Specific New Approach to Finite-Range Distorted-Wave Born Approximation*

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A new method for calculating finite-range distorted-wave Born-approximation matrix elements is fully developed. This method is based on plane-wave expansions of the distorted waves which allows a separability feature not found in the usual approach. A spin-orbit interaction is included for both the bound states and the distorted waves.

I. INTRODUCTION

A great deal of work has recently been done on finite-range distorted-wave Born approximation (DWBA).[1,2] This is due mainly to the recent increased interest in heavy-ion physics where the use of the usual zero-range approximation yields only limited success. Almost all work has been done with the methods in Ref. 3 which involve multiple integrals. Alternate methods, which replace two of the integrals with sums, have been proposed.[4,5] The purpose of this paper is to present a formalism which makes use of these methods. The methodology developed here will also allow for spin-orbit effects which is not included in the references cited. The distorted waves are expanded in plane waves which allows a separation of coordinates and, in effect, replaces integrals with sums over plane-wave states.

Mainly to define notation, the DWBA formalism is reviewed in Sec. II. The use of plane-wave sums is discussed in Sec. III, and in Sec. IV a finite-range expression is developed. A discussion of the method is given in Sec. V.

II. DWBA FORMALISM

A brief description of the DWBA formalism is given in this section. It is applied to a pickup (or, by reciprocity, stripping) reaction. In this paper only the direct mode is derived, but the method applies to any direct process involving a six-dimensional integral of the type considered here.

We write the pickup reaction symbolically as:

$$T+P\rightarrow R+D, \quad m_D>m_P .$$

The target, T, is composed of the residual nucleus, R, and the transferred particle, H; i.e.,

$$T=R+H .$$

The detected particle is

$$D=P+H .$$

The differential cross section for such a process may be written as[3,6-8]

$$\frac{d\sigma_{PD}}{d\Omega}=m_{PT}m_{DR}(2\pi\hbar^2)^{-2}k_Dk_P^{-1}\hat{j}_P^{-2}\hat{J}_T^{-2}\sum_{M_R\mu_DM_T\mu_P}|T_{M_R\mu_DM_T\mu_P}|^2, \tag{2.1}$$

where

$$T_{M_R\mu_DM_T\mu_P}=\int d\tau\,\chi^{(-)}_{\mu_D}(\vec{k}_D,\vec{r}_{DR})^*\psi_{j_D\mu_D}(\xi_D)^*\psi_{J_RM_R}(\xi_R)^*V_{PH}(r_{PH})\psi_{J_TM_T}(\xi_T)\psi_{j_P\mu_P}(\xi_P)\chi^{(+)}_{\mu_P}(\vec{k}_P,\vec{r}_{PT}), \tag{2.2}$$

χ denotes the distorted wave for the projectile or detected particle, ψ gives the internal wave function for the four particles involved (subscripts refer to total angular momentum quantum numbers and their z-axis projections), and the potential V is taken to be that which binds P and H to form D. Writing the wave functions for the detected particle and the target as a sum over cluster states and performing the appropriate overlap integrals allows (2.2) to be written as

$$T_{M_R\mu_DM_T\mu_P}=\sum_{l_{HR}J_{HR}l_{PH}}\sum_{J_{PH}j_H}A^{j_H}_{l_{HR}J_{HR}}A^{j_H}_{l_{PH}J_{PH}}\sum_{\mu_{HR}M_{HR}\mu_{PH}}\sum_{M_{PH}\mu_H}D^{J_{PH}J_{HR}\mu_P\mu_D}_{l_{PH}\mu_{PH}l_{HR}\mu_{HR}}C(l_{HR}j_HJ_{HR};\mu_{HR}\mu_HM_{HR})$$
$$\times C(l_{PH}j_HJ_{PH};\mu_{PH}\mu_HM_{PH})C(J_{HR}J_RJ_T;M_{HR}M_RM_T)C(J_{PH}j_Pj_D;M_{PH}\mu_P\mu_D) \tag{2.3}$$

with

$$D^{J_{PH}J_{HR}\mu_P\mu_D}_{l_{PH}\mu_{PH}l_{HR}\mu_{HR}} = \int d\vec{r}_{PH}\, d\vec{r}_{HR}\, \chi^{(-)}_{\mu_D}(\vec{k}_D, \vec{r}_{DR})^* \phi^{J_{PH}}_{l_{PH}\mu_{PH}j_{PH}}(\vec{r}_{PH})^* V_{PH}(r_{PH}) \phi^{J_{HR}}_{l_{HR}\mu_{HR}j_H}(\vec{r}_{HR}) \chi^{(+)}_{\mu_P}(\vec{k}_P, \vec{r}_{PT}) . \quad (2.4)$$

The A's are spectroscopic amplitudes, the ϕ's denote the appropriate relative wave function, and the coordinates are related by

$$\vec{r}_{DR} = \vec{r}_{HR} + \frac{m_P}{m_D}\vec{r}_{PH} \quad (2.5)$$

and

$$\vec{r}_{PT} = \vec{r}_{PH} + \frac{m_R}{m_T}\vec{r}_{HR} . \quad (2.6)$$

Using the usual expansions,

$$\chi^{(\pm)}_{\mu}(\vec{k}, \vec{r}) = 4\pi \sum_{LJM\mu'} C(LjJ; M\mu) C(LjJ; M+\mu-\mu'\mu')(i)^L Y^{M*}_L(\hat{k}) Y^{M+\mu-\mu'}_L(\hat{r}) u^{J(\pm)}_{Lj}(kr)/(kr) \quad (2.7)$$

and

$$\phi^J_{l\mu j}(\vec{r}) = V^J_{lj}(r)(i)^l Y^{\mu}_l(\hat{r}) , \quad (2.8)$$

in (2.4) allows a form for the T matrix element that may be evaluated numerically. It is necessary, however, to perform three-dimensional integrals. (See Ref. 1, for instance.) The method to be developed in the following sections of this paper replaces two of the integrals with sums. This is accomplished by expanding the radial part of the distorted waves [Eq. (2.7)] in Bessel functions and, by the use of projection operators, rewriting (2.4) as a sum over plane-wave matrix elements.

III. PLANE-WAVE EXPANSIONS

The convergence properties of the DWBA integrals allow one to match the distorted waves over a limited region of space, perhaps for values of the radial coordinate up to 20 or 25 fm. Matching the distorted waves by the use of a Bessel function expansion gives

$$u^{J(\pm)}_{Lj}(kr) = (kr) \sum_{n=1}^{N(L)} a^{J(\pm)}_{nLj}\, j_L(k_n r) , \quad (3.1)$$

where j_L is a spherical Bessel function, and the expansion is intended to apply only over a limited region of space (0 to R_L). The coefficients $a^{J(\pm)}_{nLj}$ may be found as overlaps of u and j_L if the k_n's are chosen in such a way as to allow the functions to form a complete orthonormal set. This could be accomplished, for instance, by choosing the k_n such that $j_L(k_n R_L) = 0$ or $j_L(k_n R_L)' = 0$, where the prime denotes derivative with respect to the argument evaluated at R_L. Either of these methods, however, would require a different set of k_n's for each L. Such a procedure would not result in the most efficient computational method. This will be shown in the next section. The use of a technique given in Ref. 9 provides a better way of expressing the distorted waves with

$$a^{J(\pm)}_{nLj} = \sum_{n'=1}^{N(L)} N^L_{nn'}\, b^{J(\pm)}_{n'Lj} , \quad (3.2)$$

where

$$b^{J(\pm)}_{nLj} = k^{-1} \int_0^{R_L} r\, dr\, u^{J(\pm)}_{Lj}(kr)\, j_L(k_n r) , \quad (3.3)$$

$$N^L_{nn'} = (\vec{O}^{L\,-1})_{nn'} , \quad (3.4)$$

and

$$O^L_{nn'} = \int_0^{R_L} r^2 dr\, j_L(k_n r)\, j_L(k_{n'} r) . \quad (3.5)$$

TABLE I. The number of plane-wave states needed to match a sampling of partial waves.

Process	"zero-range" system	L	N_i (L)	N_f (L)
^{40}Ca$(d,p)^{41}$Ca$_{g.s.}$ (E_d = 11 MeV)	^{41}Ca = n + ^{40}Ca	0	14	11
		7	12	9
		10	5	4
		20	3	3
^{12}C(^{6}Li, d)^{16}O$_{6.06}$ ($E_{^6\mathrm{Li}}$ = 18 MeV)	^{16}O$_{6.06}$ = α + ^{12}C	0	19	13
		7	17	11
		10	4	3
		20	3	2

The correctness of this method is shown in Refs. 5 and 9. The k_n's in Eqs. (3.2)–(3.5) can now be chosen to minimize the number of states needed $[N(L)]$, and they may be the same for a large range of L's. The technique just discussed allows Eq. (3.1) to be written as

$$u_{Lj}^{J(\pm)}(kr) = (kr) \sum_{n=1}^{N(L)} \sum_{n'=1}^{N(L)} N_{nn'}^{L} b_{nLj}^{J(\pm)} j_L(k_n r) . \tag{3.6}$$

With the use of

$$4\pi(i)^L j_L(k_n r) Y_L^M(\hat{\mathbf{r}}) = \int d\hat{\mathbf{k}}_n e^{i\hat{\mathbf{k}}_n \cdot \hat{\mathbf{r}}} Y_L^M(\hat{\mathbf{k}}_n) \tag{3.7}$$

the expansion may be expressed as

$$4\pi(i)^L u_{Lj}^{J(\pm)}(kr) Y_L^M(\hat{\mathbf{r}}) = (kr) \sum_{n=1}^{N(L)} \sum_{n'=1}^{N(L)} N_{nn'}^{L} b_{n'Lj}^{J(\pm)} \int d\hat{\mathbf{k}}_n e^{i\hat{\mathbf{k}}_n \cdot \hat{\mathbf{r}}} Y_L^M(\hat{\mathbf{k}}_n) . \tag{3.8}$$

This is the form that will be used in the next section to develop a full finite-range expression.

In choosing the k_n's, there are two mathematical restrictions. First, the values of k_n and k_{n+1} (for any n) must not be too close together, since that would result in two rows (or columns) in the matrix $\vec{O}$ being approximately identical which makes $\vec{O}$ almost singular. (If any two rows or columns of a matrix are identical, it is singular.) This results in calculational problems in the actual use of the method as presented. The second restriction is that for large L's, no more than one state may be included with a value $\lesssim (L/R_L)$. This would cause two rows, or columns, in the matrix $\vec{O}$ to be almost proportional which, of course, also yields a matrix close to being singular.

Probably the most economical choice of states results by realizing that there are three groups of distorted waves when they are classified by their partial wave quantum number L. Namely, those that "see" the strong interaction distorting potential, those influenced by only the Coulomb potential and a transition region between, lying about $L \simeq kR_A$ where k is the physical wave number and R_A the radius of the nucleus involved. The first two groups may be matched by using the orthonormal set of states given by the zeros of $j_0(k_n R_L)$ (orthonormal only for $L=0$). For the larger values of L in these two groups, a few of the states with small values of k_n must not be used in order to avoid the mathematical difficulty mentioned earlier. The largest value of k_n needed is roughly the largest momenta that will fit in the real part of the strong interaction potential well. $(k_n^{\max} \simeq (1/\hbar)\{2u[E_{\text{c.m.}} + |\operatorname{Re} V(r)|]\}^{1/2}|_{r=0}$, where u is the reduced mass of the system, $E_{\text{c.m.}}$ the center of mass energy and $V(r)$ the distorting potential.)

The best choice of k_n's for the "Coulomb partial waves" is probably to select values that fit in the Coulomb well between $r_{\min}$ and R_L where $r_{\min}$ is the classical radius of closest approach allowed by the centrifugal barrier. The number of states may then be made larger to obtain convergence.

Whether the method described here is a more efficient method depends strongly on how many states $[N(L)]$ are needed in the Bessel function expansions. Convergence has been tested by evaluating

$$I_{Ll}^{Jjj'}(R_L) = \int_0^{R_L} r\, dr\, u_{Lj}^{J(\pm)}(kr) V_{lj'}^{J}(r) , \tag{3.9}$$

where V is the zero-range form factor. The I's were found using both the original distorted waves and their Bessel function expansion. Convergence was claimed when the two values differed by $\simeq 1\%$. The number of states needed $[N(L)]$ is shown in Table I for various distorted waves. The u and V were generated by use of a Woods-Saxon well with the parameters given in Table II.

The properties of the functions $I(R_L)$ may also be used to determine R_L (large enough to allow convergence of the integrals).

TABLE II. The optical parameters used to describe the nuclei involved in the "matching" of Table I. The potential W describes an imaginary interaction of the surface type. In the p-^{41}Ca system, a spin-orbit term of strength 8 MeV was also used. The parameters for ^{40}Ca$(d,p)^{41}$Ca$_{\text{g.s.}}$ are from Ref. 12 and those for ^{12}C-(^{6}Li, d)^{16}O$_{6.06}$ are from Ref. 2.

	Distorted waves						
System	V	r_{0R}	a_R	W	r_{0I}	a_I	r_{0C}
d-^{40}Ca	112	1.00	0.90	18.0	1.55	0.47	1.25
p-^{41}Ca$_{\text{g.s.}}$	51.5	1.20	0.65	11.0	1.25	0.47	1.25
^{6}Li-^{12}C	80.0	1.30	0.80	5.0	1.50	0.70	2.00
d-^{16}O$_{6.06}$	80.0	1.00	0.72	5.0	1.59	0.63	1.30

Bound states ("zero-range form factor")				
System	Binding energy	r_{0R}	a_R	r_{0C}
n-^{40}Ca	−8.37	1.20	0.65	1.20
α-^{12}C	−1.10	1.30	0.70	1.25

IV. FINITE-RANGE MATRIX ELEMENTS

The results of Secs. II and III are now used to develop a full finite-range expression for Eq. (1.1). Using Eqs. (2.5), (2.6), (2.7), and (3.8) allows (2.4) to be written as

$$D^{J_{PH}J_{HR}\mu_P\mu_D}_{l_{PH}\mu_{PH}l_{HR}\mu_{HR}} = \sum_{L_PL_DJ_PJ_D} \sum'_{M_PM_D\mu'_P\mu'_D} C(L_Pj_PJ_P;M_P\mu_P)C(L_Pj_PJ_P;M_P+\mu_P-\mu'_P\mu'_P)C(L_Dj_DJ_D;M_D\mu_D)$$
$$\times C(L_Dj_DJ_D;M_D+\mu_D-\mu'_D\mu'_D)Y^{M_P*}_{L_P}(\hat{\vec{k}}_P)Y^{M_D}_{L_D}(\hat{\vec{k}}_D)$$
$$\times \sum_{n_Pn'_Pn_Dn'_D} N^{L_P}_{n_Pn'_P}N^{L_D}_{n_Dn'_D}b^{J_P(\pm)}_{n'_PL_Pj_P}b^{J_D(\pm)}_{n'_DL_Dj_D}\int d\hat{\vec{k}}_{n_P}d\hat{\vec{k}}_{n_D}Y^{M_P+\mu_P-\mu'_P}_{L_P}(\hat{\vec{k}}_{n_P})Y^{M_D+\mu_D-\mu'_D*}_{L_D}(\hat{\vec{k}}_{n_D})$$
$$\times B^{J_{PH}J_{HR}}_{l_{PH}\mu_{PH}l_{HR}\mu_{HR}}(\vec{k}_{n_P},\vec{k}_{n_D}), \tag{4.1}$$

where

$$B^{J_{PH}J_{HR}}_{l_{PH}\mu_{PH}l_{HR}\mu_{HR}}(\vec{k}_{n_P},\vec{k}_{n_D}) = \int d\vec{r}_{PH}\,d\vec{r}_{HR}\,e^{-i\vec{q}_2\cdot\vec{r}_{HR}}e^{i\vec{q}_1\cdot\vec{r}_{PH}}\phi^{J_{PH}*}_{l_{PH}\mu_{PH}j_H}(\vec{r}_{PH})V_{PH}(r_{PH})\phi^{J_{HR}}_{l_{HR}\mu_{HR}j_H}(\vec{r}_{HR}), \tag{4.2}$$

$$\vec{q}_1 = \vec{k}_{n_P} - \frac{m_P}{m_D}\vec{k}_{n_D}, \tag{4.3}$$

and

$$\vec{q}_2 = \vec{k}_{n_D} - \frac{m_R}{m_T}\vec{k}_{n_P}. \tag{4.4}$$

One advantage of the present methodology is that the potential (V_{PH}) may now be eliminated by using the Schrödinger equation and the Fourier transform of $\phi^{J_{PH}}_{l_{PH}\mu_{PH}j_H}$.[11] This gives for Eq. (4.2)

$$B^{J_{PH}J_{HR}}_{l_{PH}\mu_{PH}l_{HR}\mu_{HR}}(\vec{k}_{n_P},\vec{k}_{n_D}) = \left[\mathcal{E}_{PH} - \frac{(m_P+m_H)}{2m_Pm_H}\hbar^2q_1^2\right]\int d\vec{r}_{PH}e^{i\vec{q}_1\cdot\vec{r}_{PH}}\phi^{J_{PH}*}_{l_{PH}\mu_{PH}j_H}(\vec{r}_{PH})$$
$$\times \int d\vec{r}_{HR}\,e^{-i\vec{q}_2\cdot\vec{r}_{HR}}\phi^{J_{HR}}_{l_{HR}\mu_{HR}j_H}(\vec{r}_{HR}). \tag{4.3}$$

Expanding the exponentials and performing the angular integrations yields

$$B^{J_{PH}J_{HR}}_{l_{PH}\mu_{PH}l_{HR}\mu_{HR}}(\vec{k}_{n_P},\vec{k}_{n_D}) = F^{J_{PH}}_{l_{PH}j_H}(q_1)G^{J_{HR}}_{l_{HR}j_H}(q_2)Y^{\mu_{PH}*}_{l_{PH}}(\hat{\vec{q}}_1)Y^{\mu_{HR}}_{l_{HR}}(\hat{\vec{q}}_2)q_1^{l_{PH}}q_2^{l_{HR}}, \tag{4.4}$$

where

$$F^{J_{PH}}_{l_{PH}j_H}(q_1) = 4\pi\left[\mathcal{E}_{PH} - \frac{(m_P+m_H)}{2m_Pm_H}\hbar^2q_1^2\right]\left[\int_0^\infty r_{PH}^2dr_{PH}\,j_{l_{PH}}(q_1r_{PH})V^{J_{PH}}_{l_{PH}j_H}(r_{PH})\right]\Big/q_1^{l_{PH}} \tag{4.5}$$

and

$$G^{J_{HR}}_{l_{HR}j_H}(q_2) = 4\pi\left[\int_0^\infty r_{HR}^2dr_{HR}\,j_{l_{HR}}(q_2r_{HR})V^{J_{HR}}_{l_{HR}j_H}(r_{HR})\right]\Big/q_2^{l_{HR}}. \tag{4.6}$$

It is necessary to place Eq. (4.4) in a form that allows the integrals in Eq. (4.1) to be done ($\int d\hat{k}_{n_P}$ and $\int d\hat{k}_{n_D}$). This is accomplished by expanding the functions F and G in terms of Legendre polynomials and by using[3]

$$q^l\,Y^m_l(\hat{\vec{q}}) = (4\pi)^{1/2}\sum_{\lambda=0}^{l}\sum_{\nu}\hat{\lambda}^{-1}\binom{2l+1}{2\lambda}^{1/2}(aq')^{l-\lambda}(bq'')^\lambda C(l-\lambda\lambda l;m-\nu\nu)Y^{m-\nu}_{l-\lambda}(\hat{\vec{q}}')Y^\nu_\lambda(\hat{\vec{q}}''), \tag{4.7}$$

where

$$\vec{q} = a\vec{q}' + b\vec{q}'', \tag{4.8}$$

$$\binom{x}{y} = x![y!(x-y)!]^{-1} \tag{4.9}$$

and

$$\hat{\lambda}=\sqrt{2\lambda+1}\,. \tag{4.10}$$

Performing the $\hat{k}_{n_P}$ and $\hat{k}_{n_D}$ integrals, using standard Racah algebra techniques and choosing k_P along the z axis gives the final form for Eq. (1.1) as

$$\frac{d\sigma}{d\Omega}=[4(4\pi)^4]^{-1}E_P^{cm^{-1}}E_D^{cm^{-1}}k_D\,k_P^{-1}\hat{j}_P^{-2}H(\theta)\,, \tag{4.11}$$

where

$$H(\theta)=\sum_{J_{HR}\mu_P\mu_D}\sum_{\nu\geq 0}(2-\delta_{\nu 0})\left|\sum_{l_{HR}l_{PH}J_{PH}j_H}(-)^{j_H+J_{PH}}\hat{l}_{HR}{}^2\hat{l}_{PH}{}^2\hat{j}_D\hat{J}_{PH}A^{j_H}_{l_{HR}J_{HR}}A^{j_H}_{l_{PH}J_{PH}}\right.$$
$$\left.\times\sum_{L_D}\hat{L}_D I^{L_D}_{\nu\mu_P\mu_D}P^{\nu}_{L_D}(\cos\theta)\right|^2, \tag{4.12}$$

$$I^{L_D}_{\nu\mu_P\mu_D}=\sum_{L_PJ_PJ_D}\hat{J}_P\hat{L}_P C(L_Dj_DJ_D;\nu\mu_D)C(L_Pj_PJ_P;O\mu_P)C(J_PJ_{HR}J_D;-\mu_P\mu_P-\mu_D-\nu)K^{L_PL_D}_{J_PJ_D}\,, \tag{4.13}$$

$$K^{L_PL_D}_{J_PJ_D}=\sum_P\hat{P}^2\begin{Bmatrix}P & L_P & L_D\\ J_{PH} & j_P & j_D\\ J_{HR} & J_P & J_D\end{Bmatrix}W(l_{PH}l_{HR}J_{PH}J_{HR};Pj_H)\sum_{L\Lambda_D\Lambda_P}(-)^{\Lambda_P}\hat{L}^2\hat{\Lambda}_D\hat{\Lambda}_P C(\Lambda_DLL_D;OO)C(\Lambda_PLL_P;OO)$$
$$\times W(L_PL_D\Lambda_P\Lambda_D;PL)$$
$$\times\sum_{\lambda_1\lambda_2}\sum_{n_Dn_D'n_Pn_P'}N^{L_P}_{n_Pn_P'}N^{L_D}_{n_Dn_D'}b^{J_P}_{n_PL_Pj_P}b^{J_D}_{n_DL_Dj_D}M^{J_{PH}j_HJ_{HR}}_{Ll_{PH}l_{HR}}(k_{n_P}k_{n_D})k_{n_P}^{(l_{PH}-\lambda_1)}k_{n_D}^{(l_{HR}-\lambda_2)}$$
$$\times\left(\frac{-m_R}{m_T}k_{n_P}\right)^{\lambda_2}\left(\frac{-m_P}{m_D}k_{n_D}\right)^{\lambda_1}\begin{pmatrix}2 & l_{PH}\\ 2 & \lambda_1\end{pmatrix}^{1/2}\begin{pmatrix}2 & l_{HR}\\ 2 & \lambda_2\end{pmatrix}^{1/2}\begin{Bmatrix}P & l_{PH} & l_{HR}\\ \Lambda_P & l_{PH}-\lambda_1 & \lambda_2\\ \Lambda_D & \lambda_1 & l_{HR}-\lambda_2\end{Bmatrix}$$
$$\times C(l_{HR}-\lambda_2\lambda_1\Lambda_D;OO)C(l_{PH}-\lambda_1\lambda_2\Lambda_P;OO)\,, \tag{4.14}$$

$$M^{J_{PH}j_HJ_{HR}}_{Ll_{PH}l_{HR}}(k_{n_P}k_{n_D})=\int_{-1}^{+1}F^{J_{PH}*}_{l_{PH}j_H}(q_1)G^{J_{HR}}_{l_{HR}j_H}(q_2)P_L(\mu)d\mu\,, \tag{4.15}$$

$$q_1=(k_{n_P}{}^2+m_P{}^2m_D{}^{-2}k_{n_D}{}^2-2\mu\,m_Pm_D{}^{-1}k_{n_P}k_{n_D})^{1/2}\,,$$

and

$$q_2=(k_{n_D}{}^2+m_R{}^2m_T{}^{-2}k_{n_P}{}^2-2\mu\,m_Rm_T{}^{-1}k_{n_P}k_{n_D})^{1/2}\,.$$

It may be seen that

$$|L_P-L|\leq l_{PH}+l_{HR} \tag{4.16}$$

and

$$|L_D-L|\leq l_{PH}+l_{HR}\,. \tag{4.17}$$

Thus, for given L_P and L_D, many values of L are required (in general). The advantage of using the same states (k_n's) for a wide range of partial waves is now clear. The k_n's which appear in the M_L functions are the same instead of differing for each L_P-L_D pair. This, of course, results in a considerable calculational advantage. There are, however, still the two groups of partial waves which require different k_n's, but the range of L for which the M_L functions must be found twice is small $[2\times(l_{PH}+l_{HR})]$.

Equations (4.11)–(4.15) probably represent the best form for actual use of the formalism as developed here.

V. DISCUSSION

The methodology as given in this paper has been fully tested numerically and incorporated into the computer code MERCURY. Two examples of finite-range calculations are shown in Figs. 1 and 2. These results are compared both to zero range and to previous finite-range work using other methods. In Fig. 3 is shown the result of chang-

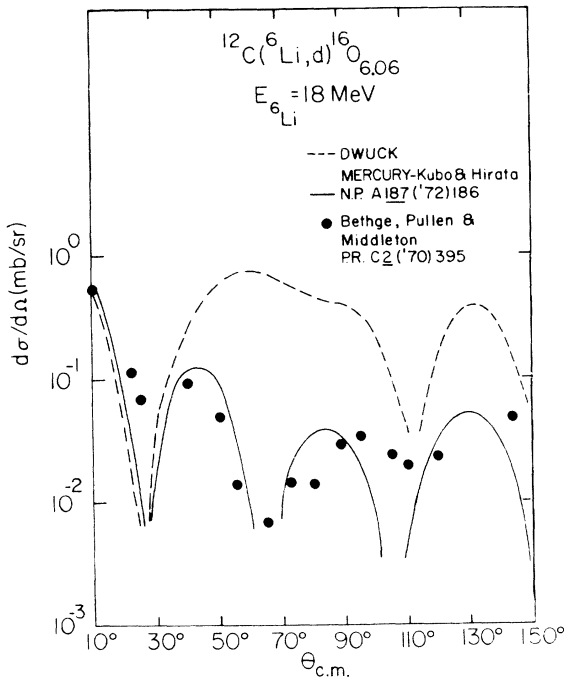

FIG. 1. A finite-range calculation for the process indicated is compared to zero range as calculated by the computer code DWUCK (Ref. 10). The run time on a CDC 6500 was 26 sec (129 sec) for DWUCK (MERCURY). As indicated, the finite-range results were identical to those of Ref. 2 which used the method of Ref. 3. The parameters are given in Table II.

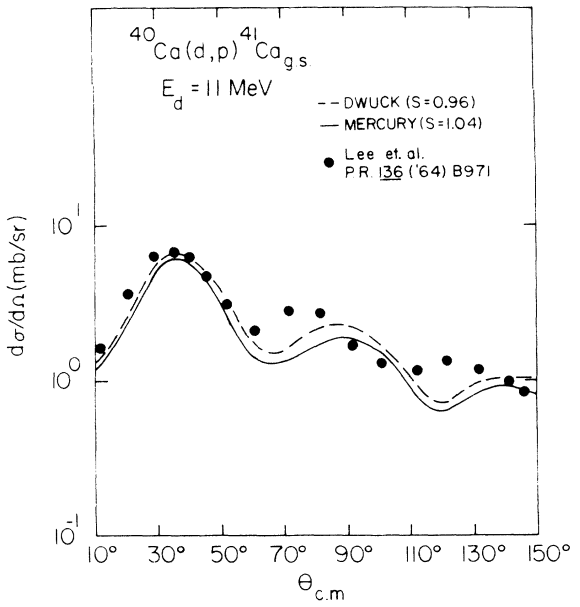

FIG. 2. Another finite-range calculation is compared to zero range as given by DWUCK. The run time on a CDC 6500 was 47 sec (242 sec) for DWUCK (MERCURY). The finite-range results are identical to those of Ref. 12 which replaced the δ function of zero range by a Gaussian. The parameters are given in Table II.

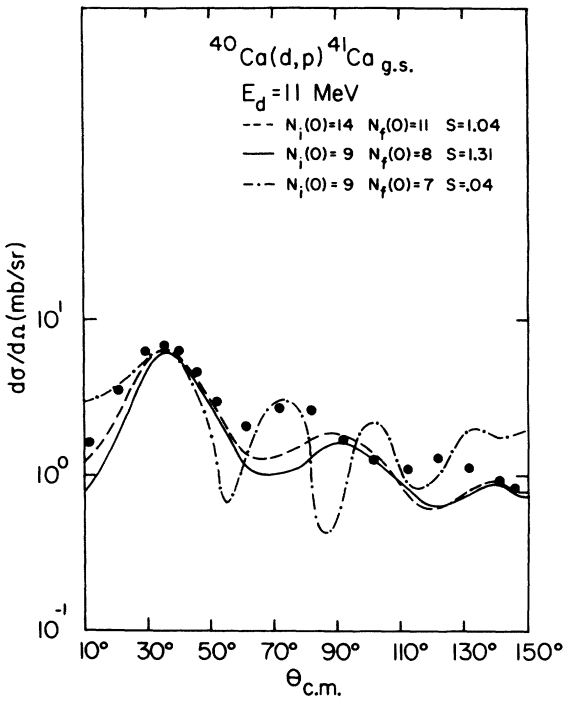

FIG. 3. The effect of using different numbers of plane-wave states. Those shown are for $L_P = L_D = 0$. The other partial waves were represented by fewer plane-wave states as discussed in the text.

ing the number of plane-wave states. Note particularly, the difference in normalization given by the spectroscopic factor S.

It is felt that the method presented in this paper provides a more efficient way of calculating finite-range matrix elements. As may be seen in Table I, integrals are replaced, for some partial waves, by two- and three-term sums. The method would cease to be effective only when the number of terms in the plane-wave expansions approach the number of points needed to perform an integral. This would be the case only at very high energies or with the use of extremely large strengths for the distorting potentials.

It is planned to use the methods of this paper in a multinucleon transfer formalism and also in a coupled-channels Born-approximation approach.

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