Generalization of the Jost Function and Its Application to Off-Shell Scattering

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An off-shell generalization of the Jost function is developed within the framework of the differential-equation approach to the off-shell T matrix. Irregular solutions of the inhomogeneous Schrödinger-like equation that occurs in this approach are introduced, and their behavior at the origin is used to define an off-shell Jost function. The half-off-shell T matrix is expressed directly in terms of the off-shell Jost function. It is shown how the fully off-shell T matrix for a particular partial wave can be expressed simply in terms of a single integral involving the irregular solution for that partial wave. An integral equation for the irregular solution is developed, and used to derive an integral representation for the off-shell Jost function. Iteration of the integral equation leads to a series of successive approximations to the T matrix. The formalism is applied to several examples, including a boundary-condition model.

I. INTRODUCTION

The concept of the Jost[1] function was originally introduced as a basis for examining the properties of the S matrix that arise in the scattering of s waves from a central potential. The Jost function is determined by the behavior of the so-called irregular solutions of the radial Schrödinger equation at the origin. For large values of the radial coordinate these solutions behave like $e^{\pm ikr}$, where k is the wave number. The solution of the Schrödinger equation which is well behaved at the origin, the so-called regular solution, can be expressed in terms of the irregular solutions. The Jost function, as well as the regular and irregular solutions, have played a central role in the general problem of constructing a potential from a knowledge of the phase shifts.[2,3] The Jost function itself can be expressed in terms of the phase shifts and the bound-state energies.[3] Besides being related to the irregular solutions of the radial Schrödinger equation, it can be shown[4] that the Jost function for a local potential is identical to the Fredholm determinant of the integral form of the radial Schrödinger equation. Thus it is tied in with the conventional theory of integral equations. Detailed treatments of the Jost function, as well as extensive references to the literature, can be found in several of the more recent texts on scattering theory.[5,6] Here we shall follow the conventions of Newton.[5]

In general the Jost function, corresponding to a particular partial wave, has two very important properties: Its phase is the negative of the phase shift for the partial wave, and its zeros in the upper half of the k plane correspond to the energies of the bound states which occur in the partial wave. Thus, the Jost function for a two-particle system is directly related to the observables of the system.

As is well known, the phase shifts for a two-particle system can be used to describe the elastic, or on-shell, scattering amplitudes for the system. In a many-particle system the various pairs of particles do not scatter elastically from each other, and therefore knowledge of the on-shell two-particle scattering amplitudes is not sufficient to determine the properties of the many-particle system. What are needed are the off-shell amplitudes. In many theories, these off-shell amplitudes are expressed as matrix elements of a transition or T operator. This set of matrix elements is usually referred to as the T matrix. The T matrix plays a role in the theories of three-particle systems,[7] nucleon-nucleon bremsstrahlung,[8] nuclear matter,[9] and finite nuclei.[10] The T matrix is also used for the evaluation of cluster coefficients in quantum statistical mechanics.[11] The theory of some solid-state systems can also be formulated in terms of the T matrix.[12]

Here we shall present a theory of the T matrix based on a generalization of the Jost function. This generalization, which we shall refer to as the off-shell Jost function, will be formulated in terms of the van Leeuwen-Reiner[13] approach to the T matrix. In their approach the T matrix is obtained from an inhomogeneous form of the Schrödinger equation, in which the inhomogeneous term is proportional to a free wave. In this equation there appear two momenta, k and q, where k is an on-shell momentum, related to the energy by $E=k^2$, and q is an off-shell momentum. When $q=k$ the equation reduces to the conventional Schrödinger equation. In analogy to the theory of the Jost function, we shall define irregular solutions of the inhomogeneous equation, which for large r behave like $e^{\pm iqr}$. An off-shell generalization of the Jost function arises by considering the behavior of these irregular solutions for small r. We shall show that the half-off-shell T matrix can be expressed directly in terms of the off-shell Jost function. The fully off-shell T matrix will be expressed in terms of a single integral involving an irregular solution. We shall also show that these irregular solutions can be obtained from the solution of an integral equation of the Volterra type. This integral equation will be used to write down on integral representation for the off-shell Jost function. All of the elements of this off-shell theory go over into corresponding elements of the theory of the ordinary Jost function, when one goes on shell; i.e., when $q=k$.

The formal results outlined above, and presented in detail in Sec. II, are applied in Sec. III to several examples; namely, the square-well potential, the exponential potential, the boundary-condition model (BCM) with an external exponential potential, and the Hulthén potential. In each case, the irregular solutions of the off-shell theory and the off-shell Jost function are derived. For all but the square-well potential the fully off-shell T matrix is determined. The results for the exponential and Hulthén potentials are much simpler than those given previously.[14,15] The results for the BCM mentioned above have not been given before.

A brief summary and discussion of the results is given in Sec. IV. The Bessel functions used here follow Messiah's[16] conventions. Throughout we work in units in which $\hbar^2/2m$ is unity.

II. OFF-SHELL JOST FUNCTION

The two-particle transition operator $T(s)$ can be obtained as the solution of the equation

$$T(s)=V+V(s-H_0)^{-1}T(s)\,, \tag{2.1}$$

where V is the two-particle potential, s is a complex energy parameter, and H_0 is the kinetic energy operator. The operator $T(s)$ can be obtained from the solution of a differential equation. Following van Leeuwen and Reiner,[13] we define a wave operator $\Omega(s)$ according to the relation

$$\Omega(s)=1+(s-H_0)^{-1}T(s)\,. \tag{2.2}$$

It then follows from (2.1) and (2.2) that

$$T(s)=V\Omega(s) \tag{2.3}$$

and

$$(s-H_0-V)\Omega(s)=s-H_0\,. \tag{2.4}$$

Writing out (2.4) in a mixed representation we have

$$[s+\nabla^2-V(r)]\langle\vec{r}\,|\,\Omega(s)|qlm\rangle =(s-q^2)(2/\pi)^{1/2}j_l(qr)Y_{lm}(\hat{r})\,, \tag{2.5}$$

where we have introduced the free waves

$$\langle\vec{r}|qlm\rangle=(2/\pi)^{1/2}j_l(qr)Y_{lm}(\hat{r}); \tag{2.6}$$

$j_l(qr)$ is the usual spherical Bessel function and $Y_{lm}(\hat{r})$ is a spherical harmonic. The boundary condition that must be imposed on the solution of (2.5) for large r can be obtained from (2.2). We have

$$\langle\vec{r}\,|\,\Omega(s)\,|\,qlm\rangle=(2/\pi)^{1/2}j_l(qr)Y_{lm}(\hat{r}) +\int\langle\vec{r}|G_0(s)|\,\vec{r}'\rangle d\vec{r}'\langle\vec{r}'|T(s)|qlm\rangle\,, \tag{2.7}$$

in which we insert the well-known representation for the free Green's function,[16]

$$\langle\vec{r}|G_0(s)|\,\vec{r}'\rangle=-k\sum_{l,m}j_l(kr_<)h_l^{(+)}(kr_>)Y_{lm}(\hat{r})Y^*_{lm}(\hat{r}')\,, \tag{2.8}$$

and then let r become large. We find, using (2.6), that

$$\langle\vec{r}\,|\,\Omega(s)|qlm\rangle \underset{r\to\infty}{\sim} (2/\pi)^{1/2}(qr)^{-1}Y_{lm}(\hat{r})[\,\sin(qr-\tfrac{1}{2}l\pi)-\tfrac{1}{2}\pi q\langle klm\,|T(s)|qlm\rangle e^{i(kr-\frac{1}{2}l\pi)}]\,. \tag{2.9}$$

In (2.8), we have set

$$s=k^2+i\epsilon\,,\ \ 0<\epsilon\ll1\,. \tag{2.10}$$

We see that the half-off-shell T matrix element $\langle klm|T(s)|qlm\rangle$ can be obtained from the asymptotic limit of the solution of (2.5). By comparing the on-shell ($q=k$) version of (2.9) with the well-known asymptotic form of the solution of the Schrödinger equation, it is easy to show that the on-shell T matrix elements have the normalization

$$\langle klm|T(s)|\,klm\rangle=-(2/\pi k)e^{i\delta_l(k)}\sin\delta_l(k)\,, \tag{2.11}$$

where δ_l is the phase shift for the lth partial wave.

It is convenient at this point to introduce a set of functions related to the spherical Bessel, Neumann, and Hankel functions by

$$\begin{aligned} &u_l(z)=zj_l(z)\,,\\ &v_l(z)=zn_l(z)\,,\\ &w_l^{(\pm)}(z)=zh_l^{(\pm)}(z)=v_l(z)\pm iu_l(z)\,. \end{aligned} \tag{2.12}$$

Since the potential is central, we can write

$$\langle\vec{r}\,|\,\Omega(s)|qlm\rangle=(2/\pi)^{1/2}(qr)^{-1}\phi_l(k,q,r)Y_{lm}^{(\hat{r})}\,, \tag{2.13}$$

which upon substitution into (2.5) gives

$$\left(k^2+\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}-V(r)\right)\phi_l(k,q,r)=(k^2-q^2)u_l(qr)\,. \tag{2.14}$$

We shall now show that the solution of (2.14) can be related to the solutions of the equation

$$\left(k^2+\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}-V(r)\right)f_l(k,q,r) =(k^2-q^2)e^{i\frac{1}{2}l\pi}w_l^{(+)}(qr)\,. \tag{2.15}$$

The solution of (2.15) that is of interest has the asymptotic normalization

$$f_l(k,q,r)\underset{r\to\infty}{\sim} e^{iqr}\,. \tag{2.16}$$

It follows from (2.15) and (2.16) that when $q=\pm k$, this function goes over into the two irregular solutions of the Schrödinger equation, which enter into the theory of the ordinary Jost function[5]; i.e.,

$$f_l(\pm k,r)=f_l(k,\pm k,r)\,. \tag{2.17}$$

It now follows in a straightforward manner from (2.9) and (2.12)–(2.17) that we can write

$$\phi_l(k,q,r)=-\tfrac{1}{2}\pi q T_l(k,q;s)e^{-i(1/2)l\pi}f_l(k,r)+(1/2i)[e^{-i(1/2)l\pi}f_l(k,q,r)-e^{i(1/2)l\pi}f_l(k,-q,r)]\,, \tag{2.18}$$

where

$$T_l(k,q;s)=\langle klm|T(s)|qlm\rangle\,. \tag{2.19}$$

The half-off-shell T matrix element $T_l(k,q;s)$ is determined by the behavior of the f_l's for small r. Since, in general, for small r, the centrifugal barrier term in (2.15) will dominate over the potential, it is expected that f_l will behave as r^{-l} as r approaches zero. This can be shown more carefully by assuming a power series solution for f_l, and by requiring that the potential either have a simple pole at the origin or be analytic there. We define an off-shell Jost function by

$$f_l(k,q)=\frac{q^l e^{-i(1/2)l\pi}(2l+1)}{(2l+1)!!}\lim_{r\to 0} r^l f_l(k,q,r)\,. \tag{2.20}$$

We have normalized the off-shell Jost function $f_l(k,q)$ so that when $q=k$ it becomes the ordinary Jost function.[5] It now follows from (2.18) and (2.20) that

$$T_l(k,q;s)=\left(\frac{k}{q}\right)^l\frac{f_l(k,q)-f_l(k,-q)}{\pi i q f_l(k)}\,, \tag{2.21}$$

where

$$f_l(k)=f_l(k,k)\,, \tag{2.22}$$

and is the ordinary Jost function. Thus the half-off-shell T matrix elements can be expressed directly in terms of the off-shell Jost function. By using the relations which exist between the fully off-shell T matrix elements and the half-off-shell T matrix elements, one can write the off-shell T matrix in terms of the off-shell Jost function. For example, rewriting (2.21) of Ref. 17 with our normalization and notation, we have

$$T_l(p,q;s)=\frac{1}{q^2-p^2}\Big[(q^2-s)T_l(q,p;p^2+i\epsilon)-(p^2-s)T_l(p,q;q^2+i\epsilon)-(p^2-s)(q^2-s) \times\int_0^\infty x^2dx\,\frac{T_l(p,x;x^2+i\epsilon)T_l^*(q,x;x^2+i\epsilon)}{x^2-s}\left(\frac{1}{x^2-q^2-i\epsilon}-\frac{1}{x^2-p^2-i\epsilon}\right)\Big] \tag{2.23}$$

This relation assumes there are no bound states.

It follows from (2.15) and (2.16) that

$$f_l(k,-q,r)=f_l^*(k,q,r)\qquad (k,q,\text{ and } r \text{ real})\,, \tag{2.24}$$

which in turn implies from (2.20) that

$$f_l(k,-q)=f_l^*(k,q)\qquad (k \text{ and } q \text{ real})\,. \tag{2.25}$$

It is well known[5] that the phase of the Jost function is the negative of the phase shift; i.e.,

$$f_l(k)=|f_l(k)|\,e^{-i\delta_l(k)}\,. \tag{2.26}$$

Combining this with (2.21) and (2.25), we see that for k and q real, the phase of the half-off-shell T matrix element is the phase shift.

Using standard Green's function techniques, it is possible to derive the following integral equation:

$$f_l(k,q,r)=e^{i(1/2)l\pi}w_l^{(+)}(qr) -k^{-1}\int_r^\infty dr'\,[u_l(kr)v_l(kr')-v_l(kr)u_l(kr')] \times V(r')f_l(k,q,r')\,. \tag{2.27}$$

An integral representation for the off-shell Jost function is obtained by combining (2.20) and (2.27) to yield

$$f_l(k,q)=1+k^{-1}(q/k)^l e^{-i(1/2)l\pi}\times\int_0^\infty dr\,u_l(kr)V(r)f_l(k,q,r)\,. \qquad (2.28)$$

It follows from (2.15) that if the energy k^2 is everywhere large compared to the potential, then $f_l(k,q,r)$ should be well approximated by the solution of (2.15) with no potential; i.e., we expect

$$f_l(k,q,r)\underset{k\to\infty}{\sim} e^{i(1/2)l\pi}w_l^{(+)}(qr)\,. \qquad (2.29)$$

Thus at high energies the iterative solution of (2.27) should be of practical value in determining the off-shell function $f_l(k,q,r)$, and from (2.28) the off-shell Jost function. One also expects the right-hand side of (2.29) to be a good approximation to $f_l(k,q,r)$ when the centrifugal barrier term in (2.15) dominates the potential.

Besides the possibility of obtaining the off-shell T matrix from (2.23), it is also possible to use (2.3). Combining (2.3), (2.6), and (2.13), we have

$$T_l(p,q;s)=\langle plm|T(s)|qlm\rangle=\frac{2}{\pi pq}\int_0^\infty dr\,u_l(pr)V(r)\phi_l(k,q,r)\,. \qquad (2.30)$$

Inserting (2.18) in (2.30), we obtain

$$T_l(p,q;s)=T_l(k,q;s)[1-Y_l(p,k,k)]+\frac{(k/q)^l}{\pi iq}[\,Y_l(p,q,k)-Y_l(p,-q,k)]\,, \qquad (2.31)$$

where

$$Y_l(p,q,k)=1+p^{-1}(q/k)^l e^{-i(1/2)l\pi}\times\int_0^\infty dr\,u_l(pr)V(r)f_l(k,q,r)\,. \qquad (2.32)$$

One can easily check that (2.31) reduces to an identity when $p=k$ by comparing (2.32) and (2.28), and by using (2.21).

The results of this section can be generalized in a natural way to the boundary-condition model.[18] In a BCM the logarithmic derivative of the radial wave function is specified at a certain distance $r=c$ from the origin, and no detailed assumptions are made about the potential within the boundary-condition radius c. We shall specify the boundary condition by the equation

$$\Psi_l(c)=b_l\Psi_l'(c)\,. \qquad (2.33)$$

Throughout, primes will denote differentiation with respect to the argument. The case $b_l=0$ corresponds to a hard core within the radius c. It it shown in Ref. 19 that the T matrix for the BCM can be obtained in the form

$$T_l(p,q;s)=t_l(p,q;s)+t_l^{(1)}(p,q;s)\,, \qquad (2.34)$$

where $t_l(p,q;s)$ is the T matrix for the pure BCM (no outside forces), and is given in the half-off-shell case by the simple formula[19]

$$t_l(k,q;s)=\frac{2g_l(q)}{\pi q d_l(k)} \qquad (2.35)$$

Here

$$d_l(k)=w_l^{(+)}(kc)-kb_l w_l^{(+)\prime}(kc)\,, \qquad (2.36)$$

and

$$g_l(q)=u_l(qc)-qb_l u_l'(qc)=\frac{d_l(q)-(-)^l d_l(-q)}{2i}\,. \qquad (2.37)$$

The other part of the T matrix is the contribution from the potential outside the boundary-condition radius and is given by

$$t_l^{(1)}(p,q;s)=\frac{2}{\pi pq}\int_c^\infty dr\left(u_l(pr)-\frac{g_l(p)}{d_l(k)}w_l^{(+)}(kr)\right)\times V(r)\phi_l(k,q,r)\,, \qquad (2.38)$$

where ϕ_l is the solution of (2.14) with the boundary condition (2.33).

The expression (2.18) for ϕ_l is still valid for $r\geqslant c$, and when combined with (2.33) leads to the following result:

$$T_l(k,q;s)=\frac{d_l(q)F_l(k,q)-(-)^l d_l(-q)F_l(k,-q)}{\pi iqd_l(k)F_l(k)}\,, \qquad (2.39)$$

where

$$F_l(k,q)=e^{-i(1/2)l\pi}\frac{f_l(k,q,c)-b_l f_l'(k,q,c)}{d_l(q)}\,, \qquad (2.40)$$

and

$$F_l(k)=F_l(k,k)\,. \qquad (2.41)$$

We have chosen to write the formulas this way so that for large k, F_l has the behavior

$$F_l(k,q)\underset{k\to\infty}{\sim}1\,. \qquad (2.42)$$

This follows from (2.29) and (2.36). Clearly, $F_l(k,q)$ plays the role of an off-shell Jost function for the potential outside $r=c$. If there is no outside potential $F_l(k,q)=1$, and (2.39) reduces to (2.35).

The integral equation (2.27) is still valid for $r\geqslant c$, and can be used to derive an integral representation for F_l. The result is

$$F_l(k,q)=1+k^{-1}[d_l(k)/d_l(q)]e^{-i(1/2)l\pi}e^{i\eta_l(k)}\times\int_c^\infty dr[\cos\eta_l(k)u_l(kr)+\sin\eta_l(k)v_l(kr)]\times V(r)f_l(k,q,r)\,, \tag{2.43}$$

where η_l is the phase shift for the pure BCM, and is given by the on-shell version of (2.35), which is normalized as in (2.11). It is easy to see that the factor in square brackets under the integral sign in (2.43) is the radial Schrödinger wave function for the pure BCM. It is also not hard to show that when the BCM is turned off, i.e., $b_l=0$, $c\to 0$, (2.43) reduces to (2.28).

By inserting (2.18) into (2.38), and using (2.34), (2.35), and (2.37), it can be shown that

$$t_l^{(1)}(p,q;s)=t_l^{(1)}(k,q;s)-T_l(k,q;s)Y_l(p,k,k)+\frac{1}{\pi iqd_l(k)}[d_l(q)Y_l(p,q,k)-(-)^l d_l(-q)Y_l(p,-q,k)]\,, \tag{2.44}$$

with

$$Y_l(p,q,k)=1+p^{-1}[d_l(k)/d_l(q)]e^{-i(1/2)l\pi}\int_c^\infty dr\left(u_l(pr)-\frac{g_l(p)}{d_l(k)}w_l^{(+)}(kr)\right)V(r)f_l(k,q,r)\,. \tag{2.45}$$

When the BCM is turned off, (2.44) and (2.45) reduce to (2.31) and (2.32).

We shall illustrate the results of this section in the next section by presenting some explicit examples.

III. SOME EXAMPLES

A. Square Well

The simplest example is that of the square-well potential, given by

$$V(r)=-V_{0>},\quad 0<r<a$$
$$=0,\quad r>a\,. \tag{3.1}$$

It is straightforward to solve (2.15) and carry out the limit in (2.20). The results are

$$f_l(k,q,r)=e^{i(1/2)l\pi}\left(A_l u_l(Kr)+B_l v_l(Kr)+\frac{k^2-q^2}{K^2-q^2}w_l^{(+)}(qr)\right),$$

$$0<r\leqslant a=e^{i(1/2)l\pi}w_l^{(+)}(qr),\quad r\geqslant a, \tag{3.2}$$

$$K^2=V_0+k^2,$$

$$A_l=\frac{V_0}{(K^2-q^2)K}W[v_l(Ka),w_l^{(+)}(qa)]\,,$$
$$B_l=-\frac{V_0}{(K^2-q^2)K}W[u_l(Ka),w_l^{(+)}(qa)]\,, \tag{3.3}$$

and

$$f_l(k,q)=(q/K)^l B_l+\frac{k^2-q^2}{K^2-q^2}\,, \tag{3.4}$$

One can easily verify that (2.29) is valid for this potential. The half-off-shell T matrix for the square well is obtained from (2.21) and (3.4). The relations (2.39), (2.40), and (3.2) can be used to construct the half-off-shell T matrix for a BCM, in which the external potential is a square well. The fully off-shell T matrices are derivable from (2.31), (2.32), (2.44), (2.45), and (3.2). We do not bother to write these results out since they have been given before,[13] and do not appear any simpler in our formalism.

B. Exponential Potential

The s-wave T matrix for the potential

$$V(r)=-(z_0^2/4a^2)e^{-r/a} \tag{3.5}$$

has been worked out before,[14] but it is instructive to rederive the results in terms of the formalism of Sec. II. The new expressions are much simpler than those given previously. Following the techniques of Ref. 14, one can show that the solution of (2.15) for $l=0$ is

$$f_0(k,q,r)=e^{iqr}{}_1F_2(1;1-ika-iqa,1+ika-iqa;-\tfrac{1}{4}z_0^2e^{-r/a})\,, \tag{3.6}$$

where ${}_1F_2$ is a special case of the generalized hypergeometric function defined by

$${}_mF_n(\alpha_1,\ldots,\alpha_m;\beta_1,\ldots,\beta_n;z)=\sum_{j=0}^\infty\frac{(\alpha_1)_j\cdots(\alpha_m)_j}{(\beta_1)_j\cdots(\beta_n)_j}\frac{z^j}{j!}\,. \tag{3.7}$$

This result is interesting in that each term of the power series in (3.6) corresponds to an iteration of the integral equation (2.27). The power series converges for all values of the independent variable. The integral in (2.32) can be evaluated by integrating the series term by term to give

$$Y_0(p,q,k)=1+\sum_{n=1}^{\infty}\frac{k^2a^2+(n-iqa)^2}{(1-ika-iqa)_n(1+ika-iqa)_n[p^2a^2+(n-iqa)^2]}(-\tfrac{1}{4}z_0^2)^n. \tag{3.8}$$

Combining (2.31) and (3.8) gives a much simpler result for the T matrix than that given previously.[14]

The half-off-shell T matrix, for the BCM in which (3.5) is the external potential, can be obtained from (2.39), (2.40), and (3.6). The integral in (2.45) can be performed by integrating the series term by term, which yields

$$Y_0(p,q,k)=1+(pa)^{-1}[d_0(k)/d_0(q)]e^{iqc}\sum_{n=1}^{\infty}\frac{[k^2a^2+(n-iqa)^2](-\frac{1}{4}z_0^2e^{-c/a})^n}{(1-ika-iqa)_n(1+ika-iqa)_n}$$
$$\times\left(\frac{pa\cos pc+(n-iqa)\sin pc}{p^2a^2+(n-iqa)^2}+\frac{\sin pc-pb_0\cos pc}{(1-ikb_0)(ika+iqa-n)}\right). \tag{3.9}$$

The results for this BCM have not been given before. All of the series (3.6), (3.8), and (3.9) converge for all values of z_0^2; moreover, for reasonable values of z_0^2 they converge rapidly enough to be of practical value in numerical work.

C. Hulthén Potential

This potential is given by

$$V(r)=V_0\frac{e^{-r/a}}{1-e^{-r/a}}. \tag{3.10}$$

It falls off exponentially for large r and has an r^{-1} singularity at the origin. Following the techniques of Ref. 15 it is not hard to show that

$$f_0(k,q,r)=e^{iqr}\left(1+\frac{ABe^{-r/a}}{(1+\sigma)(C+\sigma)}\,{}_3F_2(1,1+A+\sigma,1+B+\sigma;2+\sigma,1+C+\sigma;e^{-r/a})\right), \tag{3.11}$$

where ${}_3F_2$ is a special case of (3.7) and

$$\begin{aligned}A&=-ika+i(V_0a^2+k^2a^2)^{1/2},\\ B&=-ika-i(V_0a^2+k^2a^2)^{1/2},\\ C&=1-2ika,\\ \sigma&=ika-iqa.\end{aligned} \tag{3.12}$$

It can also be shown using Ref. 15 that

$$f_0(k,q)=\frac{\Gamma(1+\sigma)\Gamma(C+\sigma)}{\Gamma(1+A+\sigma)\Gamma(1+B+\sigma)}, \tag{3.13}$$

thus the off-shell Jost function for the Hulthèn potential has a very simple expression. This can be written in terms of an infinite product by using the well-known representation for the gamma function

$$\frac{1}{\Gamma(1+z)}=e^{\gamma z}\prod_{n=1}^{\infty}(1+z/n)e^{-z/n}. \tag{3.14}$$

The result is

$$f_0(k,q)=\prod_{n=1}^{\infty}\left(1+\frac{V_0a^2}{k^2a^2-q^2a^2+n(n-2iqa)}\right). \tag{3.15}$$

When $q=k$ (3.13) and (3.15) reduce to the known results for the Hulthèn potential Jost function.[5] In evaluating the fully off-shell T matrix it is convenient to rewrite (2.32) in the form

$$Y_0(p,q,k)=f_0(k,q)+(k^2-p^2)p^{-1}\times\int_0^{\infty}dr\sin pr\,[f(k,q,r)-e^{iqr}]. \tag{3.16}$$

This result is obtained by combining (2.15) and (2.32) and by carrying out an integration by parts.

Putting (3.11) into (3.16), we find

$$Y_0(p,q,k)=f_0(k,q)+\frac{(k^2a^2-p^2a^2)AB}{(1+\sigma)(C+\sigma)}\sum_{n=0}^{\infty}\frac{(1+A+\sigma)_n(1+B+\sigma)_n}{(2+\sigma)_n(1+C+\sigma)_n}\frac{1}{p^2a^2+(iqa-n-1)^2}. \tag{3.17}$$

(2.31) and (3.17) can be combined to give the fully off-shell T matrix for the Hulthèn potential.

IV. SUMMARY AND DISCUSSION

We have shown how the Jost function can be generalized in a natural way, so as to allow a treatment of off-shell scattering. All of our equations have been written in terms of local potentials; however, it is clear that they can easily be extended to nonlocal potentials as well. Besides making the analysis of the exactly solvable examples more transparent, our results lead to a series of successive approximations to the T matrix. Iteration of (2.27) and subsequent substitution into (2.28) and (2.21) gives the half-off-shell T matrix as the ratio of two power series in the potential strength. These series should converge rapidly for high energies and/or high angular momenta.

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