Operator Averages in a Shell-Model Basis*

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Using the theory of groups the many-particle average of the product of one-body operators is determined in terms of the one-body average of the products of these operators. With this all the energy moments of the noninteracting system are derived in terms of the one-particle moments. For the interacting systems the third and fourth moments are derived.

I. INTRODUCTION

Recently there has been success in calculating the density of states in nuclei by assuming that the density per unity energy of the finite set of shell-model states for a given number of nucleons is Gaussian.[1] The mean and width of the Gaussian for m nucleons can be determined directly from the mean and width of the one- and two-particle system; that is the mean and width for m particles is a simple known function of m.

However, for extracting more detailed information such as occupation probabilities, spin cutoff factors,[2] and low-lying energy spectra more moments of the density are needed, as well as averages of product of different operators. In this paper we develop a general method for obtaining the m-particle averages of products of one-particle operators as a function of m- and the one-particle averages of the products of these operators. In particular, we apply the method to obtain *all* the moments of a noninteracting system in terms of the one-particle moments and also to derive the third and fourth central moments of an interacting system. The third and fourth moments have also been derived recently using diagrammatic methods.[3]

Our attack makes use of the rotation group in $2N+1$ dimension, $R(2N+1)$, where N is the number of single-particle orbits in which the nucleons are moving. This group is the set of all transformations on the quantum states of fermions restricted to N orbits. Our final result leaves no trace of its ancestry in $R(2N+1)$ so more than likely it can be derived by some other, probably combinatorial, method. However, the group $R(2N+1)$ has had only a trifle of attention[4] in nuclear spectroscopy. In the course of our work we have derived some important properties and geometric quantities of the group which are discussed in detail in the two appendixes.

In the next section the general properties of $R(2N+1)$ are discussed. In Sec. III the average of a general operator is derived using group theory. In Sec. IV this result is used to find the average of a product of different traceless one-body operators. In Secs. V and VI use is made of the average of the product of operators to determine the energy moments of a noninteracting and interacting system, respectively.

II. GROUP STRUCTURE

All the possible states of a system of fermions confined to a quantum system with N single-particle states (N even) form a basis for the spinor representation of the rotation group in $2N+1$ dimensions, $R(2N+1)$. In order to understand this fact it is convenient to introduce fermion creation operators for each single-particle state, A_i, $i=1,\ldots,N$ and destruction operators, $B_i=A_i^\dagger$, which obey anticommutation rules:

$$[A_i, B_j]_+ = \delta_{i,j}, \tag{1a}$$

$$[A_i, A_j]_+ = [B_i, B_j]_+ = 0. \tag{1b}$$

Consider the set of operators

$$\{2^{-1/2}A_i, 2^{-1/2}B_i, F_i, E_{ij}, C_{ij}, D_{ij}, i, j=1,\ldots,N\}, \tag{2a}$$

where

$$F_i \equiv \tfrac{1}{2}[A_i, B_i]_-, \tag{2b}$$

$$E_{ij} \equiv A_i B_j, \quad i \neq j, \tag{2c}$$

$$C_{ij} \equiv A_i A_j, \quad i<j, \tag{2d}$$

$$D_{ij} \equiv B_j B_i, \quad i<j. \tag{2e}$$

Using the anticommutation rules (1) it is easy to see that the commutation relations of the operators (2) with each other give back a member of the set. Hence this set of operators form a Lie algebra. Since no operator in the set commutes with all the other operators, the operators generate a semisimple Lie group. Since there are $N(2N+1)$ operators in the set, the Lie group has order $N(2N+1)$. And since the subset of N operators

$$\{F_i=1,\ldots,N\} \tag{3}$$

form the maximal set of commuting operators,

the Lie group has rank N. Therefore the set of operators (2) must generate the Lie group of rotations in $2N+1$ dimension, $R(2N+1)$.[5]

The operators F_i, $i=1,\ldots,N$ commute with each other and therefore can be diagonalized in a particular irreducible representation (IR) of $R(2N+1)$. If the IR is M dimensional then for each operator F_i, there will be M eigenvalues f_i^α, $\alpha=1,\ldots,M$. The collection of the N eigenvalues for a particular α form an N-dimensional vector called the weight

$$f^\alpha=(f_1^\alpha,f_2^\alpha,\ldots,f_N^\alpha),\quad \alpha=1,\ldots,M. \tag{4}$$

Two weights are equal if in the difference all components are zero; that is

$$f^{\alpha_1}-f^{\alpha_2}=(f_1^{\alpha_1}-f_1^{\alpha_2},\ldots,f_N^{\alpha_1}-f_N^{\alpha_2})=(g_1,\ldots,g_N) \tag{5}$$

and all the g_i, $i=1,\ldots,N$ are zero. If the first nonzero component is positive then f^{α_1} is said to be larger than f^{α_2}; if the first nonzero component is negative, then f^{α_1} is said to be smaller than f^{α_2}. The largest weight is unique and labels the IR. If we call this maximum weight $\bar{f}$, then for $R(2N+1)$ the components $\bar{f}_i$, $i=1,\ldots,N$ can be only a positive integer, positive odd half integer, or zero subject to the conditions that[6]

$$\bar{f}_1\geq\bar{f}_2\geq\cdots\geq\bar{f}_N\geq 0. \tag{6}$$

The set of all possible states of the quantum system is given by

$$|0\rangle;\ A_i|0\rangle, i=1,\ldots,N;\ A_iA_j|0\rangle, 1\leq i<j\leq N;\ \ldots;\ A_1A_2\cdots A_N|0\rangle, \tag{7}$$

where $|0\rangle$ is the zero particle state (the vacuum). The total number of states is 2^N. It is clear that these states are eigenstates of the operators F_i given in (2b), and that the only values that the eigenvalues f_i can take are $\pm\frac{1}{2}$. If an orbit i is filled than $f_i=\frac{1}{2}$; if it is not, then $f_i=-\frac{1}{2}$. Hence the maximum possible weight is that with $f_i=\frac{1}{2}$ for all i. Also it is the only weight which satisfies the conditions (6). Hence the set of states in (7) form a basis for the spinor IR of $R(2N+1)$ which has maximum weight given by:

$$\bar{f}=(\tfrac{1}{2},\tfrac{1}{2},\ldots,\tfrac{1}{2})\equiv\Omega. \tag{8}$$

We refer to the spinor maximum weight by the symbol Ω for convenience.

We can go further and ask how does a general operator transform under $R(2N+1)$. The most general operator will be a linear combination of the operators

$$A_{i_1}\cdots A_{i_p}B_{i_1}\cdots B_{i_s} \tag{9}$$

for all values of p and s such that $0\leq p\leq N$, $0\leq s\leq N$. Commuting these operators with the F_i we see that the eigenvalues f_i can take on only the values $\pm1,0$. Hence the maximum weight is that for which all f_i equal to 1. However in general the operators are reducible; more than one IR of $R(2N+1)$ is contained in this set of operators. But from this weight content and the condition (6) the possible IR's are the scalar IR which has $\bar{f}_i=0$ for all i and the antisymmetric IR's of rank t which have $\bar{f}_i=1$ for $1\leq i\leq t$ and $\bar{f}_i=0$ for $t<i\leq N$. We denote these IR's as Λ_l which is defined as

$$\Lambda_l=>\begin{cases}\bar{f}_i=1 & 1\leq i\leq 2l \quad (0\leq l\leq\frac{1}{2}N) \\ \bar{f}_i=0 & 2l<i\leq N \quad (\frac{1}{2}N<l\leq N)\end{cases}\qquad \begin{cases}1\leq i\leq 2(N-l)+1 \\ 2(N-l)+1<i\leq N.\end{cases} \tag{10}$$

The dimension of these IR's is

$$D_l=\binom{2N+1}{2l}, \tag{11}$$

where $\binom{a}{b}$ is the binomial coefficient:

$$\binom{a}{b}=\frac{a!}{(a-b)!\,b!}. \tag{12}$$

A subgroup of $R(2N+1)$ which is of great interest to us is the group of unitary transformations in N dimensions, $U(N)$, which is generated by the subset of operators,

$$\{F_i,E_{ij};i,j=1,\ldots,N\}. \tag{13}$$

The IR's of $U(N)$ are labeled by N integers h_i, $i=1,\ldots,N$, with

$$h_1\geq h_2\geq\cdots\geq h_N\geq 0. \tag{14}$$

The h_i are the number of fermions which are symmetrically grouped together. However, since all the creation operators must be antisymmetric with each other and all the destruction operators antisymmetric with each other, the IR's relevant to this paper are those with $0\leq h_i\leq 2$. Hence we need only two parameters to characterize an IR of $U(N)$. We designate an IR of $U(N)$ by $\lambda_{\nu,\mu}$, where

$$\lambda_{\nu,\mu}=>\begin{cases}h_i=2 & \text{for } 1\leq i\leq\nu+\mu \\ h_i=1 & \text{for } \nu+\mu<i\leq N-2\nu \\ h_i=0 & \text{for } N-2\nu<i\leq N.\end{cases} \tag{15a}$$

The parameters ν,μ are restricted by the conditions[1]

$$\begin{aligned}&\nu+\mu=\text{integer};\\ &2\nu \text{ non-negative integer},\ 0\leq|\mu|\leq\nu.\end{aligned} \tag{15b}$$

The dimension of these IR's is

$$d(\nu,\mu)=\frac{N-2\nu+1}{N+1}\binom{N+1}{\nu+\mu}\binom{N+1}{\nu-\mu}. \tag{16}$$

The set of states in (7) are reducible with respect to the U(N) subgroup. In fact the states with a definite number of particles m transform among themselves under the unitary transformation and form a basis for the IR's of U(N) which have $\nu=-\mu$ and $m=N-2\nu$ and dimension

$$d\left(\frac{N-m}{2},\frac{m-N}{2}\right)=\binom{N}{m}. \tag{17}$$

Of course the total number of states for all m must be 2^N the dimension of the spinor IR of R($2N+1$). Indeed we have the identity

$$\sum_{m=0}^{N}\binom{N}{m}=2^N. \tag{18}$$

Because we shall use these IR's often we designate these IR's of U(N) which are contained in the spinor IR of R($2N+1$) by

$$\omega_m\equiv\lambda_{(N-m)/2,\,(m-N)/2}. \tag{19}$$

The operators which have been reduced with respect to R($2N+1$) and transform like a definite IR Λ_l of R($2N+1$) will be reducible with respect to U(N) also. The allowed IR's of U(N) will be those which have

$$\begin{aligned}&0\leq\nu\leq(l,N-l)_<,\\&0\leq|\mu|\leq\nu,\end{aligned} \tag{20}$$

where $(d,e)_<$ means the lesser of the two integers d and e. These IR's of U(N) appear once and only once for a given IR of R($2N+1$).

III. OPERATOR AVERAGES

The fact that all the states of the quantum system belong to the same IR of R($2N+1$) will be helpful in determining how the averages of operators depend on the number of particles. By average we mean

$$\langle O\rangle^m=\binom{N}{m}^{-1}\sum_{\alpha}\langle\Omega\omega_m\alpha|O|\Omega\omega_m\alpha\rangle, \tag{21}$$

where Ω is the IR of R($2N+1$), ω_m the IR of U(N) and α labels the $\binom{N}{m}$ states in the IR ω_m.

The operator O can be reduced with respect to R($2N+1$):

$$O=\sum_{l=0}^{N}O^l, \tag{22}$$

where the operator O^l transforms like the IR of R($2N+1$) called Λ_l and defined in (10). The sum over l will in practice have some known upper limit which is less than N.

Since the average in (21) sums over all the states of U(N) it depends only on the properties of the IR of R($2N+1$). In the Appendix I it is shown that the average is

$$\langle O\rangle^m=\sum_l F(-m,-l;-N;2)\langle O^l\rangle^0, \tag{23}$$

where F is the hypergeometric function given by

$$F(-m,-l;-N;2)=\sum_p\binom{N}{p}^{-1}\binom{l}{p}\binom{m}{p}(-2)^p \tag{24a}$$

$$=(-1)^lF(m-N,-l;-N;2) \tag{24b}$$

$$=(-1)^mF(-m,l-N;-N;2) \tag{24c}$$

$$=F(-l,-m;-N;2). \tag{24d}$$

From (24b) we see that the operators with l even are symmetric under particles going to holes and antisymmetric for l odd.

The average can be determined from (23) if the vacuum expectation value of the reduced operators are known. We can, in fact, project out the operator O^l from O. If O has an even number of fermions we label it O_+; if an odd number we label it O_-. We then define an operator C such that

$$CO_\pm=\tfrac{1}{4}\sum_s\{[A_s,[B_s,O_\pm]_\mp]_\mp+[B_s,[A_s,O_\pm]_\mp]_\mp\} \tag{25}$$

that is the double commutator is used for an even number of fermions and the double anticommutator for an odd number of fermions.

The operators $O^l_\pm$ are eigenoperators of C

$$CO^l_+=lO^l_+, \tag{26a}$$

$$CO^l_-=(l-\tfrac{1}{2})O^l_-; \tag{26b}$$

this fact is proved in Appendix II.

Using (26) we can construct projection operators

$$P^l_+=\sum_r\binom{C}{r}\binom{r}{l}(-1)^{r+l}, \tag{27a}$$

$$P^l_-=\sum_r\binom{C}{r}\binom{r}{l-\frac{1}{2}}(-1)^{r+l} \tag{27b}$$

such that

$$O^l_\pm=P^l_\pm O_\pm. \tag{27c}$$

Since in the vacuum expectation value only the operator with an even number of fermions will contribute we have

$$\langle O\rangle^m=\sum_l F(-m,-l;-N;2)\langle P^l_+O\rangle^0. \tag{28}$$

From (24a) and (26a) the sum over l can be per-

formed:

$$\langle O\rangle^m=\sum_r\binom{N}{r}^{-1}\binom{m}{r}(-2)^r\left\langle\binom{C}{r}O\right\rangle^0. \tag{29}$$

Hence the average can be expressed in terms of the vacuum expectation of certain contractions of the original operator. In the next section we shall use this result to derive the average for the product of one-body operators.

IV. AVERAGE OF A PRODUCT OF ONE-BODY OPERATORS

In this section we shall derive the average of the product of p one-body operators for a fixed number of particles m in terms of the averages for one particle of different arrangements of the products. With this result the average for products of two-body and in fact for arbitrary operators can be deduced.

Every one-body operator can be reduced with respect to U(N) in the following way[1]:

$$F=\sum_{\alpha\beta}\langle\alpha|F|\beta\rangle A_\alpha B_\beta=\mathfrak{F}+n\langle F\rangle^1, \tag{30a}$$

where n is the number operator

$$n=\sum_\alpha A_\alpha B_\alpha \tag{30b}$$

and $\mathfrak{F}$ is

$$\mathfrak{F}=F-n\langle F\rangle^1 \tag{30c}$$

and $\langle F\rangle^1$ is just the average in the one particle state

$$\langle F\rangle^1=N^{-1}\sum_\alpha\langle\Omega\omega_1\alpha|F|\Omega\omega_1\alpha\rangle. \tag{30d}$$

The operator $\mathfrak{F}$ transforms like the Λ_1 IR of R($2N+1$) and the $\lambda_{1,0}$ IR of U(N). [See (11) and (15a).] $\mathfrak{F}$ is a one-body operator but it is the traceless part of F. That is

$$\langle\mathfrak{F}\rangle^m=0 \tag{31a}$$

and

$$\langle F\rangle^m=m\langle F\rangle^1. \tag{31b}$$

We need only consider the product of p traceless one-body operators. Using (29) we have

$$\langle\mathfrak{F}_1\mathfrak{F}_2\cdots\mathfrak{F}_p\rangle^m=\sum_{r=0}^{m}\binom{N}{r}^{-1}\binom{m}{r}(-2)^r\times\left\langle\binom{C}{r}\mathfrak{F}_1\mathfrak{F}_2\cdots\mathfrak{F}_p\right\rangle^0. \tag{32}$$

Now the product of one-body operators is an operator with an even number of fermions so the double commutator in (25) is used. In the operation of C on the product of these operators there will be the operation of C on each operator and also contractions between operators with a factor of $-\frac{1}{2}$. Take as an example two operators, using the summation convention for repeated indices,

$$\begin{aligned}C\mathfrak{F}_1\mathfrak{F}_2&=(C\mathfrak{F}_1)\mathfrak{F}_2+\mathfrak{F}_1(C\mathfrak{F}_2)\\&\quad-\tfrac{1}{2}\{[B_s,\mathfrak{F}_1][\mathfrak{F}_2,A_s]+[\mathfrak{F}_1,A_s][B_s,\mathfrak{F}_2]\}\\&=2\mathfrak{F}_1\mathfrak{F}_2-\tfrac{1}{2}({}_s\mathfrak{F}_1\mathfrak{F}_2^s+\mathfrak{F}_1^s{}_s\mathfrak{F}_2),\end{aligned} \tag{33}$$

where we have used the result of (26a) and have defined

$${}_s\mathfrak{F}\equiv[B_s,\mathfrak{F}], \tag{34}$$

$$\mathfrak{F}^s\equiv[\mathfrak{F},A_s]. \tag{35}$$

Since the $\mathfrak{F}$'s transform like Λ_1 of R($2N+1$), the $\mathfrak{F}^s$ and ${}_s\mathfrak{F}$ do also. This follows because A_s and B_s are generators of R($2N+1$) and hence cannot change the IR of operators. However, they do change the number of fermions from even to odd, so we have by (26b)

$$C\mathfrak{F}^s=\tfrac{1}{2}\mathfrak{F}^s, \tag{36a}$$

$$C\,{}_s\mathfrak{F}=\tfrac{1}{2}\,{}_s\mathfrak{F}. \tag{36b}$$

Applying C again we get

$$\begin{aligned}C^2\mathfrak{F}_1\mathfrak{F}_2&=2[(C\mathfrak{F}_1)\mathfrak{F}_2+\mathfrak{F}_1(C\mathfrak{F}_2)-\tfrac{1}{2}({}_s\mathfrak{F}_1\mathfrak{F}_2^s+\mathfrak{F}_1^s{}_s\mathfrak{F}_2)]-\tfrac{1}{2}[(C\,{}_s\mathfrak{F}_1)\mathfrak{F}_2^s+{}_s\mathfrak{F}_1C\mathfrak{F}_2^s+C\mathfrak{F}_1^s{}_s\mathfrak{F}_2+\mathfrak{F}_1^sC({}_s\mathfrak{F}_2)]\\&\quad+(-\tfrac{1}{2})^2[{}_{s'}{}_s\mathfrak{F}_1\mathfrak{F}_2^{s's}+{}_s\mathfrak{F}_1^{s'}{}_{s'}\mathfrak{F}_2^s+{}_{s'}\mathfrak{F}_1^s{}_s\mathfrak{F}_2^{s'}+\mathfrak{F}_1^{s's}{}_{s'}{}_s\mathfrak{F}_2],\end{aligned} \tag{37a}$$

where

$${}_{s'}{}_s\mathfrak{F}=\{B_{s'},{}_s\mathfrak{F}\}=0 \tag{37b}$$

$$\mathfrak{F}^{s's}=\{\mathfrak{F}^s,A_{s'}\}=0 \tag{37c}$$

$${}_{s'}\mathfrak{F}^s=\{B_{s'},\mathfrak{F}^s\}=\langle s'|\mathfrak{F}|s\rangle. \tag{37d}$$

Again we see that C operates on each term individually and also contracts between terms introducing a factor $(-\frac{1}{2})$. Also we note that there are two kinds of contractions. The contraction

$$\mathfrak{F}_1^s{}_s\mathfrak{F}_2 \tag{38}$$

we call normal, since it is the contraction which would occur if the operators were normal ordered (in the sense of Wick's theorem). The other

$${}_s\mathfrak{F}_1\mathfrak{F}_2^s \tag{39}$$

we call backward contractions.

Using (26a) and (36), (37a) becomes

$$C^2\mathcal{F}_1\mathcal{F}_2 = 2[2\mathcal{F}_1\mathcal{F}_2 - \tfrac{1}{2}({}_s\mathcal{F}_1\mathcal{F}_2^s + \mathcal{F}_1{}_s\mathcal{F}_2)] - \tfrac{1}{2}[{}_s\mathcal{F}_1\mathcal{F}_2^s + \mathcal{F}_1^s{}_s\mathcal{F}_2] + (\tfrac{1}{2})^2[{}_{s'}{}_s\mathcal{F}_1\mathcal{F}_2^{s's} + {}_s\mathcal{F}_1^{s'}{}_{s'}\mathcal{F}_2^s + {}_{s'}\mathcal{F}_1^s{}_s\mathcal{F}_2^{s'} + \mathcal{F}_1^{s's}{}_{s'}{}_s\mathcal{F}_2]. \quad (40)$$

We can now generalize the result. We know that each time C operates on the product of p operators with k contractions it has one term which gives those operators back multiplied by $(p-k)$ and another term which has $k+1$ contractions and a factor of $(-\frac{1}{2})$. If we denote by d_k^p the sum of terms with p operators and k contractions we have the recursion relation

$$Cd_k^p = (p-k)d_k^p - \frac{k+1}{2}d_{k+1}^p. \quad (41)$$

If we start from $k=0$ and apply the operators $\binom{C}{r}$ we will then get a linear combination d_i^p, with i running from $i=0$ to $i=r$. From the relation (41) it follows that this linear combination is

$$\binom{C}{r}d_0^p = \sum_{i=0}^{s}(-\tfrac{1}{2})^i\binom{p-i}{r-i}d_i^p. \quad (42)$$

If we take the vacuum expectation of the term d_i^p there will be more normal contractions, however no additional backward contractions. Let D_t^p be the sum of terms with all p operators contracted but with t backward contractions. The vacuum expectation value of d_i^p will give rise to terms with backward contractions from $t=0$ to $t=i$. The total number of contractions must be p so the remaining $p-t$ contractions must be normal. Operators with t contractions will appear $\binom{p-t}{p-i}$ times. That is,

$$\langle d_i^p\rangle^0 = \sum_{t=0}^{i}\binom{p-t}{p-i}D_t^p. \quad (43)$$

Inserting this into (42) and performing the sum over i

$$\left\langle\binom{C}{r}d_0^p\right\rangle^0 = \sum_{i,t}(-\tfrac{1}{2})^i\binom{p-i}{r-i}\binom{p-t}{p-i}D_t^p = \sum_t(-\tfrac{1}{2})^r\binom{p-t}{r-t}D_t^p. \quad (44)$$

Inserting this into (32) and doing the sum over r explicitly we have

$$\langle\mathcal{F}_1\mathcal{F}_2\cdots\mathcal{F}_p\rangle^m = \binom{N}{m}^{-1}\sum_t\binom{N-p}{m-t}D_t^p. \quad (45)$$

The D_t^p contains terms in which the p traceless one-body operators are contracted with t backward contractions. A backward contraction can have one of two effects. One is that it can separate the operators into partitions in which a subset of say k of the p operators are contracted with each other, excluding the other $p-k$ operators. The other is that it can rearrange the order of the operators.

Say there are τ partitions. There must be τ backward contractions to produce these partitions. Within each partition the operators can be rearranged. If the operators appear in natural order, that is labels increasing in value from left to right [as on the left-hand side of (45)], there are no rearrangements. If there are two groups of increasing labels there is one rearrangement, and so on. For a partition with i operators there can be s rearrangements where $0 \leq s \leq i-2$. With each rearrangement a minus sign is picked up. This follows from the fact that for a rearrangement two backward contractions are needed. After the first contraction, the next contraction must pass a single fermion rather than a one-body operator; hence a minus sign.

The total number of backward contractions t is then the sum of those used to make partitions and rearrangements,

$$t = \tau + \sum_{j=1}^{\tau}s_j \quad (47)$$

and (45) becomes

$$\langle\mathcal{F}_1\cdots\mathcal{F}_p\rangle^m = \binom{N}{m}^{-1}\sum_{t=1}^{p-1}\binom{N-p}{m-t}\sum N^\tau \times\prod_{j=1}^{\tau}(-1)^{s_j}\left\langle\prod_{q=1}^{i_j}\mathcal{F}_{k_{jq}}\right\rangle^1. \quad (48a)$$

The sum is over all partitions and rearrangements under conditions

$$t = \tau + \sum_{j=1}^{\tau}s_j, \quad p = \sum_{j=1}^{\tau}i_j. \quad (48b)$$

For each rearrangement for a given partition there is a rearrangement which corresponds to the matrix elements of each operator replaced by the transpose of the matrix elements. Thus for a term with $t=t_1$ there is a term with $t=p-t_1$ in which all the operators are replaced by their transpose. From the symmetry of the binomial factor in (48) and the fact that an odd number of rearrangements are negative, the following relationship holds between the m-particle average of p operators and the average of their transpose:

$$\langle\mathcal{F}_1\mathcal{F}_2\cdots\mathcal{F}_p\rangle^m = (-1)^p\langle\tilde{\mathcal{F}}_1\tilde{\mathcal{F}}_2\cdots\tilde{\mathcal{F}}_p\rangle^{N-m}. \quad (48c)$$

Hence if the operators are all symmetric the m-particle average and the m-hole average are equal

for an even number of operators and the negative of each other for an odd number of operators.

V. CENTRAL MOMENTS OF A NONINTERACTING SYSTEM

If particles are moving in a potential well but are not interacting the Hamiltonian is just a one-body operator,

$$H_0=\sum_{i=1}^{A}\frac{p_i^2}{2m}+U_i\,, \tag{49}$$

where U_i is the central potential which the ith particle feels and A is the total number of particles. The average energy of the system depends linearly on m:

$$\langle H_0\rangle^m=m\langle H_0\rangle^1\,. \tag{50a}$$

The higher central moments are defined as

$$\mu_p^m=\langle(H_0-\langle H_0\rangle^m)^p\rangle^m=\langle(H_0-n\langle H_0\rangle)^p\rangle^m=\langle\mathcal{H}_0^p\rangle^m\,, \tag{50b}$$

where $\mathcal{H}_0$ is the traceless part of the Hamiltonian. We can then use the general result (48) for evaluating these central moments as a function of m. In this case the p operators will all be equal

$$\mathcal{F}_1=\mathcal{F}_2=\cdots=\mathcal{F}_p=\mathcal{H}_0\,. \tag{51}$$

Partitions which differed only in the labels of the operators partitioned all become equal. Let κ_1 be the number of partitions each with one operator, κ_2 the number with two operators, and so on up to κ_p, so that

$$\sum_{i=1}^{p} i\kappa_i=p\,. \tag{52}$$

The number of terms corresponding to this partition is

$$\mathfrak{N}_p=p!\prod_{i=1}^{p}\frac{1}{(i!)^{\kappa_i}\kappa_i!}\,. \tag{53}$$

Another way of saying it is that given p distinct operators there are $\mathfrak{N}_p$ ways to make the partition $(\kappa_1,\ldots,\kappa_p)$.

Also rearrangements would become equivalent. The number of ways to make s rearrangements of i operators is given by Euler numbers[7]:

$$A_{i,s}=\sum_{k=0}^{s}(k+1)^{i-1}\binom{i}{s-k}(-1)^{k+s} \tag{54}$$

which have the symmetry

$$A_{i,s}=A_{i,i-s-2}\,. \tag{55a}$$

This symmetry follows from the fact that the one-body averages are invariant under cyclic permutations of the operators.

Using this symmetry the central moments for m particles are related to the central moments of one particle in the following way:

$$\mu_p^m=p!\binom{N}{m}^{-1}\sum_{t=0}^{[p/2]}\left[\binom{N-p}{m-t}+(1-\delta_{t,p/2})(-1)^p\binom{N-p}{N-m-t}\right]\sum N^{\Sigma_i\kappa_i}\prod_{i=1}^{p}\frac{1}{\kappa_i!}\prod_{j=1}^{\kappa_i}(-1)^{s_{ij}}A_{i,s_{ij}}\left(\frac{\mu_i^1}{i!}\right)^{\kappa_i}, \tag{55b}$$

where

$$[\tfrac{1}{2}p]=\begin{cases}\frac{1}{2}p\,, & p \text{ even}\\ (p-1)/2\,, & p \text{ odd}.\end{cases} \tag{55c}$$

The second summation is over rearrangements and all distinct partitions subject to the condition

$$\begin{aligned}p&=\sum_{i}^{p}\kappa_i i\,,\\ t&=\sum_{i}^{p}\kappa_i+\sum_{i=1}^{p}\sum_{j=1}^{\kappa_i}s_{ij}\,.\end{aligned} \tag{55d}$$

It is interesting to examine how the moments vary for large systems, that is N large. From (55) we see that for a given t the terms with s_{ij} all zero will dominate because of the factor $N^{\Sigma_i\kappa_i}$. For a large number of particles, but small compared to N, the t with highest powers of m will contribute the most. Hence for p even $t=\frac{1}{2}p$ will dominate which means that the partition $\kappa_i=0$, $i\neq 2$, $\kappa_2=\frac{1}{2}p$ will be the largest term. For p odd $t=(p-1)/2$ will dominate which means the partition $\kappa_i=0$, $i\neq 2,3$, $\kappa_2=(p-3)/2$, $\kappa_3=1$.

Asymptotically the moments become

$$\mu_p^m\sim m^{p/2}(p-1)!!\,(\mu_2^1)^{p/2}\,, \quad p \text{ even}; \tag{56a}$$

$$\mu_p^m\sim m^{(p-1)/2}[(p-1)/6]\,p!!\,(\mu_2^1)^{(p-3)/2}\mu_3^1\,, \quad p \text{ odd}. \tag{56b}$$

The width is given by the $p=2$ central moment

$$\mu_2^m=\sigma_m^2\,. \tag{57}$$

In terms of the width these central moments become

$$\mu_p^m\sim(p-1)!!\,\sigma_m^p\,, \quad p \text{ even}; \tag{58a}$$

$$\mu_p^m\sim\frac{1}{\sqrt{m}}\frac{(p-1)}{6}\,p!!\,\sigma_m^p\,\mu_3^1(\mu_2^1)^{-3/2}\,, \quad p \text{ odd}. \tag{58b}$$

Hence as m becomes very large the odd moments

go to zero and the even moments go to moments derived from a Gaussian distribution of the density of levels as a function of energy. Thus in the limit of large systems the density of levels approach a normal distribution. This asymptotic result for noninteracting particles has also been derived by French.[8]

VI. CENTRAL MOMENTS OF AN INTERACTING SYSTEM

The Hamiltonian for nucleons in a central field and interacting pairwise is

$$H = H_0 + V , \tag{59a}$$

where V is the residual two-nucleon interaction:

$$V = \sum_{i<j} V_{ij} . \tag{59b}$$

Again we separate the trace and the traceless part of the operator. However, we further separate the traceless part of the interaction into a part which transforms like the IR $\lambda_{1,0}$ and $\lambda_{2,0}$ of U(N)[1]:

$$H = n\langle H_0\rangle^1 + \frac{n(n-1)}{2}\langle V\rangle^2 + \mathcal{H}_0 + (n-1)\overline{\mathfrak{V}} + \mathfrak{V} , \tag{60a}$$

where

$$\mathcal{H}_0 = H_0 - n\langle H_0\rangle^1 , \tag{60b}$$

$$\overline{\mathfrak{V}} = \frac{1}{N-2}\left[\sum_{\beta}\langle\alpha\beta|V|\gamma\beta\rangle_a A_\alpha B_\gamma - \frac{n}{N\alpha}\sum_{\beta}\langle\alpha\beta|V|\alpha\beta\rangle_a\right] , \tag{60c}$$

$$\mathfrak{V} = V - \frac{(n-1)}{(N-2)}\sum_{\beta}\langle\alpha\beta|V|\gamma\beta\rangle_a A_\alpha B_\gamma + \frac{n(n-1)}{2(N-2)}\langle V\rangle^2 . \tag{60d}$$

The matrix element of V in the above is the antisymmetrized matrix element and $\alpha,\beta,\gamma,\delta$ label the quantum numbers of the single-particle states. The operator $\overline{\mathfrak{V}}$ is a one-body traceless operator derived by taking one trace of the interaction V and it transforms with respect to U(N) in the same way as $\mathcal{H}_0$.

The mean energy is given by

$$\langle H\rangle^m = m\langle H_0\rangle^1 + \frac{m(m-1)}{2}\langle V\rangle^2 \tag{61a}$$

and the central moments are

$$\begin{aligned}\mu_p^m &\equiv \langle (H - \langle H\rangle^m)^p\rangle^m \\ &= \langle [\mathcal{H}_0 + (n-1)\overline{\mathfrak{V}} + \mathfrak{V}]^p\rangle^m \\ &= \langle [\mathcal{H}_0 + (m-1)\overline{\mathfrak{V}} + \mathfrak{V}]_m^p\rangle^m \\ &= \langle [\mathcal{H}(m) + \mathfrak{V}]^p\rangle^m .\end{aligned} \tag{61b}$$

In the above we have defined

$$\mathcal{H}(m) \equiv \mathcal{H}_0 + (m-1)\mathfrak{V} , \tag{61c}$$

a useful step because $\mathcal{H}$ transforms like $\lambda_{1,0}$ of U(N) and is a function of the number of particles m.

The general result given in (48) can be used to calculate the central moments (61b) because the traceless part of the Hamiltonian can be written as a linear combination of at most a product of two traceless one-body operators. However, since the interaction must be separated this way the central moments for $p>2$ cannot be written in terms of the one-body and two-body central moments alone, but other types of products of matrix elements are needed as well.

Using (48), S. Ayik has derived the central moments for the entire Hamiltonian for $p \leq 4$. For these moments there are certain matrix combinations which appear. For this reason it is convenient to define the following matrices:

$$\langle ab|\mathfrak{w}|cd\rangle \equiv \langle ad|\mathfrak{v}|cb\rangle_a , \tag{62a}$$

$$\langle ab|\mathfrak{X}|cd\rangle \equiv \langle b|\mathcal{H}|a\rangle\langle c|\mathcal{H}|d\rangle , \tag{62b}$$

$$\langle a|\mathfrak{y}|c\rangle \equiv \sum_{\substack{d<e\\ b}}\langle ab|\mathfrak{v}|de\rangle_a\langle de|\mathfrak{v}|cb\rangle_a , \tag{62c}$$

$$\langle ab|\mathfrak{z}|cd\rangle = \langle b|\mathcal{H}|a\rangle\langle c|\mathfrak{y}|d\rangle , \tag{62d}$$

$$\langle ab|\mathfrak{u}|cd\rangle = \sum_{e,f}\langle db|\mathfrak{w}|ef\rangle\langle ef|\mathfrak{w}|ca\rangle , \tag{62e}$$

$$\langle ab|\mathfrak{q}|cd\rangle = \sum_{e,f}\langle ad|\mathfrak{w}|ef\rangle\langle ef|\mathfrak{w}|cb\rangle . \tag{62f}$$

In the moments the traces of the products of these matrices will come in with a sum over all single-particle states. For example

$$\mathrm{tr}\mathfrak{w}^3 = \sum_{\substack{abcd\\ ef}}\langle ab|\mathfrak{w}|cd\rangle\langle cd|\mathfrak{w}|ef\rangle\langle ef|\mathfrak{w}|ab\rangle . \tag{63}$$

The Hamiltonian H conserves the angular momentum. Hence if the single-particle basis is spherical and the states are coupled to a definite angular momentum, the Hamiltonian matrix will be diagonal with respect to these angular momenta. The same is true of the matrices in (62); if ab and cd are coupled each to a definite angular momentum, the matrices will be diagonal with respect to this angular momentum.

The third central moment is given by

$$\mu_3^m = \binom{N}{m}^{-1}\sum_{q=3}^{6}\sum_{t=1}^{[q/2]}\times\left[\binom{N-q}{m-t} + (-1)^q(1-\delta_{t,q/2})\binom{N-q}{m+t-q}\right]M_t^q , \tag{64}$$

where the M_t^q are given in Table I. For display

simplicity we have dropped the explicit m label on $\mathcal{H}(m)$ and have simply written $\mathcal{H}$. The fourth central moment is given by

$$\mu_4^m = \binom{N}{m}^{-1} \sum_{q=4}^{8} \sum_{t=1}^{[q/2]} \times \left[\binom{N-q}{m-t} + (-1)^q (1-\delta_{t,q/2}) \binom{N-q}{m+t-q} \right] K_t^q \tag{65}$$

with K_t^q tabulated in Table II. Nomura has derived these results for nucleons moving in at most two spherical orbits.[9]

If $\bar{\mathcal{U}}$ is not small compared to $\mathcal{U}$, then the central moments tend to the Gaussian limit for a large interacting system using arguments similar to those used in deriving the asymptotic limit for the noninteracting system of particles.

For averaging with a fixed number of particles in spherical orbits $\bar{\mathcal{U}}$ vanishes. In that case the moments will not tend to their Gaussian limit. For example μ_3^m tends to

$$\mu_3^m \sim \frac{\mathrm{tr}\mathcal{W}^3}{N^3} \left\{ \frac{\langle \mathcal{U}^2 \rangle^2}{2} \right\}^{-3/2} \sigma_m^3 . \tag{66}$$

Only if the trace of $\mathcal{W}$ cubed falls off slower than N^3 will the Gaussian limit be reached asymptotically.

VII. SUMMARY

Using the group theory of the rotation group in $2N+1$ dimensions, we have derived the averages for a product of operators. The m particle averages of the product of p traceless one-body operators depend only on the one-body traces, but with the operators partitioned in different one-particle averages and with different ordering of the operators within a one-particle average. The average is given by a sum of all possible partitions and rearrangements weighted by a binomial factor which depends on m and on the total number of partitions and rearrangements. From this all the energy moments for a noninteracting system are derived and it is shown that for large systems the moments are those of a Gaussian system. The third and fourth moments of an interacting system are derived and it is shown that for large systems the moments go to a Gaussian limit only if the one-body trace of the interaction is not small.

These results will be used in studying the nuclear spectroscopy of nuclei with many valence nucleons. The moments for a given configuration (configuration averaging) are also being derived by the method derived in this paper and will be published soon.[10]

Table I. The coefficients M_t^q needed to calculate the many-particle third moment μ_3^m for interacting systems are tabulated.

q	t	M_t^q
3	1	$N\langle \mathcal{H}^3 \rangle^1$
4	1	0
	2	$\frac{3N(N-1)}{2}\langle \mathcal{H}^2 \mathcal{U} \rangle^2$
5	1	0
	2	$\frac{3N(N-1)}{2}\langle \mathcal{H}\mathcal{U}^2 \rangle^2$
6	1	0
	2	$\frac{N(N-1)}{2}\langle \mathcal{U}^3 \rangle^2$
	3	$\mathrm{tr}\mathcal{W}^3$

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Table II. The coefficients K_t^q needed to calculate the many-particle fourth moment μ_4^m for interacting systems are tabulated.

q	t	K_t^q
4	1	$N\langle \mathcal{H}^4 \rangle^1$
	2	$3N^2\langle \mathcal{H}^2 \rangle^1 \langle \mathcal{H}^2 \rangle^1 - 4N\langle \mathcal{H}^4 \rangle^1$
5	1	0
	2	$2N(N-1)\langle \mathcal{H}^3 \mathcal{U} \rangle^2$
6	1	0
	2	$N(N-1)(2\langle \mathcal{H}^2 \mathcal{U}^2 \rangle^2 + \langle \mathcal{H}\mathcal{U}\mathcal{H}\mathcal{U} \rangle^2)$
	3	$-3N^2(N-1)\langle \mathcal{H}^2 \rangle^1 \langle \mathcal{U}^2 \rangle^2 + 12\,\mathrm{tr}(\mathcal{H}^2 \mathcal{Y}) - \mathrm{tr}(\mathcal{X}\mathcal{W}^2)$
		$+ 2N(N-1)\langle \mathcal{H}^2 \mathcal{W}^2 \rangle^2 + 4N(N-1)\langle \mathcal{H}\mathcal{U}\mathcal{H}\mathcal{U} \rangle^2$
7	1	0
	2	$2N(N-1)\langle \mathcal{H}\mathcal{U}^3 \rangle^2$
	3	$4N(N-1)\langle \mathcal{H}\mathcal{U}^3 \rangle^2 + 12\,\mathrm{tr}(\mathcal{W}\mathcal{S}) - 12\,\mathrm{tr}(\mathcal{H}\mathcal{W}^3)$
8	1	0
	2	$\frac{N(N-1)}{2}\langle \mathcal{U}^4 \rangle^2$
	3	$-2\,\mathrm{tr}\mathcal{Y}^2 - \mathrm{tr}(\mathcal{W}^2 \mathcal{U}) + 4\,\mathrm{tr}(\mathcal{U}^2 \mathcal{Q})$
	4	$3\frac{N^2(N-1)^2}{4}\langle \mathcal{U}^2 \rangle^2 \langle \mathcal{U}^2 \rangle^2 + 2N(N-1)\langle \mathcal{U}^4 \rangle^2$
		$-8\,\mathrm{tr}\mathcal{Y}^2 - 4\,\mathrm{tr}(\mathcal{W}^2 \mathcal{U}) + 4\,\mathrm{tr}(\mathcal{U}^2 \mathcal{Q}) + 3\,\mathrm{tr}\mathcal{W}$

APPENDIX I

A generalized Wigner-Eckart theorem applies to a tensor operator which transforms in a definite way under R(2N+1). That is, given a matrix element of a tensor operator it can be factored into a Clebsch-Gordan coefficient (CGC) which reduces the Kronecker product of two IR's of R(2N+1) into an IR of R(2N+1), and a reduced matrix element which does not depend on the quantum numbers which label the basis states of the IR of R(2N+1). Given a tensor operator $T^{\Lambda_I}_{\lambda_\nu,\mu}$ we have

$$\langle \Omega\omega_m\alpha | T^{\Lambda_I}_{\lambda_\nu,\mu\beta} | \Omega\omega'_m\alpha' \rangle = C^{\Omega\Lambda_I\Omega}_{\omega_m'\lambda_\nu,\mu\omega_m} C^{\omega'_m\lambda_\nu,\mu\omega_m}_{\alpha'\beta\alpha} \langle \Omega \| T^{\Lambda_I} \| \Omega \rangle . \tag{A1}$$

In this formula we have also used Racah's theorem,[11] which states that since U(N) is a subgroup which has a given IR $\lambda_{\nu,\mu}$ appearing only once for an IR Λ_I of R(2N+1), the Clebsch-Gordan coefficient can be factored into the CGC for the subgroup U(N), $C^{\omega_m'\lambda_\nu,\mu\omega_m}_{\alpha'\beta\alpha}$ and a unitary scalar coefficient (USC) $C^{\Omega\Lambda_I\Omega}_{\omega_m'\lambda_\nu,\mu\omega_m}$ which does not depend on the quantum numbers which label the basis states of U(N).

If we take the trace of the matrix element in (A1), that is set $\alpha=\alpha'$ and sum over α only the scalar IR of U(N) will survive:

$$\sum_\alpha C^{\omega_m'\lambda_\nu,\mu\omega_m}_{\alpha\beta\alpha} = \binom{N}{m} \delta_{\nu,0}\delta_{\mu,0}\delta_{\beta,0}\delta_{\omega_m\omega_m'} . \tag{A2}$$

Hence

$$\langle T^{\Lambda_I}_{\lambda_\nu,\mu\beta} \rangle^m = \delta_{\nu,0}\delta_{\mu,0}\delta_{\beta,0} C^{\Omega\Lambda_I\Omega}_{\omega_m\lambda_\nu,0\omega_m} \langle \Omega \| T^{\Lambda_I} \| \Omega \rangle = \frac{C^{\Omega\Lambda_I\Omega}_{\omega_m\lambda_0,0\omega_m}}{C^{\Omega\Lambda_I\Omega}_{\omega_0\lambda_0,0\omega_0}} \langle T^{\Lambda_I}_{\lambda_\nu,\mu\beta} \rangle^0 , \tag{A3}$$

where in the last line we have converted to the vacuum expectation value, the m dependence is in the unitary scalar coefficient.

We can determine the USC appearing in (A3) by taking a special case of an operator which transforms like Λ_I but whose m-particle average is easy to evaluate. Such an operator is simply

$$O^{\Lambda_I} = \prod_{i=1}^{I} [A_i, B_i]_- = \sum_{x=0}^{I} (-1)^I (-2)^x \sum_{1<i_1<i_2\cdots<I} \prod_{k=1}^{x} (A_{i_k} B_{i_k}) . \tag{A4}$$

The m-particle average of this operator is

$$\langle O^{\Lambda_I} \rangle^m = \sum_{x=0}^{I} (-1)^I (-2)^x \sum \left\langle \prod_{k=1}^{k} A_{i_k} B_{i_k} \right\rangle^m = \sum_x (-1)^I (-2)^x \binom{N}{m}^{-1} \binom{N-x}{m-x} \binom{I}{x} . \tag{A5}$$

Hence

$$\frac{C^{\Omega\Lambda_I\Omega}_{\omega_m\lambda_0,0\omega_m}}{C^{\Omega\Lambda_I\Omega}_{\omega_0\lambda_0,0\omega_0}} = \binom{N}{m}^{-1} \sum_x (-2)^x \binom{N-x}{m-x} \binom{I}{x} \tag{A6}$$

which can be shown to be equal to the expression given in Eq. (24) in the text.

If we take the normalization to be

$$C^{\Omega\Lambda_I\Omega}_{\omega_0\lambda_0,0\omega_0} = 2^{-N/2} , \tag{A7}$$

the USC can be shown to satisfy the orthonomality conditions:

$$\binom{N}{I} \sum_m \binom{N}{m} C^{\Omega\Lambda_I\Omega}_{\omega_0\lambda_0,0\omega_m} C^{\Omega\Lambda_{I'}\Omega}_{\omega_m\lambda_0,0\omega_m} = \delta_{I,I'} \tag{A8}$$

and the inverse relations:

$$\binom{N}{m} \sum_I \binom{N}{I} C^{\Omega\Lambda_I\Omega}_{\omega_m\lambda_0,0\omega_m} C^{\Omega\Lambda_I\Omega}_{\omega_m'\lambda_0,0\omega_m} = \delta_{m,m'} . \tag{A9}$$

Hence the USC are completely symmetric with respect to the interchange of I and m.

Using similar arguments the USC can also be derived for other IR of U(N). In general

$$C^{\Omega\Lambda_I\Omega}_{\omega_m\lambda_\nu,\mu\omega_{m'}} = \delta_{m';m+2\mu} C^{\Omega\Lambda_I\Omega}_{\omega_m\lambda_\nu,\mu\omega_{m+2\mu}} \tag{A10}$$

and

$$C^{\Omega\Lambda_I\Omega}_{\omega_m\lambda_\nu,\mu\omega_{m+2\mu}} = [2^{-N+2\nu}]^{1/2} F(-m+\nu-\mu, -I+\bar{\nu}; -N+2\nu; 2) , \tag{A11a}$$

where

$$\bar{\nu}=\begin{cases}\nu & \text{for } 2\nu \text{ even}\\ \nu+\frac{1}{2} & \text{for } 2\nu \text{ odd}.\end{cases} \tag{A11b}$$

These USC satisfy the orthogonality relations:

$$\binom{N-2\nu}{l-\bar{\nu}}\sum_m\binom{N-2\nu}{m-\nu+\mu}C^{\Omega\Lambda_l\Omega}_{\omega_m\lambda_{\nu,\mu}\omega_{m+2\mu}}C^{\Omega\Lambda_{l'}\Omega}_{\omega_m\lambda_{\nu,\mu}\omega_{m+2\mu}}=\delta_{l,l'} \tag{A12}$$

and

$$\binom{N-2\nu}{m-\nu+\mu}\sum_l\binom{N-2\nu}{l-\bar{\nu}}C^{\Omega\Lambda_l\Omega}_{\omega_m\lambda_{\nu,\mu}\omega_{m+2\mu}}C^{\Omega\Lambda_l\Omega}_{\omega_{m'}\lambda_{\nu,\mu}\omega_{m'+2\mu}}=\delta_{m,m'}. \tag{A13}$$

APPENDIX II

An operator with an even number ($=2l$) of fermions (both creation and destruction operators, A_i, B_j) or an odd number ($=2l-1$) all in antisymmetric combination belongs to the IR Λ_l of R($2N+1$). An operator which transforms like an IR $\lambda_{\nu,\mu}$ of U(N) must have at least 2ν fermions; if such an operator has exactly 2ν fermions these fermions must necessarily all be in an antisymmetric combination. This follows because fermions in a symmetric combination are just a constant (zero or one) and therefore invariant under U(N) and thus the operator would not transform like $\lambda_{\nu,\mu}$ but rather like $\lambda_{\nu-1,\mu}$ if any two of the fermions were in a symmetric combination. Hence an operator which transforms like $\lambda_{\nu,\mu}$ under U(N) and has 2ν fermion transforms like Λ_ν of R($2N+1$) if 2ν is an even integer and $\Lambda_{\nu+1/2}$ if 2ν is an odd integer.

Let us denote such an operator as

$$O^{\bar{\nu}}(\nu,\mu), \tag{B1}$$

where $\bar{\nu}$ is defined in (A11b). Define the operator

$$O^l(\nu,\mu)=[2^{2\nu-N}]^{1/2}F(-n+\nu+\mu,-l+\bar{\nu};-N+2\nu;2)O^{\bar{\nu}}(\nu,\mu), \tag{B2}$$

where F is the hypergeometric function given in (24), and the argument m the number of particles has been replaced by the number operator n. Since n is a scalar under U(N), the operator $O^l(\nu,\mu)$ transforms like $\lambda_{\nu,\mu}$ under U(N). However, under R($2N+1$) it transforms like Λ_l. To see this we take the matrix element of $O^l(\nu,\mu)$:

$$\langle\Omega\omega_{m'}\alpha'|O^l(\nu,\mu)|\Omega\omega_m\alpha\rangle=\delta_{m',m+2\mu}C^{\Omega\Lambda_l\Omega}_{\omega_m\lambda_{\nu,\mu}\omega_{m+2\mu}}\langle\Omega\omega_m\alpha'|O^{\bar{\nu}}(\nu,\mu)|\Omega\omega_m\alpha\rangle \tag{B3}$$

which follows simply from the fact that the states are eigenfunctions of the operator n and from (A11). Thus comparing (B3) with the generalized Wigner-Eckart theorem (A1) we see that the matrix element of this operator has the dependence on l like a tensor operator of R($2N+1$) of IR Λ_l.

Chang, French, and Thio[1] have shown that

$$\sum_s A_s O^{\bar{\nu}}(\nu,\mu)B_s=(-1)^{2\nu}(n-\nu-\mu)O^{\bar{\nu}}(\nu,\mu), \tag{B4a}$$

$$\sum_s B_s O^{\bar{\nu}}(\nu,\mu)A_s=(-1)^{2\nu}(N-n-\nu+\mu)O^{\bar{\nu}}(\nu,\mu). \tag{B4b}$$

These relations follow from the fact that the operators are completely traceless with respect to U(N). We have then

$$\sum_s A_s O^l(\nu,\mu)B_s=[2^{2\nu-N}]^{1/2}F(-n+\nu+\mu+1,-l+\bar{\nu};-N+2\nu;2)(-1)^{2\nu}(n-\nu-\mu)O^{\bar{\nu}}(\nu,\mu),$$

$$\sum_s B_s O^l(\nu,\mu)A_s=[2^{2\nu-N}]^{1/2}F(-n+\nu+\mu-1,-l+\bar{\nu};-N+2\nu;2)(-1)^{2\nu}(N-n-\nu+\mu)O^{\bar{\nu}}(\nu,\mu). \tag{B5}$$

Using the definition of C given in the text we have

$$CO^l(\nu,\mu)=\tfrac{1}{2}[NO^l(\nu,\mu)-(-1)^{2\nu}\sum_s(A_s O^l B_s+B_s O^l A_s)]. \tag{B6}$$

Using the property of the hypergeometric function,

$$\begin{aligned}&NF(-n+\nu+\mu,-l+\bar{\nu};-N+2\nu;2)-(N-n-\nu+\mu)F(-n+\nu+\mu-1,-l+\bar{\nu};-N+2\nu;2)\\&\quad-(n-\nu-\mu)F(-n+\nu+\mu+1,-l+\bar{\nu};-N+2\nu;2)=2(l-\bar{\nu}+\nu)F(-n+\nu+\mu,-l+\bar{\nu};-N+2\nu;2)\end{aligned} \tag{B7}$$

we will get the result (26) quoted in the text.

From the definition (B2) we see that $O^l(\nu, \mu)$ has the symmetry property

$$O^l(\nu, \mu) = (-1)^{n-\nu-\mu} \begin{cases} O^{N-l}(\nu, \mu), & 2\nu \text{ even} \\ O^{N-l+1}(\nu, \mu), & 2\nu \text{ odd}. \end{cases} \tag{B8}$$

Under the particle-hole transformation

$$\tilde{A}_s = -B_s,$$

$$\tilde{B}_s = -A_s, \tag{B9}$$

the operators go into their hermitian conjugate except for an l-dependent phase:

$$\tilde{O}^l(\nu, \mu) = (-1)^l \{ O^l(\nu, \mu) \}^\dagger. \tag{B10}$$

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