}$$

where

$$\vec{K} \equiv K(\vec{K}_i + \vec{K}_f)/|\vec{K}_i + \vec{K}_f|\,. \tag{2.10}$$

The form (2.9) is especially useful at large angles, and includes also the form (2.2) as a special case.

With (2.6) and (2.7), the amplitude becomes

$$f = f_B + f_c\,, \tag{2.11}$$

where

$$f_B = -\frac{1}{2\pi} \int e^{i\vec{q} \cdot \vec{r}}\, V(\vec{r})\, d^3r \tag{2.12}$$

is the usual Born amplitude, and f_c is the correction to f_c, given explicitly as

$$f_c = -\frac{1}{2\pi} \int d^3r\, e^{-i\vec{K}_f \cdot \vec{r}} V(\vec{r}) \times \int d^3r'\, G^{(+)}(\vec{r}, \vec{r}\,') V(\vec{r}\,') e^{i\vec{K}_i \cdot \vec{r}\,'}\,. \tag{2.13}$$

In (2.12), we used the momentum transfer vector $\vec{q}$ defined by

$$\vec{q} = \vec{K}_i - \vec{K}_f\,, \tag{2.14}$$

which assumes the values $0 \le q \le 2K$ for given K. In terms of the scattering angle Θ,

$$q = 2K \sin\tfrac{1}{2}\Theta\,. \tag{2.15}$$

The scattering cross section is given by

$$\sigma(K, \Theta) = |f_B + f_c|^2\,. \tag{2.16}$$

Evidently, the evaluation of f_c is considerably more difficult than f_B or some of the other approximate amplitudes obtained with (2.2), even when a reasonably simple form for Ψ_i^{out} is used. However, in some cases, f_c can be simplified, as will be discussed in detail below and in Sec. III. As an example, consider the free Green's function $G_0^{(+)}$ in the place of $G^{(+)}$, where

$$G_0^{(+)} = (E + i\epsilon - T)^{-1} = -\frac{1}{2\pi} \frac{e^{iK|\vec{r} - \vec{r}\,'|}}{|\vec{r} - \vec{r}\,'|}\,. \tag{2.17}$$

That is, $G_0^{(+)}$ depends only on the variables $|\vec{r}-\vec{r}\,'|$, and suggests that perhaps a more convenient variable to use in f_c may be the new set $(\vec{u}, \vec{v})$ defined by

$$\vec{u}=\vec{r}-\vec{r}\,' \quad \text{and} \quad \vec{v}=\tfrac{1}{2}(\vec{r}+\vec{r}\,') \tag{2.18}$$

with

$$d^3r\, d^3r' = d^3u\, d^3v\,. \tag{2.19}$$

The exponent function in f_c is rewritten as

$$\vec{K}_i\cdot\vec{r}\,'-\vec{K}_f\cdot\vec{r}=\vec{q}\cdot\vec{v}-\vec{K}_a\cdot\vec{u}, \tag{2.20}$$

where

$$\vec{K}_a=\tfrac{1}{2}(\vec{K}_i+\vec{K}_f), \quad \text{with } \vec{q}\cdot\vec{K}_a=0\,. \tag{2.21}$$

Thus, (2.13) becomes

$$f_c=-\frac{1}{2\pi}\int d^3v\, e^{i\vec{q}\cdot\vec{v}}\int d^3u\, e^{-i\vec{K}_a\cdot\vec{u}}G^{(+)}(\vec{u},\vec{v}) \times V(\vec{v}+\tfrac{1}{2}\vec{u})V(\vec{v}-\tfrac{1}{2}\vec{u})\,. \tag{2.22}$$

For $G_0^{(+)}$ of (2.17), and also $G_{\text{sc}}^{(+)}$ derived previously[4] in the semiclassical approximation and to be considered further in Sec. III, the reduction of (2.22) to a much simpler form depends on the particular form for $V(\vec{v}+\frac{1}{2}\vec{u})V(\vec{v}-\frac{1}{2}\vec{u})$. For example, we will show explicitly that f_c reduces trivially to a two-dimensional integral if the potential $V(r)$ assumes the form

$$V(r)=\frac{g}{2}\,e^{-Ar^2}(1+\rho r^2+\rho' r^4+\cdots+\rho^{(N)}r^{2N}) \tag{2.23}$$

and if we use either $G_0^{(+)}$ or $G_{\text{sc}}^{(+)}$ in the place of $G^{(+)}$, where these approximate Green's functions depend only on the scalar variables u and v. The difficulty in the evaluation of (2.22) arises essentially from the angular dependence of $\vec{u}\cdot\vec{v}$. We will return to this problem in Sec. III.

Before considering the problem of evaluating (2.22), we study briefly an alternate approximation to f_c which can be derived with the form (2.6). As in (2.2), we adopt the ansatz (2.9) for Ψ_i^{out} and assume that φ is a slowly varying function of $\vec{r}$. Then, neglecting the $(T\varphi)$ term, we obtain immediately, from (2.3), (2.6), and (2.9), the equation satisfied by φ,

$$i\vec{K}\cdot\vec{\nabla}_{\vec{r}}\varphi - V\varphi \approx V e^{i\vec{q}_i\cdot\vec{r}}, \tag{2.24}$$

where

$$\vec{q}_i=\frac{\vec{q}}{2}-(K-K_a)\frac{\vec{K}_a}{K_a}\,. \tag{2.25}$$

The boundary condition on φ is

$$\varphi\to 0 \quad \text{as} \quad z_a\to -\infty\,, \tag{2.26}$$

where z_a is the z axis along the $\hat{K}_a$ direction. [(2.26) seems to be more convenient than the asymptotic condition at $z=-\infty$ where $\hat{z}$ is parallel to the initial $\vec{K}_i$ direction.] The solution of (2.24) is given by

$$\varphi\approx\varphi^{\text{eik}}=\frac{i}{K}\,e^{i\chi_a}\int_{-\infty}^{z_a} dz_a{}'V' e^{i\vec{q}_i\cdot\vec{r}\,'}e^{-i\chi_a'}, \tag{2.27}$$

where

$$\chi_a=-\frac{1}{K}\int_{-\infty}^{z_a} V' dz_a{}'\,. \tag{2.28}$$

The primed quantities in (2.27) are dependent on the variables $r_a{}'=(x_a{}^2+y_a{}^2+z_a{}'^2)^{1/2}$, and the subscript a denotes the variables in the coordinate system in which the z_a axis is parallel to $\hat{K}_a(=\hat{K})$. Using the solution (2.27), we have

$$f\approx f_B+f_c^{\text{eik}}, \tag{2.29}$$

where

$$f_c^{\text{eik}}=-\frac{i}{2\pi K}\int d^2b_a\int_{-\infty}^{\infty} dz_a\int_{-\infty}^{z_a} H_f H_i\, dz_a{}'\,, \tag{2.30}$$

with

$$H_i=e^{-\chi_a'}V(r_a')e^{i\vec{q}_i\cdot\vec{r}_a'}\,, \qquad H_f=e^{i\chi_a}V(r_a)e^{i\vec{q}_f\cdot\vec{r}_a}\,, \tag{2.31}$$

and

$$\vec{q}_f\equiv\vec{K}-\vec{K}_f=\vec{q}/2+(K-K_a)\vec{K}_a/K_a\,.$$

Note that H_i is not a Hermitian conjugate of H_f (even with $\vec{r}_a\to\vec{r}_a'$). Equation (2.30) shows explicitly that the momentum transfers at the collision points $\vec{r}_a$ and $\vec{r}_a'$ are $\vec{q}_f/2$ and $\vec{q}_i/2$, respectively, with $\vec{q}_i+\vec{q}_f=\vec{q}$. This is a direct consequence of the choice of Ψ_i^{out} we made in (2.9). The particle is to travel in the $\hat{K}_a$ direction between the two hard collisions, accompanied by soft collisions which give rise to the distortion factor $\exp(\pm i\chi_a)$. Both these effects are seen to be important in producing accurate amplitudes at large values of q. (See Sec. IV.) It turns out that, although (2.30) usually involves a triple-integral and thus harder to evaluate than (2.22), it has the advantage that almost any form of potentials can be used.

It is interesting to note that (2.30) is consistent with the eikonal approximation on $G^{(+)}$, with

$$G^{\text{eik}}=-\frac{i}{2\pi K}\,\delta(\vec{b}_a-\vec{b}_a')\theta(z_a-z_a')e^{i\vec{K}\cdot\vec{u}}e^{i\chi_a-i\chi_a'}\,. \tag{2.32}$$

A more general form of f_c^{eik}, in which the separation of $\vec{q}$ into $\vec{q}_i$ and $\vec{q}_f$ is not specified, has also been worked out.[6]

III. APPROXIMATE AMPLITUDES

We consider in this section several approximate forms of the amplitude derived from (2.6) and (2.11), which will then be tested in Sec IV. For convenience, we adopt the Gaussian form for $V(r)$ which was also used in the earlier studies,[1] given by

$$V(r) = \frac{g}{2}\, e^{-Ar^2}(1+\rho r^2) \tag{3.1}$$

with the parameters chosen such that the diffraction maxima are produced at moderate angles, at $q\approx 0$ and $q\approx 1.5$ in the appropriate units. That is, we set

$$\begin{aligned} &g=-0.4 \quad (\text{and also } -0.2),\\ &A=0.2, \quad \rho=0.3, \end{aligned} \tag{3.2}$$

and

$$K=2.0 \quad (=K_i=K_f).$$

Aside from its effectiveness in testing the various approximate amplitudes, the form (3.1) turns out to be extremely convenient in the evaluation of f_c. We will also study the Yukawa form for $V(r)$ to test procedures which may be applicable to situations where some of the simplifying features discussed in Sec. II (related to the $\vec{u}\cdot\vec{v}$ term) are not present. We now list the approximate amplitudes which are being studied.

A. Born and Glauber Amplitudes

The Born amplitude is given when we set $\Phi_i=1$ in (2.2), as

$$f_B = -g\int_0^\infty r dr\, e^{-Ar^2}(1+\rho r^2)\sin(qr)/q\,, \tag{3.3}$$

which is real for the set of parameters (3.2). The Glauber amplitude is obtained from (2.2) with

$$\Phi_i \approx \Phi_{\text{GL}} = \exp\left[-\frac{i}{K}\int_{-\infty}^{z_a} V(r_a')\,dz_a'\right], \tag{3.4}$$

where z_a and z_a' are chosen along the $\hat{K}_a$ direction. Thus,

$$f_{\text{GL}} = -iK\int_0^\infty b\,db\, J_0(qb)(e^{i\chi_{\text{GL}}}-1), \tag{3.5}$$

where

$$\chi_{\text{GL}} = -\frac{1}{K}\int_{-\infty}^{\infty} V(r_a)\,dz_a\,. \tag{3.6}$$

Both f_B and f_{GL} were extensively studied in Ref. 1 for the potential (3.1) and for the Yukawa form. We simply note that f_{GL} of (3.5) involves *two* integrations, one for χ_{GL} and the other over the impact parameter b, and that χ_{GL} does not seem to depend on the momentum transfer q. This is the main simplifying feature of the Glauber amplitude.

B. Second Born Amplitude and $G_0^{(+)}$

The exact second Born amplitude is given for the potential (3.1) by

$$f_{c,B} = \frac{1}{4\pi^2}\int d^3v\, e^{i\vec{q}\cdot\vec{v}} \times \int d^3u\, e^{iKu - i\vec{K}_a\cdot\vec{u}}\, V(\vec{v}+\tfrac12\vec{u})V(\vec{v}-\tfrac12\vec{u}) \tag{3.7}$$

and thus

$$f\approx f_B + f_{c,B} = f_{B2}\,. \tag{3.8}$$

The potential factor in (3.7) is

$$V(\vec{v}+\tfrac12\vec{u})V(\vec{v}-\tfrac12\vec{u}) = \frac{g^2}{4}\, e^{-2Av^2 - Au^2/2}F\,,$$

where

$$\begin{aligned} &F=F_0+F_1\,,\\ &F_0 = 1+2\rho(v^2+\tfrac14 u^2)+\rho^2(v^2+\tfrac12 u^2v^2+\tfrac{1}{16}u^4)\,,\\ &F_1 = -\rho^2(\vec{u}\cdot\vec{v})^2\,. \end{aligned} \tag{3.9}$$

Both the exponent and F_0 in (3.9) depend only on the scalar variables u and v, so that the angular dependence on $\hat{u}$ and $\hat{v}$ is entirely contained in the factors $e^{i\vec{q}\cdot\vec{v}}$ and $e^{-i\vec{K}_a\cdot\vec{u}}$. Therefore, the angular integrations over $d\Omega_{\vec{u}}$ and $d\Omega_{\vec{v}}$ in $f_{c,B}$ can be carried out immediately for the F_0 part. On the other hand, the F_1 part in (3.9) involves the angle between $\hat{u}$ and $\hat{v}$, and the angular integrations become more tedious. The details are given in the Appendix. Thus, using the result (A2) and (A7) of the Appendix, we have

$$\int d\Omega_{\vec{u}}\, d\Omega_{\vec{v}}\, e^{i\vec{q}\cdot\vec{v} - i\vec{K}_a\cdot\vec{u}}F(u,v;\vec{u}\cdot\vec{v}) \equiv M(u,v;K_a,q)$$

and finally

$$f_{c,B} = \frac{g^2}{16\pi^2}\int_0^\infty v^2 dv\, e^{-2Av^2} \times \int_0^\infty u\,du\, e^{iKu - Au^2/2} M(u,v;K_a,q)\,. \tag{3.10}$$

The double integral in $f_{c,B}$ can now readily be done numerically, and thus, for $V(r)$ of the form (3.1), the second Born amplitude f_{B2} is as simple to evaluate as f_{GL}. Note that this simplification is brought about by the simple u dependence of $G_0^{(+)}$ as well as the particular form (3.1) for $V(r)$, and the only complication was the presence of the $(\vec{u}\cdot\vec{v})^2$ term, which nevertheless could be treated exactly.

C. Semiclassical Green's Function

The Green's function in the semiclassical approximation, with the full interaction V, was given earlier[4] by

$$G_{\rm sc}^{(+)} = -\frac{1}{2\pi u}\, e^{iQ(v)u}\,, \tag{3.11}$$

where

$$Q(v) = [K^2 - 2V(v)]^{1/2} = \text{real}\,. \tag{3.12}$$

It was shown there that the resulting amplitude

$$f \approx f_{\rm sc} = f_B + f_{c,\rm sc} \tag{3.13}$$

is extermely accurate at all angles up to $q \lesssim K$ considered there. Since $Q(v)$ depends only on the scalar variable v, the evaluation of $f_{c,\rm sc}$ will involve exactly the same angular integrations as for $f_{c,B}$ above, where $Q(v)$ is replaced simply by K in $G_0^{(+)}$. The complication with the $(\vec{u}\cdot\vec{v})^2$ factor also appears in $f_{c,\rm sc}$, but the final form of $f_{c,\rm sc}$ will again involve a double integral, as

$$f_{c,\rm sc} = \frac{g^2}{16\pi^2}\int_0^\infty v^2 dv\, e^{-2Av^2} \times \int_0^\infty u du\, e^{iQ(v)u - Au^2/2} M(u, v; K_a, q)\,. \tag{3.14}$$

Again, $f_{\rm sc}$ is as simple to evaluate as $f_{\rm GL}$ and f_{B2} when $V(r)$ is of the Gaussian form (3.1) or (2.23).

D. Angle-Averaging Procedure

We have seen above that both f_{B2} and $f_{\rm sc}$ can be simplified because of the way the factor $(\vec{u}\cdot\vec{v})$ appears in the amplitude integrals. Obviously, for $V(r)$ other than the Gaussian form, f_c will be extremely hard to evaluate accurately even if $G_0^{(+)}$ or $G_{\rm sc}^{(+)}$ is introduced in f_c. Therefore, it is of interest to consider possible ways of further simplifying f_c. One possibility is to take an average of $(\vec{u}\cdot\vec{v})^2$ over the angles, *independent* of the $\vec{q}\cdot\vec{v}$ factor in the exponents $e^{i\vec{q}\cdot\vec{v}}$, i.e.,

$$(\vec{u}\cdot\vec{v})^2 \to \alpha^2(uv)^2\,, \quad \alpha = 1/\sqrt{3}\,. \tag{3.15}$$

[The symmetry of $V(\vec{v}+\frac{1}{2}\vec{u})V(\vec{v}-\frac{1}{2}\vec{u})$ allows only terms of even powers in $(\vec{u}\cdot\vec{v})$.] In fact, for $q=0$ and with (3.1) for $V(r)$, (3.15) is exact. This is of course an accident, because F of (3.9) only involves $(\vec{u}\cdot\vec{v})^2$ and not of higher powers. For $q>0$, however, we may try a q-dependent α factor, for example, of the form

$$\alpha_q \to \alpha_q' = \frac{1}{\sqrt{3}}\left(1 - \frac{q}{2K}\right)\,. \tag{3.16}$$

The forms such as (3.15) and (3.16) will simplify the f_c integrals for any potentials to the same degree f_{B2} and $f_{\rm sc}$ are. In particular, for (3.1), we have with (3.16),

$$f_{\rm sc}(\alpha_q) = f_B + f_{c,\rm sc}(\alpha_q)\,, \tag{3.17}$$

where

$$f_{c,\rm sc}(\alpha_q) = \frac{g^2}{16\pi^2}\int_0^\infty v^2 dv\, e^{-2Av^2}\int_0^\infty u du\, e^{iQ(v)u - Au^2/2} \times (M_0 - \alpha_q{}^2\rho^2 u^2 v^2)\,. \tag{3.18}$$

E. Eikonal Approximation to $G_{\rm sc}^{(+)}$

Another method of evaluating f_c without the complications caused by the $\vec{u}\cdot\vec{v}$ term is to adopt $f_c^{\rm eik}$ of (2.30). The expression is fairly complicated and in general involves fivefold integrations, including the integral involved in the phase function χ_a. We can introduce, however, a simplification similar to that which appears in $G_{\rm sc}^{(+)}$; i.e., in (2.30), we let

$$\chi_a - \chi_a' \approx R(v_a)u_z\,, \tag{3.19}$$

where

$$u_z = z_a - z_a' \geq 0\,,\quad v_a = (r_a{}^2 + u_z{}^2/4 - z_a u_z)^{1/2}\,,\quad r_a = (z_a{}^2 + b_a{}^2)^{1/2}\,, \tag{3.20a}$$

and

$$R(v_a) = [K^2 - 2V(v_a)]^{1/2} - K \approx -V(v_a)/K\,. \tag{3.20b}$$

We also write

$$\vec{q}_i\cdot\vec{r}_a' + \vec{q}_f\cdot\vec{r}_a = \vec{q}\cdot\vec{v} + (K - K_a)u_z\,. \tag{3.21}$$

Since $\vec{z}_a$ is orthogonal to $\vec{q}$, we have

$$\vec{q}\cdot\vec{v}_a = \vec{q}\cdot\vec{b}_a = qb_a\cos\varphi_b$$

and the φ_b integration in d^2b_a can be carried out. Therefore, with (3.20) and (3.21), we can reduce (2.30) into the form

$$f_{c,\rm sc}^{\rm eik} \approx -\frac{i}{K}\int_0^\infty b db\, J_0(qb)\int_{-\infty}^\infty dz_a \times \int_0^\infty du_z\, V(r_a)V(r_a')e^{iRu_z + i(K-K_a)u_z}\,, \tag{3.22a}$$

where

$$r_a' = (r_a{}^2 + u_z{}^2 - 2z_a u_z)^{1/2}$$

and r_a, u_z, v_a are given by (3.20). The form (3.22) is much simpler than (2.30) and involves only triple integrals, which is one more integral than the Glauber $f_{\rm GL}$ and the semiclassical $f_{\rm sc}$. However, unlike with $f_{\rm sc}$, $f_{\rm sc}^{\rm eik}$ in the form (3.22) for f_c is now applicable to all forms of potentials without the $\vec{u}\cdot\vec{v}$ complication, and presumably $f_{\rm sc}^{\rm eik}$ should be an improvement over $f_{\rm GL}$ at larger angles. This will be tested in Sec. IV. Obviously, (3.22) corresponds to the semiclassical approximation in which $G_{\rm sc}^{(+)}$ is replaced by

$$G_{\rm sc}^{\rm eik} = -\frac{i}{2\pi K}\,\delta(\vec{b}_a - \vec{b}_a')\theta(z_a - z_a')e^{i\vec{K}\cdot\vec{u}}\,e^{-iV(v_a)u_z/K}\,, \tag{3.23}$$

which should then be compared with (2.32). Incidentally, $f_{\rm sc}^{\rm eik}$ at $q=0$ should be similar to $f_{\rm GL}$; the difference is mainly in the phase $Q(v_a)$ compared with $\chi_{\rm GL}$. The two amplitudes are quite different at large q. A slightly improved form is also obtained as[7]

$$f^{\rm eik} = f_{\rm sc}^{\rm eik}[R \to R' = R + V(b)bq/K^2]\,. \tag{3.22b}$$

F. q Dependence of $f_{\rm sc}$ Through K_a

It is of some interest to study the q dependence of f_c coming from the wave vector $\vec{K}_a$ in the intermediate state. For $q>0$, K_a deviates appreciably from K, where

$$K_a = (K^2 - \tfrac{1}{4}q^2)^{1/2}$$

We denote the amplitude $f_{\rm sc}$ with K_a replaced by K as

$$f_{\rm sc}^{(0)} = f_{\rm sc}(K_a \to K)\,. \tag{3.24}$$

This will be compared with f and $f_{\rm sc}$.

G. Three-Particle Green's Functions in the Semiclassical Approximation

Finally, we note that the approximations such as (3.13) and (3.22) can also be developed for systems with more than two particles. The necessary semiclassical Green's function $G_{\rm sc}^{(+)}$ for the three-particle system has been given recently,[4] and possible generalizations to other many-particle systems are fairly straightforward. However, the number of variables which are involved in the many-particle $G_{\rm sc}^{(+)}$ is very large, and it is essential for their applications to simplify the f_c integrals along the line discussed in this section. Thus, for example, we have for the three-particle system described by the Hamiltonian

$$H = -\frac{P_{\vec{R}}^2}{2M} - \frac{\vec{p}_{\vec{r}}^{\,2}}{2m} + W(\vec{r}, \vec{R})\,, \tag{3.25}$$

where m and M are the reduced masses of the particles 1 and 2, and (1 +2) and 3, respectively,

TABLE I. The real and imaginary parts of the various amplitudes and the cross sections are given for the Gaussian potential (3.1) with the parameters (3.2) and $g=-0.4$, $K=2.0$. $f_{\rm ex}$ = the exact amplitude; f_B is the first Born amplitude; f_{B2} is the second Born amplitude; $f_{\rm sc}$ is the semiclassical amplitude; $f_{\rm sc}^{(0)}$ is the same as $f_{\rm sc}$, but with K_a replaced by K; $f_{\rm msi}$ is the improved amplitude obtained in Ref. 1 which includes the $T\Phi_i$ correction.

	q	$f_{\rm ex}$	f_B	f_{B2}	$f_{\rm GL}$	$f_{\rm sc}$	$f_{\rm sc}^{(0)}$	$f_{\rm msi}$
Ref	0.0	6.305	6.440	6.535	6.213	6.293	6.293	6.312
	0.4	4.634	4.786	4.837	4.586	4.625	4.650	4.637
	0.8	1.664	1.825	1.798	1.691	1.660	1.718	1.657
	1.2	0.0653	0.1802	0.1247	0.116	0.0662	0.1132	0.059
	1.4	−0.1484	−0.0727	−0.1186		−0.1471	−0.1198	
	1.6	−0.1629	−0.1252	−0.1539	−0.143	−0.1619	−0.1552	−0.152
	1.8	−0.1052	−0.0975	−0.1086		−0.1056	−0.1141	
	2.0	−0.0464	−0.0567	−0.0551	−0.055	−0.0485	−0.0641	−0.035
Imf	0.0	1.319	0.000	1.338	1.305	1.299	1.299	1.320
	0.4	1.106	0.000	1.131	1.096	1.088	1.090	1.113
	0.8	0.628	0.000	0.663	0.631	0.615	0.628	0.638
	1.2	0.1976	0.000	0.2318	0.218	0.1927	0.2173	0.199
	1.4	0.0650	0.000	0.0932		0.0638	0.0891	
	1.6	−0.0092	0.000	0.0106	0.013	−0.0076	0.0134	−0.010
	1.8	−0.0378	0.000	−0.0270		−0.0347	−0.0215	
	2.0	−0.0380	0.000	−0.0351	−0.031	−0.0349	−0.0304	−0.026
$\lvert f\rvert^2$	0.0	41.49	41.48	44.49	40.30	41.29	41.29	41.59
	0.4	22.70	22.91	24.67	22.23	22.58	22.81	22.74
	0.8	3.163	3.332	3.671	3.259	3.135	3.346	3.153
	1.2	0.0434	0.0325	0.0693	0.0610	0.0415	0.0600	0.0431
	1.4	0.0262	0.0053	0.0228	0.0200	0.0257	0.0223	
	1.6	0.0266	0.0157	0.0238	0.0205	0.0263	0.0243	0.0233
	1.8	0.0125	0.0095	0.0125	0.0109	0.0124	0.0135	
	2.0	0.0036	0.0032	0.0043	0.0040	0.0036	0.0050	0.0019

we have

$$G_{sc}^{(+)} = -\frac{(m/M)^{3/2}}{8\pi^2}\,\frac{Q^2}{\rho_u^{\,2}}\,iH_2^{(1)}(Q\rho), \tag{3.26}$$

where

$$Q = \{2[E - W(\vec{v}, \vec{V})]\}^{1/2} = \text{real},$$

$$\rho_u = (mu^2 + MU^2)^{1/2},$$

and

$$\vec{u} = \vec{r} - \vec{r}', \quad \vec{v} = (\vec{r} + \vec{r}')/2$$

$$\vec{U} = \vec{R} - \vec{R}', \quad \vec{V} = (\vec{R} + \vec{R}')/2.$$

Details of the study of $G_{sc}^{(+)}$ given by (3.26) will be presented elsewhere with specific applications. We only note that the amplitude with $G_{sc}^{(+)}$ of (3.26) will contain integrations over 12 variables, and, even with the Gaussian form of the potential for W, some further drastic approximations are required for the evaluation of f_c.

IV. RESULT OF THE CALCULATION

We now test the various approximate amplitudes derived in Sec. III by numerically evaluating them and comparing them with the Glauber amplitude f_{GL} and the exact amplitude f_{ex}, and also with some of the "improved" results obtained earlier.

Table I contains the exact amplitude obtained by the partial wave analysis[1] for the potential (3.1) with the parameter values specified in (3.2) and $g = -0.4$. The second Born amplitude f_{B2}, the Glauber amplitude f_{GL}, and the semiclassical amplitudes f_{sc} are listed, both their real and imaginary components and the differential cross sections. Often, compensating errors in $\text{Re}f_t$ and $\text{Im}f_t$ give misleading results, where the subscript t denotes any one of the approximate amplitudes. Evidently, f_{sc} is extremely accurate for all values of $q \lesssim K$, both for $\text{Re}f_{sc}$ and $\text{Im}f_{sc}$. For comparison, we have also included the amplitude f_{msi} from Ref. 1, which was the best of all the approximations studied there and contains the correction term $(T\Phi_i)$.

TABLE II. The potential and energy parameters are the same as in Table I; the angle-averaged forms are used. $\alpha_q' = 1/\sqrt{3}\,(1 - q/2K)$. The values are for $|f_t|^2$.

q	f_{ex}	$f_{sc}(\alpha_q = 1)$	$f_{sc}(\alpha_q = 1/\sqrt{3})$	$f_{sc}(\alpha_q = 0)$	$f_{sc}(\alpha_q')$	$f_{sc}(\alpha_q'')$
0.0	41.49	41.57	41.29	41.29	41.29	41.58
0.4	22.70	22.82	22.61	22.53	22.59	22.80
0.8	3.163	3.221	3.167	3.150	3.160	3.206
1.2	0.0434	0.0445	0.0516	0.0562	0.0539	0.0477
1.4	0.0262	0.0265	0.0306	0.0328	0.0319	0.0292
1.6	0.0266	0.0293	0.0287	0.0285	0.0286	0.0288
1.8	0.0125	0.0153	0.0134	0.0126	0.0128	0.0135
2.0	0.0036	0.0052	0.0039	0.0034	0.0035	0.0037

The second Born amplitude f_{B2} in Table I shows that the double hard collision alone is not sufficient to improve the amplitude f_B, although $|f_{B2}|^2$ at large angles is quite reasonable and consistently better than $|f_{GL}|^2$ for $q \gtrsim 1.4$. Comparison between $|f_{B2}|^2$ and $|f_{sc}|^2$ indicates the importance of the distortion effect in $G_{sc}^{(+)}$ which is not present in $G_0^{(+)}$. f_{B2} is not reliable for $q \lesssim 1.2$, as expected, again because of the effect of many soft collisions. The apparent agreement of $|f_B|^2$ with $|f_{ex}|^2$ at small values of q is fortuitous as $f_{c,B}$ immediately destroys the agreement at small q. (The f_{B2} quoted in Ref. 1 was obtained by eikonal approximation on $G_0^{(+)}$, and thus differs from f_{B2} given here.)

One of the conspicuous changes in f_{sc} when compared with the Glauber form f_{GL} and f_c^{eik} of (2.30), is the appearance of $\vec{K}_a$ defined in (2.21). The inclusion of this factor may improve the large-angle behavior, as K_a deviates appreciably from K as q increases. Therefore, we have evaluated $f_{sc}^{(0)}$ of (3.24) by modifying the program for f_{sc}. The comparison between $|f_{sc}|^2$ and $|f_{sc}^{(0)}|^2$ indicates that the changes brought about by the replacement $K_a \to K$ seem to oscillate as q is varied, and even to go in the wrong direction, insofar as the corresponding change in f_{GL} is concerned. This result may be model-dependent, however.

For potentials other than the Gaussian form (3.1), the second part of the amplitude, $f_{c,sc}$, is difficult to evaluate because of the presence of the terms $(\vec{u}\cdot\vec{v})^n$, with n an even integer. To extend the usefulness of $G_{sc}^{(+)}$ to other potentials, we have introduced the angle-averaging procedure in Sec. III, which replaces $\vec{u}\cdot\vec{v}$ by $\alpha_q uv$, so that the resulting integrals for f_c become two dimensional $(dudv)$. We have thus evaluated $f_{c,sc}(\alpha_q)$ of (3.18) for different choices of α_q, i.e.,

$$\alpha_q = 0,\ \frac{1}{\sqrt{3}},\ \text{and } 1.0 \tag{4.1a}$$

and also

$$\alpha_q \approx \alpha_q' = \frac{1}{\sqrt{3}}\left(1 - \frac{q}{2K}\right) \tag{4.1b}$$

$$\alpha_q \approx \alpha_q'' = \left(1 - \frac{q^2}{2K^2}\right) = \cos\Theta. \tag{4.1c}$$

The result is given in Table II. Of course, $\alpha_q = 1/\sqrt{3}$ is exact at $q = 0$ because of the particular forms for V given by (3.1), and we have $f_{c,sc}(\alpha_q) \doteq f_{c,sc}$ in that case. For $q > 0$, $f_{c,sc}(\alpha_q)$ seems to favor the smaller α_q, and $\alpha_q \approx \alpha_q'$ of (4.1b) seems to fit $f_{c,sc}$ better for the whole range $0 \leqslant q \leqslant K$. The q dependence of α_q' and α_q'' is completely arbitrary and adjusted to roughly fit the trend. Note that for $q \lesssim 1.2$, the improvement over f_{GL} is quite significant.

TABLE III. The notations are the same as in Table I. The Gaussian potential is used with $g=-0.2$, and $K=2.0$. The values are for $|f_t|^2$.

q	f_{ex}	f_{sc}	f_{B2}	f_{GL}	$f_{\text{sc}}(\alpha_q=1/\sqrt{3})$	$f_{\text{sc}}(\alpha_q')$
0.0	10.44	10.43	10.64	10.30	10.43	10.43
0.4	5.74	5.735	5.867		5.739	5.737
0.8	0.812	0.814	0.848	0.867	0.818	0.817
1.2	0.007 63	0.007 54	0.009 17	0.009 93	0.008 35	0.008 43
1.4	0.003 07	0.003 09	0.002 83	0.002 25	0.003 40	0.003 50
1.6	0.005 08	0.005 06	0.004 88	0.004 22	0.005 23	0.005 22
1.8	0.002 71	0.002 69	0.002 70	0.002 46	0.002 81	0.002 74
2.0	0.000 82	0.000 82	0.000 86	0.000 84	0.000 87	0.000 82

Table III contains the result for the Gaussian potential (3.1) with the same parameter values (3.2), except that $g=-0.2$. This corresponds in effect to higher-energy scattering, and the forward peak is more pronounced than the case with $g=-0.4$. Again, f_{sc} gives an excellent representation of f_{ex} for all $q \lesssim K=2.0$, while $f_{\text{sc}}(\alpha_q')$ gives a much improved amplitude compared with f_{GL}.

To further test the effectiveness of the angle-averaging procedure with $\alpha_q \approx \alpha_q'$, we have applied the same procedure with the parameter α_q to the Yukawa potential case

$$V=\tfrac{1}{2}g' e^{-Br_c}/r_c\ , \tag{4.2}$$

where

$$r_c=(r^2+c^2)^{1/2}\ .$$

The parameters are chosen as before[1]

$$g'=-0.4\ ,\quad B=0.5\ ,$$
$$c=10^{-3}\approx 0\ ,\quad \text{and } K=1.0\ . \tag{4.3}$$

The result is given in Table IV. This is a fairly low-energy scattering, so that f_{GL} is not expected to be very effective. Of course, the exact evaluation of $f_{c,\text{sc}}$ using $G_{\text{sc}}^{(+)}$ in this case is not easy, and we can only infer its effectiveness through the prediction with $\alpha_q=1/\sqrt{3}$ and $\alpha_q \approx \alpha_q'=1/\sqrt{3} \times(1-q/2K)$. Note that $f_{\text{sc}}(\alpha_q)$ is remarkably accurate at $q \lesssim 1.0$, with $K=1.0$. We emphasize that the calculation of $f_{\text{sc}}(\alpha_q)$ is as easy in this case as the Gaussian case and also the Glauber amplitude. Note that $f_{\text{sc}}(\alpha)$ and $f_{\text{sc}}(\alpha_q')$ turn out to be approximately the same.

TABLE IV. The Yukawa potential (4.2) is used, with the parameters specified in (4.3). $K=1.0$.

q	f_{ex}	f_{GL}	$f_{\text{sc}}(\alpha_q')\approx f_{\text{sc}}(\alpha=1/\sqrt{3})$	f_B
0.0	2.618	2.392	2.602	2.530
0.2	1.953	1.771	1.964	1.903
0.4	0.994	0.877	0.989	0.951
0.6	0.457	0.396	0.452	0.430
0.8	0.218	0.188	0.214	0.202
1.0	0.113	0.098	0.109	0.102

Now, we turn to the second method of simplifying the $f_{c,\text{sc}}$ integral using $G_{\text{sc}}^{\text{eik}}$ in the place of $G_{\text{sc}}^{(+)}$. Explicitly, we have evaluated (3.22) for $f_{c,\text{sc}}^{\text{eik}}$ in the case of the potential (3.21). The result is summarized in Table V. The result with $G_{\text{sc}}^{\text{eik}}$ shows that for small $q \lesssim 0.4$, where the Glauber amplitude is expected to be reliable, $f_{\text{sc}}^{\text{eik}}$ is very similar to f_{GL}, as is clear from (3.22). On the other hand, $|f_{\text{sc}}^{\text{eik}}|^2$ falls below $|f_{\text{ex}}|^2$, with a consistently better fit, in the region $0.4 \lesssim q \lesssim 1.6$. For $q \gtrsim 1.6$, however, $f_{\text{sc}}^{\text{eik}}$ seems to be less reliable. In spite of the physically attractive nature of the approximations introduced for $f_{\text{sc}}^{\text{eik}}$, the improvement is only marginal for large q regions, and the result further suggests that the straight-line propagation of the waves along the $\hat{K}_a$ direction during the interval between the two hard collisions may be too stringent a condition. Such a restriction is not present in $G_{\text{sc}}^{(+)}$. A slightly more general form of the amplitude than (3.22), in which $\vec{q}_i$ and $\vec{q}_f$ are not predetermined, can be given,[6] but involves additional integrations. Incidentally, the extra phase factor $\exp[i(K-K_a)u_z]$ in $f_{\text{sc}}^{\text{eik}}$ appears as a part of the momentum transfer at the points $\vec{r}$ and $\vec{r}'$, and has a nonnegligible effect on the amplitude (and improves it). A slightly improved $|f^{\text{eik}}|^2$ of (3.22b) is also given.

V. CONCLUSION

We have shown[4] earlier that f_{sc} of (3.13) with the semiclassical $G_{\text{sc}}^{(+)}$ is very effective for large-

TABLE V. The eikonal approximation to $G_{\text{sc}}^{(+)}$ is used to calculate the amplitude for the Gaussian potential (3.1) with (3.2), and $g=-0.4$, $K=2.0$.

q	$\text{Re} f_{\text{sc}}^{\text{eik}}$	$\text{Im} f_{\text{sc}}^{\text{eik}}$	$\lvert f_{\text{sc}}^{\text{eik}}\rvert^2$	$\lvert f_{\text{sc}}(\alpha_q')\rvert^2$	$\lvert f_{\text{ex}}\rvert^2$	$\lvert f_{\text{eik}}\rvert^2$
0.0	6.206	1.292	40.27	41.29	41.29	40.27
0.4	4.558	1.077	21.94	22.59	22.70	22.16
0.8	1.634	0.601	3.03	3.160	3.163	3.182
1.2	0.074	0.187	0.0403	0.0539	0.0434	0.0428
1.4	−0.132	0.064	0.0216	0.0319	0.0262	0.0243
1.6	−0.147	−0.001	0.0215	0.0286	0.0266	0.0262
1.8	−0.094	−0.025	0.0094	0.0128	0.0125	0.0135
2.0	−0.041	−0.025	0.0024	0.0035	0.0036	0.0050

angular and -energy ranges, when the integrals in $f_{c,\mathrm{sc}}$ can readily be carried out. For most of the nuclear collisions, the form factors are usually given in Gaussian forms, (2.23), so that f_{sc} is simply integrable. Incidentally, note that the potential in the $Q(v)$ factor in $G_{\mathrm{sc}}^{(+)}$ need not be a Gaussian; only the $V(r)V(r')$ factor in $f_{c,\mathrm{sc}}$ has to be of the form (2.23) to simplify the integrals.

When the potential is not Gaussian, we have shown here that further simplification of $f_{c,\mathrm{sc}}$ is possible, either by the angular averaging and obtain $f_{\mathrm{sc}}(\alpha_q)$ of (3.17), or by the eikonalization of $G_{\mathrm{sc}}^{(+)}$ and obtain $G_{\mathrm{sc}}^{\mathrm{eik}}$ and $f_{\mathrm{sc}}^{\mathrm{eik}}$ of (3.22). Equation (3.17) involves a double integral, just as with f_{GL}, while (3.22) requires triple integrations. In both cases, the result is not as accurate as the exact f_{sc}, but (3.17) still shows improvements over f_{GL}. We did not attempt to actually evaluate f_{sc} directly for non-Gaussian potentials, but, since f_{sc} has already been shown to be quite effective, additional works to further simplify the procedure, such as that explored here, would be useful for many applications.

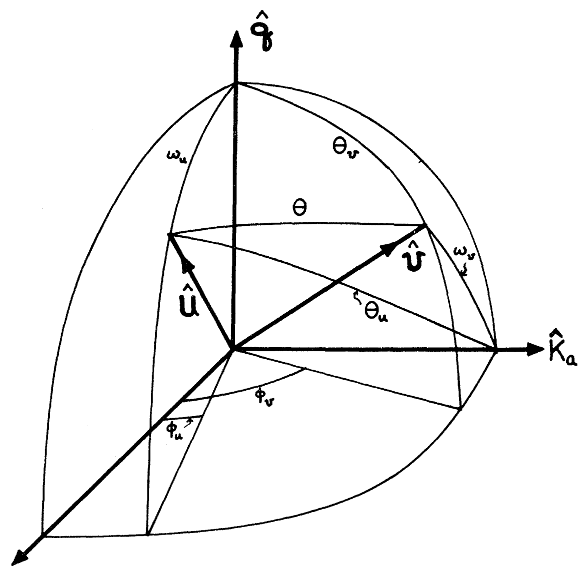

FIG. 1. The coordinate systems are chosen for the angular integrations $d\Omega_{\vec{u}}$ and $d\Omega_{\vec{v}}$ in the amplitude $f_{c,\mathrm{sc}}$.

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APPENDIX. ANGULAR INTEGRALS FOR f_{B2} AND f_{sc}

We explicitly carry out the angular integrals which are involved in the amplitudes (3.6) and (3.16) for the Gaussian potential (3.1). The method can be applied to cases in which more complicated polynomials are involved, as in most of the nuclear form factors. The various vectors are conveniently defined in Fig. 1. The integral of concern here is

$$M \equiv \int_{-1}^{1} d\cos\theta_u \int_0^{2\pi} d\phi_u \int_{-1}^{1} d\cos\theta_v \int_0^{2\pi} d\phi_v\, e^{iqv\cos\theta_v}\, e^{-iK_a u\cos\theta_u}[F_0(u,v;\rho)-\rho^2u^2v^2\cos^2\theta]$$
$$=M_0+M_1, \tag{A1}$$

where

$$M_0=4\pi^2F_0(u,v;\rho)\left(\int_{-1}^{1} d\cos\theta_v\, e^{iqv\cos\theta_v}\right)\left(\int_{-1}^{1} d\cos\theta_u\, e^{-iK_a u\cos\theta_u}\right)$$
$$=16\pi^2F_0(u,v;\rho)\left(\frac{\sin qv}{qv}\right)\left(\frac{\sin K_a u}{K_a u}\right) \tag{A2}$$

with

$$F_0=1+2\rho(v^2+\tfrac{1}{4}u^2)+\rho^2(v^4+\tfrac{1}{2}u^2v^2+\tfrac{1}{16}u^4).$$

The evaluation of M_1 is more involved. First we set, from Fig. 1,

$$\cos\theta=\cos\theta_v\cos\omega_u-\sin\theta_v\sin\omega_u\cos(\phi_u-\phi_v) \tag{A3a}$$
$$=\cos\theta_u\cos\omega_v-\sin\theta_u\sin\omega_v\cos(\phi_u-\phi_v). \tag{A3b}$$

On the other hand, using the fact that $\vec{K}_a\cdot\vec{q}=0$, we have

$$\cos\omega_u=\cos\theta_u\cos(\hat{K}_a,\hat{q})-\sin\theta_u\sin(\hat{K}_a,\hat{q})\cos\phi_u$$
$$=-\sin\theta_u\cos\phi_u \tag{A4a}$$

and

$$\cos\omega_u = -\sin\theta_v \cos\phi_v \ . \tag{A4b}$$

Therefore, $\cos\theta$ becomes

$$\cos\theta = -\cos\theta_u \sin\theta_v \cos\phi_v - \sin\theta_u \cos(\phi_u - \phi_v)(1 - \sin^2\theta_v \cos^2\phi_v)^{1/2} \ . \tag{A5}$$

Now, using the integrals

$$\int_0^{2\pi} \cos\phi_u \, d\phi_u = 0 \quad \text{and} \quad \int_0^{2\pi} \cos^2\phi_u \, d\phi_u = \pi \ ,$$

we have, after the $d\phi_u$ integration,

$$\int_0^{2\pi} d\phi_u \cos^2\theta = \pi[2\cos^2\theta_u \sin^2\theta_v \cos^2\phi_v + \sin^2\theta_u (1 - \sin^2\theta_v \cos^2\phi_v)]$$

and, finally with the $d\phi_v$ integration,

$$\int_0^{2\pi} d\phi_u \int_0^{2\pi} d\phi_v \cos^2\theta = \pi^2[1 + \cos^2\theta_u + \cos^2\theta_v - 3\cos^2\theta_u \cos\theta_v] \ . \tag{A6}$$

Now, the $d\cos\theta_u$ and $d\cos\theta_v$ integrations in $M1$ can be carried out to give

$$M_1 = -(8\pi^2)\rho^2 u^2 v^2 \left[\left(\frac{\sin K_a u}{K_a u} \right) \left(\frac{\sin qv}{qv} + \frac{\cos qv}{(qv)^2} - \frac{\sin qv}{(qv)^3} \right) - \left(\frac{\sin K_a u}{K_a u} + 2\frac{\cos K_a u}{(K_a u)^2} - 2\frac{\sin K_a u}{(K_a u)^3} \right) \left(\frac{\sin qv}{qv} + 3\frac{\cos qv}{(qv)^2} - 3\frac{\sin qv}{(qv)^3} \right) \right] \ . \tag{A7}$$

The sum of (A2) and (A7) gives M. Obviously, we can carry out the angular part of the integrals in f_c for a more general potential of the form (2.23) so that f_c in this case can again be reduced to a double integral involving du and dv. Such a reduction does not seem possible for non-Gaussian potentials and we have to resort either to the angle-averaging procedure of (3.16) or to an eikonal approximation such as (3.22). Extensions of the angular integrations given here to cases with more general Gaussian potentials of the form (2.23) are now straightforward.

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