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The degenerate spin-flip doublet $(3^+/2, 5^+/2)$ of ${}^{9}_{\Lambda}$Be

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The energy of the degenerate doublet $(3^+/2, 5^+/2)$ of ${}^{9}_{\Lambda}$Be, treating it as a partially nine-body system in the $\Lambda\alpha\alpha$ cluster model, has been calculated in the variational Monte Carlo framework. A simplified treatment, with the central two-body Urbana type ΛN and the three-body dispersive and two-pion exchange ΛNN forces along with the central two- and three-body correlations, is found to be adequate in explaining the energy of observed γ-ray transition from the excited degenerate doublet to the ground state. The hypernucleus ${}^{9}_{\Lambda}$Be is highly deformed and has an oblate shape in the excited state. The results of the present work are consistent with the earlier three-body cluster model analyzes of ${}^{9}_{\Lambda}$Be.

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I. INTRODUCTION

Recently, Shoeb [1,2] and coworkers [3–5] have analyzed the ground-state binding energies of s- and p-shell hypernuclei and excited states of p-shell hypernuclei in the α cluster model using the variational Monte Carlo (VMC) method. In these analyzes [1–4], a phenomenological dispersive three-body $\Lambda\alpha\alpha$ force of Yukawa shape was proposed in analogy with the one suggested in explaining the spectra of ^{12}C using the three-body cluster $\alpha\alpha\alpha$ potential [6,7]. The phenomenological dispersive three-body $\Lambda\alpha\alpha$ force [1–4] along with the appropriate $\Lambda\alpha$, $\alpha\alpha$, and $\Lambda\Lambda$ potentials explains the ground- and excited-state energies of ${}^{9}_{\Lambda}$Be and ${}^{10}_{\Lambda\Lambda}$Be. Moreover, a particle stable 4^+ state [4] of ${}^{10}_{\Lambda\Lambda}$Be has been predicted where the contribution of a dispersive three-body $\Lambda\alpha\alpha$ force was found to have negligibly small owing to large separation of $\alpha\alpha$ clusters. The need to incorporate a repulsive dispersive three-body force has been pointed out not only in the α-cluster model analysis [8] but investigated in detail in the earlier microscopic study [9]. In contrast to our analyzes [1–5], Hiyama *et al.* [10] have found the modified odd/even state ΛN potential important in explaining the binding energy and excited state data of $S = -1$ and -2 hypernuclei. Although these potentials have been microscopically calculated but were tailored to adjust the Λ-cluster potentials so that ground-state or ground- and excited-state energies of hypernuclei containing clusters are reproduced thus rendering these as phenomenological ones. In the calculation of the properties of ${}^{9}_{\Lambda}$Be in the $\Lambda\alpha\alpha$ model, using the Faddeev method, Cravo, Fonseca, and Koike [11] needs neither a dispersive force nor modified odd/even state ΛN potential. In the past few years, other cluster model analyzes [12,13] have also been performed for the binding energy of s-shell hypernuclei. Here it is noteworthy to point out that the oldest calculation for the excited state of ${}^{9}_{\Lambda}$Be is by Ali, Murphy, and Bodmer [14].

In all the above-mentioned analyzes, α clusters are treated as rigid entities devoid of structure and thus the effects of the role played by the dynamical correlations among the baryons are not explicitly manifested in the energy calculation. Therefore, such studies are deficient in accommodating the realities of the internal structure and seem, therefore, far from satisfactory. Earlier, Shoeb, Usmani, and Bodmer [15] have calculated the ground-state energy of ${}^{9}_{\Lambda}$Be, treating it as a partially nine-body system in the $\Lambda + 2\alpha$ model. This work presupposes the existence of the αs structure. The two-body NN correlations within αs were explicitly incorporated. However, the effect of NN correlations, where each α contributes a nucleon, is simulated through $\alpha\alpha$ correlation. Thus, antisymmetrization between two αs has been ignored. However, soft repulsive core in the $\alpha\alpha$ potential [7] simulates the effect of NN antisymmetrization in the wave function. The three-body ΛNN correlations were included in the trial wave function but ΛN space-exchange correlations were ignored. From this study it was concluded that dispersive three-body ΛNN force and space-exchange ΛN potential cannot be determined uniquely. The presence of one masks the determination of other. However, B_Λ of ${}^{9}_{\Lambda}$Be is satisfactorily explained.

Recently, Usmani and his coworker [16] have investigated the effect of space-exchange ΛN correlations for Argonne v_{18} NN potential on the B_Λ, Λ binding in ${}^{5}_{\Lambda}$He and found it to be negligibly small. Thus ignoring of space-exchange correlations in our earlier work [15] turned out to be justified.

The success of partially nine-body problem within the $\Lambda + 2\alpha$ model in explaining the B_Λ of ${}^{9}_{\Lambda}$Be, though for a single set of potential parameters, motivated us to apply it to analyze the experimental [17] binding energy $B_\Lambda = 3.67$ MeV of the degenerate doublet $(3^+/2, 5^+/2)$ of ${}^{9}_{\Lambda}$Be. Following the spirit of the earlier work [15], for the simplicity of the calculation, we have chosen simple baryon-baryon potentials along with the corresponding simple correlation functions. We have also calculated the magnetic and quadrupole moments to gain further insight into the structure of the hypernucleus ${}^{9}_{\Lambda}$Be. To our knowledge, this is the first application of VMC method to the calculation of the excited state. Preliminary results on the excited degenerate doublet of ${}^{9}_{\Lambda}$Be were presented in the earlier work [18]. The Hamiltonian and expressions for the input potentials used can be found in the earlier work [15,19–21] despite that we give these here for ready reference.

For the calculation of energy of ${}^{9}_{\Lambda}$Be, now we have two competing models: classical $\Lambda + 2\alpha$ model and partially

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nine-body $\Lambda + 2\alpha$ model. In the former, internal structure is ignored while the latter takes into account the relevant correlations consistent with the philosophy of the model. Here it will be interesting to make a comparative discussion of two approaches and in Sec. V we have reserved a subsection for this purpose.

The article is organized as follows: In the next section we describe the Hamiltonian of ${}^{5}_{\Lambda}$He and of hypernucleus ${}^{9}_{\Lambda}$Be in the α cluster model along with the potential models for ΛN, NN, and $\alpha\alpha$. For the three-body ΛNN we use dispersive and two-pion exchange forces. The construction of trial wave function is discussed in Sec. III and the energy calculation and moments are presented in Sec. IV. In Sec. V, we present the results and discussion. The last section is devoted to the summary of our work.

II. HAMILTONIANS IN THE α CLUSTER MODEL AND POTENTIAL MODELS

Although ground state of ${}^{5}_{\Lambda}$He, as a $(A = 5)$ five-body system, has been analyzed extensively [15,19–21] in VMC framework but ${}^{9}_{\Lambda}$Be, as a partially nine-body problem [15], is yet to be investigated in detail. Therefore, we are interested in making a combined and a detailed study of the energies [17,22] of ground state of ${}^{5}_{\Lambda}$He and ground and excited states of ${}^{9}_{\Lambda}$Be by choosing the same sets of potential parameters. The hypernucleus of mass number A consists of $(A-1)$ nucleons and a Λ particle. The hypernuclear Hamiltonian H_H^A for A particles system, in general, can be written as the sum of H_C^{A-1}, Hamiltonian of the $(A-1)$ nucleons of the core nucleus and Λ particle Hamiltonian H_Λ:

$$H_H^A = H_C^{A-1} + H_\Lambda. \tag{1}$$

The nuclear Hamiltonian, H_C^{A-1} for the two-body NN interaction $V_{NN}(r_{ij})$ for α clusters nuclear core is given by

$$H_C^{A-1} = \sum_{i=1}^{A-1} K_N(i) + \sum_{i<j}^{A-1} V_{NN}(r_{ij}) + V_{\alpha\alpha}^{(l)}(r_{\alpha_1\alpha_2}) \tag{2}$$

and the Λ particle Hamiltonian, H_Λ is written as

$$H_\Lambda = K_\Lambda + \sum_{i=1}^{A-1} V_{\Lambda N}(r_{\Lambda i}) + \sum_{i<j}^{A-1} V_{ij\Lambda}(r_{i\Lambda}, r_{j\Lambda}), \tag{3}$$

where K_a is the kinetic energy operator for particle $a(=N, \Lambda)$ and $V_{\Lambda N}$ and $V_{ij\Lambda}$ are the two-body ΛN and three-body ΛNN potentials, respectively. The Hamiltonian for ${}^{5}_{\Lambda}$He is obtained from Eqs. (1), (2), and (3) on restricting baryon number $A = 5$ and suppressing $\alpha\alpha$ potential. We calculate the energy of the degenerate doublet $(3^+/2, 5^+/2)$ of ${}^{9}_{\Lambda}$Be in the $\Lambda\alpha\alpha$ cluster model. These states are presumably built on the first excited state $J_C = 2^+$ of ^{8}Be core nucleus that is coupled state of $L_C = 2$, and $S_C = 0$. The coupling of $0s\Lambda$ particle of spin $s_\Lambda = 1/2$ to the $J_C = 2^+$ of ^{8}Be core in ${}^{9}_{\Lambda}$Be gives rise the spin-flip doublet $(3^+/2, 5^+/2)$. The measured energy spacing $\sim$0.03 MeV of the doublet is attributed to a very weak spin-orbit force that has been ignored here.

The Hamiltonian of the $A(=9)$ baryons system ${}^{9}_{\Lambda}$Be in the $2\alpha + \Lambda$ is given as:

$$H_H^9 = \sum_{i=1}^{A-1} K_N(i) + \sum_{i<j}^{4} V_{NN}(r_{ij}) + \sum_{i<j\, i=5}^{A-1} V_{NN}(r_{ij}) + V_{\alpha\alpha}^{(l)}(r_{\alpha_1\alpha_2}) + K_\Lambda + \sum_{i=1}^{4} V_{\Lambda N}(r_{\Lambda i}) + \sum_{i=5}^{A-1} V_{\Lambda N}(r_{\Lambda i}) + \sum_{i<j}^{A-1} V_{ij\Lambda}(r_{i\Lambda}, r_{j\Lambda}), \tag{4}$$

where labels 1 to 8 specify the nucleons and α_1 and α_2 two α particles. V_{xy} denotes the potential for a pair of particles $xy(=NN, \Lambda N, \alpha\alpha)$ and in the case of $\alpha\alpha$, $V_{\alpha\alpha}^{(l)}$ is potential in the relative angular momentum $l = 2$ for excited state. The three-body potential $V_{ij\Lambda}(r_{i\Lambda}, r_{j\Lambda}) = V_{\Lambda NN}^{D} + V_{\Lambda NN}^{2\pi}$. The contribution of $\langle V_{\Lambda NN}^{D} \rangle$ to the energy is quite significant as shown in Refs. [15,19], neglecting it over binds the ${}^{9}_{\Lambda}$Be. The Hamiltonian of the ground state is recovered for $V_{\alpha\alpha}^{(l)}$ in the relative angular momentum $l = 0$ state.

A. Baryon-baryon and $\alpha\alpha$ potentials

The baryon-baryon, $\alpha\alpha$, and three-body potentials used here are reasonable as these are motivated from the meson-exchange model and in case of phenomenological ones are constrained by the experimental data of the relevant pairs of particles. All these potentials have a soft repulsive core and in the case of three-body forces, a cut-off radius has been introduced to simulate the short-range behavior that is yet to be understood.

1. ΛN potential

An Urbana-type central spin-dependent and space-exchange ΛN potential $V_{\Lambda N}$ consistent with the Λp-scattering data has been employed. This potential [15,19–21] for hypernuclei with spin zero core in the relative ΛN s state has the following form;

$$V_{\Lambda N} = (1 - \epsilon + \epsilon P_x)\tilde{V}_{\Lambda N}^{0}, \tag{5}$$

where

$$\tilde{V}_{\Lambda N}^{0} = V_{2\pi} = W(r) - \bar{V} T_\pi^2(r) \tag{6}$$

with $\bar{V}$(=6.15 MeV) as the spin-average strength, P_x is the Majorana space-exchange operator for Λ and nucleon with strength ϵ(=0.25) determined from the Λp-scattering data, $W(r)$ is a Woods-Saxon repulsive core, which is given as

$$W(r) = W_0\left[1 + \exp\left(\frac{r-R}{d}\right)\right]^{-1} \tag{7}$$

with $W_0 = 2137$ MeV, $R = 0.5$ fm, $d = 0.2$ fm, and $T_\pi(r)$ is the one-pion exchange tensor potential shape modified with a cutoff,

$$T_\pi(r) = \left[1 + \frac{3}{\mu r} + \frac{3}{(\mu r)^2}\right]\frac{\exp(-\mu r)}{\mu r}[1 - \exp(-\hat{c}r^2)]^2 \quad (8)$$

with $\mu = 0.7$ fm^{-1} and the cut-off parameter $\hat{c} = 2.0$ fm^{-2}.

2. *NN potential*

For the NN pair we use the central, spin-isospin-independent Malfliet-Tjon (MT) potential [23] that gives reasonable ground-state energy (−31.20 MeV) and rms radius (1.42 fm) for ^{4}He within VMC approach. This potential has the form;

$$V_{NN}(r) = \frac{\hbar c}{r}[7.39\exp(-3.11r) - 2.93\exp(-1.55r)], \quad (9)$$

We have also used Volkov NN central potential [24];

$$V_{NN}(r) = 144.86\exp\left[-\left(\frac{r}{0.82}\right)^2\right] - 83.34\exp\left[-\left(\frac{r}{1.60}\right)^2\right], \quad (10)$$

which gives a binding energy of 30.34 MeV and an rms radius of 1.51 fm for ^{4}He, consistent with the experimental values. The Volkov NN potential induces much weaker correlations compared to the MT one. The energy for ^{8}Be core is taken as sum of the energy of 2α plus resonance energy ≈0.1 MeV.

3. *αα potential*

We employ Ali-Bodmer [25] $\alpha\alpha$ potential that fits $\alpha\alpha$-scattering data and has been modified by Fedorov and Jensen [7]. The $\alpha\alpha$ potential $V^{(l)}_{\alpha\alpha}(r)$ in the angular-momentum state l is given by

$$V^{(l)}_{\alpha\alpha}(r) = V^{(l)}_{\rm rep}\exp\left\{-\left[\frac{r}{\beta^{(l)}_{\rm rep}}\right]^2\right\} - V^{(l)}_{\rm att}\exp\left\{-\left[\frac{r}{\beta^{(l)}_{\rm att}}\right]^2\right\}, \quad (11)$$

where $V^{(l)}_i$ and $\beta^{(l)}_i$ are the strength and range parameters in the relative l state, respectively, for i = rep (att). The parameters are listed in Table I. A finite size $V_{\rm Coul}(r)$ potential is added in Eq. (11) while performing the actual calculation for the energy. Here we remark that Filikhin and Gal [26] have used 120.0 instead of 125.0 as the coefficient of the repulsive part. We have also repeated our calculation taking 120.0 as the soft repulsive core and found that separation of the degenerate doublet from the ground state is less at most approximately by 6% than for the ones reported here.

TABLE I. Potential parameters of Ali and Bodmer [25] from fit to $\alpha\alpha$-scattering data.

Angular momentum l	$V^{(l)}_{\rm rep}$ (MeV)	$\beta^{(l)}_{\rm rep}$ (fm)	$V^{(l)}_{\rm att}$ (MeV)	$\beta^{(l)}_{\rm att}$ (fm)
0	125.0	1.53	30.18	2.85
2	20.0	1.53	30.18	2.85

B. Three-body ΛNN potentials

The phenomenological dispersive three-body ΛNN potential that we shall be using in our analysis is a representation of the suppression of the TPE ΛN potential arising from modification ("dispersion") of the intermediate Σ, N^*, Δ, . . . components by the medium (a "second" nucleon N_2). The two types of phenomenological dispersive ΛNN potentials [19,20] have been constructed to resolve the over binding of $^5_\Lambda$He. The spin-dependent dispersive three-body ΛNN potential for spin-zero core nuclei (e.g., $^5_\Lambda$He, $^9_\Lambda$Be) is equivalent to the spin-independent force that in our case corresponds to;

$$V^D_{\Lambda NN} = W_d T^2_\pi(r_{\Lambda 1})T^2_\pi(r_{\Lambda 2}), \quad (12)$$

where the strength parameter W_d has repulsive nature. The contribution of this force by design is repulsive for all distances and for all the ΛN correlations included in the wave function.

The two-pion exchange three-body ΛNN potential [27] has, for s-shell hypernucleus $^5_\Lambda$He, the following form:

$$V^{2\pi}_{\Lambda NN} = C_p[1 + (3\cos^2\theta_{1\Lambda 2} - 1)\hat{T}_\pi(r_{\Lambda 1})\hat{T}_\pi(r_{\Lambda 2})] \times \hat{Y}(r_{\Lambda 1})\hat{Y}(r_{\Lambda 2}), \quad (13)$$

where $\hat{T}_\pi(r_{\Lambda i}) = [1 + \frac{3}{\mu r} + \frac{3}{(\mu r)^2}][1 - \exp(-\hat{c}r^2)]$, $\hat{Y}(r_{\Lambda i}) = \frac{\exp(-\mu r)}{\mu r}[1 - \exp(-\hat{c}r^2)]$, and $\theta_{1\Lambda 2}$ is the angle between the arms $\mathbf{r}_{\Lambda 1}$ and $\mathbf{r}_{\Lambda 2}$. The coefficient C_p is also not well determined quantity due to the ambiguity in the coupling constant and other approximations made in deriving Eq. (13). Further, lack of knowledge in the short range behavior of $V^{2\pi}_{\Lambda NN}$ is circumvented by introducing a cut-off distance $\hat{c}$. There is no unique way to fix it and is rather a arbitrary number. Hence we have chosen a range of values for both: $C_p = 1$ and 2 MeV, $\hat{c} = 1$, 2, and 3 fm^{-2}, which are not very far off from those given in Ref. [27]. The contribution of $V^{2\pi}_{\Lambda NN}$ is sensitive to $\hat{c}$ and has a highly nonlinear behavior. These values of C_p and $\hat{c}$ have a moderating effect on the strength W_d of the dispersive three-body ΛNN potential when the two are used together.

III. TRIAL WAVE FUNCTION

In the construction of good trial wave function, in the variational calculation, care should be taken to incorporate the physics relevant to describe the state of the system under investigation and, moreover, it should be reasonably efficient to compute. The trial wave function for $^9_\Lambda$Be in the state (J, J_z), ignoring space-exchange correlations, is the product of central two-body correlation functions f_{xy} and for $\alpha\alpha$ it is in the relative angular-momentum state l(=0 or 2), three-body correlations $f_{\Lambda NN}$ and the ls coupled function $(y_{lm}(\Omega_{\alpha_1\alpha_2}) \otimes \chi^{1/2}{}_{m_z})_{JJ_z}$, a appropriate combination of $\chi^{1/2}{}_{m_z}$, spin function of Λ particle and $y_{lm}(\Omega_{\alpha_1\alpha_2})$ spherical harmonic for relative motion of two αs:

$$\Psi_H^{(9)}(J,J_z)=\left[\prod_{i=1}^{4} f_{\Lambda N}(r_{\Lambda i})\right]\left[\prod_{i=5}^{A-1} f_{\Lambda N}(r_{\Lambda i})\right] \times\left[\prod_{i<j}^{4} f_{NN}(r_{ij})\right]\left[\prod_{i<j}^{A-1}{}_{i=5} f_{NN}(r_{ij})\right] \times\left[\prod_{i<j}^{A-1}{}_{i=1} f_{\Lambda NN}(r_{i\Lambda},r_{j\Lambda})\right] \times f_{\alpha\alpha}^{(l)}(r_{\alpha_1\alpha_2})(y_{lm}(\Omega_{\alpha_1\alpha_2})\otimes\chi^{1/2}{}_{m_z})_{JJ_z}. \quad (14)$$

The two-body correlation functions $f_{\Lambda N}$, f_{NN}, and $f_{\alpha\alpha}^{(l)}(r)$, as usual, are spin-independent and are obtained with the method developed by the Urbana group, from the solution of the Schrödinger-type equation for the appropriate relative angular-momentum state. A Schrödinger-type equation, which contains the effective two-body potential through which the variational parameters enter is solved for each pair of particles. The forms of $f_{\Lambda N}$, f_{NN}, and $f_{\alpha\alpha}^{(l)}$ have the asymptotic behavior required by the full A-body Schrö dinger equation, namely

$$f_{BN}\sim r^{-\nu_{BN}}\exp(-\kappa_{BN}r) \quad (15)$$

with the appropriate products of the $f\prime s$ then having the asymptotic behavior $\sim r^{-1}\exp(-\kappa_B r)$ if the ν_{BN} are chosen appropriately ($\nu_{\Lambda N}=0.125$, $\nu_{NN}=0.292$, and $\nu_{\alpha\alpha}=0.5$ for ${}^{9}_{\Lambda}$Be). The results are quite insensitive to the precise values of the ν_{BN} if the variational parameters are optimized for any given choice.

The three-body correlations $f_{\Lambda NN}$ have the following forms:

$$f_{\Lambda NN}=f_{\Lambda NN}^{D}f_{\Lambda NN}^{2\pi},$$

where

$$f_{\Lambda NN}^{D}=1-\alpha\tilde{Y}(r_{1\Lambda})\tilde{Y}(r_{2\Lambda}), \quad (16)$$

is appropriate for $V_{\Lambda NN}^{D}$ and

$$f_{\Lambda NN}^{2\pi}=1-\beta(3\cos^2\theta_{1\Lambda 2}-1)\tilde{Y}(r_{1\Lambda})\tilde{Y}(r_{2\Lambda}), \quad (17)$$

for $V_{\Lambda NN}^{2\pi}$. $\tilde{Y}(r)$ are the Yukawa functions, as defined in Eq. (13) with the range and cut-off parameters $\tilde{\mu}$ and $\tilde{c}$, respectively. The $\tilde{\mu},\tilde{c}$ and the correlation strengths α and β are variational parameters. Here we may point out that $f_{\Lambda NN}$ correlations that make negligibly small contribution for the triad ΛNN, where a participating nucleon is from each α, have been ignored. The wave function $\Psi_H^{(9)}$ depends on a total of 13 variational parameters $\kappa_{\Lambda N}$, $c_{\Lambda N}$, $a_{\Lambda N}$, $R_{\Lambda N}$, $s_{\Lambda N}$, κ_{NN}, c_{NN}, a_{NN}, R_{NN}, $\kappa_{\alpha\alpha}$, $c_{\alpha\alpha}$, $a_{\alpha\alpha}$, and $R_{\alpha\alpha}$ for ${}^{9}_{\Lambda}$Be ground state and exactly the same number of variational parameters for excited state. After excluding the parameters related to $\alpha\alpha$ correlation from the above set we are left with the nine variational parameters in the wave function for ${}^{5}_{\Lambda}$He.

IV. ENERGY CALCULATION AND MOMENTS

The energy $-B_\Lambda(J,J_z)$ for a hypernucleus of baryon number A is the difference of energy of hypernucleus in the state $\Psi_H^{(A)}(J,J_z)$ and of the nuclear core in the state $\Psi_C^{(A-1)}(J_C,M_C)$ and is written as:

$$-B_\Lambda(J,J_z)=\frac{\langle\Psi_H^{(A)}(J,J_z)|H_H^A|\Psi_H^{(A)}(J,J_z)\rangle}{\langle\Psi_H^{(A)}(J,J_z)|\Psi_H^{(A)}(J,J_z)\rangle} -\frac{\langle\Psi_C^{(A-1)}(J_C,M_C)|H_C^{A-1}|\Psi_C^{(A-1)}(J_C,M_C)\rangle}{\langle\Psi_C^{(A-1)}(J_C,M_C)|\Psi_C^{(A-1)}(J_C,M_C)\rangle} \quad (18)$$

The estimates for the energy were made for 100,000 points. The two terms in Eq. (18) were separately calculated and optimized with respect to variational parameters that are different for the two wave functions. The second term in Eq. (18) is $-31.20(-30.34)$ MeV for ^{4}He and$-62.3(-60.58)$ MeV for ^{8}Be core for MT(Volkov) NN potential.

The variational parameters entering through the two- and three-body correlations involved in the wave function are varied to optimize the energy using the standard optimizing routine. The $\kappa_{\Lambda N}$, κ_{NN}, and $\kappa_{\alpha\alpha}$ and $s_{\Lambda N}$ are the parameters on which the energy depends sensitively.

The space-exchange energy is calculated by exchanging the coordinate of the Λ with each nucleon in turn in the wave function $\Psi_H^{(A)}$ and is written as:

$$\langle\Psi_H^{(A)}(\mathbf{r_1},\mathbf{r_2}\ldots,\mathbf{r_{(A-1)}};\mathbf{r}_\Lambda)| \times\sum_{i=1}^{A-1}V_{\Lambda N}(r_{\Lambda i})P_x(i\Lambda)|\Psi_H^{(A)}(\mathbf{r_1},\mathbf{r_2}\ldots,\mathbf{r_{(A-1)}};\mathbf{r}_\Lambda)\rangle = \langle\Psi_H^{(A)}(\mathbf{r_1},\mathbf{r_2}\ldots,\mathbf{r_{(A-1)}};\mathbf{r}_\Lambda)| \times\sum_{i=1}^{A-1}V_{\Lambda N}(r_{\Lambda i})|\Psi_H^{(A)}(\mathbf{r_1},\mathbf{r_2}\ldots,\mathbf{r}_\Lambda\ldots,\mathbf{r_{(A-1)}};\mathbf{r_i})\rangle. \quad (19)$$

In evaluating the above integral, it should be noted that on exchanging of the coordinate $\mathbf{r}_\Lambda$ of the Λ with $\mathbf{r_i}$ of a ith nucleon in all the eight terms, the $\mathbf{R}_{\text{c.m.}}$, the radius vector of the center of mass (c.m.) of the system due to the mass difference in the masses m_Λ and m_i of the two particles shifts to new position $\mathbf{R}'_{\text{c.m.}}$ such that $\mathbf{R}'_{\text{c.m.}}-\mathbf{R}_{\text{c.m.}}=\frac{(m_\Lambda-m_i)(\mathbf{r}_i-\mathbf{r}_\Lambda)}{(A-1)m_N+m_\Lambda}$. The center of mass is restored to its original position by giving an appropriate displacement to each of the particles in the wave function.

A. Magnetic and quadrupole moments

To explore further structure of the hypernucleus ${}^{9}_{\Lambda}$Be in the state (J,J_z) we have calculated the magnetic and quadrupole moments. The expression for the magnetic moment operator μ in the unit of nuclear magneton (μ_0) for a hypernucleus of charge Z is given by;

$$\mu_{J,J_z} = \left[\sum_{i=1}^{Z}(\mathbf{l_i} + g_p\mathbf{s_i}) + g_n\sum_{i=Z+1}^{A-1}\mathbf{s_i} + g_\Lambda\mathbf{s}_\Lambda\right], \quad (20)$$

where $\mathbf{l_i}$ is the angular momentum of i^{th} proton, $g_x(x = n, p, \Lambda)$ is the gyromagnetic ratio of the particle x and $\mathbf{s_i}(i = \text{nucleon}, \Lambda)$ is the spin angular-momentum operator of particle i.

The expectation value of Eq. (20) for the angular-momentum state $(J, J_z = J)$ of ${}^9_\Lambda$Be is written as

$$\mu_{J,J} = \frac{1}{4(J+1)}[J(J+1)(2g_\Lambda + 1) + s_\Lambda(s_\Lambda + 1) \times (2g_\Lambda - 1) - J_C(J_C+1)(2g_\Lambda) + L_C(L_C+1)], \quad (21)$$

where $g_\Lambda = -1.226$.

The quadrupole moment in the unit of e fm^2 using the expression:

$$\langle Q\rangle_{J,J} = \langle\Psi_H^{(9)}(J, J_z)|Q|\Psi_H^{(9)}(J, J_z)\rangle_{J_z=J}, \quad (22)$$

where the quadrupole moment operator is given by

$$Q = \sum_{i=1}^{2}\left(3z_i^2 - r_i^2\right) + \sum_{i=5}^{6}\left(3z_i^2 - r_i^2\right) \quad (23)$$

with summation index i running over coordinates of protons in the two αs. The distances of protons are being measured from the center of mass of the two αs.

V. RESULTS AND DISCUSSION

The experimental data for the systems ${}^5_\Lambda$He and ${}^9_\Lambda$Be are given in Refs. [17,22]. Prior to the analysis of the energy of the degenerate doublet ($3^+/2$, $5^+/2$) of ${}^9_\Lambda$Be we need to recalculate the ground state energy of ${}^5_\Lambda$He and ${}^9_\Lambda$Be. The parameters of the two-body ΛN force are $\bar{V} = 6.15$ MeV and $\epsilon = 0.25$ and the strength W_d of dispersive force is adjusted for the sets of combinations of C_p and $\hat{c}$ to fit B$_\Lambda$ of ${}^5_\Lambda$He for a chosen NN potential. These parameters are then employed to analyze the ground and excited states of ${}^9_\Lambda$Be. We will first discuss the detailed results of the calculation for the energy for MT NN potential of all the systems included here and only quote the final results for the Volkov potential [24].

A. Ground states of ${}^5_\Lambda$He and ${}^9_\Lambda$Be

The optimized energy $-B_\Lambda$ of ${}^5_\Lambda$He for the ΛN potential Eq. (5) in conjunction with $W_d = 0.012$ MeV of the dispersive force Eq. (12) for $C_p(\hat{c}) = 0(0)$ is close to the experimental value. For other sets of $C_p(\hat{c})$, the results of our calculations are shown in Table II. From the table we note that as cut-off radius changes from 1 to 3 fm^{-2}, the contribution of $V^{2\pi}_{\Lambda NN}$ changes from moderately repulsive to moderately attractive and consequently, a minor adjustment in the strength W_d is made to produce the B$_\Lambda$ of ${}^5_\Lambda$He close to the experimental.

The energy of ${}^9_\Lambda$Be is optimized for all the three-body ΛNN potentials parameters sets which fit B$_\Lambda$ of ${}^5_\Lambda$He. The $\alpha\alpha$ potential of Ali and Bodmer [25] in $l = 0$ was employed.

TABLE II. VMC results for ${}^5_\Lambda$He for Urbana type ($\bar{V} = 6.15$ MeV) ΛN and Malfliet-Tjon NN potentials. The last column gives the total energy with statistical error. The third and fourth columns have the values of average kinetic $\langle T\rangle$ and two-body potential $\langle V_{BN}\rangle$ energies while the contribution of dispersive $\langle V^D_{\Lambda NN}\rangle$ and two-pion exchange $\langle V^{2\pi}_{\Lambda NN}\rangle$ energies are given in the fifth and sixth columns for combinations of dispersive and two-pion exchange three body ΛNN potential strength parameters, given in the first two columns. All the potential strengths and the energies are in MeV and $\hat{c}$ in fm^{-2}. Experimental $B_\Lambda = 3.12$ MeV.

W_d	$C_p(\hat{c})$	$\langle T\rangle$	$-\langle V_{BN}\rangle$	$\langle V^D_{\Lambda NN}\rangle$	$\langle V^{2\pi}_{\Lambda NN}\rangle$	$-E \pm \Delta E$
0.012	0 (0)	83.91	120.68	2.35	0.00	34.42 ± 0.04
0.009	1 (1)	84.67	121.51	1.89	0.64	34.31 ± 0.06
0.010	1 (2)	84.49	121.09	2.27	0.09	34.24 ± 0.06
0.0115	1 (3)	84.65	121.17	2.70	−0.60	34.42 ± 0.05
0.006	2 (1)	84.67	121.51	1.26	1.29	34.29 ± 0.06
0.009	2 (2)	86.53	122.95	2.20	−0.23	34.45 ± 0.05
0.016	2 (3)	82.49	117.57	3.95	−3.23	34.36 ± 0.07

Thus there is no free potential parameter in the calculation of ${}^9_\Lambda$Be. In all the cases, listed in Table III, the theoretical B_Λ values are found to lie between 6.30 and 6.90 MeV, which are consistent with the experimental value 6.71 MeV. Further, $\mathbf{R}_{\alpha\alpha}$ varies between 3.7 and 4.0 fm, which is more than twice the rms radius of an α particle and thus justifying the internal consistency of the α cluster model.

B. Degenerate doublet ($3^+/2$, $5^+/2$) of ${}^9_\Lambda$Be

The same potential parameters sets as employed for the ground state of B_Λ of ${}^9_\Lambda$Be were utilized to calculate the energy of the degenerate doublet ($3^+/2$, $5^+/2$) of ${}^9_\Lambda$Be. The excitation energy of the degenerate doublet has been calculated for the first time in the VMC approach using the $\alpha\alpha$ potential [7,25] in the relative $l = 2$ state. The results of calculation for optimum values of variational parameters are tabulated in Table IV for ($3^+/2$, 3/2) state. The energy is found to lie between −3.32 and −3.89 MeV which is 2.89 to 3.14 MeV higher than the calculated ground-state energy of ${}^9_\Lambda$Be given in Table III. These values are close to those obtained in the $\Lambda\alpha\alpha$ cluster model calculations [1]. Essentially the same results are obtained for the ($5^+/2$, 5/2) state. The separation energy $\Delta E_\Lambda[=E({}^9_\Lambda\text{Be}^*) - E({}^9_\Lambda\text{Be})]$, the difference between excited- and ground-states energies for all the combinations of the three-body ΛNN potentials, listed in the column third of Table V for MT NN potential, is in good agreement with the observed emitted γ-ray [17] provided small spin-orbit splitting is ignored. We have also calculated the quadrupole moment for ${}^9_\Lambda$Be* to explore its structure. Its value for ($3^+/2$, 3/2) state is found to vary from −7.12 to −6.24 e fm^2, and for ($5^+/2$, 5/2) it is approximately 50% higher in magnitude than for ($3^+/2$, 3/2). The hypernucleus ${}^9_\Lambda$Be is highly deformed and has an oblate shape. The values of quadrupole moment are comparable to those calculated in the $\Lambda\alpha\alpha$ cluster model by Shoeb [1]. Further, $R_{\alpha\alpha}$ is marginally larger than that for the ground state as it should be due to the centrifugal barrier in the relative $l = 2$ state.

TABLE III. VMC results for the ground state ($1^+/2, 1/2$) of ${}^{9}_{\Lambda}$Be. $\mathbf{R}_{\alpha\alpha}$ (in fm), the $\alpha\alpha$ rms radii for various potential parameters sets, are given in the last column. The energy -62.3 MeV, which is the sum of energies of two αs plus $\approx$0.1 MeV for the resonance energy of ^{8}Be, is to be subtracted from the energy given in the seventh column to get the calculated energy $-B_\Lambda$ of ${}^{9}_{\Lambda}$Be. Other quantities are the same as in Table II. Experimental $B_\Lambda = 6.71$ MeV.

W_d	$C_p(\hat{c})$	$\langle T\rangle$	$-\langle V_{BN}\rangle$	$\langle V^D_{\Lambda NN}\rangle$	$\langle V^{2\pi}_{\Lambda NN}\rangle$	$-E \pm \Delta E$	$\mathbf{R}_{\alpha\alpha}$
0.012	0 (0)	160.61	233.07	3.26	0.00	69.20 ± 0.07	3.98
0.009	1 (1)	159.94	232.69	2.77	1.07	68.91 ± 0.07	3.96
0.010	1 (2)	161.22	233.87	3.39	0.43	68.83 ± 0.06	3.73
0.0115	1 (3)	159.46	231.92	3.66	-0.15	68.95 ± 0.08	3.80
0.006	2 (1)	163.08	236.11	2.31	1.83	68.89 ± 0.08	3.74
0.009	2 (2)	160.51	233.05	3.47	-0.01	69.08 ± 0.07	3.78
0.016	2 (3)	158.50	229.53	5.99	-3.56	68.60 ± 0.08	3.77

For all the three choices of $\hat{c}$ discussed above it should be noted that for a given C_p, the contribution $\langle V^{2\pi}_{\Lambda NN}\rangle$ is highly nonlinear for the ground and excited states of ${}^{9}_{\Lambda}$Be as was found above in Table II for ${}^{5}_{\Lambda}$He. It changes from moderately repulsive to attractive as $\hat{c}$ changes from 1 to 3 fm^{-2}.

We have also carried out the calculation of the energy using Volkov NN potential [24] for the same sets of combinations $C_p(\hat{c})$ as given in Table V. This potential induces weaker correlations compared to MT NN potential. However, the results for separation energy ΔE_Λ, listed in the fourth column of Table V are not very different from those obtained for the MT potential. Thus both the central NN potentials with soft repulsive core give equally good fit to the separation energy of the ground and excited states of ${}^{9}_{\Lambda}$Be.

From the analytical expression Eq. (21), the magnetic moment μ_{J,J_z} in the unit of μ_0 for ${}^{9}_{\Lambda}$Be in various states has been calculated and is given as:

$$\mu_{1^+/2,1/2} = \mu_\Lambda = -0.613, \quad \mu_{3^+/2,3/2} = 1.268, \quad \text{and}$$
$$\mu_{5^+/2,5/2} = 0.387.$$

These are exactly the same as calculated by Cravo, Fonseca, and Koike [11].

C. Cluster model $\Lambda\alpha\alpha$ versus partially nine-body system for ${}^{9}_{\Lambda}$Be

In our earlier work [15], the investigation of the ground state of ${}^{9}_{\Lambda}$Be through partially nine-body problem within the $\Lambda\alpha\alpha$ model and its application here to the excited state, where nuclear degrees of freedom in the αs are assumed from the outset, is in essence an extension of the $\Lambda\alpha\alpha$ cluster model [1–4,8–11,13,14] in which rigid αs are devoid of structure. The three-body cluster model for the analysis of the energy of ${}^{9}_{\Lambda}$Be uses $\Lambda\alpha$ and $\alpha\alpha$ potentials constrained to fit the properties of ${}^{5}_{\Lambda}$He and low-energy $\alpha\alpha$-scattering phase shifts. The potential for $\Lambda\alpha$ pair and realistic $\alpha\alpha$ potential for $l = 0$ and 2 take into account the NN and ΛN forces with all its complications and their corresponding correlations represent a sort of average over all types of the NN and ΛN correlations compatible with the relevant baryon-baryon interactions. In partially nine-body problem, the ΛN and NN central forces, taken explicitly into account within each α, are simple, reasonable, and consistent with the expectation of meson-exchange model and satisfactorily explain the relevant data. Further, dispersive and two-pion exchange three-body ΛNN forces within each α and between nucleons where each α contributes a nucleon are included. The trial wave function contains simple correlations compatible with the interactions. However, in both the models, the antisymmetrization between two αs is simulated through soft repulsive core in the $\alpha\alpha$ potential. The cluster model calculations [1–4,8,10] of energy for the ground and excited states of α-cluster hypernuclei give results in agreement with partially nine-body problem. The calculated quadrupole moments for the two models under discussion also agree with each other. Here it will be worth mentioning to remark that a very old cluster model variational calculation of Ali, Murphy, and Bodmer [14] with very simple interactions and correlations yields energy of degenerate doublet ($3^+/2, 5^+/2$)

TABLE IV. VMC results for the excited state ($3^+/2, 3/2$) of ${}^{9}_{\Lambda}$Be. The last column has value of the quadrupole moment in the unit of e fm^2. Other quantities are the same as in the preceding table. Experimental $B_\Lambda = 3.67$ MeV.

W_d	$C_p(\hat{c})$	$\langle T\rangle$	$-\langle V_{BN}\rangle$	$\langle V^D_{\Lambda NN}\rangle$	$\langle V^{2\pi}_{\Lambda NN}\rangle$	$-E \pm \Delta E$	$\mathbf{R}_{\alpha\alpha}$	$-\langle Q\rangle_{(3^+/2,3/2)}$
0.012	0 (0)	161.83	231.07	3.12	0.00	66.12 ± 0.08	4.08	7.12
0.009	1 (1)	169.06	238.75	2.83	1.09	65.77 ± 0.07	4.11	6.52
0.010	1 (2)	160.92	230.51	3.14	0.52	65.93 ± 0.07	3.91	6.41
0.0115	1 (3)	160.95	230.25	3.48	-0.13	65.95 ± 0.06	3.93	6.62
0.006	2 (1)	160.07	229.66	1.86	1.80	65.93 ± 0.07	3.89	6.27
0.009	2 (2)	163.14	232.09	2.94	-0.18	66.19 ± 0.07	3.95	6.24
0.016	2 (3)	164.65	232.22	5.62	-3.67	65.62 ± 0.08	3.96	6.64

TABLE V. Separation energy ΔE_Λ (in MeV) of ${}^9_\Lambda$Be for the MT and Volkov NN potentials. Other quantities are the same as in the preceding table. Experimental $\Delta E_\Lambda = 3.04$ MeV.

W_d	$C_p(\hat{c})$	Theoretical ΔE_Λ	
		Malfliet-Tjon [23]	Volkov [24]
0.012	0 (0)	3.08	3.07
0.009	1 (1)	3.14	3.09
0.010	1 (2)	2.90	3.01
0.0115	1 (3)	3.00	3.09
0.006	2 (1)	2.96	3.11
0.009	2 (2)	2.89	3.06
0.016	2 (3)	2.98	3.07

of ${}^9_\Lambda$Be close to the one evaluated here. In view of the simplicity of α-cluster model and its remarkable success in explaining the observed energy of ${}^9_\Lambda$Be, makes it an obvious and serious alternative to a partially nine-body model. Moreover, an $\Lambda\alpha\alpha$ model is expected to be superior for the excited state as compared to the ground state because the $\alpha\alpha$ separation will be larger for $l = 2$ than for $l = 0$ and probably with less resulting distortion in the presence of a Λ. Nonetheless, the partially/full nine-body problem will still be a preferred option over the cluster model approach because of the two counts: first, in exploring the effect of interplay between various components of the ΛN forces and their definite contributions to the properties of the system as investigated in the earlier work [15] and, second, it is these microscopic calculations that ultimately guide in improving/modifying the cluster model calculations as is indicated from the earlier analyzes [3,10].

VI. SUMMARY

We conclude that we have made, to our knowledge the first application of VMC method to analyze the degenerate doublet ($3^+/2, 5^+/2$) of ${}^9_\Lambda$Be, treating it as a partially nine-body system in the $\Lambda\alpha\alpha$ cluster model using the simple potentials and correlation functions. The Urbana ΛN potential consistent with the Λp-scattering data along with the dispersive ΛNN or dispersive plus two-pion exchange ΛNN forces are adequate to explain the emission of γ-ray of energy 3.04 MeV from the degenerate doublet to the ground state of ${}^9_\Lambda$Be. In the absence of any definite knowledge about the strength and cut-off radius of two-pion exchange ΛNN, it is not possible, at present, to make a unique separation of the contributions of ΛNN forces. Further, ${}^9_\Lambda$Be in the excited state is highly deformed and has an oblate shape and the quadrupole moment in $3^+/2$ state is less in magnitude than for the calculated in $5^+/2$. The earlier three-body $\Lambda\alpha\alpha$ cluster model approach also gives results for ${}^9_\Lambda$Be consistent with the partially nine-body model. Notwithstanding cluster model approach provides a simple and serious alternative to the A-body approach in explaining the properties of hypernuclei, the partially/full nine-body model with realistic forces still remains and will be an indispensable preferred choice for extracting information about the contributions of the various components of ΛN forces and in studying the effect of NN correlations on these and ultimately for dictating the improvement and supplementing the cluster model calculations.

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