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Interaction Between Identical Nuclei*

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The nature of the interaction between two identical nuclei having spins zero is studied based on a microscopic theory of complex collisions. The "internucleus" potential is defined such that it appears consistently in the equation of motion and the T matrix of the elastic scattering channel. It is shown that by a suitable integral transformation a strongly nonlocal potential such as that of the resonating-group method can be transformed into an l-dependent local potential with a weak nonlocal residual interaction. An application of the theory is made to a two-α particle system. The theoretical potential shows the similar l dependence to the phenomenological potential. The importance of the l dependence of the real part of the potential is discussed in relation to the elastic scattering phenomena of heavy-ion reactions.

1. INTRODUCTION

This paper is concerned with the interaction between two identical nuclei having spins zero, and particularly the existence of the inner strong repulsion from a theoretical point of view.

Experimentally the α-α interaction is known to have a strong repulsive potential for relative angular momentum $l=0$ and 2 states.[1] Also, the analysis of the $^{12}C+^{12}C$ resonance by the molecular-like-configuration model requires a strong repulsive potential in order to prevent low relative angular momentum states from strongly coupling with compound states.[2] However, in general, elastic scattering experiments of heavier nuclei do not provide sufficient data to confirm the existence of such a repulsion. This situation can be understood if one remembers the existence of the absorptive potential which is due to the coupling to reactions and compound processes: The imaginary potential located near the nuclear surface will make the short-range bahavior of the real potential insignificant, and the absorption inside the nucleus acts effectively like a repulsive potential by damping the wave function for small partial waves.

In order to investigate the internucleus potential theoretically, there are two important points to be made. Firstly, the definition of the potential has to be consistent. In other words, a theoretical potential should appear consistently in the equation of motion and the T matrix. Unfortunately, most of the potentials, which have been calculated by using the definition analogous to the Heitler-London method, i.e.,

$$V(r)=E(r)-E(r=+\infty), \tag{1.1}$$

where $E(r)$ is the energy expectation value of the system constrained to be separated by a distance r, do not satisfy the consistency requirement unless giving the corresponding mass parameter simultaneously.

Secondly, if one wants to compare the theoretical potential to an experimental one, it is necessary to choose an appropriate representation in which the interaction can be expressed as local as possible. The studies of the α-α system by the resonating-group method have shown how the strongly nonlocal interaction acts like a singular local potential.[3] The strong nonlocality of the interaction has been understood as a consequence of the Pauli principle, and interpreted as the origin of the effective repulsive core.[4,5]

Recently, a calculation of the $^{16}O+^{16}O$ potential based on the generator-coordinate-type method has been made.[6] The theoretical potential shows a strong repulsion at a short separation distance. The repulsive potential appearing in the calculation has a quite different nature from the repulsion of the potential defined by Eq. (1.1)[7-9] which is a consequence of the saturation property of nuclear matter. The diverging potential has been understood as a consequence of the breaking symmetry, by which we mean that the symmetric total Hamiltonian is broken to a nonsymmetric cluster Hamiltonian.[10] In this paper, we study the origin of the repulsive potential and its relation to the other method by making use of integral transformations between basis functions for various representations.

In the next section (Sec. 2) we review the theory based on Refs. 6 and 10. We restrict ourselves to the elastic channel by using the projection-operator formalism.[11] In Sec. 3 the resonating-group method and the generator-coordinate method are unified by an integral transformation. In Sec. 4 the nature of the interaction is studied, and the importance of the l dependence of the singular potential is discussed. In Sec. 5 the theory is applied to the α-α potential assuming no internal

excitation. The l-dependent potential obtained gives qualitative agreement with the phonomenological potential.[1,4] In the last section (Sec. 6) three characteristic features of the theoretical potential are discussed in relation to experimental facts:
(a) existence of the strong repulsive potentials for certain partial waves,
(b) l dependence of the potential, and
(c) change of the behavior of the potential at a certain l.

2. THEORY

Let us define the projection operator P, which projects out the elastic channel of two identical nuclei having spins zero, by

$$P=\int d\vec{k}\,|\mathcal{A}\vec{k}\rangle I^{-1}(\vec{k},\vec{k}')\langle\mathcal{A}\vec{k}'|\,d\vec{k}', \tag{2.1}$$

and its orthogonal complement by $Q\equiv\mathcal{A}-P$, where $\mathcal{A}$ is the antisymmetrization operator of the total system. The representation basis $|\mathcal{A}\vec{k}\rangle$, which corresponds to the state of two identical nuclei in the c.m. frame with the relative momentum $\vec{k}$ and the internal states ϕ, can be written explicitly as

$$\langle\vec{\eta},\xi^{(1)},\xi^{(2)}|\mathcal{A}\vec{k}\rangle=c\mathcal{A}_{12}\left(\frac{e^{i\vec{k}\cdot\vec{\eta}}}{(2\pi)^{3/2}}\phi(\xi^{(1)})\phi(\xi^{(2)})\right), \tag{2.2}$$

where $\vec{\eta}$ is the relative coordinate between the two nuclei and $\xi^{(1)}$, $\xi^{(2)}$ are sets of internal coordinates of the two nuclei. The operator $\mathcal{A}_{12}$, which is a sum of permutations of nucleons belonging to different nuclei, antisymmetrizes the total system assuming each internal state is already antisymmetrized. The normalization constant c is fixed so that the basis function is normalized,

$$\langle\mathcal{A}\vec{k}|\,\mathcal{A}\vec{k}'\rangle\sim\delta(\vec{k},\vec{k}'), \tag{2.3}$$

in the asymptotic (large $|\vec{k}|$) region, where the "symmetrized" δ function $\delta(\vec{k},\vec{k}')$ has been defined by

$$\delta(\vec{k},\vec{k}')=\tfrac{1}{2}[\delta(\vec{k}-\vec{k}')+\delta(\vec{k}+\vec{k}')] \tag{2.4}$$

in order to satisfy the permutation symmetry of bosons. The projection-operator condition $P^2=P$ requires that the function $I^{-1}(\vec{k},\vec{k}')$ should satisfy the inverse relation

$$\int I^{-1}(\vec{k},\vec{k}'')\langle\mathcal{A}\vec{k}''|\,\mathcal{A}\vec{k}'\rangle\,d\vec{k}''=\int\langle\mathcal{A}\vec{k}|\,\mathcal{A}\vec{k}''\rangle I^{-1}(\vec{k}'',\vec{k}')d\vec{k}''=\delta(\vec{k},\vec{k}'). \tag{2.5}$$

In Appendix I we show that the "overlap integral" $\langle\mathcal{A}\vec{k}|\,\mathcal{A}\vec{k}'\rangle$ can be written as

$$\langle\mathcal{A}\vec{k}|\,\mathcal{A}\vec{k}'\rangle=\delta(\vec{k},\vec{k}')-\mathcal{L}(\vec{k},\vec{k}'),$$

where $\mathcal{L}(\vec{k},\vec{k}')$ is a Hilbert-Schmidt-type kernel, and the existence of the unique inverse $I^{-1}(\vec{k},\vec{k}')$ is shown in Appendix II.

According to Ref. 11 the equation of motion for the elastic channel is given by

$$(E-\tilde{H}_{PP})\Psi_P=0, \tag{2.6}$$

where the "effective" Hamiltonian is defined by

$$\tilde{H}_{PP}=H_{PP}+H_{PQ}\frac{1}{E^{(+)}-H_{QQ}}H_{QP} \tag{2.7}$$

with $\Psi_P=P\Psi$, $H_{PP}=PHP$, etc. The T matrix for this channel can be written as

$$T_{fi}=\langle(\tilde{H}_{PP}^{\dagger}-E)\chi_f,\,\Psi_{P(i)}^{(+)}\rangle, \tag{2.8}$$

where the scattering-state solution is given by

$$\Psi_{P(i)}^{(+)}=\chi_i+\frac{1}{E^{(+)}-\tilde{H}_{PP}}(\tilde{H}_{PP}-E)\chi_i. \tag{2.9}$$

Here, the asymptotic-state wave functions for the initial (χ_i) and the final (χ_f) states are assumed to be described by normalized noninteracting wave packets.[12]

From Eq. (2.1) the projected wave function can be written either as

$$\Psi_P=\int|\mathcal{A}\vec{k}\rangle I^{-1}(\vec{k},\vec{k}')g(\vec{k}')d\vec{k}\,d\vec{k}', \tag{2.10}$$

or

$$\Psi_P=\int|\mathcal{A}\vec{k}\rangle f(\vec{k})d\vec{k}, \tag{2.11}$$

where we have written

$$g(\vec{k})=\langle\mathcal{A}\vec{k}|\,\Psi\rangle, \tag{2.12}$$

$$f(\vec{k})=\int I^{-1}(\vec{k},\vec{k}')g(\vec{k}')d\vec{k}'. \tag{2.13}$$

Let us call these "state functions." From Eq. (2.6) we can write the equations of motion as

$$\int h^{L}(\vec{k},\vec{k}')g(\vec{k}')d\vec{k}'=Eg(\vec{k}) \tag{2.14}$$

for the state function $g(\vec{k})$, or

$$\int h^{R}(\vec{k},\vec{k}')f(\vec{k}')d\vec{k}'=Ef(\vec{k}) \tag{2.15}$$

for $f(\vec{k})$, where the Hamiltonian kernels are defined by

$$h^{L}(\vec{k},\vec{k}')=\int\langle\mathcal{A}\vec{k}|\,\tilde{H}|\,\mathcal{A}\vec{k}''\rangle I^{-1}(\vec{k}'',\vec{k}')d\vec{k}'', \tag{2.16}$$

$$h^{R}(\vec{k},\vec{k}')=\int I^{-1}(\vec{k},\vec{k}'')\langle\mathcal{A}\vec{k}''|\,\tilde{H}|\,\mathcal{A}\vec{k}'\rangle\,d\vec{k}''. \tag{2.17}$$

Because of the non-orthogonality of the basis functions, we have obtained two sets of the equations of motion for the state functions $g(\vec{k})$ and

$f(\vec{k})$ which are not independent but related by Eq. (2.13). It is also possible to represent Eq. (2.6) by a set of orthonormalized basis functions

$$|\mathcal{A}\vec{k}) = \int |\mathcal{A}\vec{k}'\rangle I^{-1/2}(\vec{k}', \vec{k}) d\vec{k}' . \tag{2.18}$$

Then, the equation of motion for the state function defined by

$$\varphi(\vec{k}) = (\mathcal{A}\vec{k}|\,\Psi\rangle \tag{2.19}$$

is given by

$$\int h(\vec{k}, \vec{k}')\varphi(\vec{k}')d\vec{k}' = E\varphi(\vec{k}), \tag{2.20}$$

where $h(\vec{k}, \vec{k}') \equiv (\mathcal{A}\vec{k}|\,\tilde{H}|\,\mathcal{A}\vec{k}')$. While this representation provides a symmetric expression for the kernel, we find that the reduction for the kernel h^L or h^R is much simpler than for h.

In order to reduce the Hamiltonian kernel h^L, we write the Hamiltonian integral as

$$\langle \mathcal{A}\vec{k}|\,\tilde{H}|\,\mathcal{A}\vec{k}'\rangle = c\langle \vec{k}|\,\tilde{H}|\,\mathcal{A}\vec{k}'\rangle, \tag{2.21}$$

using the relationship, $[\mathcal{A}, \tilde{H}] = 0$. Furthermore, we separate the original Hamiltonian into

$$H = t + h_1 + h_2 + v_0, \tag{2.22}$$

where t is the relative kinetic energy and h_1, h_2 are the internal energy operators of two nuclei. The interaction v_0 is the sum of nucleon-nucleon interactions V_{ij}, restricted to those which interact between two nucleons belonging to different nuclei,

$$v_0 = \sum_{i\in \text{nucleus I}} \sum_{i\in \text{nucleus II}} V_{ij} . \tag{2.23}$$

From Eqs. (2.19)–(2.21) we obtain

$$\langle \mathcal{A}\vec{k}|\,\tilde{H}|\,\mathcal{A}\vec{k}'\rangle = \mathcal{K}(\vec{k})\langle \mathcal{A}\vec{k}|\,\mathcal{A}\vec{k}'\rangle + c\langle \vec{k}|\,v\,|\mathcal{A}\vec{k}'\rangle, \tag{2.24}$$

where we have written

$$\mathcal{K}(\vec{k}) = \frac{\hbar^2}{2\mu}k^2 + 2\epsilon_0, \tag{2.25}$$

$$v = v_0 + HQ\frac{1}{E^{(+)} - H_{QQ}}QH . \tag{2.26}$$

Here we assume that the internal state $\phi(\xi^{(i)})$ is the ground-state eigenfunction of h_i ($i = 1$ or 2) with eigenvalue ϵ_0. From Eq. (2.16) we have

$$h^L(\vec{k}, \vec{k}') = \mathcal{K}(\vec{k})\delta(\vec{k}, \vec{k}') + \mathcal{V}^L(\vec{k}, \vec{k}') \tag{2.27}$$

with

$$\mathcal{V}^L(\vec{k}, \vec{k}') = \int c\langle \vec{k}|\,v\,|\,\mathcal{A}\vec{k}''\rangle I^{-1}(\vec{k}'', \vec{k}')d\vec{k}'' . \tag{2.28}$$

Therefore, we obtain the reduced equation of motion for the state function $g(\vec{k})$:

$$\mathcal{K}(\vec{k})g(\vec{k}) + \int \mathcal{V}^L(\vec{k},\vec{k}')g(\vec{k}')d\vec{k}' = Eg(\vec{k}) . \tag{2.29}$$

Using the fact that the asymptotic states χ_i and χ_f are described by noninteracting wave packets, we can write the T matrix as

$$T_{fi} = \int f^{(0)}_{\vec{k}_f}(\vec{k})^*\mathcal{V}^L(\vec{k}, \vec{k}')g^{(+)}_{\vec{k}_i}(\vec{k}')d\vec{k}\,d\vec{k}', \tag{2.30}$$

where

$$f^{(0)}_{\vec{k}_f}(\vec{k}) = \delta(\vec{k}, \vec{k}_f), \tag{2.31}$$

$$g^{(+)}_{\vec{k}_i}(\vec{k}) = \langle \mathcal{A}\vec{k}|\,\Psi^{(+)}_i\rangle = \delta(\vec{k}, \vec{k}_i) + (\text{outgoing wave}) . \tag{2.32}$$

Since the appearance of the function $\mathcal{V}^L(\vec{k},\vec{k}')$ in the equation of motion (2.29) and the T matrix (2.30) with boundary conditions (2.31) and (2.32) is consistent with a "potential" in the quantum theory of scattering, we will call this function the "potential" between two nuclei.

Equation (2.30) shows that as far as the scattering state is concerned we need only the knowledge of $g^{(+)}(\vec{k})$, because the normalization of scattering states is performed at the asymptotic region while the normalization of bound states requires the knowledge of two solutions, $g(\vec{k})$ and $f(\vec{k})$, in order to satisfy the condition

$$1 = \int f^*(\vec{k})g(\vec{k})d\vec{k} .$$

It is also possible to write the T matrix in terms of $f^{(+)}(\vec{k})$ using Eq. (2.13). However, in general the solution $f(\vec{k})$ is not unique because of the nonlinear independence of the set of basis functions (see Appendix II). Besides, we will find the inconvenience of the formula using $f(\vec{k})$ in the later section.

3. INTEGRAL TRANSFORMATION

In the last section, the equation of motion has been represented in the relative momentum space of two nuclei. (We will refer to this representation as "$\vec{k}$ rep.") Since the parameter $\vec{k}$ in $\vec{k}$ rep corresponds to the relative momentum of two nuclei, we can expect that by a Fourier transformation the equations will be transformed into those in the conjugate coordinate space ("$\vec{x}$ rep"):

$$|\mathcal{A}\vec{x}) = \int \frac{d\vec{k}}{(2\pi)^{3/2}} e^{-i\vec{k}\cdot\vec{x}}|\,\mathcal{A}\vec{k}\rangle, \tag{3.1}$$

or explicitly

$$\langle \vec{\eta}, \xi^{(1)}, \xi^{(2)}|\,\mathcal{A}\vec{x}\rangle = c\mathcal{A}_{12}[\delta(\vec{x} - \vec{\eta})\phi(\xi^{(1)})\phi(\xi^{(2)})] . \tag{3.2}$$

Since the redundant coordinate $\vec{x}$ is not affected by the operator $\mathcal{A}_{12}$ and is conjugate to $\vec{k}$, it seems to have more physical meaning as the relative coordinate than the coordinate $\vec{\eta}$, which is a superposition of nucleon coordinates and is affected by

the antisymmetrization.

We note that the Fourier transform of $f(\vec{k})$ in $\vec{k}$ rep coincides with the solution of the resonating-group method,

$$\langle \vec{\eta}, \xi | \Psi \rangle = \int \hat{f}(\vec{x}) \langle \vec{\eta}, \xi | \mathcal{A}\vec{x} \rangle d\vec{x}$$
$$= c\mathcal{A}_{12}[\hat{f}(\vec{\eta})\phi(\xi^{(1)})\phi(\xi^{(2)})], \quad (3.3)$$

where

$$\hat{f}(\vec{x}) = \int \frac{d\vec{k}}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}} f(\vec{k}). \quad (3.4)$$

The state function $\hat{f}(\vec{x})$ is the solution of the integrodifferential equation (see Appendix III),

$$-\frac{\hbar^2}{2\mu}\nabla^2 \hat{f}(\vec{x}) + \int \mathcal{V}^R(\vec{x}, \vec{x}')\hat{f}(\vec{x}')d\vec{x}' = \epsilon \hat{f}(\vec{x}), \quad (3.5)$$

where the transformed interaction kernel $\mathcal{V}^R(\vec{x}, \vec{x}')$ should coincide with that of the resonating-group method. The kernel $\mathcal{V}^R$ has been calculated within the framework of the resonating-group method for a two-α-particle system.[13] It is shown that $\mathcal{V}^R(\vec{x}, \vec{x}')$ is strongly nonlocal due to the antisymmetrization.

Instead of making the Fourier transformation, we can make the following integral transformation for the set of representation basis ("$\vec{r}$ rep")[14]:

$$|\mathcal{A}\vec{r}\rangle = \int \left(\frac{4\pi}{\nu}\right)^{3/4} e^{-k^2/2\nu} \frac{e^{-i\vec{k}\cdot\vec{r}}}{(2\pi)^{3/2}} |\mathcal{A}\vec{k}\rangle d\vec{k}, \quad (3.6)$$

or explicitly

$$\langle \vec{\eta}, \xi^{(1)}, \xi^{(2)} | \mathcal{A}\vec{r} \rangle = c\mathcal{A}_{12}\left[\left(\frac{\nu}{2\pi}\right)^{3/4} e^{-(1/4)\nu(\vec{r}-\vec{\eta})^2} \times \phi(\xi^{(1)})\phi(\xi^{(2)})\right], \quad (3.7)$$

where ν is an additional parameter which we will fix later in order to obtain a "maximal local" potential. This transformation has several important properties: The basis function (3.6) has a finite norm while the overlap integrals of $\vec{k}$ and $\vec{x}$ rep contain both δ functions and bounded functions. This finiteness of the norm of basis functions makes it easier to evaluate the order of magnitude of various kernels when we need to make approximations to them. Furthermore, the basis function $\langle \vec{\eta}, \xi | \mathcal{A}\vec{r} \rangle$ can be given by the harmonic-oscillator shell-model wave function when the parameter ν is appropriately chosen. In spite of these advantages of $\vec{r}$ rep, it also has a disadvantage: Because of the nonunitarity of the transformation, the inverse transformation is not a definite integral, and the existence of the state function (Appendix III)

$$\hat{f}(\vec{r}) = \int \left(\frac{\nu}{4\pi}\right)^{3/4} e^{k^2/2\nu} \frac{e^{i\vec{k}\cdot\vec{r}}}{(2\pi)^{3/2}} f(\vec{k})d\vec{k} \quad (3.8)$$

cannot be proved generally,[15] unless $f(\vec{k})$ converges faster than $e^{-k^2/2\nu}$ for large $|\vec{k}|$. We note that the solution $g(\vec{r})$ always exists and is unique as far as Ψ exists.

Above all, we expect that the transformation can localize strongly nonlocal interaction in $\vec{x}$ rep. In general, any nonlocal interaction can be rewritten by a momentum-dependent local potential,

$$v(\vec{x}, \vec{x}') = \sum_n \sum_{\{\mu_i\}} \frac{1}{n!} v_{\mu_1 \cdots \mu_n}(\vec{x}) \frac{\partial}{\partial x_{\mu_1}} \cdots \frac{\partial}{\partial x_{\mu_n}} \delta(\vec{x} - \vec{x}'),$$
$$\mu_i = (x, y, z). \quad (3.9)$$

On the other hand, a potential which is local in $\vec{r}$ rep,

$$\mathcal{V}(\vec{r}, \vec{r}') = U(r)\delta(\vec{r}, \vec{r}'), \quad (3.10)$$

is transformed to a momentum-dependent potential in $\vec{x}$ rep (Appendix III),

$$\mathcal{V}(\vec{x}, \vec{x}') = U\left(\vec{x} + \frac{1}{\nu}\vec{\nabla}\right)\delta(\vec{x}, \vec{x}'). \quad (3.11)$$

While the momentum dependence of the potential (3.11) is more restricted than Eq. (3.9), we expect that by choosing an appropriate value for the parameter ν we will be able to include the nonlocality of the interaction due to the antisymmetrization. In the next section, we show that the transformation (3.6) leads to a maximal local potential for a certain ν.

We note that the mathematical details of the properties of transformations between $\vec{k}$, $\vec{x}$, and $\vec{r}$ rep are summarized in Appendix III.

4. NATURE OF THE INTERACTION

In this section, we will consider the equation of motion for $g(\vec{r})$ in $\vec{r}$ rep:

$$-\frac{\hbar^2}{2\mu}\nabla^2 g(r) + \int \mathcal{V}(\vec{r}, \vec{r}')g(\vec{r}')d\vec{r}' = Eg(\vec{r}), \quad (4.1)$$

and especially the nature of the interaction

$$\mathcal{V}(\vec{r}, \vec{r}') = \int u(\vec{r}, \vec{r}'')I^{-1}(\vec{r}'', \vec{r}')d\vec{r}'', \quad (4.2)$$

$$u(\vec{r}, \vec{r}') = c\langle \vec{r} | v | \mathcal{A}\vec{r}' \rangle. \quad (4.3)$$

Let us first examine the overlap integral. From the Schwartz inequality, we have

$$-N(r)N(r') \leq \langle \mathcal{A}\vec{r} | \mathcal{A}\vec{r}' \rangle \leq N(r)N(r'), \quad (4.4)$$

where we have written

$$N(r) = [\langle \mathcal{A}\vec{r} | \mathcal{A}\vec{r} \rangle]^{1/2}. \quad (4.5)$$

In order to avoid the singular behavior of the inverse $I^{-1}(\vec{r}, \vec{r}')$ in case that $N(r)N(r') \sim 0$ for all $\vec{r}$ or $\vec{r}'$, we employ a set of "normalized basis

functions"

$$|\tilde{\mathcal{A}}\vec{r}\rangle = \frac{1}{N(r)}|\mathcal{A}\vec{r}\rangle . \tag{4.6}$$

Then, we have

$$-1 \leq \tilde{I}(\vec{r},\vec{r}') \leq 1, \quad \tilde{I}(\vec{r},\vec{r}) = 1, \tag{4.7}$$

where we have written the "normalized overlap integral" as

$$\tilde{I}(\vec{r},\vec{r}') = \frac{1}{N(r)}\langle \mathcal{A}\vec{r}|\,\mathcal{A}\vec{r}'\rangle \frac{1}{N(r')} . \tag{4.8}$$

It is also feasible to express the interaction integral in terms of the normalized basis functions,

$$\tilde{u}(\vec{r},\vec{r}') = \int \tilde{v}(\vec{r},\vec{r}'')\tilde{I}(\vec{r}'',\vec{r}')d\vec{r}'' , \tag{4.9}$$

where those kernels have been defined by

$$\tilde{u}(\vec{r},\vec{r}') = \frac{1}{N(r)}u(\vec{r},\vec{r}')\frac{1}{N(r')}, \tag{4.10}$$

$$\tilde{v}(\vec{r},\vec{r}') = \frac{1}{N(r)}v(\vec{r},\vec{r}')N(r') . \tag{4.11}$$

Since the overlap integral $\tilde{I}(\vec{r},\vec{r}')$ is expected to be a peaked function at $\vec{r}\simeq\vec{r}'$ (also $\vec{r}\simeq-\vec{r}'$ for a two-identical-particle system), and normalized as $\tilde{I}(\vec{r},\vec{r})=1$, the function $\tilde{v}(\vec{r},\vec{r}')$ can be considered to have a similar behavior as $\tilde{u}(\vec{r},\vec{r}')$ unless it is singular, by which we mean a quickly varying or diverging function of $\vec{r}$ and $\vec{r}'$. In fact $\tilde{u}(\vec{r},\vec{r}')$ can be a diverging function of $\vec{r}$ as we will see later. We now wish to show that $\tilde{u}(\vec{r},\vec{r}')$ can be separated into a regular and a diverging part, and the diverging term can be expressed by a local potential while the regular part gives a weakly nonlocal potential.

From Eq. (4.3) with Eq. (2.22), the direct part of the interaction integral $\tilde{u}_0$ can be written as

$$\begin{aligned}\tilde{u}_0(\vec{r},\vec{r}') &= c\langle\vec{r}|\,v_0|\,\mathcal{A}\vec{r}'\rangle \\ &= \Delta E(\vec{r},\vec{r}') + \Delta T(\vec{r},\vec{r}'),\end{aligned} \tag{4.12}$$

where the first term of the right-hand side of Eq. (4.12),

$$\Delta E(\vec{r},\vec{r}') = \langle\tilde{\mathcal{A}}\vec{r}|\,H|\,\tilde{\mathcal{A}}\vec{r}\rangle - (2\epsilon_0+\epsilon_{\rm rel})\tilde{I}(\vec{r},\vec{r}'), \tag{4.13}$$

corresponds to the difference of the total energy expectation value of an interacting and a separated state. Here $\epsilon_{\rm rel}$ is the kinetic energy of the relative motion of basis function $|\mathcal{A}\vec{r}\rangle$ which is described by a Gaussian function. The quantity (4.13) (mainly the diagonal component) has been calculated as a so-called "static" potential assuming implicitly that

$$\Delta T(\vec{r},\vec{r}') = -c\langle\vec{\tilde{r}}|\,t|\,\tilde{\mathcal{A}}\vec{r}'\rangle + \epsilon_{\rm rel}\tilde{I}(\vec{r},\vec{r}') \tag{4.14}$$

is small at the static limit.[7-9] However, as we will see, the term (4.14) can be a diverging function when $|\vec{r}|\to 0$.

Now, let us consider Eq. (4.14) which we can write

$$\Delta T(\vec{r},\vec{r}') = \frac{\hbar^2}{2\mu}\frac{1}{N(r)}\nabla_r{}^2\langle\mathcal{A}\vec{r}|\,\mathcal{A}\vec{r}'\rangle\frac{1}{N(r')} + \epsilon_{\rm rel}\tilde{I}(\vec{r},\vec{r}'), \tag{4.15}$$

using the properties that $t|\vec{r}\rangle = -(\hbar^2/2\mu)\nabla_r{}^2|\vec{r}\rangle$ (Appendix III). From Eq. (4.14), it is obvious that $\Delta T(\vec{r},\vec{r}')$ vanishes at the asymptotic region $|\vec{r}|, |\vec{r}'|\to\infty$. In order to consider the short-range behavior of ΔT, we expand the overlap integral in partial waves

$$\tilde{I}(\vec{r},\vec{r}') = \sum_l f_l(r,r')P_l(\cos\theta), \tag{4.16}$$

which is possible because of the rotational invariance of $\langle\mathcal{A}\vec{r}|\,\mathcal{A}\vec{r}'\rangle$ in case of spin-zero particles. The lth partial-wave component of Eq. (4.15) is then given by

$$\begin{aligned}\frac{\hbar^2}{2\mu}\Big[&\left(\frac{N''}{N}+\frac{2N'}{rN}-\frac{l(l+1)}{r^2}\right)f_l(r,r') \\ &+2\left(\frac{N'}{N}+\frac{1}{r}\right)f_l'(r,r')+f_l''(r,r')+\epsilon_{\rm rel}f_l(r,r')\Big],\end{aligned} \tag{4.17}$$

where $N'\equiv dN(r)/dr$, $f_l'\equiv\partial f_l(r,r')/\partial r$, etc. By expanding the explicit form of $\langle\mathcal{A}\vec{r}|\,\mathcal{A}\vec{r}'\rangle$ into partial waves, we can obtain the lth partial-wave component of $\tilde{I}(\vec{r},\vec{r}')$ as

$$f_l(r,r') = \frac{1}{N(r)}h_l(r,r')\frac{1}{N(r')}, \tag{4.18}$$

where

$$h_l(r,r') = r^l\left(\sum_{m,n}h_{mn}^l r^{2m}r'^{2n}\right)r'^l . \tag{4.19}$$

Also we can write

$$[N(r)]^2 = \sum_k a_k r^{2k}, \tag{4.20}$$

where

$$a_k = \sum_{l,m,n}{}' h_{mn}^l \quad (k=l+m+n) . \tag{4.21}$$

By writing

$$f_l(r,r') = \sum_m g_l^m(r)f_l^m(r'), \tag{4.22}$$

where

$$g_l^m(r) = \frac{r^{l+2m}}{N(r)},$$

$$f_l^m(r') = \left(\sum_n h_{mn}^l r'^{2n+l}\right)\Big/N(r'), \tag{4.23}$$

Eq. (4.17) is expressed by

$$\Delta T_l(r,r')=\frac{\hbar^2}{2\mu}\sum_m \frac{2m(2m+2l+1)}{r^2} g_l^m(r) f_l^m(r') . \tag{4.24}$$

Now, we consider two cases in order to evaluate the behavior of Eq. (4.24) for small r:

case I: $N(r)\neq 0$ when $r\to 0$ (or $a_0\neq 0$);

case II: $N(r)\to 0$ when $r\to 0$ (or $a_0=0$).

First, we consider case I. For the limit $r\to 0$, leading terms of $f_l(r,r')$ and $\Delta T_l(r,r')$ are

$$f_l(r,r')\to g_l^0(r) f_l^0(r') , \tag{4.25}$$

$$\Delta T_l(r,r')\to \frac{\hbar^2}{2\mu}\frac{2(2l+3)}{r^2} g_l^1(r) f_l^1(r') . \tag{4.26}$$

Since $g_l^1(r)=r^{l+2}/N(r)$ and $N(r)\neq 0$, when $r\to 0$, $\Delta T_l(r,r')$ is a regular function for small r. Next, we consider case II assuming

$$a_n=0 \quad \text{for } n<2M,$$

and $a_{2M}\neq 0$. For the small r, we can write

$$N(r)\to a_{2M}{}^{1/2} r^{2M}\left(1+\frac{a_{2M+1}}{2a_{2M}} r^2\right), \tag{4.27}$$

$$f_l(r,r')\to \frac{r^{l-2M}}{a_{2M}{}^{1/2}}\left(\sum_m r^{2m} f_l^m(r')\right). \tag{4.28}$$

Since $f_l(r,r')$ should be finite at $r\to 0$, $f_l^m(r')$ should vanish for $m<M-\frac{1}{2}l$ in the case that $l-2M<0$. Then the leading terms of $f_l(r,r')$ and $\Delta T_l(r,r')$ are

$$f_l(r,r')\to \frac{1}{a_{2M}{}^{1/2}} f_l^{M-l/2}(r') , \tag{4.29}$$

$$\Delta T_l(r,r')\to \frac{\hbar^2}{2\mu}\frac{2M(2M+1)-l(l+1)}{r^2} \times \frac{1}{a_{2M}{}^{1/2}} f_l^{M-l/2}(r') \tag{4.30}$$

for $l<2M$, and

$$f_l(r,r')\to g_l^0(r) f_l^0(r') , \tag{4.31}$$

$$\Delta T_l(r,r')\to \frac{\hbar^2}{2\mu}\frac{2(2l+3)}{r^2} g_l^1(r) f_l^1(r') \tag{4.32}$$

for $l>2M$. Equation (4.30) shows that $\Delta T_l(r,r')$ is a singular function of r when $l<2M$, while it is regular for the case $l>2M$. The divergence of $\Delta T_l(r,r')$ comes from the first term of Eq. (4.17), because from Eqs. (4.27) and (4.29) the first term converges as

$$\left(\frac{N''}{N}+\frac{2N'}{rN}-\frac{l(l+1)}{r^2}\right) f_l(r,r') \to \frac{2M(2M+1)-l(l+1)}{r^2}\frac{1}{a_{2M}{}^{1/2}} f_l^{M-l/2}(r') \tag{4.33}$$

for small r. On the other hand the other terms converge to the order of constant. Therefore, it is possible to express the interaction corresponding to the singular part by a local l-dependent potential

$$U_l^S(r)=\frac{\hbar^2}{2\mu}\left(\frac{N''}{N}+\frac{2N'}{rN}-\frac{l(l+1)}{r^2}\right). \tag{4.34}$$

Since the nature of the function $N(r)$ depends on the choice of a transformation parameter ν as well as wave functions of identical nuclei, we can conclude that the $\Delta T(\vec{r},\vec{r}')$ term of the interaction, which is consequence of breaking of the symmetry of the total Hamiltonian into the cluster Hamiltonian [Eq. (2.22)], will be transformed to a manifestly local potential by choosing an appropriate transformation constant.

By now, we have shown that the interaction integral (4.12) may be divided into a singular and a regular part:

$$\tilde{u}_0(\vec{r},\vec{r}')=\tilde{u}_S(\vec{r},\vec{r}')+\tilde{u}_R(\vec{r},\vec{r}') , \tag{4.35}$$

where the singular term converges to

$$\tilde{u}_S(\vec{r},\vec{r}')\to \sum_l^{<2M} U_l^S(r) f_l(r,r') P_l(\cos\theta) \tag{4.36}$$

for small r. Now, let us consider the regular part which we write

$$\tilde{u}_R(\vec{r},\vec{r}')=U_R(\vec{r},\vec{r}')\tilde{I}(\vec{r},\vec{r}') , \tag{4.37}$$

where $U_R(\vec{r},\vec{r}')=\tilde{u}_R(\vec{r},\vec{r}')/\tilde{I}(\vec{r},\vec{r}')$. From the exchange symmetry of two identical nuclei, we find that it is convenient to introduce a set of new coordinates $\vec{r}_1\equiv\vec{r}-\vec{r}'/\sqrt{2}$ and $\vec{r}_2\equiv\vec{r}+\vec{r}'/\sqrt{2}$. Then, we have the following relationships for $U_R(\vec{r}_1,\vec{r}_2)$:

$$U_R(\vec{r}_1,\vec{r}_2)=U_R(\vec{r}_2,\vec{r}_1) ,$$
$$U_R(-\vec{r}_1,-\vec{r}_2)=U_R(\vec{r}_1,\vec{r}_2) . \tag{4.38}$$

Now, we assume that $U_R(\vec{r}_1,\vec{r}_2)$ is a smooth function of $\vec{r}_1$ and $\vec{r}_2$. Then we can expand $U_R(\vec{r}_1,\vec{r}_2)$ near $\vec{r}_1=0$:

$$U_R(\vec{r}_1,\vec{r}_2)=U_R(0,\vec{r}_2) +\frac{1}{2}\sum_{ij} r_{1_i} r_{1_j}\left[\partial_{1_i}\partial_{1_j} U_R(\vec{r}_1,\vec{r}_2)\right]_{\vec{r}_1=0}+\cdots . \tag{4.39}$$

The first term $U(0,\vec{r}_2)=U(\vec{r},\vec{r})$ corresponds to the local potential

$$\tilde{v}_R(\vec{r},\vec{r}')=\int U_R(\vec{r},\vec{r})\tilde{I}(\vec{r},\vec{r}'')\tilde{I}^{-1}(\vec{r}'',\vec{r}')d\vec{r}'' = U_R(\vec{r},\vec{r})\delta(\vec{r},\vec{r}') , \tag{4.40}$$

and the other terms contribute to the nonlocal interaction. When $U(\vec{r}_1,\vec{r}_2)$ is smooth {i.e.,

$[\partial_{1_i}\partial_{1_j}U_R(\vec{r}_1,\vec{r}_2)]_{\vec{r}_1=0}$ is small} and the overlap integral $\tilde{I}(\vec{r},\vec{r}')$ is sharply peaked [i.e., $r_{1_i}r_{1_j}\times I(\vec{r},\vec{r}')$ is small], we can neglect these nonlocal terms. Since there is no trivial choice of the transformation parameter ν in order to satisfy these conditions, the parameter we will choose in the next section is quite empirical, and we can show our choice, which gives case II [i.e., $N(r)\to 0$, when $r\to 0$], is fairly good after all.

5. APPLICATION TO THE α-α SYSTEM

Now we wish to apply our method of calculating a potential for a simple system. To study the short-range behavior of nucleus-nucleus interaction, the α-particle system is ideal. Since the reaction channels of the system open at the relative kinetic energy around 35 MeV, we can expect that in the low-energy elastic scattering the "direct" v_0 term is dominant. Actually the experimental data are well analyzed without an imaginary potential for the partial waves $l=0$, 2, and 4.[1]

To proceed with a practical calculation, we need to give a nucleon-nucleon potential and an internal wave function for the α particle. We employ the Volkov II potential[16] and for the internal wave function a single Slater determinant filled with four nucleons in the (IS) harmonic-oscillator orbit as has been employed in Ref. 16. Then, the overlap integral in $\vec{r}$ rep with the transformation parameter ν [Eq. (3.6)] is given by

$$\langle \mathcal{A}\vec{r}|\,\mathcal{A}\vec{r}'\rangle = \Delta(\vec{r},\vec{r}') - L(\vec{r},\vec{r}')\,, \tag{5.1}$$

where we have written

$$\Delta(\vec{r},\vec{r}') = \tfrac{1}{2}\{\exp[-\tfrac{1}{8}\nu(\vec{r}-\vec{r}')^2]+\exp[-\tfrac{1}{8}\nu(\vec{r}+\vec{r}')^2]\}, \tag{5.2}$$

$$\begin{aligned} L(\vec{r},\vec{r}') = 2\beta\Big\{&\exp\Big[-\tfrac{1}{8}\nu\frac{(\vec{r}-\vec{r}')^2}{(1+\frac{1}{3}\alpha)}\Big]\exp\Big[-\tfrac{1}{8}\nu\frac{(\vec{r}+\vec{r}')^2}{(1+3\alpha)}\Big]\\ &+\exp\Big[-\tfrac{1}{8}\nu\frac{(\vec{r}-\vec{r}')^2}{(1+3\alpha)}\Big]\exp\Big[-\tfrac{1}{8}\nu\frac{(\vec{r}+\vec{r}')^2}{(1+\frac{1}{3}\alpha)}\Big]\Big\}\\ &-3\gamma\exp\Big[-\tfrac{1}{8}\nu\frac{(\vec{r}-\vec{r}')^2}{(1+\alpha)}\Big]\exp\Big[-\tfrac{1}{8}\nu\frac{(\vec{r}+\vec{r}')^2}{(1+\alpha)}\Big], \end{aligned} \tag{5.3}$$

with[17]

$$\alpha=\frac{\nu}{\nu_{\mathrm{He}}},\qquad \beta=\left(\frac{16\alpha}{(3+\alpha)(1+3\alpha)}\right)^{3/2},$$

and

$$\gamma=\left(\frac{4\alpha}{(1+\alpha)^2}\right)^{3/2}. \tag{5.4}$$

The harmonic-oscillator constant for the α particle is taken as $\nu_{\mathrm{He}}=0.535\ \mathrm{fm}^{-2}$. In Fig. 1, we plot $N(r)$ as a function of r for $\alpha=0.4$, 1.0, and 1.6. $\alpha=1.0$ corresponds to case II of the last section and we can expect the maximal local potential for the $\Delta T(\vec{r},\vec{r}')$ term [Eq. (4.14)] in this case.

In our model wave function, the choice of the parameter $\alpha=1$ corresponds to a single-Slater-determinant basis function. As we have mentioned in the last section, it is not necessary that this choice also gives the maximal local potential for the $\Delta E(\vec{r},\vec{r}')$ term. At least we can tell from our model basis function that:
(i) the overlap integral $\langle \mathcal{A}\vec{r}|\,\mathcal{A}\vec{r}'\rangle$ is a peaked function at $\vec{r}_1=0$ and $\vec{r}_2=0$,
(ii) the quantity $c\langle\vec{r}\,|H|\,\mathcal{A}\vec{r}'\rangle/I(\vec{r},\vec{r}')$ is a smooth function of $\vec{r}_1$ and $\vec{r}_2$,
while the limit $\alpha\to\infty$, in which case the equation of motion converges to the resonating-group method, gives a sharper overlap integral but nonsmooth value $c\langle\vec{r}|H|\mathcal{A}\vec{r}'\rangle/I(\vec{r},\vec{r}')$. Although there is no *a priori* reason to choose $\alpha=1$, we have calculated the l-dependent local potential by taking $\alpha=1$. According to Eq. (4.39), we make the following ansatz:

$$\tilde{u}_l(r,r')\cong U_l(r)\tilde{I}_l(r,r')\,, \tag{5.5}$$

as the zeroth approximation, where

$$U_l(r)=\tilde{u}_l(r,r)/\tilde{I}_l(r,r)\,, \tag{5.6}$$

which is feasible to include the l dependence of the singular part of the interaction integral [Eq. (4.33)]. In Fig. 2, we compare the exact $[\tilde{u}_l(r,r')]$ and the approximate $[U_l(r)\tilde{I}_l(r,r')]$ interaction integrals for $l=2$ [Fig. 2(a)] and $l=4$ [Fig. 2(b)]. For $l=0$ and $l=6$, 8, the behavior of these functions is similar to that of $l=2$ and $l=4$, respectively. The over-all features of the exact and approximate functions are quite similar, especial-

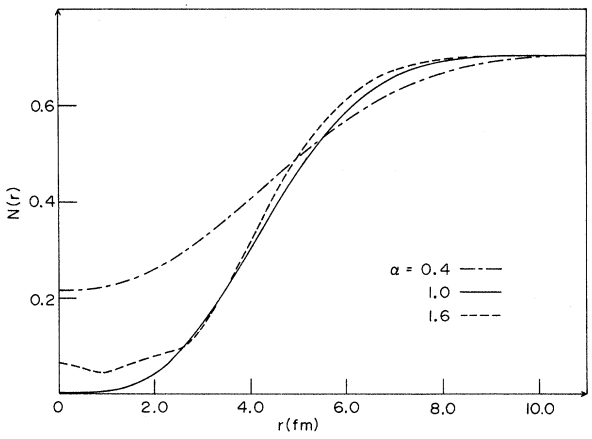

FIG. 1. Square root of the diagonal overlap integral [Eq. (4.5)] as a function of the separation distance, taking the parameter $\alpha=0.4$, 1.0, and 1.6.

ly at the short and the long separation distances. We note that this agreement is significant compared to the corresponding resonating-group interaction for which the nonlocal term is very large and the approximation [Eq. (5.5)] breaks down. From the definition of the interaction kernel, the approximation [Eq. (5.5)] gives the local interaction:

$$\tilde{v}_l(r,r') = U_l(r)\frac{1}{rr'}\delta(r-r').$$

In Fig. 3, we show the local potential $U_l(r)$ defined by Eq. (5.6) for $l=0, 2, \ldots, 8$. When we compare the theoretical potential to the experimental one,[1] we will find some significant behavior which both potentials have:
(i) For $l=0, 2$, there appears a strong repulsion whereas for $l=4, 6, 8$ there does not;
(ii) the potential for $l=2$ is deeper than that for $l=0$, and the potentials of $l=4, 6, 8$ are much deeper than those for $l=0, 2$.
Of course, there are also disagreements:
(i) For $l=4, 6, 8$, the potentials should be much deeper than we have obtained;
(ii) for $l=6, 8$, the potentials should have longer range than for $l=0, 2, 4$.
These may be because of the relatively poor approximation for $l=4, 6, 8$ compared to that for $l=0, 2$, and the neglect of polarization effect which becomes important for the partial waves

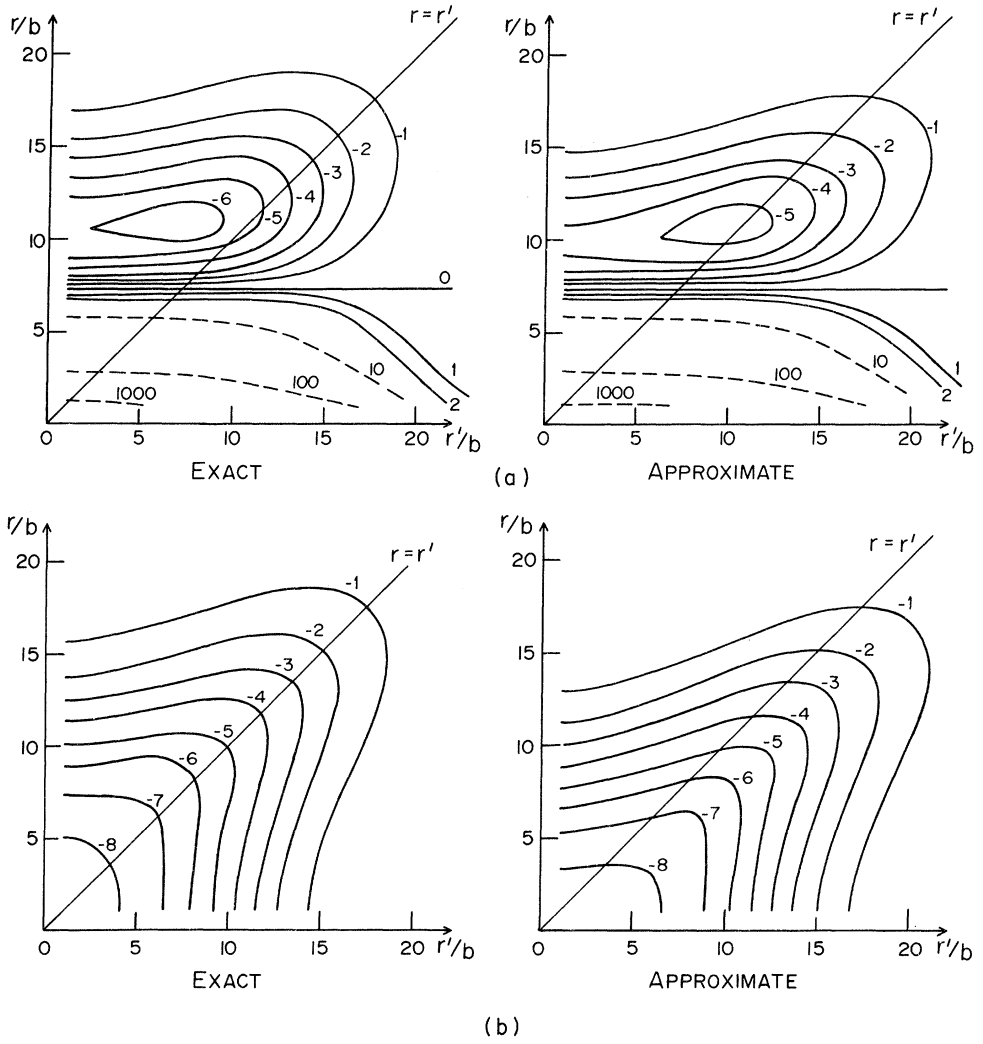

FIG. 2. (a) Comparison of the exact and the approximate interaction integrals for the partial-wave component $l=2$. The difference between two neighboring contours is 1.0 arbitrarily unit except for the three broken lines (10, 100, and 1000). The parameter b is taken to be 1.32 fm. (b) Comparison of the exact and the approximate interaction integrals for $l=4$. The difference between two neighboring contours is 1.0 arbitrarily unit. The parameter b is taken to be 1.32 fm.

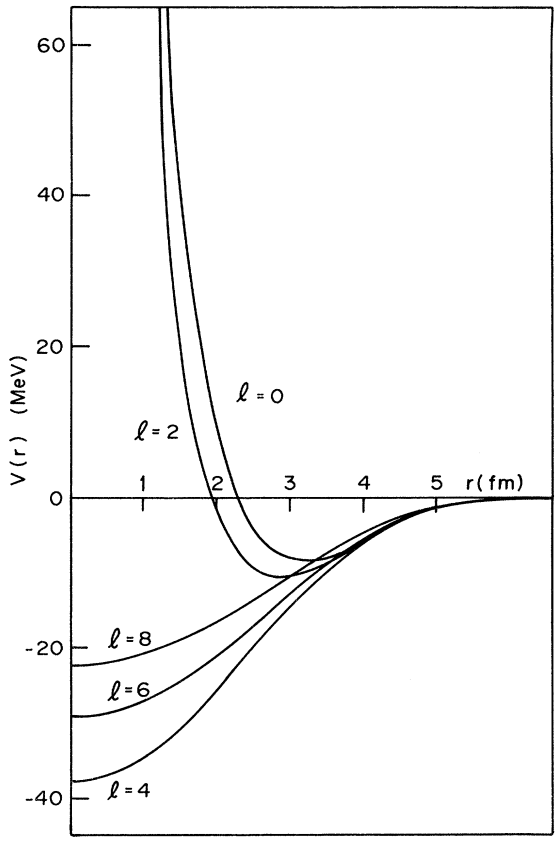

FIG. 3. Real local potential of the $\alpha+\alpha$ system *without* the centrifugal and Coulomb interactions.

$l=6, 8$. While there will be many more corrections such as nonlocal effects and intrinsic wave functions, the potential we have obtained seems quite significant in the sense that even our simple model can reproduce over-all behavior of the α-α interaction.

6. DISCUSSION

Based on the results of the last two sections, let us discuss a general property of the potential between identical nuclei. Before giving any conclusion we have to remember that the results obtained in the last two sections are restricted in the case that the coupling to other channels does not modify the elastic-channel potential too much, or in the nuclear-surface region where the coupling is supposed to be small.

A. Existence of the Strong Repulsion

According to Eq. (4.30), we can expect the appearance of the strong repulsive potential for partial waves $l<2M$, where M depends on the internal wave function of the nucleus. For example, if we employ a single Slater determinant with harmonic-oscillator single-particle states, we can easily obtain that $M=2$ for ^{4}He, $M=8$ for ^{16}O etc. We note that this repulsive potential is not the same kind as we can expect from the saturation property of nuclear matter, which will appear in the $\Delta E(\vec{r}, \vec{r}')$ term of Eq. (4.12). But it corresponds to what the resonating-group method can predict.[4] However, as we mentioned in the Introduction, it is questionable how these repulsive potentials can be physically significant for heavier nuclei.

B. l-Dependence of the Potential

The angular momentum dependence of our potential comes from $\Delta T(\vec{r}, \vec{r}')$ and $\Delta E(\vec{r}, \vec{r}')$ for states with $l<2M$ and from $\Delta E(\vec{r}, \vec{r}')$ for states $l>2M$. The l dependence of the $\Delta T(\vec{r}, \vec{r}')$ term has a shape similar to the centrifugal potential at short separation distances, but with the opposite sign, i.e., the potential is less repulsive for larger l. On the other hand, the $\Delta E(\vec{r}, \vec{r}')$ has a comparably moderate l dependence,[9] i.e., it has a finite value in the limit $|\vec{r}|\to 0$, and it is more attractive for smaller l. Thus, when $l<2M$, the potential is more attractive for larger l. This agrees with the l dependence of the phenomenological potential.[18]

C. Change in the Behavior of the Potential

At $l=2M$, the repulsive potential in $\Delta T(\vec{r}, \vec{r}')$ disappears, and in the α-particle case the potentials for $l>2M$ are much deeper than the potentials for $l<2M$. Also, the l dependence of the potential changes its sign after l passes $2M$, i.e., less attractive for larger l ($l>2M$). If we can expect a similar behavior of the potential for the ^{16}O nucleus, it will occur around $l=16$. Since the states with $l>16$ will attract more strongly than those with $l<16$, it might be possible that the shape resonance will appear at the energy which corresponds to the quasibound states with $l\gtrsim 16$. (The peaks at 21, 25, and 29 MeV of the ^{16}O+^{16}O elastic scattering excitation function[19] can be those states with $l=16$, 18, and 20, respectively.)

These characteristic properties of our l-dependent potential seem to be quite favorable for explaining elastic scattering phenomena of identical nuclei, although we do not take into account the coupling with other channels. According to the recent calculation,[20] this effect leads to a considerably large change of the potential under the assumption of the adiabatic distortion. However, it is not easy to understand what the adiabatic distortion means for a system of two nuclei in

which we cannot separate the intrinsic and relative energy in an *a priori* way. This is the essential difference between the many-nucleus system and the many-atom system in which we can factorize the nucleonic wave function from the electronic (or intrinsic) wave function.[10] This complexity also leads to a misunderstanding of the $\Delta T(\vec{r}, \vec{r}')$ term, which looks unphysical from our knowledge of the molecular system.[21] Of course, no one can deny the importance of the coupling with other channels for a system like heavy-ion scattering. Therefore, it is important to investigate the condition of adiabaticity in the framework of the microscopic theory.

After all, we ask ourselves how our potential can have any physical significance in spite of the neglect of many important effects. We do not have any answer but hope; since the l dependence of the α-α potential justifies our calculation, within the lowest order we can expect the similar l dependence for any internucleus potential, and it may or may not be a good approximation of the full potential.

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APPENDIX I. OVERLAP INTEGRAL IN $\vec{k}$ rep

The overlap integral in $\vec{k}$ rep has been defined by

$$\langle \mathcal{A}\vec{k} | \mathcal{A}\vec{k}' \rangle = \int \Phi^*_{\mathcal{A}\vec{k}}(x)\Phi_{\mathcal{A}\vec{k}'}(x)dx, \tag{AI.1}$$

where the basis function $\Phi_{\mathcal{A}\vec{k}}(x)$ has been given by Eq. (2.2). The integration in Eq. (AI.1) should be taken over all the nucleon coordinates including sums over spin—τ-spin degrees of freedom. From Eqs. (AI.1) and (2.2), we write the integral explicitly:

$$\langle \mathcal{A}\vec{k} | \mathcal{A}\vec{k}' \rangle = c^2 \frac{[(A/2)!]^2}{A!} \sum_P{}' \epsilon_P \int \frac{e^{-i\vec{k}\cdot\vec{\eta}}}{(2\pi)^{3/2}} \phi^*(\xi^{(1)})\phi^*(\xi^{(2)}) \frac{e^{+i\vec{k}\cdot\vec{\eta}_P}}{(2\pi)^{3/2}} \phi(\xi_P^{(1)})\phi(\xi_P^{(2)}) d\vec{\eta}\, d\xi^{(1)}\, d\xi^{(2)}, \tag{AI.2}$$

where $\mathcal{A}_{12}$ is replaced by its explicit form

$$\sum_P{}' \epsilon_P \frac{[(A/2)!]^2}{A!} P,$$

in which the summation $\sum'$ should be taken over those permutations which permute nucleons belonging to different nuclei. The permuted coordinates $\{\vec{\eta}_P, \xi_P^{(1)}, \xi_P^{(2)}\} \equiv P\{\vec{\eta}, \xi^{(1)}, \xi^{(2)}\}$ are linear functions of the original set of coordinates $\{\vec{\eta}, \xi^{(1)}, \xi^{(2)}\}$, because $\vec{\eta}$, $\xi^{(1)}$, and $\xi^{(2)}$ are the linear functions of nucleon coordinates $\vec{x}_1, \ldots, \vec{x}_A$. Then we can write

$$\begin{aligned} &e^{i\vec{k}'\cdot\vec{\eta}_P} = e^{i\alpha_P \vec{k}'\cdot\vec{\eta}} e^{ig_P(\xi^{(1)}, \xi^{(2)})}, \\ &\phi(\xi_P^{(i)}) = \phi^{(i)}(\vec{\eta}, \xi^{(1)}, \xi^{(2)}), \end{aligned} \tag{AI.3}$$

where α_P is a scalar number and $g_P(\xi^{(1)}, \xi^{(2)})$ is a linear function of $\xi^{(1)}$ and $\xi^{(2)}$. From Eqs. (AI.2) and (AI.3), we obtain

$$\langle \mathcal{A}\vec{k} | \mathcal{A}\vec{k}' \rangle = c^2 \frac{[(A/2)!]^2}{A!} \sum_P{}' \epsilon_P \int \frac{e^{-i\vec{k}\cdot\vec{\eta}}}{(2\pi)^{3/2}} \frac{e^{i\alpha_P\vec{k}'\cdot\vec{\eta}}}{(2\pi)^{3/2}} h_P(\vec{\eta})\, d\vec{\eta}, \tag{AI.4}$$

where we have integrated over $\xi^{(1)}$ and $\xi^{(2)}$ first:

$$h_P(\vec{\eta}) = \int \phi^*(\xi^{(1)})\phi^*(\xi^{(2)}) e^{ig_P(\xi^{(1)}, \xi^{(2)})} \phi^{(1)}(\vec{\eta}, \xi^{(1)}, \xi^{(2)})\phi^{(2)}(\vec{\eta}, \xi^{(1)}, \xi^{(2)}) d\xi^{(1)} d\xi^{(2)}. \tag{AI.5}$$

In general $h_P(\vec{\eta})$ is a bounded function of $\vec{\eta}$, because the internal-state wave function $\phi^{(i)} \times (\vec{\eta}, \xi^{(1)}, \xi^{(2)})$ is bounded. There are two exceptions, i.e., when $P = 1$ (identity) or the one which interchanges all the nucleons of each identical nucleus mutually (we write $P = -1$). In these cases

$h_{\pm 1}=1$ and $\alpha_{\pm 1}=\pm 1$. Therefore, we can write

$$\langle \mathcal{A}\vec{k}|\mathcal{A}\vec{k}'\rangle = c^2\frac{[(A/2)!]^2}{A!}[\delta(\vec{k}-\vec{k}')+\delta(\vec{k}+\vec{k}') - 2\mathcal{L}(\vec{k},\vec{k}')], \quad \text{(AI.6)}$$

where $\mathcal{L}(\vec{k},\vec{k}')$ is a bounded function of $\vec{k}$ and $\vec{k}'$. Taking the normalization constant as

$$c^2=\frac{1}{2}\frac{A!}{[(A/2)!]^2},$$

we obtain

$$\langle \mathcal{A}\vec{k}|\mathcal{A}\vec{k}'\rangle = \delta(\vec{k},\vec{k}')-\mathcal{L}(\vec{k},\vec{k}'), \quad \text{(AI.7)}$$

where

$$\delta(\vec{k},\vec{k}')=\tfrac{1}{2}[\delta(\vec{k}-\vec{k}')+\delta(\vec{k}+\vec{k}')]. \quad \text{(AI.8)}$$

APPENDIX II. EFFECTS OF THE ANTISYMMETRIZATION

In order to study the effect of the antisymmetrization, let us define the projection operator p, which projects out the *nonsymmetrized* elastic channel, by

$$p=\int d\vec{k}\,|\vec{k}\rangle\langle\vec{k}|, \quad \text{(AII.1)}$$

where $|\vec{k}\rangle$ is related to $|\mathcal{A}\vec{k}\rangle$ by

$$|\mathcal{A}\vec{k}\rangle = c\mathcal{A}|\vec{k}\rangle, \quad \text{(AII.2)}$$

with the normalization condition

$$\langle\vec{k}|\vec{k}'\rangle=\delta(\vec{k}-\vec{k}'). \quad \text{(AII.3)}$$

We also define the orthogonal complement of p by $q\equiv 1-p$. Using these projection operators, the relationship $\mathcal{A}^2=\mathcal{A}$ can be written as a set of coupled equations:

$$\mathcal{A}_{pp}\mathcal{A}_{pp}+\mathcal{A}_{pq}\mathcal{A}_{qp}=\mathcal{A}_{pp} \quad \text{etc.} \quad \text{(AII.4)}$$

Assuming the existence of ψ_p^λ, which satisfies

$$\mathcal{A}_{pp}\psi_p^\lambda=\lambda\psi_p^\lambda, \quad \text{(AII.5)}$$

we have

$$\lambda^2\psi_p^\lambda+\mathcal{A}_{pq}\mathcal{A}_{qp}\psi_p^\lambda=\lambda\psi_p^\lambda, \quad \text{(AII.6)}$$

or

$$(\lambda-\lambda^2)\|\psi_p^\lambda\|^2=\|\mathcal{A}_{qp}\psi_p^\lambda\|^2. \quad \text{(AII.7)}$$

Therefore the eigenvalue of Eq. (AII.5) is bounded by

$$0\leq\lambda\leq 1 \quad (\text{equality when } \mathcal{A}_{qp}\psi_p^\lambda=0). \quad \text{(AII.8)}$$

From Eqs. (AII.1) and (AII.5), we obtain

$$\int\langle\vec{k}|\mathcal{A}|\vec{k}'\rangle\psi_p^\lambda(\vec{k}')d\vec{k}'=\lambda\psi_p^\lambda(\vec{k}), \quad \text{(AII.9)}$$

where we have defined $\psi_p^\lambda(\vec{k})\equiv\langle\vec{k}|\psi_p^\lambda\rangle$. Also, the kernel $\langle\vec{k}|\mathcal{A}|\vec{k}'\rangle$ is related to the overlap integral by

$$\langle\mathcal{A}\vec{k}|\mathcal{A}\vec{k}'\rangle=c^2\langle\vec{k}|\mathcal{A}|\vec{k}'\rangle. \quad \text{(AII.10)}$$

Therefore, we obtain

$$\int\langle\mathcal{A}\vec{k}|\mathcal{A}\vec{k}'\rangle\psi_p^\lambda(\vec{k}')d\vec{k}'=\lambda c^2\psi_p^\lambda(\vec{k}). \quad \text{(AII.11)}$$

According to Eq. (AI.7), Eq. (AII.11) can be written as

$$\int\mathcal{L}(\vec{k},\vec{k}')\psi_p^\lambda(\vec{k}')d\vec{k}'=(1-\lambda c^2)\psi_p^\lambda(\vec{k}) \quad \text{(AII.12)}$$

assuming $\psi_p^\lambda(\vec{k})$ is symmetric $[\psi(\vec{k})=\psi(-\vec{k})]$. Since $\mathcal{L}(\vec{k},\vec{k}')$ is of the Hilbert-Schmidt type, the existence of solutions $\psi_p^\lambda(\vec{k})$ is assured and the kernel $\mathcal{L}(\vec{k},\vec{k}')$ can be given by

$$\mathcal{L}(\vec{k},\vec{k}')=\sum_\lambda(1-\lambda c^2)\psi_p^\lambda(\vec{k})\psi_p^{\lambda *}(\vec{k}'). \quad \text{(AII.13)}$$

It is straightforward to obtain the inverse:

$$I^{-1}(\vec{k},\vec{k}')=\delta(\vec{k},\vec{k}')+\sum_{\lambda\neq 0}\frac{1-\lambda c^2}{\lambda c^2}\psi_p^\lambda(\vec{k})\psi_p^{\lambda *}(\vec{k}'). \quad \text{(AII.14)}$$

To be precise, $I^{-1}(\vec{k},\vec{k}')$ is *not* the inverse of $\langle\mathcal{A}\vec{k}|\mathcal{A}\vec{k}'\rangle$, but the inverse of

$$I(\vec{k},\vec{k}')=\delta(\vec{k},\vec{k}')-\sum_{\lambda\neq 0}(1-\lambda c^2)\psi_p^\lambda(\vec{k})\psi_p^{\lambda *}(\vec{k}'). \quad \text{(AII.15)}$$

The "redundancy-free" overlap integral $I(\vec{k},\vec{k}')$ is related to the original overlap integral by

$$\langle\mathcal{A}\vec{k}|\mathcal{A}\vec{k}'\rangle=\int I(\vec{k},\vec{k}'')J(\vec{k}'',\vec{k}')d\vec{k}'', \quad \text{(AII.16)}$$

where the kernel, defined by

$$J(\vec{k},\vec{k}')=\delta(\vec{k},\vec{k}')-\sum_{\substack{a\\(\lambda_a=0)}}\psi_p^{\lambda a}(\vec{k})\psi_p^{\lambda a *}(\vec{k}'), \quad \text{(AII.17)}$$

projects nonzero eigenstates of the integral Eq. (AII.9). Since the zero-eigenvalue solution $\psi_p^{0a}(\vec{k})$ comes from the linear dependence of the set of basis functions, the operator J acts as the projector of the space spanned by $|\mathcal{A}\vec{k}\rangle$ onto a subspace of linear independent basis functions.

APPENDIX III. TRANSFORMATIONS OF THE REPRESENTATION SPACES

In this Appendix, we summarize the transformations between three representations,

$$\langle\vec{\eta},\xi|\mathcal{A}\vec{k}\rangle=c\mathcal{A}_{12}\left[\frac{e^{i\vec{k}\cdot\vec{\eta}}}{(2\pi)^{3/2}}\phi(\xi^{(1)})\phi(\xi^{(2)})\right] \quad (\vec{k}\text{ rep}), \quad \text{(AIII.1)}$$

$$\langle\vec{\eta},\xi|\mathcal{A}\vec{x}\rangle = c\mathcal{A}_{12}[\delta(\vec{x}-\vec{\eta})\phi(\xi^{(1)})\phi(\xi^{(2)})] \quad (\vec{x}\text{ rep}), \tag{AIII.2}$$

$$\langle\vec{\eta},\xi|\mathcal{A}\vec{r}\rangle = c\mathcal{A}_{12}\left[\left(\frac{\nu}{\pi}\right)^{3/4} e^{-(\nu/2)(\vec{r}-\eta)^2} \times\phi(\xi^{(1)})\phi(\xi^{(2)})\right] \quad (\vec{r}\text{ rep}). \tag{AIII.3}$$

The transformation between $\vec{k}$ and $\vec{x}$ rep is the Fourier transformation:

$$|\mathcal{A}\vec{x}\rangle = \int d\vec{k}\,\frac{e^{-i\vec{k}\cdot\vec{x}}}{(2\pi)^{3/2}}|\mathcal{A}\vec{k}\rangle, \tag{AIII.4}$$

$$|\mathcal{A}\vec{k}\rangle = \int d\vec{x}\,\frac{e^{i\vec{k}\cdot\vec{x}}}{(2\pi)^{3/2}}|\mathcal{A}\vec{x}\rangle; \tag{AIII.5}$$

and between $\vec{k}$ and $\vec{r}$ rep is given by

$$|\mathcal{A}\vec{r}\rangle = \left(\frac{4\pi}{\nu}\right)^{3/4}\int e^{-k^2/2\nu}\,\frac{e^{-i\vec{k}\cdot\vec{r}}}{(2\pi)^{3/2}}|\mathcal{A}\vec{k}\rangle\,d\vec{k}, \tag{AIII.6}$$

$$|\mathcal{A}\vec{k}\rangle = \left(\frac{\nu}{4\pi}\right)^{3/4}\int e^{k^2/2\nu}\,\frac{e^{i\vec{k}\cdot\vec{r}}}{(2\pi)^{3/2}}|\mathcal{A}\vec{r}\rangle\,d\vec{r}. \tag{AIII.7}$$

Combining these two sets of transformations, we obtain the transformation between $\vec{x}$ and $\vec{r}$ rep:

$$|\mathcal{A}\vec{r}\rangle = \left(\frac{\nu}{\pi}\right)^{3/4}\int e^{-(\nu/2)(\vec{r}-\vec{x})^2}|\mathcal{A}\vec{x}\rangle\,d\vec{x}, \tag{AIII.8}$$

$$|\mathcal{A}\vec{x}\rangle = \left(\frac{\nu}{4\pi}\right)^{3/4}\int e^{k^2/2\nu}e^{-i\vec{k}\cdot(\vec{r}-\vec{x})}|\mathcal{A}\vec{r}\rangle\,d\vec{r}\,d\vec{k}, \tag{AIII.9}$$

where the integration over $\vec{k}$ should be carried out after the integration over $\vec{r}$.

We list the transformations of state functions and various kernels below, when the basis functions are transformed as

$$|\mathcal{A}\vec{s}\rangle = \int T(\vec{s},\vec{t})|\mathcal{A}\vec{t}\rangle\,d\vec{t}, \tag{AIII.10}$$

$$|\mathcal{A}\vec{t}\rangle = \int T^{-1}(\vec{s},\vec{t})|\mathcal{A}\vec{s}\rangle\,d\vec{s}. \tag{AIII.11}$$

From

$$|\Psi\rangle = \int f(\vec{s})|\mathcal{A}\vec{s}\rangle\,d\vec{s} = \int f(\vec{t})|\mathcal{A}\vec{t}\rangle\,d\vec{t}, \tag{AIII.12}$$

the state function is transformed as[22]

$$f(\vec{s}) = \int f(\vec{t})T^{-1}(\vec{s},\vec{t})\,d\vec{t}, \tag{AIII.13}$$

$$f(\vec{t}) = \int f(\vec{s})T(\vec{s},\vec{t})\,d\vec{s}. \tag{AIII.14}$$

Also, from the definition of $g(\vec{s})$ and $g(\vec{t})$, we obtain

$$g(\vec{s}) = \int T^*(\vec{s},\vec{t})g(\vec{t})\,d\vec{t}, \tag{AIII.15}$$

$$g(\vec{t}) = \int T^{-1*}(\vec{s},\vec{t})g(\vec{s})\,d\vec{s}. \tag{AIII.16}$$

A kernel defined by

$$\mathcal{O}^L(\vec{s},\vec{s}') = \int \mathcal{O}(\vec{s},\vec{s}'')I^{-1}(\vec{s}'',\vec{s}')\,d\vec{s}'' \tag{AIII.17}$$

is transformed as

$$\mathcal{O}^L(\vec{t},\vec{t}') = \int T^{-1}(\vec{s},\vec{t})\mathcal{O}^L(\vec{s},\vec{s}') \times T^*(\vec{s}',\vec{t}')\,d\vec{s}\,d\vec{s}', \tag{AIII.18}$$

and its inverse transformation is

$$\mathcal{O}^L(\vec{s},\vec{s}') = \int T^*(\vec{s},\vec{t})\mathcal{O}^L(\vec{t},\vec{t}')T^{-1}(\vec{s}',\vec{t}')\,d\vec{t}\,d\vec{t}'. \tag{AIII.29}$$

As a special case of those transformations, we consider the case when an operator is a "one-body" operator:

$$\mathcal{O}^L(\vec{s},\vec{s}') = O(\vec{s})\delta(\vec{s},\vec{s}'). \tag{AIII.20}$$

The transformed operator is also given by a "one-body" operator

$$\mathcal{O}^L(\vec{t},\vec{t}') = \hat{O}(\vec{t})\delta(\vec{t},\vec{t}'), \tag{AIII.21}$$

where $\hat{O}(\vec{t})$ is defined by

$$\hat{O}(\vec{t})T^{-1*}(\vec{s},\vec{t}) = O(\vec{s})T^{-1*}(\vec{s},\vec{t}). \tag{AIII.22}$$

For example, the kinetic-energy operator, which is given by

$$+\frac{\hbar^2}{2\mu}k^2\delta(\vec{k},\vec{k}') \tag{AIII.23}$$

in $\vec{k}$ rep, is given by

$$-\frac{\hbar^2}{2\mu}\nabla_x^2\delta(\vec{x},\vec{x}') \quad \text{in } \vec{x}\text{ rep} \tag{AIII.24}$$

and

$$-\frac{\hbar^2}{2\mu}\nabla_r^2\delta(\vec{r},\vec{r}') \quad \text{in } \vec{r}\text{ rep}. \tag{AIII.25}$$

When a potential is local in $\vec{r}$ rep

$$\mathcal{U}^L(\vec{r},\vec{r}') = U(\vec{r})\delta(\vec{r},\vec{r}'); \tag{AIII.26}$$

then the potential in $\vec{x}$ rep is momentum-dependent,

$$\mathcal{U}^L(\vec{x},\vec{x}') = U\left(\vec{x}+\frac{1}{\nu}\vec{\nabla}_x\right)\delta(\vec{x},\vec{x}'), \tag{AIII.27}$$

assuming the function $U(\vec{r})$ is a regular function of $\vec{r}$.

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[22]We write the transformed function of $f(\vec{t})$ as $f(\vec{s})$. The reader should remember that $f(\vec{s})$ and $f(\vec{t})$ represent the same state in $\vec{s}$ and $\vec{t}$ rep, respectively, and the functions $f(\vec{s})$ and $f(\vec{t})$, in general, may be different.