Boson-Expansion Method and the Collective Negative-Parity States in Even-Even Spherical Nuclei

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Based on the conventional shell-model Hamiltonian with pairing, quadrupole, and octupole effective forces, the quadrupole-octupole collective anharmonicities are studied by the boson-expansion method. The Belyaev-Zelevinski expansion convergence is improved by supplementing it with a canonical transformation of the random-phase-approximation phonons, determined from the minimum condition for the ground-state energy. Calculations for the one level case show that the anharmonic terms corresponding to the new representation are much smaller than the harmonic-like terms. Explicit expressions for the first 2^+, 3^-, and quintuplet ($1^-,2^-,3^{-\prime},4^-,5^-$) energies and wave functions as well as for $B(E2)$ ($2^+ \to 0^+$) and $B(E3)$ ($3^- \to 0^+$ and $3^{-\prime} \to 0^+$) are derived by a perturbation treatment. The quintet ordering predicted by the previous calculation is reproduced by a rough estimation. Also, expressions for the static quadrupole moments of the first 2^+ and 3^- states are given.

1. INTRODUCTION

In recent years it was shown that the conventional shell-model Hamiltonian with pairing, quadrupole, and octupole effective forces is quite suitable to describe, in the framework of the so-called random-phase approximation (RPA), the experimental systematics of even-even spherical nuclei concerning the first-excited collective states $2^+, 3^-$ and their corresponding $E2$ and $E3$ transition probabilities to the ground state.[1-6] According to RPA, the low-lying collective levels are described in terms of coherent-harmonic modes (phonons).

Due to the harmonic character of the phonon the multiphonon states are degenerate. Other consequences of the harmonic approximation are the vanishing values for the crossover transitions and for the diagonal-phonon matrix elements. For example, the two-quadrupole and the quadrupole-octupole two-phonon states consist of a degenerate triplet $0^+, 2^+, 4^+$ and a degenerate quintuplet $1^-, 2^-, 3^-, 4^-, 5^-$, respectively; the electric $E2$ and $E3$ crossover transitions ($2^{+\prime} \to 0^+$ and $3^{-\prime} \to 0^+$), as well as the $M1$ transitions ($2^{+\prime} \to 2^+$ and $3^{-\prime} \to 3^-$) and the quadrupole moments for the 2^+ and 3^- states are zero.

Real nuclei deviate appreciably from these simple regularities.[7-14] In order to describe these deviations we have to improve the harmonic approximation by including higher effects, the so-called anharmonic effects.

Extending the idealized description for even-even nuclei given by RPA to the neighboring odd-even nuclei, by considering the independent quasiparticle-phonon model besides the discrepancies due to the inadequate description of even-even nuclei, new types of contradictions appear: the odd-even mass difference and the pushing-up effect of quasiparticle states in the vicinity of phonon states (for Tc isotopes). These facts call for the necessity of a self-consistent treatment of the quasiparticle and the phonon concepts.[15]

In the framework of the microscopic theory of the anharmonic effects in even-even spherical nuclei two methods have been extensively developed.

A. Linearization Procedure

Replacing the particle-number operators by their average on the ground state of the noninteracting system, the equations of motion, associated for example with the particle-hole-like excitations, become closed. Consequently, the eigenvalues of the Hamiltonian are obtained directly by the diagonalization of the transposed matrix defining the closed system-of-motion equations.[16, 17] This method was extended to the study of the anharmonic effects on collective RPA states in Ref. 18. There the linearization is achieved by estimating the matrix elements of the quadratic terms in the two quasiparticle operators appearing in the motion equations, with the help of the RPA collective states as intermediate states (spectral decomposition procedure). From these expansions one keeps the terms containing one nonvanishing RPA factor. Using the proper equation of motion the other factors are factorized by the same technique relative to a single term. In this way one obtains the dispersion equations for the energy levels which differ from the corresponding RPA dispersion equations by an additional term which reflects the anharmonic contribution.

B. Boson-Expansion Method

The idea underlying this method, initiated by Belyaev and Zelevinsky (BZ),[19] is that a coupled pair of fermion operators can be expanded in a series of boson operators with the coefficients chosen in such a way that the commutators of the pair fermion operators are consistently reproduced. The expansion terms are distinguished by a parameter which establishes the order of the approximation. Consequently, the Hamiltonian can be written as a series containing a convergence parameter. Although we have a criterium for the selection of the most important terms of the Hamiltonian, any truncation yields the following kind of error: every eigenstate of a truncated Hamiltonian has "unphysical" components, i.e., states which do not have the right antisymmetry properties. The convergence of BZ expansion in the sense of the Pauli principle is very slow. This deficiency is removed by the Marumori (M) expansion, which removes at each step of approximation the "unphysical" states.[20,21] Consequently, the M expansion satisfies the Pauli principle term by term. However, this expansion converges more slowly. The M expansion can be directly obtained from the BZ expansion supplemented with a canonical transformation of pure bosons, i.e., with a deformation of phonon amplitudes, determined in such a way that the Pauli-principle constraints be fulfilled.[22] [Details about the equivalence of the two expansions (BZ and M) are given in Ref. 23.]

Another way to improve the RPA calculations was suggested by Hara[24] and consists in changing the "size" of phonons by including some of the phonon-vacuum correlations. In this way the quasiparticles composing the phonons are no longer pure quasibosons, their fermion character being to some extent restored. Alternatively, the change of the phonon "size" can be achieved by using a BZ expansion for the particle-hole pair of operator commutators, containing a constant term y^0[25] which can be fixed either by the particle-number conservation in a certain state or by a stability condition for the static quadrupole momentum of the 2^+ state.[26] For $y^0 \neq 0$, the expansion has better convergence properties than that corresponding to $y^0 = 0$. The effect of such a parameter can be simulated by a canonical transformation of the phonon operators followed by a normal ordering of the BZ expansion written in terms of the "new" phonons (see Sec. 3).

A very important problem in treating the collective states by a BZ expansion is how to choose the canonical transformation from the pure boson representation to the normal modes representation in such a way that the collective branch of excitations is more or less separated from the remaining states.[22,27]

Up to now, only the anharmonic effects of the quadrupole collective branch have been explicitly studied. This has been done in connection with the description of the triplet $0^+, 2^+, 4^+$ considered as a two-quadrupole phonon state.

However, there are some experimental data concerning low-lying collective spectra which could not be explained by ignoring the octupole anharmonicities. For example, the observed experimental levels $1^-, 3^-, 5^-$ in ^{114}Cd and ^{150}Sm and $1^-, 3^-, 5^-, 4^-$ in ^{148}Sm, as well as the corresponding ratios $B(E3; 3^{-1} \to 0)/B(E3; 3^- \to 0)$,[28] have been interpreted in the frame of the phenomenological model[29] as members of the quadrupole-octupole quintuplet. The γ decay, which follows the decay of the 60-day and 15-min isomers of ^{124}Sb, indicates[12] that the second 3^- level of ^{124}Te is a member of a quintuplet. Also, it has been shown, by using the coupled-channel theory, that the collective levels $1^-, 3^-, 5^-$ in ^{110}Pd excited by inelastic proton scattering might result from one-octupole–one-quadrupole phonon vibrations.[11] Also, it is worth mentioning that the coupling of the quadrupole and the octupole vibrations can induce a nonvanishing static quadrupole moment for the octupole vibrational state 3^-.[30]

In a first paper[31] we developed a semimicroscopic theory for the description of the quintet obtained from the combination of one quadrupole phonon and one octupole phonon. In addition to the conventional shell-model Hamiltonian with pairing, quadrupole, and octupole forces we added a new term H_{23} which represents a simple quadrupole-octupole interaction similar to the term introduced in the phenomenological treatment.[29,32] The spectral-decomposition procedure[18] has been used for the treatment of the anharmonic terms produced by H_{23}. The theory showed limited agreement with experiment and the phenomenological model.

In another paper,[33] based on the same shell-model Hamiltonian, we have given a complete microscopic description of the quintet. We have treated all anharmonic terms, appearing in the equation of motion of the two-quasiparticle-quadrupole and two-quasiparticle-octupole operators, by the spectral-decomposition procedure. Explicit expressions for the quintet splitting and for the relevant $B(E3)$ values have been given.

In the present paper, based on the same Hamiltonian, we studied the quadrupole-octupole couplings by the boson-expansion method. In Sec. 2, the Hamiltonian is written in the quasiparticle representation and then, in the usual way, we define the RPA modes described by the phonon operators.

In Sec. 3 the two-quasiparticle operators of rank 2 and 3 are expanded in terms of quadrupole and octupole phonon operators, following the Belyaev-Zelevinsky method. Consequently, the Hamiltonian is written in terms of phonon operators. We truncate the expansion, neglecting the terms higher than the fourth order.

It is well known that the RPA base leads to a slowly converging series and for large quadrupole and octupole strength parameters the use of fourth-order Hamiltonian becomes unreasonable. On the other hand, the inclusion of the higher-order terms gives much trouble when one wants to use the model for practical purposes.

Thus, it is necessary to change the initial base so as to improve the convergence of the expansion. In order to do this, we make use of the remark that the BZ expansion is determined up to a canonical transformation. Indeed, suppose that the phonon operators are subjected to a linear and canonical transformation and the two-quasiparticle operators are correspondingly rewritten in terms of the new bosons. One easily verifies that the new series associated with the two-quasiparticle operators still preserves the commutation relations. This arbitrariness can be removed by fixing the parameters of the canonical transformation by one of the following alternatives:

(a) Writing the Hamiltonian in terms of the new phonons, the ground-state energy is renormalized due to the normal ordering with respect to the new phonon vacuum. Fixing the linear transformation by taking for its parameters the values which assure a minimum value for the ground-state energy, one expects that the corresponding anharmonic terms become smaller than those appearing in the usual BZ expansion.

(b) Cancelling the coefficients of the cross terms responsible for the ground-state–two-phonon-states correlations, one obtains two equations defining the parameters of the canonical transformation. In this way the old harmonic structure of phonons is modified by accounting for an important part of anharmonicity.

(c) Equations for the canonical-transformation coefficients can be obtained by the minimum conditions for the quadrupole moment of 2^+ states, which is very sensitive to the change of the anharmonicity.

(d) One can impose the constraint that the average number of particles in the new correlated vacuum of phonons be conserved and fix the 2^+ level energy.

Thus, fixing the canonical transformation by one of the alternatives (a)–(d), the amplitudes of anharmonic terms become small and so, the new harmonic states can be considered as unperturbed states. In Sec. 4 the single-phonon states, 2^+, 3^- as well as the two quadrupole-octupole phonon states are obtained by a perturbation treatment. Section 5 is devoted to the study of the electric transition probabilities $2^+\to 0^+$, $3^-\to 0^+$, and $3^-\to 0^+$ and the static quadrupole moments of the first 2^+ and 3^- states. The last section summarizes the results.

2. HAMILTONIAN AND THE RPA PHONONS

The conventional shell-model Hamiltonian with pairing, quadrupole, and octupole effective forces in the quasiparticle representation is

$$H = U + \sum_{\alpha} E_a a_\alpha^\dagger a_\alpha - \tfrac{1}{2} \sum_{\substack{L=2,3\\ q}} X_L Q_{Lq} Q_{Lq}^\dagger , \tag{2.1}$$

where the Greek letters α, β, ... stand for the shell-model quantum numbers $|n_a l a j a m_\alpha\rangle = |a, m_\alpha\rangle \equiv |\alpha\rangle$, while E_a expresses the energy of the quasiparticle states $|\alpha\rangle$. The term U is the Hamiltonian averaged with respect to the quasiparticle vacuum state $|0\rangle$; $a_\alpha^\dagger(a_\alpha)$ is the creation (annihilation) operator of one quasiparticle in the state $|\alpha\rangle$; and X_2, X_3 are the strength parameters of the quadrupole and octupole forces, respectively. The other factors appearing in our model Hamiltonian are

$$\begin{aligned} Q_{Lq} &= \sum g_L(ac)[\xi_{(-)L}(ac)A_{Lq}^{(+)}(ac) + \eta_{(-)L+1}(ac)B_{Lq}^{(+)}(ac)] ,\\ g_2(ab) &= \left(\frac{4\pi}{5}\right)^{1/2} \frac{\hat{j}_a}{2} \langle j_a \| Y_2 \| j_b\rangle \langle a|r^2|b\rangle / \langle a|r^2|a\rangle_m ,\\ g_3(ab) &= \left(\frac{4\pi}{7}\right)^{1/2} \frac{\hat{j}_a}{3} \langle j_a \| Y_3 \| j_b\rangle \langle a|r^3|b\rangle / \tfrac{1}{2} \sum_{j_a<j_b} \langle a|r^3|b\rangle , \end{aligned} \tag{2.2}$$

where $\xi_{(\pm)}$, $\eta_{(\pm)}$ depend only on Bogoliubov-Valatin transformation parameters (U, V):

$$\begin{aligned} \xi_{(-)L}(ac) &= \tfrac{1}{2}[U_a V_c + (-1)^L U_c V_a] ,\\ \eta_{(-)L}(ac) &= \tfrac{1}{2}[U_a U_c + (-1)^L V_a V_c] , \end{aligned} \tag{2.3}$$

and $A_{Lq}^{(+)}$, $B_{Lq}^{(+)}$ are the two-quasiparticle tensor operators of rank L defined by

$$\begin{aligned} A_{Lq}^{(\pm)}(ac) &= A_{Lq}^\dagger(ac) \pm (-1)^q A_{L-q}(ac) ,\\ B_{Lq}^{(\pm)}(ac) &= B_{Lq}^\dagger \pm (-1)^q B_{L-q}(ac) ,\\ A_{Lq}^\dagger(ac) &= \sum_{m_\alpha, m_\gamma} C^{j_a\ j_c\ L}_{m_\alpha\ m_\gamma\ q} a_\alpha^\dagger a_\gamma^\dagger ,\\ B_{Lq}^\dagger(ac) &= \sum C^{j_a\ j_c\ L}_{m_\alpha\ -m_\gamma\ q} s_\gamma a_\alpha^\dagger a_\gamma , \quad s_\gamma = (-1)^{j_c - m_\gamma} . \end{aligned} \tag{2.4}$$

To find exactly the eigenstates of the Hamiltonian (2.1) is a very difficult task. All the formalisms proposed up to now are based on separating, more or less explicitly, the Hamiltonian (2.1) into two terms, one of them being negligible compared with the other one considering their physical effects estimated in a representation induced by two kinds of excitations: quasiparticles (particle-like modes) and phonons (collective modes). One obtains the dominant effects either by an iterative procedure[18] or by an explicit series of progressively decreasing[19] terms. The starting approach of all techniques as well as that of the present paper is the RPA, whose basic points are sketched below.

Ignoring the fermion character of the quasiparticle operators, one can build, in the manner of (2.4), the operators $\mathring{A}^{(\pm)}_{Lq}$ and $\mathring{B}^{(\pm)}_{Lq}$. Approximating the true operators A, B involved in the Hamiltonian by the above operators (the result will be denoted by $\mathring{H}$) one can define a superposition of $\mathring{A}^{(+)}$, $\mathring{A}^{(-)}$ called phonon operators

$$\mathring{C}^\dagger_{Lq}(i)=\sum_{a,b}[\lambda_{L+}(i,ab)\mathring{A}^{(+)}_{Lq}(ab)+\lambda_{L-}(i,ab)\mathring{A}^{(-)}_{Lq}(ab)], \quad L=2,3 \quad (2.5)$$

having a boson character

$$[\mathring{C}_{nq}(i),\mathring{C}^\dagger_{n'q'}(i')]=\delta_{nn'}\delta_{ii'}\delta_{qq'} \quad (2.6)$$

and describing harmonic oscillations of an ideal system associated with $\mathring{H}$:

$$[\mathring{H},\mathring{C}^\dagger_{nq}(i)]=\mathring{\omega}_n(i)\mathring{C}^\dagger_{nq}(i). \quad (2.7)$$

The relations (2.6) and (2.7) completely determine the phonons, i.e., the harmonic energies

$$\mathfrak{F}_L(\mathring{\omega}_L,X_L)\equiv 1-4X_L\sum\frac{\xi_{(-)L}{}^2(ab)g_L{}^2(ab)E(ab)}{E^2(ab)-\mathring{\omega}_L{}^2}=0, \quad E(ab)=E_a+E_b \quad (2.8)$$

and the λ_L amplitudes

$$\lambda_{L-}(i,ab)=X_L\mathring{\rho}_L(i)\xi_{(-)L}(ab)g_L(ab)E(ab)\times[E^2(ab)-\mathring{\omega}_L{}^2(i)]^{-1},$$
$$\lambda_{L+}(i,ab)=\mathring{\omega}_L(i)\lambda_{L-}(i,ab)/E(ab), \quad (2.9)$$

where

$$\mathring{\rho}_L(i)=\{8X_L{}^2\mathring{\omega}_L(i)\sum\xi_{(-)L}{}^2(ab)g_L{}^2(ab)\times[E^2(ab)-\mathring{\omega}_L{}^2(i)]^{-2}\}^{-1/2}. \quad (2.10)$$

Here, the index i stands for ordering the roots given by (2.8); i equal to zero corresponds to collective phonons.

Defining the phonon vacuum as

$$\mathring{C}_{Lq}(i)|0\rangle_0=0, \quad (2.11)$$

the eigenstates of $\mathring{H}$ can be simply generated by acting successively with $\mathring{C}^\dagger_L$ upon the vacuum state. For example the collective states which will be used in what follows are:

$$|1_nq\rangle_0=\mathring{C}^\dagger_{nq}(0)|0\rangle_0, \quad n=2,3,$$
$$|1_21_2JM\rangle_0=\frac{1}{\sqrt2}\sum_q C^{2}_{q}{}^{2}_{M-q}{}^{J}_{M}\mathring{C}^\dagger_{2M-q}(0)\mathring{C}^\dagger_{2q}(0)|0\rangle_0,$$
$$|1_31_3JM\rangle_0=\frac{1}{\sqrt2}\sum_q C^{3}_{q}{}^{3}_{M-q}{}^{J}_{M}\mathring{C}^\dagger_{3M-q}(0)\mathring{C}^\dagger_{3q}(0)|0\rangle_0,$$
$$|1_21_3JM\rangle_0=\sum_q C^{3}_{q}{}^{2}_{M-q}{}^{J}_{M}\mathring{C}^\dagger_{2M-q}(0)\mathring{C}^\dagger_{3q}(0)|0\rangle_0,$$
$$|(1_21_2J)1_3k\mu\rangle_0=\sum_M C^{J}_{M}{}^{3}_{\mu-M}{}^{k}_{\mu}|1_21_2JM\rangle_0\otimes|1_3\mu-M\rangle_0,$$
$$|(1_31_3J)1_3k\mu\rangle_0=N_3(kJ)\sqrt2\times\sum C^{J}_{M}{}^{3}_{\mu-M}{}^{k}_{\mu}|1_31_3JM\rangle_0\otimes|1_3\mu-M\rangle_0. \quad (2.12)$$

Concerning the states $|(1_31_3J)1_3k\mu\rangle$ we should note that the states having the same k but different J are not orthogonal. Two cases are to be distinguished: (a) $k\neq3$. Conventionally, we shall use the smallest J able to couple with 3 to the final angular momentum k; (b) $k=3$. We shall use the following two independent states of different seniority (v)

$$|v=1\rangle=|(1_31_30)1_33\mu\rangle, \quad (2.13)$$
$$|v=3\rangle=\alpha_2|(1_31_32)1_33\mu\rangle+\alpha_4|(1_31_34)1_33\mu\rangle,$$
$$\alpha_2=\left(\frac{671}{1911}\right)^{1/2}, \quad \alpha_4=-\left(\frac{580}{1911}\right)^{1/2}.$$

The normalization factor $N_3(kJ)$ is given by

$$N_3(kJ)=[2+4(2J+1)(-1)^{k+1}W(33k3;JJ)]^{-1/2}. \quad (2.14)$$

It is easy to verify that, at this stage of approximation, the values of the quintet splitting, the reduced electromagnetic transition probability $3^{-\prime}\to0^+$, and the static quadrupole moments of first 2^+ and 3^- states vanish.

The amount by which the Hamiltonian H differs from $\mathring{H}$, which we shall call "anharmonic part" of H, will be responsible for the nonvanishing values of the above mentioned observables. Of course, the anharmonic part of H is determined by the difference between A, B and $\mathring{A}, \mathring{B}$, respectively. In the next section using the boson expansion we shall write explicitly the anharmonic part due to the quadrupole and octupole branches. Thus, we assume that this truncation yields the main effects on the observables which we intend to describe.

3. HAMILTONIAN EXPANSION: CANONICAL TRANSFORMATION OF PHONONS

As usual,[19] we assume that the commutation relations of the operators A and B completely characterize the physical aspect of the problem. Consequently, we shall try to construct "equivalent" operators, i.e., which obey the same commutation properties as A and B, and which are series of $\mathring{A}^\dagger$ and $\mathring{A}$ operators. The entire infinite series should be taken into account because only then is the Pauli principle effect completely restored.[22,23] In this paper we shall evaluate only the effect of the first-order anharmonic terms of the quadrupole and octupole type.

Thus, we truncated the A and B series so that the quadrupole and octupole terms appearing in the commutators of A and B are reproduced in the first order of approximation. Reversing the relation (2.6), $\mathring{A}$ can be written as a superposition of RPA phonon operators and finally the expressions which we obtained for A and B are the following:

$$A_{nq}^{(+)}(ab)=\sum_{\substack{p,p'=2,3\\ m,s,r=1,2\\ i,j,k}} \mathcal{Y}_{n,\,ijk}^{pp',\,msr}(ab)\Big((-1)^{r+s}\{[\mathring{C}_{p'}^{(-)m+1}(i)\mathring{C}_{p}^{(-)s+1}(j)]_{k'}\mathring{C}_{k''}^{(-)r+1}(k)\}_{nq}$$
$$+(-1)^{m+1}\{\mathring{C}_{k''}^{(-)r+1}(k)[\mathring{C}_{p}^{(-)s+1}(j)\mathring{C}_{p'}^{(-)m+1}(i)]_{k'}\}_{nq}\Big)+4\sum_{i}\lambda_{n-}(i,ab)\mathring{C}_{nq}^{(+)}(i)\,, \tag{3.1}$$

$$B_{nq}^{(+)}(ab)=\sum_{\substack{p,p'=2,3\\ r,s=1,2\\ i,j}} X_{n,\,ij}^{pp',rs}(ab)[\mathring{C}_{p}^{(-)r+1}(i)\mathring{C}_{p'}^{(-)s+1}(j)]_{nq}\,,\quad \mathring{C}_{Lq}^{(\pm)}=\mathring{C}_{Lq}^{\dagger}\pm(-1)^{q}\mathring{C}_{L-q}\,, \tag{3.2}$$

$$\begin{aligned}
&\mathcal{Y}_{n,\,ijk}^{pp',\,msr}(ab)=4\sum(1+p_{ab}^{(n)})\,\theta(bb'')Z_{k'k''}(abb')Z_{pp'}(ab'b'')\lambda_{p'(-)m}(i,b'b'')\lambda_{p(-)s}(j,ab'')\lambda_{k''(-)r}(k,b'b)\,,\\
&X_{n,\,ij}^{pp',rs}(ab)=4\sum[1+(-1)^{n}p_{ab}^{(n)}]Z_{pp'}(abb')\,\theta(ab')\lambda_{p(-)r}(i,ab')\lambda_{p'(-)s}(j,bb')[\delta_{n2}\delta_{pp'}+\delta_{n3}(\delta_{p2}\delta_{p'3}+\delta_{p3}\delta_{p'2})](-)^{s+1}\,,\\
&Z_{nn'}(abb')=\hat{n}\hat{n}'\,\theta(ab)W(nj_{a}n'j_{b};j_{b'}n'')\,,\quad \hat{n}=(2n+1)^{1/2}\,,\quad n''=n\delta_{n'2}+(5-n)\delta_{n'3}\,,\\
&k'=|p-p'|+2\,,\quad k''=k'\delta_{n2}+(5-k')\delta_{n3}\,,\quad \theta(ab)=(-1)^{j_a+j_b}\,.
\end{aligned} \tag{3.3}$$

Here, we denote by $p^{(n)}$ the permutation operator

$$p_{ab}^{(n)}\mathfrak{F}(ab)=(-1)^{n+1}\theta(ab)\mathfrak{F}(ba)$$

and by $W(abcd;ef)$ the Racah coefficient. For the tensorial product we have used the definition

$$[\mathring{C}_{n_1}^{(-)k_1}(i)\mathring{C}_{n_2}^{(-)k_2}(j)]_{n_3q_3}=\sum_{q_1q_2} C_{q_1\,q_2\,q_3}^{n_1\,n_2\,n_3}\mathring{C}_{n_1q_1}^{(-)k_1}(i)\mathring{C}_{n_2q_2}^{(-)k_2}(j)\,. \tag{3.4}$$

As for the independent quasiparticle Hamiltonian, it can be written

$$\sum_{\alpha}E_{a}a_{\alpha}^{\dagger}a_{\alpha}=\sum_{\substack{ab\\ n,p}}E_{a}\mathring{A}_{n\mu}^{\dagger}(ab)\mathring{A}_{n\mu}(ab)+\kappa \tag{3.5}$$

and then by means of (2.6) as a function of phonon operators. Here, κ is a constant generated by the zeroth order of the BZ expansion associated with the operators $a_{\alpha}^{\dagger}a_{\alpha}$. Its value will be fixed below.

Now, using (3.1), (3.2), and (3.5) in (2.1), the resulting Hamiltonian H depends only on quadrupole and octupole phonon operators. Neglecting the terms higher than the quartic terms, one obtains:

$$H=U'+H_2+H_3+H_4\,, \tag{3.6}$$

where

$$U'=U+\sum_{i}\frac{\mathring{\omega}_2(i)}{2}\Big[1-8\sum_{a,b}\lambda_{2-}^{2}(i,a,b)\Big]+\sum_{i}\frac{\mathring{\omega}_3(i)}{2}\Big[1-8\sum_{ab}\lambda_{3-}^{2}(i,a,b)\Big]\,, \tag{3.7}$$

$$H_2=\sum_{i,q}\mathring{\omega}_2(i)\mathring{C}_{2q}^{\dagger}(i)\mathring{C}_{2q}(i)+\sum_{i,q}\mathring{\omega}_3(i)\mathring{C}_{3q}^{\dagger}(i)\mathring{C}_{3q}(i)\,, \tag{3.8}$$

$$H_3=\sum h^{(3)}\binom{pp',rs}{n,ijk}(\{[\mathring{C}_{p}^{(-)r+1}(i)\mathring{C}_{p'}^{(-)s+1}(j)]_{n}\mathring{C}_{n}^{(+)}(k)\}_{0}+\text{h.a.})\,, \tag{3.9}$$

$$H_4=\sum h_{1,n}^{(4)}\begin{pmatrix}pp'rs\\ p_1p_1'r's'\\ ijkl\end{pmatrix}\{[\mathring{C}_{p}^{(-)r+1}(i)\mathring{C}_{p'}^{(-)s+1}(j)]_{n}[\mathring{C}_{p_1}^{(-)r'+1}(k)\mathring{C}_{p_1'}^{(-)s'+1}(l)]_{n}\}_{0}$$
$$+\sum h_{2,n}^{(4)}\binom{pp'msr}{ijkl}\Big\{\Big[\Big((-1)^{r+s}\{[\mathring{C}_{p'}^{(-)m+1}(i)\mathring{C}_{p}^{(-)s+1}(j)]_{k'}\mathring{C}_{k''}^{(-)r+1}(k)\}_{n}$$
$$+(-1)^{m+1}\{\mathring{C}_{k''}^{(-)r+1}(k)[\mathring{C}_{p}^{(-)s+1}(j)\mathring{C}_{p'}^{(-)m+1}(i)]_{k'}\}_{n}\Big)\mathring{C}_{n}^{(+)}(l)\Big]_{0}+\text{h.a.}\Big\}\,. \tag{3.10}$$

Here, we denote

$$h^{(3)}\binom{pp',rs}{n,ijk} = \sum (-1)^{n+1} \frac{X_n}{2} \hat{n} \mathring{\rho}_n(k) g_n(ab) \eta_{(-)^{n+1}}(ab) X_{n,ij}^{pp',rs}(a,b), \tag{3.11}$$

$$h_{1,n}^{(4)}\begin{pmatrix} pp'rs \\ p_1p_1'r's' \\ ijkl \end{pmatrix} = \sum (-1)^{n+1} \frac{X_n}{2} \hat{n}\, g_n(ab)\, g_n(a'b') \eta_{(-)^{n+1}}(ab) \eta_{(-)^{n+1}}(a'b') X_{n,ij}^{pp'rs}(ab) X_{n,kl}^{p_1p_1'r's'}(a'b'), \tag{3.12}$$

$$h_{2,n}^{(4)}\binom{pp'msr}{ijkl} = \sum (-1)^{n+1} \frac{X_n}{2} \hat{n} \mathring{\rho}_n(l)\, g_n(ab) \xi_{(-)^n}(ab)\, Y_{n,ijk}^{pp'msr}(ab), \tag{3.13}$$

$$k' = |p - p'| + 2, \quad k'' = k'\delta_{n2} + (5 - k')\delta_{n3}$$

In the following, we shall assume that the main effect on the quintet splitting, the $2^+ \to 0^+$ and $3^- \to 0^+$ electromagnetic transition probabilities, as well as on the 2^+ and 3^- quadrupole moments, comes from the collective part of H which we shall denote by H_0. To simplify the notations, we shall omit the indices zero which stand for the lowest roots of Eq. (2.8).

We should like to remark that H_0 (written in terms of collective variables) is of the same form as the phenomenological Hamiltonian used by Lipas.[29] The advantage of the Hamiltonian (3.6) consists in the fact that the fitting parameters used in the phenomenological treatment have now an explicit microscopic structure. Also, it is to be noticed that in Ref. 31, the quintet splitting was produced by a term like $[(\mathring{C}_3^{(-)^{r+1}} \mathring{C}_3^{(-)^{s+1}}) \mathring{C}_2^{(+)}]_0$ contained by H_3.

It is well known that the BZ expansion is slowly converging especially for nuclei lying in the transition region, where the coupling strength parameters are large. In this case the RPA base is not proper for a perturbation treatment since the anharmonic terms are large. Redefining the RPA base in a certain way, we expect the anharmonicities to become smaller than the diagonal terms and so a perturbation treatment is justified. It is easily seen that the vacuum state $|0\rangle_0$ is correlated with the two-, four-, and three-phonon states by means of quartic and cubic terms, respectively.

We try to include some correlations into the ground state by means of a canonical transformation

$$\begin{aligned} \mathring{C}_{nq}^\dagger &= a_{n1} C_{nq}^\dagger + (-1)^q a_{n2} C_{n-q}, \\ \mathring{C}_{nq} &= a_{n1} C_{nq} + (-1)^q a_{n2} C_{n-q}^\dagger, \quad n = 2, 3 \end{aligned} \tag{3.14}$$

which defines a new boson representation provided we have

$$a_{n1}{}^2 - a_{n2}{}^2 = 1, \quad n = 2, 3. \tag{3.15}$$

Inserting (3.14) in H_0 and writing the resulting terms in a normal order, according to Wick theorem applied with respect to the new vacuum $|0\rangle$ defined as

$$C_{nq}|0\rangle = 0,$$

one obtains the following expression for H_0:

$$\begin{aligned} H = \overline{U} &+ \sum_{q,n} \omega_n C_{nq}^\dagger C_{nq} + \sum S_n(\mathring{\omega}_2, \mathring{\omega}_3, X_2, X_3, a_2^{(+)}, a_2^{(-)}, a_3^{(+)}, a_3^{(-)}) \\ &\times (-1)^q [C_{nq}^\dagger C_{n-q}^\dagger + C_{n-q} C_{nq}] + \sum \overline{h}_n^{(3)}(pp'rs) N[\{(C_p^{(-)^{r+1}} C_{p'}^{(-)^{s+1}})_n C_n^{(+)}\}_0 + \text{h.a.}] \\ &+ \sum \overline{h}_{1,n}^{(4)}\binom{pp'rs}{p_1p_1'r's'} N[\{(C_p^{(-)^{r+1}} C_{p'}^{(-)^{s+1}})_n (C_{p_1}^{(-)^{r'+1}} C_{p_1'}^{(-)^{s'+1}})_n\}_0] \\ &+ \sum \overline{h}_{2,n}^{(4)}(pp'msr) N\Big[\Big(\{(-1)^{r+s}[(C_{p'}^{(-)^{m+1}} C_p^{(-)^{s+1}})_{k'} C_{k''}^{(-)^{r+1}}]_n \\ &\qquad + (-1)^{m+1}[C_{k''}^{(-)^{r+1}} (C_p^{(-)^{s+1}} C_{p'}^{(-)^{m+1}})_{k'}]_n\} C_n^{(+)}\Big)_0 + \text{h.a.}\Big]. \end{aligned} \tag{3.17}$$

Here the factors $\overline{h}_n^{(3)}$, $\overline{h}_{1,n}^{(4)}$, and $\overline{h}_{2,n}^{(4)}$ are obtained from the corresponding $h_n^{(3)}$, $h_{1,n}^{(4)}$, and $h_{2,n}^{(4)}$, multiplied by $a_p^{(-)^{r+1}} a_{p'}^{(-)^{s+1}} a_n^{(+)}$, $a_p^{(-)^{r+1}} a_{p'}^{(-)^{s+1}} a_{p_1}^{(-)^{r'+1}} \times a_{p_1'}^{(-)^{s'+1}}$, and $a_{p'}^{(-)^{m+1}} a_p^{(-)^{s+1}} a_{k''}^{(-)^{r+1}} a_n^{(+)}$, respectively, where

$$a_n^{(\pm)} = a_{n1} \pm a_{n2}, \quad n = 2, 3. \tag{3.18}$$

The symbol N indicates the normal order of the product in the square brackets. The explicit expressions of S_n coefficients as well as of the shifted zero-point vibration energy $\overline{U}$ and of the correlated harmonic-vibration energies ω_n are given in Appendix A. The constant κ introduced in Sec. 2 is fixed so that

$$\kappa + 5\mathring{\omega}_2 a_{22}{}^2 + 7\mathring{\omega}_3 a_{32}{}^2 = 0. \tag{3.19}$$

In order to determine the parameters a_{n1}, a_{n2} $(n=2, 3)$, one can use one of the alternatives (a)–(d) described in Sec. 1. A consequence of fixing the canonical transformation in the above-mentioned ways is the modification of the "size" of phonons as well as the change of the zero-point oscillation energy. In these two effects a very important part of the anharmonicities is already included and so it is to be expected that the anharmonicities are small in the new representation. This fact is explicitly proved in the case of the one-level model in Sec. 6.

The correlated RPA states $|\,\rangle$ are defined in the same manner as in (2.12), using instead of $\mathring{C}^\dagger$ phonons, the $C^\dagger$ phonons and replacing the old vacuum $|0\rangle_0$ by the new vacuum $|0\rangle$. These states constitute the base for the perturbation treatment, which will be considered in the next sections.

4. ANHARMONIC CORRECTIONS TO THE 2^+, 3^-, AND 1^-, 2^-, 3^-, 4^-, 5^- CORRELATED STATES

The aim of this section is to study the first- and second-order corrections to the energies and the wave functions of the RPA correlated states $|1_2 q\rangle$, $|1_3 q\rangle$, $|1_2 1_3 JM\rangle$, determined by the cubic and quartic terms involved in H_0.

It is easy to see that, because of the normal ordering of the anharmonic Hamiltonian the first phonon states $|1_2 q\rangle$ and $|1_3 q\rangle$ are not perturbed in the first order, but they are related to the two-phonon states $|1_2 1_2 2q\rangle$, $|1_3 1_3 2q\rangle$, and $|1_2 1_3 3q\rangle$, respectively, in the second order.

By a straightforward calculation one finds the following expressions for the corrected energies of the first quadrupole (E_2) and octupole (E_3) correlated phonon states:

$$E_2 = \omega_2 + E_2^{\mathrm{II}}\,, \tag{4.1}$$

$$E_3 = \omega_3 + E_3^{\mathrm{II}}\,, \tag{4.2}$$

where

$$\begin{aligned} E_2^{\mathrm{II}} = &-\frac{8}{5\omega_2}[3\bar{h}_2^{(3)}(22,11) - \bar{h}_2^{(3)}(22,22)]^2 \\ &-\frac{8}{7(2\omega_3-\omega_2)}\{\bar{h}_2^{(3)}(33,11) + \bar{h}_2^{(3)}(33,22) + (\tfrac{7}{5})^{1/2}[-\bar{h}_3^{(3)}(32,11) + \bar{h}_3^{(2)}(32,22)]\}^2 \equiv -\frac{2B^2}{\omega_2} - \frac{2C^2}{2\omega_3-\omega_2}\,, \end{aligned} \tag{4.3}$$

$$E_3^{\mathrm{II}} = -\frac{16}{7\omega_2}\{[-\bar{h}_2^{(3)}(33,11) + \bar{h}_2^{(3)}(33,22)](\tfrac{5}{7})^{1/2} + \bar{h}_3^{(3)}(32,11)\}^2 \equiv -\frac{D^2}{\omega_2}. \tag{4.4}$$

As can be seen from (4.2) and (4.3) the effect of the anharmonicities on 2^+ and 3^- RPA correlated states is to lower their energies.

Concerning the quintet states, their energies are affected both within the first order and the second order of perturbation. In the second order of perturbation we estimated, as in the 2^+ and 3^- states case, only the contribution which comes from the cubic terms of our Hamiltonian.

The final result is:

$$E_J = \omega_2 + \omega_3 + E_J^{\mathrm{I}} + E_J^{\mathrm{II}}\,, \quad J = 1, 2, 3, 4, 5 \tag{4.5}$$

with

$$E_J^{\mathrm{I}} = \delta_{J3}K_1 + W(32J2;32)K_2 + (-1)^{J+1}W(33J3;22)K_3\,, \tag{4.6}$$

$$\begin{aligned} E_J^{\mathrm{II}} = &\frac{D^2}{\omega_2}\delta_{J3} - \frac{2}{\omega_2}\sum_{J''=0,2,4}[\delta_{J''2}B + \hat{3}\hat{J}''W(22J3;J''3)D]^2 \\ &-\frac{4}{2\omega_3-\omega_2}\Big[\delta_{J3}\Big(\tfrac{20}{49}N_3^2(30) + \{\sum_{J''=2,4}N_3(3J'')\alpha_{J''}[\delta_{J''2} + 2\hat{2}\hat{J}''W(3333;J''2)]\}^2\Big) \\ &\qquad + N_3^2(JJ')(1-\delta_{J3})[(-1)^{J+1}\delta_{J'2} + \{1+(-1)^{J'}\}\hat{2}\hat{J}'W(33J3;J'2)]^2\Big]C^2\,, \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} K_1 = (1/\sqrt{7})\{&4\sum[(-1)^s + (-1)^{m+r+1}][-\bar{h}_{2,3}^{(4)}(23,msr) + \bar{h}_{2,2}^{(4)}(32,msr) + \bar{h}_{2,2}^{(4)}(2,3,msr) + \bar{h}_{2,3}^{(4)}(32,msr)] \\ &+\sum[(-1)^{m+s} + (-1)^{m'+s'}]\bar{h}_{1,3}^{(4)}\big(\begin{smallmatrix}32ms\\32m's'\end{smallmatrix}\big)\}\,, \end{aligned} \tag{4.8}$$

$$K_2=(1/\sqrt{5})\{20\sum[1+(-1)^{m+s}][(-1)^r-1][\bar{h}^{(4)}_{2,2}(33,msr)+\bar{h}^{(4)}_{2,3}(22,msr)]$$
$$+\sum[(-1)^{m+1}+(-1)^{s+1}][(-1)^{m'}+(-1)^{s'}][5\bar{h}^{(4)}_{1,2}(\begin{smallmatrix}33ms\\22m's'\end{smallmatrix})+\bar{h}^{(4)}_{1,2}(\begin{smallmatrix}22ms\\33m's'\end{smallmatrix})]\}, \tag{4.9}$$

$$K_3=\sqrt{7}\{4\sum[1+(-1)^{r+m+s+1}][(-1)^{r+1}\bar{h}^{(4)}_{2,3}(23,msr)+(-1)^r\bar{h}^{(4)}_{2,2}(32,msr)$$
$$+(-1)^{m+s+r}\bar{h}^{(4)}_{2,2}(23,msr)-\bar{h}^{(4)}_{2,3}(32,msr)]+\sum[(-1)^{m'+s}+(-1)^{m+s'}]\bar{h}^{(4)}_{1,3}(\begin{smallmatrix}32ms\\32m's'\end{smallmatrix})\}. \tag{4.10}$$

According to the convention made in Sec. 2 concerning the three-phonon states, the index J' appearing in the last square bracket of (4.7) has to take on the smallest value able to couple the angular momentum 3 to the angular momentum J.

To see clearly the mixing of the phonon states determined by switching on the anharmonic part of the Hamiltonian it is worthwhile to write the perturbed wave functions of the states of interest. Here we considered the most important effect, namely, the first-order correction given by the cubic terms.

$$|1_2M\rangle'=|1_2M\rangle+C^{\mathrm{I}}_{22}|1_21_2J=2,M\rangle+C^{\mathrm{I}}_{33}|1_31_3J=2,M\rangle, \tag{4.11}$$

$$|1_3M\rangle'=|1_3M\rangle+C^{\mathrm{I}}_{23}|1_21_3J=3,M\rangle, \tag{4.12}$$

$$|1_21_3JM\rangle'=|1_21_3JM\rangle+\delta_{J3}B^{\mathrm{I}}_3|1_3M\rangle+\sum_{J'=0,2,4}B^{\mathrm{I}JJ'}_{223}|(1_21_2J')1_3JM\rangle$$
$$+\delta_{J3}\sum_{J''=0,2,4}D^{\mathrm{I}}_{J''}|(1_31_3J'')1_33M\rangle+(1-\delta_{J3})E^{\mathrm{I}}_{J'J}|(1_31_3J')1_3JM\rangle. \tag{4.13}$$

Here we adopted the following notations:

$$C^{\mathrm{I}}_{22}=-\frac{B\sqrt{2}}{\omega_2},\quad C^{\mathrm{I}}_{33}=\frac{-C\sqrt{2}}{2\omega_3-\omega_2},\quad C^{\mathrm{I}}_{23}=-B^{\mathrm{I}}_3=-\frac{D}{\omega_2},$$
$$B^{\mathrm{I}JJ'}_{223}=\frac{(-1)^J\sqrt{2}}{\omega_2}[2\delta_{J'2}B+\hat{3}\hat{J}'W(22J3;J'3)D],$$
$$D^{\mathrm{I}}_0=\frac{-4\sqrt{5}}{7(2\omega_3-\omega_2)}CN_3(30),\quad D^{\mathrm{I}}_2=\frac{-2}{2\omega_3-\omega_2}\alpha_2[1+10W(3333;22)]CN_3(32), \tag{4.14}$$
$$D^{\mathrm{I}}_4=\frac{12\sqrt{5}}{2\omega_3-\omega_2}\alpha_4W(3333;42)CN_3(34),$$
$$E^{\mathrm{I}}_{J'J}=\frac{-2}{2\omega_3-\omega_2}\{(-1)^{J+1}\delta_{J'2}+[1+(-1)^{J'}]\hat{2}\hat{J}'W(33J3;J'2)\}CN_3(JJ').$$

These wave functions will be used in the next section to obtain the explicit expressions of the transition probabilities and the static quadrupole moments.

5. ELECTROMAGNETIC TRANSITION PROBABILITIES AND THE STATIC QUADRUPOLE MOMENTS

In this section we shall work out the anharmonic corrections to the $2^+\to 0^+$, $3^-\to 0^+$ transition probabilities as well as the $3^{-\prime}\to 0^+$ transition rate and the quadrupole moments of 2^+ and 3^- states. The latter have nonvanishing values, which we have already seen in the body of Sec. 2 are exclusively of anharmonic nature.

We should like to mention that the present formalism is worked out for the single-closed-shell nuclei, but it can be extended without any difficulty to the nuclei having both shells open.

As usual, we assume that the core contribution to the $E2$ and $E3$ transitions can be taken into consideration by means of the effective charges $e^{(2)}_{\mathrm{eff}}$ and $e^{(3)}_{\mathrm{eff}}$, respectively. Within the quasiparticle representation, the multipole operators responsible for the $E2$ and $E3$ transitions can be written as follows:

$$M_{Lq}=e^{(L)}_{\mathrm{eff}}\nu_LQ_{Lq},\quad L=2,3, \tag{5.1}$$

where

$$\nu_2=\langle a|r^2|a\rangle_m/\sqrt{4\pi},\quad \nu_3=\tfrac{1}{2}\sum_{j_a<j_b}\langle a|r^3|b\rangle/\sqrt{4\pi}, \tag{5.2}$$

and Q_{Lq} is given by (2.2).

The reduced EL transition probability from the state $|nk\mu\rangle$ to the state $|n'k'\mu'\rangle$ is the following:

$$B(EL,k_n\to k'_{n'})=e^{(L)2}_{\mathrm{eff}}\sum_{\mu\mu'}|\langle n'k'\mu'|Q_{Lq}|nk\mu\rangle|^2. \tag{5.3}$$

To write explicitly the relation (5.3) we need the expansion of Q_L operator in terms of phonon operators. Using (3.1) and (3.2) one finds

$$Q_{Lq}=\mathring{\rho}_L a_L^{(+)}C_{Lq}^{(+)}+\sum Q_L^A(pp',msr)\{(-)^{r+s}[(C_{p'}^{(\bar{-})^{m+1}}C_p^{(-)^{s+1}})_{k'}C_{k''}^{(\bar{-})^{r+1}}]_{Lq}+(-)^{m+1}[C_{k''}^{(\bar{-})^{r+1}}(C_p^{(-)^{s+1}}C_{p'}^{(\bar{-})^{m+1}})_{k'}]_{Lq}\}$$

$$+\sum Q_L^B(pp',mn)(C_p^{(-)^{m+1}}C_{p'}^{(\bar{-})^{n+1}})_{Lq}(\delta_{L2}\delta_{pp'}+\delta_{L3}\delta_{p3}\delta_{p'2})\,,\quad L=2,3\,, \tag{5.4}$$

where

$$Q_L^A(pp',msr)=\sum g_L(ab)\xi_{(-)^L}(ab)\mathcal{Y}_{L,000}^{pp',msr}(ab)a_{p'}^{(\bar{-})^{m+1}}a_p^{(-)^{s+1}}a_{k''}^{(\bar{-})^{r+1}}\,, \tag{5.5}$$

$$Q_L^B(pp',rs)=\sum g_L(ab)\eta_{(-)^{L+1}}(ab)X_{L,0,0}^{pp'rs}(ab)a_p^{(-)^{r+1}}a_{p'}^{(\bar{-})^{s+1}}\,, \tag{5.6}$$

$$k'=|p-p'|+2\,,\quad k''=k'\delta_{L2}+(5-k')\delta_{L3}\,.$$

Using the expansion (5.4) for $L=2$ and the wave function (4.11) one obtains the following expression for the $B(E2)$, corresponding to the $2^+\to 0^+$ transition:

$$B(E2,2^+\to 0^+)=5e_{\text{eff}}^{(2)^2}\nu_2{}^2\mathring{\rho}_2{}^2a_2^{(+)^2}(1+\beta_2)^2\,. \tag{5.7}$$

Similarly, employing (5.4) for $L=3$ and the wave function (4.12), the $B(E3)$ expression of the $3^-\to 0^+$ transition can be written

$$B(E3,3^-\to 0^+)=7e_{\text{eff}}^{(3)^2}\nu_3{}^2\mathring{\rho}_3{}^2a_3^{(+)^2}(1+\beta_3)^2\,. \tag{5.8}$$

Here, β_2 and β_3 are determined by the anharmonic components of the wave functions and the multipole operators; their explicit expressions are given in Appendix B.

In Sec. 2 it was shown that the $3^{-\prime}\to 0^+$ transition is forbidden within the RPA treatment, because of phonon-number conservation rule. Now, the anharmonic structure of Q_{3q} and 3^- wave function determines a nonvanishing value for the $3^{-\prime}\to 0^+$ transition

$$B(E3,3^{-\prime}\to 0^+)=7e_{\text{eff}}^{(3)^2}\nu_3{}^2T^2\,. \tag{5.9}$$

The explicit expression of T is given in Appendix B.

Another observable which cannot be explained in the harmonic approximation is the quadrupole moment of the first 2^+ state. Recently, a nonvanishing quadrupole moment for the first 3^- state has been pointed out in some spherical nuclei.[30] This latter fact, as well as the $3^{-\prime}$ transition probability and the quintuplet splitting are a real stimulus for our theoretical investigation of quadrupole-octupole anharmonic correlation.

Making use of the standard definition for the quadrupole-moment operator

$$P_{2q}=\nu_2Q_{2q}\,, \tag{5.10}$$

one finds by a straightforward calculation the quadrupole moment of the states $|1_2q\rangle$ (F_2) and $|1_3q\rangle$ (F_3)

$$F_L\equiv\left(\frac{16\pi}{5}\right)^{1/2}\langle 1_LL|P_{20}|1_LL\rangle=\left(\frac{16\pi}{5}\right)^{1/2}\nu_2C_{L0L}^{L2L}\mathfrak{A}_L\,,$$

$$L=2,3\,, \tag{5.11}$$

where $\mathfrak{A}_2$ and $\mathfrak{A}_3$ are listed in Appendix B. From their expressions (B.4) and (B.5) it can be seen that in addition to the terms linear in the two-phonon amplitudes and the anharmonic components of the A and B operators, quadratic terms appear in the two-phonon amplitudes. This is due to the fact that the anharmonic components of B operators relate the two-phonon mixing states.

6. QUADRUPOLE VIBRATION: ONE-LEVEL CALCULATION

In this section we shall apply the previous considerations to the one-level case where the general formulas which have been written above are very much simplified.

For the sake of simplicity we shall make the convention of omitting the level indices.

Using the notations

$$x_1=8Z_{22}\lambda_-^2\,,\quad y_1=16Z_{22}^2\lambda_-^3\,,\quad a=a_2^{(+)}\,,$$

$$h_1=\frac{\sqrt{5}}{2}X_2(g\eta x_1)^2\,,\quad h_2=X_2\frac{\sqrt{5}}{2}\mathring{\rho}_2\,, \tag{6.1}$$

one obtains the following expressions for the ground-state and 2^+ level energies and the cross-term amplitude S_2:

$$\bar{U}=U'+2\sqrt{5}\left[(4h_2-h_1)a^4-4\frac{\mathring{\omega}_2}{2E}h_2a^2+2\left(\frac{\mathring{\omega}_2}{2E}\right)^2h_1\right] \tag{6.2}$$

$$\omega_2=\frac{1}{2}\left(a^2+\frac{1}{a^2}\right)\mathring{\omega}_2+\frac{8}{\sqrt{5}}\left[(4h_2-h_1)a^4-2h_2\frac{\mathring{\omega}_2}{2E}a^2+2\left(\frac{\mathring{\omega}_2}{2E}\right)^2h_1\right], \tag{6.3}$$

$$S_2=\frac{1}{4}\left(a^2-\frac{1}{a^2}\right)\mathring{\omega}_2+\frac{4}{\sqrt{5}}\left[(4h_2-h_1)a^4-2h_2\left(\frac{\mathring{\omega}_2}{2E}\right)a^2\right]. \tag{6.4}$$

One can easily verify that the function $\bar{U}$ has three extremum points against the a variation. Two minima $a_\pm$ are symmetrically distributed with respect to a maximum point a_0:

$$a_{\pm}=\pm\left(\frac{h_2}{4h_2-h_1}\,\frac{\mathring{\omega}_2}{E}\right)^{1/2},\quad a_0=0\,. \tag{6.5}$$

The maximum point a_0 is unphysical since the boson-like commutators for the new phonons are not satisfied.

It is worth mentioning that the ω_2 value is the same for the two extremum points $a_{\pm}$. Also, the even-order terms of the expanded Hamiltonian have the same value for the two values $a_{\pm}$; the odd-order terms corresponding to the two minima differ by their sign. Since the odd-order terms do not correct the energies in the first order of perturbation treatment, we can say that in the first two orders of perturbation, the energies are not sensitive to the change of the "a" parameter sign. We shall fix the canonical transformation by choosing for the "a" parameter the value a_+.

The expressions for $\overline{U}$, ω_2, S_2 contain terms due to both the contraction of the anharmonic terms in the Hamiltonian and the nonunitarity of the canonical transformation. In the case of $\overline{U}$, the latter contribution was compensated by a suitable choice of κ (3.19).

We would like to mention the behavior of the terms in (6.3) and (6.4) coming from the contraction of the anharmonicities, which will be denoted by ω_2^{anh} and S_2^{anh}, respectively. The S_2^{anh} term vanishes in the extremum points of $\overline{U}$. This is similar to the BCS approximation case. As for the ω_2^{anh} variation, it has two minima $\overline{a}_{\pm}$ and one maximum $\overline{a}_0$.

$$\overline{a}_{\pm}=(1/\sqrt{2})a_{\pm},\quad \overline{a}_0=a_0\,. \tag{6.6}$$

These relations show the fact that the effects of the anharmonicities on the energies of the two states are different.

It is now useful to write the quadrupole part of the Hamiltonian (3.17) in the following way:

$$\begin{aligned}H={}&\overline{U}+\omega_2\sum_q C_{2q}^{\dagger}C_{2q}+S_2\sum_q(-1)^q[C_{2q}^{\dagger}C_{2-q}^{\dagger}+\text{h.a.}]+[A_{30}(C_2^{\dagger}C_2^{\dagger}C_2^{\dagger})_0+A_{21}(C_2^{\dagger}C_2^{\dagger}C_2)_0+\text{h.a.}]\\&+\{A_{40}[(C_2^{\dagger}C_2^{\dagger})_2(C_2^{\dagger}C_2^{\dagger})_2]_0+A_{31}[(C_2^{\dagger}C_2^{\dagger})_2(C_2^{\dagger}C_2)_2]_0+\text{h.a.}\}+A_{22}^{1}[(C_2^{\dagger}C_2^{\dagger})_2(C_2C_2)_2]_0+A_{22}^{2}\{[C_2^{\dagger}(C_2^{\dagger}C_2)_2]_2C_2\}_0\,,\end{aligned} \tag{6.7}$$

where

$$\begin{aligned}A_{30}&=2\sum\overline{h}^{(3)}(22,rs)\,,\quad A_{21}=\sum_{r,s=1,2}[1+(-1)^{r+s}][1+(-1)^{r+1}+(-1)^{s+1}]\overline{h}^{(3)}(22,rs)\,,\\A_{40}&=\sum\overline{h}^{(4)}_{1,2}\left(\begin{smallmatrix}22\,rs\\22\,r's'\end{smallmatrix}\right)+2\sum[(-1)^{r+s}+(-1)^{m+1}]\overline{h}^{(4)}_{2,2}(22msr)\,,\\A_{31}&=\sum[(-1)^{r+1}+(-1)^{s+1}+(-1)^{r'+1}+(-1)^{s'+1}]\overline{h}^{(4)}_{1,2}\left(\begin{smallmatrix}22\,rs\\22\,r's'\end{smallmatrix}\right)+2\sum[(-1)^{r+s}+(-1)^{m+1}][1+(-)^{r+1}+(-)^{s+1}+(-)^{m+1}]\\&\quad\times\overline{h}^{(4)}_{2,2}(22msr)\,,\\A_{22}^{1}&=\sum[(-1)^{r+s}+(-1)^{r'+s'}]\overline{h}^{(4)}_{1,2}\left(\begin{smallmatrix}22\,rs\\22\,r's'\end{smallmatrix}\right)+2\sum[(-1)^{s+1}+1][(-1)^{r+m}+1]\overline{h}^{(4)}_{2,2}(22msr)\,,\\A_{22}^{2}&=\sum[(-1)^{r}+(-1)^{s}][(-1)^{r'}+(-1)^{s'}]\overline{h}^{(4)}_{1,2}\left(\begin{smallmatrix}22\,rs\\22\,r's'\end{smallmatrix}\right)+2\sum[1+(-1)^{m+r+s+1}][2(-1)^{r+1}+(-1)^{s+1}+1]\overline{h}^{(4)}_{2,2}(22msr)\,.\end{aligned} \tag{6.8}$$

In the one-level case, the above expressions become:

$$\begin{aligned}A_{30}&=2h^{(3)}\left[a^3+\left(\frac{\mathring{\omega}_2}{2E}\right)^2\frac{1}{a}\right],\quad A_{21}=2h^{(3)}\left[3a^3-\left(\frac{\mathring{\omega}_2}{2E}\right)^2\frac{1}{a}\right],\quad h^{(3)}=h^{(3)}(22,11)\,,\\A_{40}&=(4h_2-h_1)a^4-4h_1\left(\frac{\mathring{\omega}_2}{2E}\right)a^2-(4h_2+6h_1)\left(\frac{\mathring{\omega}_2}{2E}\right)^2,\\A_{31}&=4(4h_2-h_1)a^4-8h_1\left(\frac{\mathring{\omega}_2}{2E}\right)a^2,\\A_{22}^{1}&=2(4h_2-h_1)a^4+(4h_1+8h_2)\left(\frac{\mathring{\omega}_2}{2E}\right)^2,\\A_{22}^{2}&=4(4h_2-h_1)a^4+8h_1\left(\frac{\mathring{\omega}_2}{2E}\right)^2.\end{aligned} \tag{6.9}$$

In the above relation we have neglected the terms containing powers of $(\omega_2^0/2E)$ greater than 2.

Now, let us consider κ [see Eq. (3.5)] as a pure constant. Then the ground-state energy becomes:

$$\overline{U}=U'+\kappa+\tfrac{5}{4}\,\mathring{\omega}_2(a-1/a)^2+2\sqrt{5}\,[(4h_2-h_1)a^4-4(\mathring{\omega}_2/2E)h_2a^2+2(\mathring{\omega}_2/2E)^2h_1]\,. \tag{6.10}$$

One can easily check that

$$\frac{a}{10}\frac{d\bar{U}}{da}=S_2(a)\,. \tag{6.11}$$

The relation (6.11) shows the fact that the extremum points of $\bar{U}$ are the same as the zero points of S_2.

In order to see the influence of the canonical transformation on the anharmonic terms we compared (Table I), in the case $j=15/2$, $N=6,8,12$ (particle number), the coefficients (6.9) corresponding to the following three values of a: (i) a equal to 1, which leaves the usual Belyaev-Zelevinsky expansion unchanged, (ii) a equal to a_+ given by Eq. (6.5), and (iii) a equal to one positive root of the equation

$$S_2(a)=0\,. \tag{6.12}$$

To stress the effect of the canonical transformation we have used a large value for the quadrupole strength parameter. In the cases considered here, the Eq. (6.12) has only one positive root, a'. For a given N, the data of the second row correspond to $a=a_+$, while those of the third row correspond to $a=a'$. The pairing ($G=0.23$) and quadrupole strength parameters as well as the $\mathring{\omega}_2$ and ω_2 values are given in units of single-particle energy, $\epsilon_{15/2}-\lambda$.

The $a=1$ case indicates clearly that the RPA is not a good starting approach. In spite of the fact that the fourth-order terms increase the energy of phonon, the higher-order terms are of the same order of magnitude (sometimes larger) with the diagonal terms. On the other hand, both the $a=a_+$ and $a=a'$ values enable the transfer of the anharmonicities to the second-order terms. Concerning the $a=a_+$ expansion, it is to be noted that though the smallness of the third- and fourth-order terms justifies the truncation of the Hamiltonian at the fourth-order terms, the magnitude of S_2 makes necessary the use of a large diagonalization space. This trouble is eliminated in the case of $a=a'$ expansion where $S_2=0$. The magnitude of S_2 term in the former case is determined by the arbitrary way we canceled the phonon-vacuum amplitude [see Eq. (3.19)]. Choosing the κ term from the condition that the averaged particle number be conserved and inserting its expression depending on "a" in (6.10) one expects that the ground-state minima do not drastically differ from the solutions predicted by the Hartree condition (6.12).

7. DISCUSSION

In the framework of the conventional shell-model Hamiltonian with pairing, quadrupole, and octupole forces, we attempted to describe some properties of the low-lying quadrupole-octupole spectra, namely the level energies, the electromagnetic transition probabilities, and the quadrupole moments of 2^+ and 3^- states. The formalism we adopted is the boson-expansion procedure. The expansion is performed with respect to the quasiboson-two-quasiparticle operators $(\mathring{A}^{(+)},\mathring{A}^{(-)})$ of rank 2 and 3. In Ref. 33 we truncated the self-commutators of the quadrupole and octupole-two-quasiparticle operators by keeping in their expression only the operators of rank 2 (and positive parity) and 3 (and negative parity). In this way the equations of motion of the quadrupole and octupole-like operators become closed.

Here, we determined the expansion coefficients so that these truncated commutators be satisfied

TABLE I. The values of the coefficients involved in the Hamiltonian (6.7) are listed for the case $j=\frac{15}{2}$. N stands for the number of particles we consider. The values of a different from 1 correspond to a_+ (second row) and the positive solution of Eq. (6.12) (third row). Here the pairing constant is $G=0.23$; X_2, G, $\mathring{\omega}_2$ are given in units of the single-particle energy $(\epsilon_{15/2}-\lambda)$.

N	X_2	$\mathring{\omega}_2$	ω_2	S_2	A_{30}	A_{21}	A_{40}	A_{31}	A_{22}^1	A_{22}^2	a
			2.714	1.766	0.564	1.361	0.409	2.183	1.337	2.274	1
6	3	1.11	2.732	−1.235	0.226	−0.035	−0.101	0.089	0.265	0.130	0.463
			1.702	0	0.299	0.443	0.020	0.597	0.529	0.658	0.728
			3.394	2.701	0	0	0.716	3.294	1.860	3.290	1
8	3	0.978	2.805	−1.315	0	0	−0.080	0.106	0.266	0.106	0.424
			1.667	0	0	0	0.060	0.691	0.559	0.691	0.677
			2.031	0.582	−0.729	−1.505	−0.038	0.678	0.648	0.977	1
12	3	1.415	2.461	−0.948	−0.403	−0.024	−0.168	0.032	0.261	0.203	0.582
			1.804	0	−0.540	−0.813	−0.117	0.308	0.436	0.552	0.846
			4.375	3.967	−2.302	−6.264	0.713	4.226	2.657	4.888	1
12	4	0.699	2.315	−1.037	−0.538	−0.031	−0.224	0.043	0.348	0.270	0.405
			1.582	0	−0.761	−1.241	−0.141	0.486	0.626	0.825	0.616

in the first order of approximation. This aim is touched by neglecting the recoupling terms which are regularly much smaller than the direct ones, and on the other hand do not affect too much the fourth-order Hamiltonian.

Reversing the RPA transformation and inserting the results in the expanded Hamiltonian, one obtains the expression of the fourth-order Hamiltonian in terms of phonon operators $\mathring{C}_2^\dagger$, $\mathring{C}_2$, $\mathring{C}_3^\dagger$, $\mathring{C}_3$ of collective and noncollective type. The expansion associated with the quasiparticle term $(\sum E_a a_\alpha^\dagger a_\alpha)$ contains a constant term κ accounting for the presence of the quasiparticles in the RPA ground state.

Since the quintuplet levels lie in the same energetic region as the two-quasiparticle states, the terms containing noncollective photons should be included. This part of the Hamiltonian, $H^{n\,\mathrm{coll}}$, can be made equivalent to a collective effective Hamiltonian[34] renormalizing the zeroth-, second-, and third-order terms of $H\text{-}H^{n\,\mathrm{coll}}$. Thus, we have finally to study a Hamiltonian of the same structure as H_0 which includes only the collective phonons.

As the RPA leads to a slowly converging series, the following difficulties necessarily appear: (a) We have to include more higher terms to make the truncation justifiable. This fact would very much complicate the problem if we aimed to use the model for practical purposes. (b) The solutions given by the diagonalization procedure converge very slowly with respect to the dimension of the diagonalization space. On the other hand, the dimension of the diagonalization space associated with the quadrupole and octupole phonons is rapidly increasing when the total number of phonons is enhanced.

We tried to avoid such a difficulty by changing the expansion base by a linear and canonical transformation. Ordering the Hamiltonian with respect to the vacuum of the new phonons, the ground-state energy as well as the second-order terms will be renormalized; that is to say, the zero-point oscillation energy and the "size" of phonon are changed.

Fixing the canonical transformation either by a minimum condition for the ground-state energy or by the Hartree condition, the above-mentioned effects are both directed to depressing the values of the anharmonic terms. Ignoring the presence of the quasiparticles in the ground state $(\kappa = ct)$ the two ways of fixing the transformation coincide, but they should be slightly different if the condition for the particle-number conservation is imposed.

Another way to fix the canonical transformation is to force the quadrupole moment of the 2^+ state, which is very sensitive to the change of the anharmonicity strength, to be stable. It is then expected that the anharmonic terms, at least those of quadrupole nature which are essentially determined by the correlations implying the quadrupole-moment operator, are consequently lowered.

Keeping in mind the fact that the anharmonic terms associated with the new representation are small in comparison with the energy of the "correlated" phonons we treated (for a first orientation) perturbatively, the quintuplet level energies, the transition probabilities $(E2, 2^+ \to 0^+)$, $(E3, 3^- \to 0^+)$, $(E3, 3^{-\prime} \to 0^+)$ as well as the static quadrupole moments of the first vibrational states.

The concrete results of Ref. 33 indicate that the quadrupole-like anharmonicities are dominant in determining the quintuplet levels ordering and their relative spacing. Starting from this remark let us do a rough estimation of the E_J^I value [Eq. (4.6)] keeping in the expressions of K_1, K_2, K_3 only the quadrupole-type terms and considering them of the same order of magnitude. Denoting by $\bar{h}_{2,2}^{(4)}$ their common value one obtains

$$E_J^I = -32 h_{2,2}^{(4)} [\sqrt{5}\; W(32J2;32) + \sqrt{7}\,(-1)^{J+1} W(33J3;32)] \,. \qquad (7.1)$$

Since the fourth-order term is expected to be positive, the order predicted by (7.1) is $1^-, 2^-, 5^-, 3^-, 4^-$ which is the same as that given by the similar terms in a linearization-procedure formalism.[33] This order is also predicted by the phenomenological model.[29]

In Sec. 6 we tested the behavior of the anharmonic terms at a canonical transformation by a "single j" calculation. The estimation has been done for a large quadrupole strength parameter and different values for the particle number. We listed the coefficients of the anharmonic terms for the following three values of the independent parameter characterizing the canonical transformation of the quadrupole-phonon operators a:
(i) a equals 1 which corresponds to the usual Belyaev-Zelevinsky expansion in a RPA base.
(ii) a equal to the a_+ value, which minimizes the anharmonic correction of the ground-state energy. That situation corresponds to the restriction (3.19) for the constant κ generated by the expansion of the quasiparticle term.
(iii) a equal to the a' value, which minimizes the ground-state energy. Here the κ term was considered as a pure constant. The latter two cases provide anharmonic terms much smaller than the harmonic terms. The case (iii) is more favorable than the (ii) case, because of dangerous graphs cancellation.

A computational program containing the H_0 diagonalization, the static-quadrupole-moment calculation (for 2^+ and 3^-) as well as the transition probabilities ($E2$; $2^+ \to 0^+$), ($E3$; $3^- \to 0^+$), ($E3$; $3^{-\prime} \to 0^+$) is in progress.

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APPENDIX A

In this Appendix we give the explicit expressions for the shifted zero-point energy $\bar{U}$, the correlated RPA energies ω_2, ω_3, and for the coefficients S_L:

$$\begin{aligned}\bar{U} = U' &+ 2\sqrt{5}\sum(-1)^{s+r}[2(-1)^{m+s}+(-1)^{s+1}+(-1)^{m+1}][\bar{h}^{(4)}_{22}(22msr)+\bar{h}^{(4)}_{23}(33msr)]\\ &+4\sqrt{7}\sum\{(-1)^{m+r+1}[\bar{h}^{(4)}_{22}(23m1r)+\bar{h}^{(4)}_{23}(32m1r)]+(-1)^r[\bar{h}^{(4)}_{22}(321mr)-\bar{h}^{(4)}_{23}(231mr)]\}\\ &+\sum(-1)^{m+s}\{2\sqrt{5}[\bar{h}^{(4)}_{12}\tbinom{22ms}{22m's'}+\bar{h}^{(4)}_{12}\tbinom{33ms}{33m's'}]+\sqrt{7}\,\bar{h}^{(4)}_{13}\tbinom{32ms}{32m's'}\}+\kappa+5a^2_{22}\mathring{\omega}_2+7a^2_{32}\mathring{\omega}_3\,, \end{aligned}\tag{A1}$$

$$\begin{aligned}\omega_2 = \tfrac{1}{2}(a^2_{21}+a^2_{22})\mathring{\omega}_2 &+ \sum\Big(2/\sqrt{5}\,\{[1+(-1)^{m+1}][(-1)^{s+1}+(-1)^{r+1}3]+[1+(-1)^{s+1}][1+3(-1)^{r+m}]\}\,\bar{h}^{(4)}_{2,2}(22msr)\\ &\qquad+(4\sqrt{7}/5)(-1)^r[1+(-1)^{m+1}][\bar{h}^{(4)}_{2,2}(32msr)+(-1)^{s+1}\bar{h}^{(4)}_{2,2}(23smr)]\\ &\qquad+(2\sqrt{7}/5)[(-1)^{m+r}+1]\{[(-1)^s-1]\,\bar{h}^{(4)}_{23}(32msr)+[(-1)^{s+1}+(-1)^{r+m}]\,\bar{h}^{(4)}_{23}(23smr)\}\Big)\\ &+\sum_{msm's'}\Big((\sqrt{5}/5)\,\{[(-1)^m+(-1)^s][(-1)^{m'}+(-1)^{s'}]+4(-1)^{m+s}\}\,\bar{h}^{(4)}_{1,2}\tbinom{22ms}{22m's'}\\ &\qquad\times(\sqrt{7}/5)(-1)^m[(-1)^s+(-1)^{s'}]\,\bar{h}^{(4)}_{13}\tbinom{32ms}{32m's'}\Big)\,, \end{aligned}\tag{A2}$$

$$\begin{aligned}\omega_3 = \tfrac{1}{2}(a^2_{31}+a^2_{32})\mathring{\omega}_3 &+ \sum\{(2\sqrt{7}/7)[(-1)^r+(-1)^s][(-1)^{m+1}+1]\,\bar{h}^{(4)}_{2,2}(32msr)\\ &+(2\sqrt{7}/7)[(-1)^s-1][1+(-1)^{r+m}]\,\bar{h}^{(4)}_{2,2}(23msr)+(2\sqrt{5}/7)[3(-1)^{m+r+s+1}+1][2(-1)^{s+1}+(-1)^{m+s}+1]\,\bar{h}^{(4)}_{2,3}(33msr)\\ &+(4\sqrt{7}/7)[1+(-1)^{m+1}](-1)^{r+1}\bar{h}^{(4)}_{2,3}(23msr)+(2\sqrt{7}/7)(-1)^{m+r}[(-1)^s-1]\,\bar{h}^{(4)}_{2,3}(32msr)\}\\ &+\tfrac{1}{7}\sum\Big(\sqrt{5}\,\{[(-1)^m+(-1)^s][(-1)^{m'}+(-1)^{s'}]+4(-1)^{m+s}\}\,\bar{h}^{(4)}_{1,2}\tbinom{33ms}{33m's'}\\ &\qquad+\sqrt{7}\,(-1)^s[(-1)^m+(-1)^{m'}]\,\bar{h}^{(4)}_{1,3}\tbinom{32ms}{32m's'}\Big)\,, \end{aligned}\tag{A3}$$

$$\begin{aligned}&S_2(\mathring{\omega}_2,\mathring{\omega}_3,X_2,X_3,a^{(+)}_2,a^{(-)}_2,a^{(+)}_3,a^{(-)}_3)\\ &=a_{21}a_{22}\mathring{\omega}_2+\sum\{[4(-1)^{r+m}+3(-1)^{r+1}+3(-1)^{r+m+s+1}+2(-1)^{m+1}+(-1)^{m+s}+2(-1)^{r+s}+1](\sqrt{5}/5)\,\bar{h}^{(4)}_{22}(22msr)\\ &\qquad+(\sqrt{7}/10)(-1)^{m+1}[1+(-1)^{s+s'}]\,\bar{h}^{(4)}_{1,3}\tbinom{32ms}{32m's'}+(\sqrt{5}/5)(-1)^{m+s+1}[(-1)^m+(-1)^s+(-1)^{m'}+(-1)^{s'}]\,\bar{h}^{(4)}_{1,2}\tbinom{22ms}{22m's'}\\ &\qquad+(\sqrt{7}/5)[(-1)^{r+1}+(-1)^{s+1}][(-1)^{r+m}+(-1)^{s+1}]\,\bar{h}^{(4)}_{2,3}(23msr)+(2\sqrt{7}/5)(-1)^r[(-1)^{m+1}+1]\,\bar{h}^{(4)}_{2,2}(32msr)\\ &\qquad+(2\sqrt{7}/5)(-1)^{r+m+1}[1+(-1)^{s+1}]\,\bar{h}^{(4)}_{2,2}(23msr)+(\sqrt{7}/5)[(-1)^r+(-1)^m][(-1)^{s+1}+1]\,\bar{h}^{(4)}_{2,3}(32msr)\}\,, \end{aligned}\tag{A4}$$

$$\begin{aligned}&S_3(\mathring{\omega}_2,\mathring{\omega}_3,X_2,X_3,a^{(+)}_2,a^{(-)}_2,a^{(+)}_3,a^{(-)}_3)\\ &=a_{31}a_{32}\mathring{\omega}_3+\sum\{(\sqrt{7}/14)(-1)^{s+1}[1+(-1)^{m+m'}]\,\bar{h}^{(4)}_{1,3}\tbinom{32ms}{32m's'}+[(-1)^m+(-1)^s+(-1)^{m'}+(-1)^{s'}](-1)^{m+s+1}(\sqrt{5}/7)\,\bar{h}^{(4)}_{1,2}\tbinom{33ms}{33m's'}\}\\ &\qquad\sum\{(2\sqrt{7}/7)(-1)^{r+1}[1+(-1)^{m+1}]\,\bar{h}^{(4)}_{2,3}(23msr)+(\sqrt{7}/7)[(-)^{r+s}+1][(-1)^m-1]\,\bar{h}^{(4)}_{2,2}(32msr)\\ &\qquad+(\sqrt{7}/7)[(-1)^{s+1}+1][(-1)^r+(-1)^m]\,\bar{h}^{(4)}_{2,2}(23msr)+(2\sqrt{7}/7)(-1)^{m+r+1}[1+(-1)^{s+1}]\,\bar{h}^{(4)}_{2,3}(32msr)\\ &\qquad+(\sqrt{5}/7)[3(-1)^{r+m+s+1}+3(-1)^{r+1}+4(-1)^{m+r}+2(-1)^{m+1}+2(-1)^{r+s}+(-1)^{m+s}+1]\,\bar{h}^{(4)}_{2,3}(33msr)\}\,. \end{aligned}\tag{A5}$$

APPENDIX B

This Appendix contains the anharmonic factors which characterize the $2^+ \to 0^+$ (β_2), $3^- \to 0^+$ (β_3), $3^{-\prime} \to 0^+$ (T) transition probabilities and the 2^+ ($\mathcal{Q}_2$) and 3^- ($\mathcal{Q}_3$) quadrupole moments:

$$\begin{aligned}\beta_2 = \mathring{\rho}_2^{\,-1}\Big(&\sum\{Q^A_2(22,msr)[(-1)^s+(-1)^m+2(-1)^{m+s+1}](-1)^{s+r+1}\\ &\qquad+(\sqrt{7}/5)[(-1)^{r+s+1}+(-1)^{m+r+1}]\,Q^A_2(23,msr)+(\sqrt{7}/\sqrt{5})[(-1)^r+(-1)^{r+m+1}]\,Q^A_2(32,msr)\}\\ &+\sqrt{2}\sum[C^{\mathrm{I}}_{22}Q^B_2(22,rs)+C^{\mathrm{I}}_{33}Q^B_2(33,rs)]\Big)\,, \end{aligned}\tag{B1}$$

$$\begin{aligned}\beta_3 = \mathring{\rho}_3^{\,-1}\Big(&\sum\{(-1)^{r+1}[1+(-1)^{m+1}]\,Q^A_3(23,msr)+[(-)^{r+m}+(-)^{r+s+m+1}]\,Q^A_3(32;msr)\\ &+[(-1)^r+2(-1)^{r+m+1}+(-1)^{r+s+m}](\sqrt{5}/\sqrt{7})\,Q^A_3(33,msr)\}+\sum C^{\mathrm{I}}_{23}\,Q^B_3(32,ms)\Big)\,, \end{aligned}\tag{B2}$$

$$
\begin{aligned}
T = {} & \sum Q_3^B(32, ms) + B_3^{\mathrm{I}}\Big(\mathring{\rho}_3 + \sum \{(-1)^{r+1}[1+(-1)^{m+1}]\, Q_3^A(23, msr) \\
& \qquad + (\sqrt{5}/\sqrt{7})[(-1)^r + 2(-1)^{r+m+1} + (-1)^{r+s+m}]\, Q_3^A(33, msr)\}\Big) \\
& + \sqrt{2} \sum_{J'=0,2,4} \hat{3}\hat{J}' W(2233; J'3) B_{223}^{(\mathrm{I})3J'} \sum[(-1)^{r+s} + (-1)^{m+1}][\,Q_3^A(23; msr) + Q_3^A(32; msr)] \\
& + \sqrt{2}\; B_{223}^{(\mathrm{I})32} \sum[(-1)^{r+s} + (-1)^{m+1}]\, Q_3^A(22; msr) \\
& + 2N_3(33) \sum_{J'=0,2,4} D^I_{J'}[\,\delta_{J'2} + \sqrt{2}\; \hat{2}\hat{J}' W(3333; J'2)] \sum[(-1)^{r+s} + (-1)^{m+1}]\, Q_3^A(33, msr), \qquad \text{(B3)}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}_2 = {} & \sum [(-1)^{m+1} + (-1)^{s+1}]\, Q_2^B(22, ms) \\
& + C_{22}^{\mathrm{I}}\sqrt{2}\Big(2\mathring{\rho}_2 + \sum 2\{(-1)^{s+1} + (-1)^{r+m+s+1} + (-1)^{r+1} + 3(-1)^{m+r} + 5[(-1)^{m+s}+1][1+(-1)^{r+1}]\, W(2222; 22)\} \\
& \qquad \times Q_2^A(22, msr) + (2\sqrt{7}/\sqrt{5})\,[(-1)^{r+m+s} - (-1)^{r+m}]\, Q_2^A(23, msr) + (2\sqrt{7}/\sqrt{5})[(-1)^{r+m+1} + (-1)^r] Q_2^A(32, msr)\Big) \\
& + C_{33}^{\mathrm{I}} \sum\Big(\sqrt{8}\,(-1)^{s+1}[1+(-1)^{r+m+s+1}]\, Q_2^A(33, msr) + 2\sqrt{70}\;\; W(2323; 32) \\
& \qquad \times\{[(-1)^{m+s} + (-1)^{r+1}]\, Q_2^A(23, msr) + [1+(-1)^{m+s+r+1}]\, Q_2^A(32, msr)\}\Big) \\
& + (C_{22}^{\mathrm{I}})^2\, 10\, W(2222; 22) \sum[(-1)^{m+1} + (-1)^{s+1}]\, Q_2^B(22, ms) \\
& + (C_{33}^{\mathrm{I}})^2 10\, W(2323; 32) \sum[(-1)^m + (-1)^s]\, Q_2^B(33; ms), \qquad \text{(B4)}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}_3 = {} & (\sqrt{5}/\sqrt{7}) \sum[(-1)^m + (-1)^5]\, Q_2^B(33, ms) \\
& + 2\, C_{23}^{\mathrm{I}} \sum\Big([2(-1)^{m+r} + (-1)^{m+r+s+1} + (-1)^{r+1}]\, Q_2^A(22, msr) + [1+(-1)^{r+1}][(-1)^{m+s+1} - 1] \\
& \qquad \times 5\, W(3322, 23) Q_2^A(33, msr) + (1/\sqrt{35})\{35\, W(3333, 22)[(-1)^{m+r+s} - 1] + 12(-1)^{m+r+1} \\
& \qquad + 5(-1)^s + 7(-1)^{m+r+s}\}\, Q_2^A(23, msr) + (1\sqrt{35})\{35\, W(3333, 22)[(-1)^r + (-1)^{m+s+1}] \\
& \qquad + 12(-1)^{m+r+1} + 5(-1)^s + 7(-1)^r\}\, Q_2^A(32, msr)\Big) \\
& + (C_{23}^{\mathrm{I}})^2 \sum\{\hat{2}\hat{3}\, W(2233, 23)[(-1)^{m+1} + (-1)^{s+1}]\, Q_2^B(22, ms) \\
& \qquad + 7W(2332, 33)[(-1)^m + (-1)^s]\, Q_2^B(33, ms)\}\,. \qquad \text{(B5)}
\end{aligned}
$$

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