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Bohr model as an algebraic collective model

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Developments and applications are presented of an algebraic version of Bohr's collective model. Illustrative examples show that fully converged calculations can be performed quickly and easily for a large range of Hamiltonians. As a result, the Bohr model becomes an effective tool in the analysis of experimental data. The examples are chosen both to confirm the reliability of the algebraic collective model and to show the diversity of results that can be obtained by its use. The focus of the paper is to facilitate identification of the limitations of the Bohr model with a view to developing more realistic, computationally tractable models.

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I. INTRODUCTION

The Bohr collective model [1] and its unified-model extensions by Bohr, Mottelson, and numerous colleagues [2,3] provide the essential language and phenomenological framework for understanding collective structure in nuclei. The interacting boson model (IBM) [4] has also shown the huge advantages to be gained by a model expressed in algebraic terms, which can be used to quickly characterize a large body of nuclear phenomena. We show here that the algebraic collective model (ACM) combines the advantages of both models. The ACM was recently formulated [5–7] as a computationally tractable version of Bohr's collective model [1] restricted to rotational and quadrupole vibrational degrees of freedom. Thus, it is a model with a clear physical content and, as we show, its algebraic structure makes collective model calculations in the ACM a simple routine procedure. Early applications of the Bohr model along these lines were made in Refs. [5,6,8].

Central ingredients of the ACM are (i) a spectrum generating algebra, given by the Lie algebra of an SU(1, 1) × SU(5) dynamical group and (ii) the use of modified oscillator representations of the su(1, 1) Lie algebra in terms of which collective model calculations of deformed nuclei are much more rapidly convergent than in the conventional U(5) ⊃ SO(5) harmonic-vibrational basis of Chacón *et al.* [9]. The latter basis was developed into an effective computational program by Hess and others [10,11]. The SU(1, 1) × SU(5) dynamical group capitalizes on the factorization of the Bohr model Hilbert space of square-integrable wave functions on the five-dimensional coordinate space of the Bohr model into a product of β (radial) wave functions and complementary angular wave functions, spanned by SO(5) spherical harmonics. As a result, and as we demonstrate in this paper, the ACM enables collective model calculations to be carried out quickly and easily using algebraic expressions, for the β matrix elements and SO(5)-reduced matrix elements of the model's observables, and the SO(5) Clebsch-Gordan coefficients that are now available in both exact and floating point arithmetic [12–14].

Other recent developments [15,16] have simplified the Bohr model by introducing potentials for which the β and γ degrees of freedom of the Bohr model are decoupled. A treatment based on angular-momentum projection from a canonical Cartan-Weyl basis for the SO(5) irreducible representations (irreps) has also been proposed by Baerdemacker *et al.* [17].

By its nature, the Bohr collective model restricts consideration to a small subset of nuclear dynamical degrees of freedom, which are presumed to be those that are active in corresponding subsets of low-energy nuclear states. Thus, the facility to determine the extent to which subsets of low-energy nuclear data can be realistically described in terms of such a collective model is a valuable asset in interpreting their significance. At the same time, it is important to keep sight of the fact that a good model is one that reveals its limitations as well as its successes. Thus, we do not consider the Bohr model to be an end in itself. We consider it rather as an essential step in the progression toward a realistic microscopic theory of nuclear collective phenomena. An important objective is therefore to determine those aspects of experimental data that are not realistically described by the Bohr model. We return to such considerations in the discussion section.

This paper is organized as follows. After a brief outline of the ACM version of the Bohr model, we give a summary of the algebraic expressions for a wide variety of β matrix elements and SO(5)-reduced matrix elements of model observables and how they can be combined to give the matrix elements of an essentially unlimited set of model Hamiltonians and operators of interest. The intent of this section is to gather together in one place all the results needed for the subsequent application and development of the ACM. In a section on applications, we first consider the analytically solvable submodels and adiabatic limits of the Bohr model and show how these submodels are approached, without approximation, in the ACM. We then show by illustrative applications how easy it now is to perform model calculations of coupled rotations and β and γ vibrations for potentials with arbitrary equilibrium deformations. (Calculations by Thiamova *et al.* [18] are being done separately to illustrate how the recent developments can

be utilized in IBM calculations.) Finally, in the discussion section, we consider some of the limitations of the Bohr model and what might be done about them.

II. BASIS WAVE FUNCTIONS FOR THE ACM

The coordinate space of the collective model, when restricted to quadrupole degrees of freedom, is a real five-dimensional Euclidian space, $\mathbb{R}^5$, whose five coordinates define the quadrupole moments of a model nucleus. Conversely, the set of quadrupole moments $\{q_m; m = 0, \pm1, \pm2\}$ for the model density distribution defines a suitable set of spherical polar coordinates (β, γ, Ω) for $\mathbb{R}^5$ by the expansion

$$q_m(\beta, \gamma, \Omega) = \beta\Big[\cos\gamma\,\mathcal{D}^2_{0m}(\Omega) + \frac{1}{\sqrt{2}}\sin\gamma\left(\mathcal{D}^2_{2m}(\Omega) + \mathcal{D}^2_{-2m}(\Omega)\right)\Big], \tag{1}$$

where β is a radial coordinate, Ω is a triple of Euler angles for an SO(3) rotation, and (γ, Ω) is a set of coordinates for S_4, the unit four-sphere in $\mathbb{R}^5$.

As an aside, we note that in his original model [1], Bohr introduced an expansion corresponding to that of Eq. (1) for a set of shape coordinates for a liquid-drop interpretation of his model. We prefer to use quadrupole moments, because they are well defined for any nuclear density distribution; thus, their use relaxes the necessity of supposing the nucleus to have a well-defined surface. They also have a microscopic expression in terms of many-nucleon coordinates which is important for an extension to a microscopic collective model.

The Hilbert space for the model is the space $\mathcal{L}^2(\mathbb{R}^5)$ of square-integrable functions of the $\mathbb{R}^5$ coordinates. Quadrupole moments and canonical momentum observables are interpreted as operators on the functions of this Hilbert space by

$$\begin{aligned}\hat{q}_m\psi(\beta, \gamma, \Omega) &= q_m(\beta, \gamma, \Omega)\psi(\beta, \gamma, \Omega),\\ \hat{\pi}^m\psi(\beta, \gamma, \Omega) &= -i\hbar\frac{\partial}{\partial q_m}\psi(\beta, \gamma, \Omega),\end{aligned} \tag{2}$$

for $m = 0, \pm1, \pm2$. They span a Heisenberg-Weyl Lie algebra with commutation relations

$$[\hat{q}_m, \hat{\pi}^n] = i\hbar\delta^n_m\hat{I}, \tag{3}$$

where $\hat{I}$ is the identity operator.

The (β, γ, Ω) spherical polar coordinates correspond to an expression of $\mathbb{R}^5$ as a product manifold, $\mathbb{R}^5 \simeq \mathbb{R}_+ \times S_4$, of the radial line $\mathbb{R}_+$ and the unit four-sphere. The volume element for $\mathbb{R}^5$ also factors and is given [1] by

$$dv = \beta^4 d\beta \times \sin 3\gamma\, d\gamma d\Omega, \tag{4}$$

where $d\Omega$ is the standard SO(3)-invariant volume element. An orthonormal basis for the Hilbert space $\mathcal{L}^2(\mathbb{R}^5)$ is then given by states $\{|\lambda\nu; \upsilon\alpha LM\rangle\}$ with product wave functions,

$$\Phi_{\lambda\nu;\upsilon\alpha LM}(\beta, \gamma, \Omega) = \frac{1}{\beta^2}\mathcal{R}^\lambda_\nu(\beta)\,\Psi_{\upsilon\alpha LM}(\gamma, \Omega), \tag{5}$$

constructed from a set of β wave functions $\{\mathcal{R}^\lambda_\mu\}$ and angular wave functions $\{\Phi_{\lambda\nu;\upsilon\alpha LM}\}$ that, respectively, reduce the subalgebra chains

$$\begin{array}{ccccccccc}\mathrm{SU}(1,1) & \supset & \mathrm{U}(1), & \mathrm{SO}(5) & \supset & \mathrm{SO}(3) & \supset & \mathrm{SO}(2),\\ \lambda & & \nu & \upsilon & \alpha & L & & M\end{array} \tag{6}$$

where α is a multiplicity index to label multiple occurrences of the SO(3) irrep of angular momentum L, within a given irrep of SO(5) angular momentum υ (known as seniority). Thus, the β wave functions have inner products given by

$$\int_0^\infty \mathcal{R}^\lambda_\mu(\beta)\mathcal{R}^\lambda_\nu(\beta)\,d\beta = \delta_{\mu\nu}, \tag{7}$$

and the angular wave functions are SO(5) spherical harmonics with inner products

$$\int_{S_4}\Psi^*_{\upsilon\alpha LM}(\gamma, \Omega)\,\Psi_{\upsilon'\alpha' L'M'}(\gamma, \Omega)\sin 3\gamma\, d\gamma d\Omega = \delta_{\upsilon\upsilon'}\delta_{\alpha\alpha'}\delta_{LL'}\delta_{MM'}. \tag{8}$$

Because of the orthogonality of the SO(5) spherical harmonics, we can choose a basis $\{|\lambda_\upsilon\nu; \upsilon\alpha LM\rangle\}$ with a different su(1, 1) irrep, labeled by λ_υ, for each different value of the SO(5) angular momentum (seniority) υ. The choice is, in principle, arbitrary. The standard basis of eigenstates of the harmonic spherical vibrator, for example, corresponds to the choice $\lambda_\upsilon = \upsilon + 5/2$. The so-called Davidson basis [19–21] (see also Sec. VII B) corresponds to the choice $\lambda_\upsilon = 1 + \sqrt{(\upsilon + 3/2)^2 + \beta_0^4}$, where β_0 is a constant. As we show in Secs. IV and V, the ACM gives analytical expressions for all required matrix elements in the harmonic spherical vibrator. It does not give them in a Davidson basis. However, eigenfunctions of a Bohr model Hamiltonian are much more rapidly convergent for deformed nuclei in a Davidson basis when β_0 is given the equilibrium value of β for the model. The favored choice for the ACM, which has the advantages of both the spherical vibrator basis (analytical matrix elements) and the Davidson basis (rapid convergence) is to set $\lambda_\upsilon = \lambda_0(\beta_0)$ (for υ even) and $\lambda_\upsilon = \lambda_0(\beta_0) \pm 1$ (for υ odd), with $\lambda_0(\beta_0) = 1 + \sqrt{9/4 + (a\beta_0)^4}$, where a is a scale parameter (included for later convenience). However, other choices are possible.

As this paper will demonstrate, one does not need explicit expressions for either the SO(5) spherical harmonics or the β wave functions to carry out ACM calculations. This is because all the necessary matrix elements are determined by the algebraic expressions given herein. However, the SO(5) spherical harmonics and β wave functions are easily obtained if required. An efficient computer code, based on the algorithm of Ref. [12], is available [13] for the SO(5) spherical harmonics. The β basis wave functions are simply analytic continuations of standard harmonic oscillator radial wave functions [6,7] and are given for $\lambda > 0$ and the chosen value of the scale parameter a by

$$\mathcal{R}^\lambda_\nu(\beta) = (-1)^\nu\sqrt{\frac{2\nu!a}{\Gamma(\lambda+\nu)}}(a\beta)^{\lambda-\frac{1}{2}}\mathrm{L}^{(\lambda-1)}_\nu(a^2\beta^2) \times e^{-a^2\beta^2/2}, \quad \nu = 0, 1, 2, \ldots, \tag{9}$$

where $\mathrm{L}^{(\lambda-1)}_\nu$ is a Laguerre polynomial (defined for all $\lambda > 0$). When $\lambda - 1/2$ is an integer, these wave functions form an orthonormal basis, with respect to the inner product (7), for

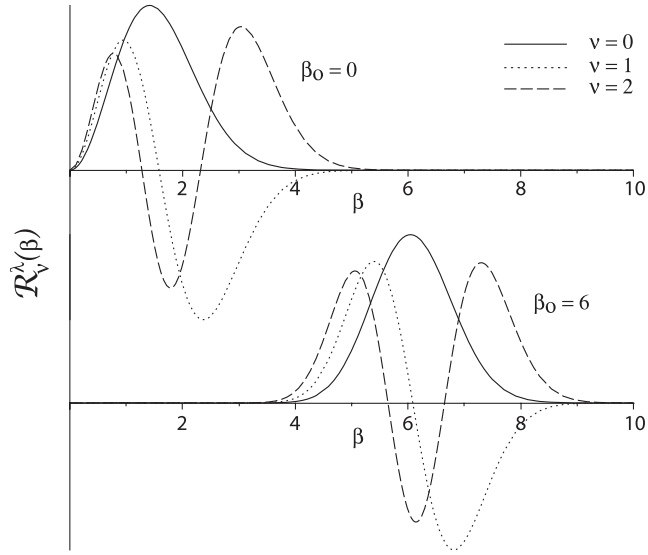

FIG. 1. Radial (β) wave functions, $\mathcal{R}^{\lambda}_{\nu}$, shown for $a = 1$, with $\lambda = \lambda_0(\beta_0)$, for $\beta_0 = 0$ and $\beta_0 = 6$. An appropriate choice of radial basis is with β_0 equal to the equilibrium β deformation, in units of $1/a$, of the nucleus under consideration.

the so-called *harmonic series* representations of the su(1, 1) Lie algebra (see the following section). [Note that we use the lower-case notation to distinguish the su(1, 1) Lie algebra from the SU(1, 1) Lie group.] For arbitrary real $\lambda > 0$, they are an orthonormal basis for a continuous-λ series of representations [7]. We refer to them simply as *modified oscillator* wave functions.

The radial wave functions $\mathcal{R}^{\lambda_0(\beta_0)}_{\nu}(\beta)$ for $\nu = 0$, 1, and 2 and $\beta_0 = 0$ and 6 are shown as functions of β in Fig. 1 to illustrate the effect of different choices of β_0. It is emphasized that a given basis of β wave functions depends on two parameters: β_0, the mean β deformation of the basis; and a, a rigidity (scale) parameter that is inversely proportional to the width of a basis wave function about its mean. One sees that a single radial wave function for a given value of $(a\beta_0)$ describes a nucleus with a range of values of a and β_0 ranging from small but highly rigid deformations to soft but large deformations.

III. THE su (1,1) LIE ALGEBRA AND ITS REPRESENTATIONS

Whereas an su(2) algebra is spanned by angular-momentum operators, J_1, J_2, J_3, with commutation relations

$$[J_1, J_2] = \mathrm{i}J_3, \quad [J_2, J_3] = \mathrm{i}J_1, \quad [J_3, J_1] = \mathrm{i}J_2, \tag{10}$$

an su(1, 1) Lie algebra [22] has basis elements X_1, X_2, X_3 with commutation relations

$$[X_1, X_2] = -\mathrm{i}X_3, \quad [X_2, X_3] = \mathrm{i}X_1, \quad [X_3, X_1] = \mathrm{i}X_2. \tag{11}$$

As for su(2), the representations of su(1, 1) of interest are most usefully constructed in terms of the combinations

$$S_0 = X_3, \quad S_{\pm} = X_1 \pm \mathrm{i}X_2, \tag{12}$$

which have commutation relations

$$[S_0, S_{\pm}] = \pm S_{\pm}, \quad [S_-, S_+] = 2S_0. \tag{13}$$

For present purposes, we require unitary irreducible representations of this algebra, in which $\{S_0, S_{\pm}\}$ are represented as operators $\{\hat{S}^{(\lambda)}_0, \hat{S}^{(\lambda)}_{\pm}\}$ on the Hilbert space of β wave functions with inner product given by Eq. (7). We also require irreps with lowest weight (corresponding to lowest energy) states. Each such representation is of the form

$$\begin{aligned} 2\hat{S}^{(\lambda)}_0 \mathcal{R}^{\lambda}_{\nu} &= (\lambda + 2\nu)\mathcal{R}^{\lambda}_{\nu}, \\ \hat{S}^{(\lambda)}_+ \mathcal{R}^{\lambda}_{\nu} &= \sqrt{(\lambda+\nu)(\nu+1)}\,\mathcal{R}^{\lambda}_{\nu+1}, \\ \hat{S}^{(\lambda)}_- \mathcal{R}^{\lambda}_{\nu} &= \sqrt{(\lambda+\nu-1)\nu}\,\mathcal{R}^{\lambda}_{\nu-1}. \end{aligned} \tag{14}$$

The lowest weight state then has a wave function $\mathcal{R}^{\lambda}_0$ that satisfies $\hat{S}^{(\lambda)}_- \mathcal{R}^{\lambda}_0 = 0$ and is an eigenstate of $2\hat{S}^{(\lambda)}_0$ with eigenvalue λ. Such an irrep is well-defined and nontrivial for any positive real value of λ and is said to have lowest weight λ.

The subset of these irreps with 2λ restricted to positive integers, known as harmonic-series representations, provides the radial wave functions for harmonic oscillator eigenstates. The more general continuous-λ irreps [23,24] provide the radial wave functions for eigenstates of central-force problems with generalized potentials of the type introduced by Davidson [19] in molecular physics. We refer to these irreps as *modified oscillator* representations. Five-dimensional versions of the Davidson potential were introduced into nuclear physics in Refs. [25] and [20].

As shown in Refs. [5,6,21] and more generally in Ref. [7], the modified-oscillator su(1, 1) irreps provide bases, with analytically defined matrix elements, for the diagonalization of many radial Hamiltonians in any dimension. They make use of realizations of the su(1, 1) operators given by

$$\hat{\mathcal{S}}^{(\lambda)}_{\pm} = \frac{1}{4}\left[\frac{1}{a^2}\frac{d^2}{d\beta^2} - \frac{\left(\lambda-\frac{3}{2}\right)\left(\lambda-\frac{1}{2}\right)}{(a\beta)^2} + (a\beta)^2 \mp \left(2\beta\frac{d}{d\beta}+1\right)\right], \tag{15}$$

$$\hat{\mathcal{S}}^{(\lambda)}_0 = \frac{1}{4}\left[-\frac{1}{a^2}\frac{d^2}{d\beta^2} + \frac{\left(\lambda-\frac{3}{2}\right)\left(\lambda-\frac{1}{2}\right)}{(a\beta)^2} + (a\beta)^2\right], \tag{16}$$

for arbitrary values of the rigidity parameter a. However, an essential characteristic of the ACM is that, while any choice of λ provides a basis of β wave functions, the calculations are much more rapidly convergent and therefore far more efficient if optimal choices are made for λ and a. We discuss the determination of optimal choices in Sec. VII B.

IV. β MATRIX ELEMENTS

In the orthonormal $|\lambda_v \nu; v\alpha LM\rangle$ bases, defined in Sec. II, the matrix elements

$$\begin{aligned} &\langle \lambda_{v'}\mu; v'\alpha' L'M' | \hat{W} | \lambda_v \nu; \alpha LM\rangle \\ &\quad = \int_{\mathbb{R}^5} \Phi_{\lambda\mu;v'\alpha'L'M'}(\beta,\gamma,\Omega)\hat{W}\Phi_{\lambda\nu;v\alpha LM}(\beta,\gamma,\Omega) \\ &\qquad \times \beta^4 d\beta \sin 3\gamma\, d\gamma d\Omega \end{aligned} \tag{17}$$

of any operator $\hat{W} = \hat{X}\hat{Y}$ that is a product of operators $\hat{X}$ and $\hat{Y}$, which act independently of the β-radial and SO(5)-angular components of a wave function, respectively, are given by the products

$$\langle \lambda_{v'}\mu; v'\alpha' L' M'|\hat{W}|\lambda_v \nu; \alpha L M\rangle = F_{\lambda_{v'}\mu;\lambda_v\nu}\Big(\beta^2 \hat{X}\frac{1}{\beta^2}\Big)\langle v'\alpha' L' M'|\hat{Y}|v\alpha L M\rangle, \quad (18)$$

where

$$F_{\lambda_{v'}\mu;\lambda_v\nu}(\hat{Z}) = \int_0^\infty \mathcal{R}_\mu^{\lambda_{v'}}(\beta)\big[\hat{Z}\mathcal{R}_\nu^{\lambda_v}(\beta)\big]\,d\beta, \quad (19)$$

and

$$F_{\lambda_{v'}\mu;\lambda_v\nu}\Big(\beta^2\hat{X}\frac{1}{\beta^2}\Big) = \int_0^\infty \frac{1}{\beta^2}\mathcal{R}_\mu^{\lambda_{v'}}(\beta)\left[\hat{X}\frac{1}{\beta^2}\mathcal{R}_\nu^{\lambda_v}(\beta)\right]\beta^4\,d\beta. \quad (20)$$

For the operators of the su(1, 1) algebra, the representation equations (14) lead directly to the identities

$$F_{\lambda\mu;\lambda\nu}(\hat{S}_0^{(\lambda)}) = \frac{1}{2}(\lambda+2\nu)\delta_{\mu,\nu}, \quad (21)$$

$$F_{\lambda\mu;\lambda\nu}(\hat{S}_+^{(\lambda)}) = \sqrt{(\lambda+\nu)(\nu+1)}\,\delta_{\mu,\nu+1}, \quad (22)$$

$$F_{\lambda\mu;\lambda\nu}(\hat{S}_-^{(\lambda)}) = \sqrt{(\lambda+\nu-1)\nu}\,\delta_{\mu,\nu-1}. \quad (23)$$

From linear combinations of these matrix elements and a recursion relation that follows from them, one also obtains the matrix elements

$$F_{\lambda\mu;\lambda\nu}(a^2\beta^2) = \delta_{\mu,\nu-1}\sqrt{(\lambda+\nu-1)\nu} + \delta_{\mu,\nu+1}\sqrt{(\lambda+\nu)(\nu+1)} + \delta_{\mu,\nu}(\lambda+2\nu), \quad (24)$$

$$F_{\lambda\mu;\lambda\nu}\Big(\beta\frac{d}{d\beta}\Big) = \delta_{\mu,\nu-1}\sqrt{(\lambda+\nu-1)\nu} - \delta_{\mu,\nu+1}\sqrt{(\lambda+\nu)(\nu+1)} - \tfrac{1}{2}\delta_{\mu,\nu}, \quad (25)$$

$$F_{\lambda\mu;\lambda\nu}\Big(\frac{1}{(a\beta)^2}\Big) = \frac{(-1)^{\mu-\nu}}{\lambda-1}\sqrt{\frac{\mu!\,\Gamma(\lambda+\nu)}{\nu!\Gamma(\lambda+\mu)}}, \qquad \text{for } \mu \geqslant \nu, \quad (26)$$

$$F_{\lambda\mu;\lambda\nu}\Big(\frac{1}{(a\beta)^2}\Big) = \frac{(-1)^{\mu-\nu}}{\lambda-1}\sqrt{\frac{\nu!\,\Gamma(\lambda+\mu)}{\mu!\Gamma(\lambda+\nu)}}, \qquad \text{for } \mu \leqslant \nu, \quad (27)$$

$$F_{\lambda\mu;\lambda\nu}\Big(\frac{1}{a^2}\frac{d^2}{d\beta^2}\Big) = \delta_{\mu,\nu-1}\sqrt{(\lambda+\nu-1)\nu} + \delta_{\mu,\nu+1}\sqrt{(\lambda+\nu)(\nu+1)} - \delta_{\mu,\nu}(\lambda+2\nu) + \big(\lambda-\tfrac{3}{2}\big) \times \big(\lambda-\tfrac{1}{2}\big)F_{\lambda\mu;\lambda\nu}(1/(a\beta)^2), \quad (28)$$

as shown, explicitly, in Refs. [6,7]. Thus, for example, one obtains the matrix elements of a Davidson-type potential, $V(\beta) = k_1\beta^2 + k_2/\beta^2$. Furthermore, matrix elements of any polynomial in β and $d/d\beta$ that is of even degree, between states of any given su(1, 1) irrep, can be obtained by summing over intermediate states, e.g.,

$$F_{\lambda\mu;\lambda\nu}\left(\beta^2\frac{d^2}{d\beta^2}\right) = \sum_\sigma F_{\lambda\mu;\lambda\sigma}\left(\beta\frac{d}{d\beta}\right)F_{\lambda\sigma;\lambda\nu}\left(\beta\frac{d}{d\beta}\right) - F_{\lambda\mu;\lambda\nu}\left(\beta\frac{d}{d\beta}\right). \quad (29)$$

[Matrix elements of functions that are sums of Gaussians, e.g., $V(\beta) = k_1\exp(-c_1\beta^2) - k_2\exp(-c_2\beta^2)$, are also evaluated without difficulty but are not considered here.]

Matrix elements of odd polynomials in β and $d/d\beta$ are obtained [7], by the so-called factorization method (reviewed, e.g., in Ref. [26]), using the factorizations

$$4\hat{\mathcal{S}}_0^{(\lambda)} = A\big(\lambda-\tfrac{1}{2}\big)A^\dagger\big(\lambda-\tfrac{1}{2}\big) - 2\lambda, \quad (30)$$

$$4\hat{\mathcal{S}}_0^{(\lambda)} = A^\dagger\big(\lambda-\tfrac{3}{2}\big)A\big(\lambda-\tfrac{3}{2}\big) - 2\lambda + 4, \quad (31)$$

$$4\hat{\mathcal{S}}_0^{(\lambda)} = A\big(-\lambda+\tfrac{3}{2}\big)A^\dagger\big(-\lambda+\tfrac{3}{2}\big) + 2\lambda - 4, \quad (32)$$

$$4\hat{\mathcal{S}}_0^{(\lambda)} = A^\dagger\big(-\lambda+\tfrac{1}{2}\big)A\big(-\lambda+\tfrac{1}{2}\big) + 2\lambda, \quad (33)$$

where $A(k)$ and $A^\dagger(k)$ are defined for a real number k by

$$A(k) := \frac{1}{a}\frac{d}{d\beta} + \frac{k}{a\beta} + a\beta, \quad A^\dagger(k) := -\frac{1}{a}\frac{d}{d\beta} + \frac{k}{a\beta} + a\beta. \quad (34)$$

These factors resemble the raising and lowering operators of a harmonic oscillator Hamiltonian. As shown in Ref. [7], they raise and lower the β wave functions according to the identities

$$A^\dagger\big(\lambda-\tfrac{1}{2}\big)\mathcal{R}_\nu^\lambda = 2\sqrt{\lambda+\nu}\,\mathcal{R}_\nu^{\lambda+1}, \quad (35)$$

$$A\big(\lambda-\tfrac{3}{2}\big)\mathcal{R}_\nu^\lambda = 2\sqrt{\lambda+\nu-1}\,\mathcal{R}_\nu^{\lambda-1}, \quad \text{for } \lambda > 1, \quad (36)$$

$$A^\dagger\big(-\lambda+\tfrac{3}{2}\big)\mathcal{R}_\nu^\lambda = 2\sqrt{\nu+1}\,\mathcal{R}_{\nu+1}^{\lambda-1}, \quad \text{for } \lambda > 1, \quad (37)$$

$$A\big(-\lambda+\tfrac{1}{2}\big)\mathcal{R}_\nu^\lambda = 2\sqrt{\nu}\,\mathcal{R}_{\nu-1}^{\lambda+1}, \quad \text{for } \nu > 1. \quad (38)$$

From them, one obtains the β matrix elements

$$F_{\lambda+1,\mu;\lambda\nu}(a\beta) = \delta_{\mu,\nu}\sqrt{\lambda+\nu} + \delta_{\mu,\nu-1}\sqrt{\nu}, \quad (39)$$

$$F_{\lambda-1,\mu;\lambda\nu}(a\beta) = \delta_{\mu,\nu}\sqrt{\lambda+\nu-1} + \delta_{\mu,\nu+1}\sqrt{\nu+1}, \quad (40)$$

$$F_{\lambda-1,\mu;\lambda\nu}\Big(\frac{1}{a\beta}\Big) = \begin{cases} 0 & \text{if } \mu > \nu, \\ (-1)^{\mu-\nu}\sqrt{\dfrac{\nu!\Gamma(\lambda+\mu-1)}{\mu!\Gamma(\lambda+\nu)}} & \text{if } \mu \leqslant \nu, \end{cases} \quad (41)$$

$$F_{\lambda+1,\mu;\lambda\nu}\Big(\frac{1}{a\beta}\Big) = \begin{cases} 0 & \text{if } \mu < \nu, \\ (-1)^{\mu-\nu}\sqrt{\dfrac{\mu!\Gamma(\lambda+\nu)}{\nu!\Gamma(\lambda+\mu+1)}} & \text{if } \mu \geqslant \nu, \end{cases} \quad (42)$$

$$F_{\lambda-1,\mu;\lambda\nu}\Big(\frac{1}{a}\frac{d}{d\beta}\Big) = \delta_{\mu,\nu}\sqrt{\lambda+\nu-1} - \delta_{\mu,\nu+1}\sqrt{\nu+1} - (\lambda-\tfrac{3}{2})F_{\lambda-1,\mu;\lambda\nu}\Big(\frac{1}{a\beta}\Big), \quad (43)$$

$$F_{\lambda+1,\mu;\lambda\nu}\Big(\frac{1}{a}\frac{d}{d\beta}\Big) = -\delta_{\mu,\nu}\sqrt{\lambda+\nu} + \delta_{\mu,\nu-1}\sqrt{\nu} + (\lambda-\tfrac{1}{2})F_{\lambda+1,\mu;\lambda\nu}\Big(\frac{1}{a\beta}\Big). \quad (44)$$

These β matrix elements are sufficient for deriving the matrix elements of any rotationally invariant polynomial (including inverse square β^2 terms) in the collective model observables $\{\hat{q}_m, \hat{\pi}^m\}$. Moreover (see further discussion in Sec. VI and in the Appendix), it is a simple matter to extend the basic ACM to include irrational functions of $\{\hat{q}_m\}$ which are polynomials in $\beta = \sqrt{\sum_m |q_m|^2}$ and $1/\beta$.

V. SO(5) MATRIX ELEMENTS

Because of the unitarity of the Wigner rotation matrices, it follows from Eq. (1) that $\sum_m |q_m|^2 = \beta^2$. Thus, β is regarded as the "length" of a vector $q \in \mathbb{R}^5$. Quadrupole moments $\{\mathcal{Q}_m; m = 0, \pm 1, \pm 2\}$ for points on the unit four-sphere S_4 are then defined by setting $\beta = 1$ in Eq. (1), i.e., by

$$\mathcal{Q}_m(\gamma, \Omega) = \cos\gamma \mathcal{D}^2_{0m}(\Omega) + \frac{1}{\sqrt{2}} \sin\gamma \left(\mathcal{D}^2_{2m}(\Omega) + \mathcal{D}^2_{-2m}(\Omega)\right). \quad (45)$$

The latter quadrupole moments satisfy the equation

$$\sum_m |\mathcal{Q}_m|^2 = 1 \quad (46)$$

and are therefore described as *unit quadrupole moments*. They are functions on S_4 that transform as a basis for a $v = 1$ irrep of SO(5). Thus, they are proportional to $v = 1$, SO(5) spherical harmonics which, from the inner product (8), are determined to be given by

$$\Psi_{112m} = \frac{\sqrt{15}}{4\pi} \mathcal{Q}_m, \quad (47)$$

These unit quadrupole moments provide the building blocks for construction of the general SO(5) spherical harmonics that are needed for computing SO(5) Clebsch-Gordan coefficients [12,13]. They are important for present purposes because, as we show, collective model observables are expressible as functions of β, $d/d\beta$, and these unit quadrupole moments.

For the purpose of calculating matrix elements of collective model observables, it is useful to introduce operators $\{\hat{\mathcal{Q}}_m\}$, which act multiplicatively on a wave function $\psi \in \mathcal{L}^2(S_4)$, such that the function $\hat{\mathcal{Q}}_m\psi$ has values given by

$$\hat{\mathcal{Q}}_m\psi(\gamma, \Omega) = \mathcal{Q}_m(\gamma, \Omega)\psi(\gamma, \Omega). \quad (48)$$

It is then meaningful to speak of the matrix elements of these operators as components of a $v = 1$, SO(5) tensor operator.

Let $\{|v\alpha LM\rangle\}$ denote the orthonormal basis states for $\mathcal{L}^2(S_4)$, which reduce the SO(5) subgroup chain of Eq. (6) and whose wave functions are SO(5) spherical harmonics. Recall that if $\{\hat{T}^v_{\alpha LM}\}$ are components of an SO(5) tensor operator that transform as the basis states $\{|v\alpha LM\rangle\}$, then, according to the Wigner-Eckart theorem (given in any book on angular momentum theory), matrix elements of these operators are expressed in terms of SO(3)-reduced matrix elements $\langle v_f\alpha_f L_f \|\hat{T}^v_{\alpha L}\| v_i\alpha_i L_i\rangle$ by the equation

$$\langle v_f\alpha_f L_f M_f|\hat{T}^v_{\alpha LM}|v_i\alpha_i L_i M_i\rangle = (L_i M_i LM|L_f M_f)\frac{\langle v_f\alpha_f L_f\|\hat{T}^v_{\alpha L}\|v_i\alpha_i L_i\rangle}{\sqrt{2L_f+1}}, \quad (49)$$

and in terms of SO(5)-reduced matrix elements $\langle v_f|||\hat{T}^v|||v_i\rangle$ by

$$\langle v_f\alpha_f L_f M_f|\hat{T}^v_{\alpha LM}|v_i\alpha_i L_i M_i\rangle = (v_i\alpha_i L_i M_i v\alpha LM|v_f\alpha_f L_f M_f)\langle v_f|||\hat{T}^v|||v_i\rangle, \quad (50)$$

where $(L_i M_i LM|L_f M_f)$ is an SO(3) Clebsch-Gordan coefficient and $(v_i\alpha_i L_i M_i v\alpha LM|v_f\alpha_f L_f M_f)$ is an SO(5) Clebsch-Gordan coefficient. Racah's factorization lemma [27] combines Eqs. (49) and (50) to give

$$\frac{\langle v_f\alpha_f L_f\|\hat{T}^v_{\alpha L}\|v_i\alpha_i L_i\rangle}{\sqrt{2L_f+1}} = (v_i\alpha_i L_i v\alpha L\|v_f\alpha_f L_f) \times \langle v_f|||\hat{T}^v|||v_i\rangle, \quad (51)$$

where $(v_i\alpha_i L_i v\alpha L\|v_f\alpha_f L_f)$, a so-called SO(3)-reduced SO(5) Clebsch-Gordan coefficient (sometimes referred to as an *isoscalar factor*), is defined by

$$(v_i\alpha_i L_i M_i v\alpha LM|v_f\alpha_f L_f M_f) = (L_i M_i\, LM|L_f M_f)(v_i\alpha_i L_i\, v\alpha L\|v_f\alpha_f L_f). \quad (52)$$

These reduced coefficients satisfy the symmetry relation [12]

$$(v_f\alpha_f L_f v\alpha L\|v_i\alpha_i L_i) = (-1)^{L_i+L-L_f}\sqrt{\frac{\dim(v_i)}{\dim(v_f)}\frac{(2L_f+1)}{(2L_i+1)}} \times (v_i\alpha_i L_i\, v\alpha L\|v_f\alpha_f L_f), \quad (53)$$

where

$$\dim(v) = \tfrac{1}{6}(v+1)(v+2)(2v+3) \quad (54)$$

is the dimension of the SO(5) irrep $[v]$. In the following, we generally omit the "SO(3)-reduced" qualifier and simply refer to the $(v_i\alpha_i L_i\, v\alpha L\|v_f\alpha_f L_f)$ coefficients as SO(5) $\supset$ SU(3) Clebsch-Gordan coefficients [or simply as SO(5) Clebsch-Gordan coefficients when the SO(3) basis is understood]).

It is shown in Refs. [6,7] that SO(5)-reduced matrix elements of the above-defined $\hat{\mathcal{Q}}$ tensor are given by

$$\langle v'|||\hat{\mathcal{Q}}|||v\rangle = \sqrt{\frac{v+1}{2v+5}}\,\delta_{v',v+1} + \sqrt{\frac{v+2}{2v+1}}\,\delta_{v',v-1}. \quad (55)$$

From this analytical expression for the reduced matrix elements of the $\hat{\mathcal{Q}}$ tensor and the analytical β matrix elements of Eqs. (39), it follows from Eqs. (49) and (50) that analytical expressions are obtained for all the matrix elements of the collective model quadrupole moments in the SU(1, 1) $\times$ SU(5)-coupled basis in terms of now-known SO(5) Clebsch-Gordan coefficients. These matrix elements are needed, in particular, for the calculation of $E2$ transition rates in the ACM.

Other important matrix elements are those of SO(3)-invariant Hamiltonians. The only SO(3)-invariant polynomials that can be formed in the elements of a traceless ($L = 2$) quadrupole matrix q are generated by the elementary invariants: $\mathrm{Tr}(q^2) \propto [q \times q]_0 \propto \beta^2$ and $\mathrm{Tr}(q^3) \propto [q \times q \times q]_0 = \beta^3 \cos 3\gamma$ [28]. It follows that any SO(3)-invariant polynomials in the $\{\mathcal{Q}_m\}$ quadrupole moments (for which $\beta = 1$) are polynomials in $\cos 3\gamma$. Thus, together with kinetic energy terms in ∇^2, whose matrix elements are given below, the important matrix elements are those of powers of $\cos 3\gamma$.

From the algorithm for generating SO(5) spherical harmonics [12], it is immediately determined that $\Psi_{3100}(\gamma, \Omega)$ is proportional to $\cos 3\gamma$. The inner product (8) determines its normalization and gives

$$\Psi_{3100}(\gamma, \Omega) = \frac{3}{4\pi} \cos 3\gamma. \tag{56}$$

More generally, it follows from the identity

$$\int_0^{\pi/3} P_n(\cos 3\gamma) P_m(\cos 3\gamma) \sin 3\gamma \, d\gamma = \frac{1}{3} \int_{-1}^{+1} P_n(z) P_m(z) \, dz = \frac{2}{3(2n+1)} \tag{57}$$

for Legendre polynomials $P_n(z)$ that

$$\Psi_{3n,100}(\gamma, \Omega) = \frac{1}{4\pi} \sqrt{3(2n+1)} P_n(\cos 3\gamma). \tag{58}$$

Thus, the reduced matrix elements of powers of $\cos 3\gamma$ are obtained from those of the spherical harmonics. For example, the SO(5)-reduced matrix elements of $\cos 3\gamma$ are given by the equations (see below)

$$\langle v+3 ||| \hat{P}_1 ||| v \rangle = \sqrt{\frac{35(v+1)(v+2)(v+3)}{2(2v+5)(2v+7)(2v+9)}}, \tag{59}$$

$$\langle v+1 ||| \hat{P}_1 ||| v \rangle = \sqrt{\frac{45v(v+1)(v+4)}{2(2v+1)(2v+5)(2v+7)}}, \tag{60}$$

and the symmetry relation

$$\frac{\langle v_1 ||| \hat{\Psi}_{v_2} ||| v_3 \rangle}{\langle v_3 ||| \hat{\Psi}_{v_2} ||| v_1 \rangle} = \sqrt{\frac{\dim(v_3)}{\dim(v_1)}}, \tag{61}$$

where $\hat{P}_1$ is the operator defined by $\hat{P}_1 \Psi(\gamma, \Omega) = \cos 3\gamma \, \Psi(\gamma, \Omega)$.

Other special cases of SO(5) spherical harmonics that are known in explicit form will be given in a following publication [29] in which Eqs. (59) and (60) will also be obtained as special cases of the following general result. First observe that the matrix element $\langle v_3 ||| \hat{\Psi}_{v_2} ||| v_1 \rangle$ is identically zero unless $v_1 + v_2 + v_3$ is even and unless the triangle inequality, $|v_1 - v_2| \leqslant v_3 \leqslant v_1 + v_2$, is satisfied. The first condition is a selection rule that arises because an SO(5) spherical harmonic, $\Psi_{v\alpha LM}$, is either symmetric or antisymmetric under inversion in $\mathbb{R}^5$ according to whether $(-1)^v$ is even or odd. The second selection rule is the standard SO(5) tensor-coupled product rule. By inspection of a large number of numerically computed reduced matrix elements, the nonzero SO(5)-reduced matrix elements of a general SO(5) spherical harmonic were found to satisfy the empirical formula

$$\langle v_3 ||| \hat{\Psi}_{v_2} ||| v_1 \rangle = \frac{1}{4\pi} \frac{\left(\frac{\sigma}{2}+1\right)!}{\left(\frac{\sigma}{2}-v_3\right)! \left(\frac{\sigma}{2}-v_1\right)! \left(\frac{\sigma}{2}-v_2\right)!} \sqrt{\frac{(2v_1+3)(2v_2+3)}{(v_3+2)(v_3+1)}} \times \sqrt{\frac{(\sigma+4)(\sigma-2v_3+1)!(\sigma-2v_1+1)(\sigma-2v_2+1)!}{(\sigma+3)!}}, \tag{62}$$

where $\sigma = v_1 + v_2 + v_3$. This formula has been derived algebraically in special cases. For example, the equation for the stretched matrix elements $\langle v+w ||| \hat{\Psi}_w ||| v \rangle$, of which Eq. (59) is an example, is relatively easy to derive. Moreover, the validity of Eq. (62) has been established extensively, for v_2 up to 10, by numerical calculation. Thus, it holds in all cases in which it is likely to be used in nuclear collective model calculations. However, it remains to be proved analytically in general.

VI. COMBINED MATRIX ELEMENTS

From the above results, it follows that the matrix elements of the basic observables, $\{\hat{q}_m, \hat{\pi}_m\}$, of the collective model, relative to the orthonormal basis $\{|\lambda_v \nu; v\alpha LM\rangle\}$ defined in Sec. II, are obtained from the SO(5)-reduced matrix elements

$$\langle \lambda' \mu; v' ||| \hat{q} ||| \lambda \nu; v \rangle = F_{\lambda'\mu;\lambda\nu}(\beta) \left[\delta_{v',v+1} \sqrt{\frac{v+1}{2v+5}} + \delta_{v',v-1} \sqrt{\frac{v+2}{2v+1}} \right], \tag{63}$$

$$\langle \lambda' \mu; v+1 ||| \hat{\pi} ||| \lambda \nu; v \rangle = -i\hbar F_{\lambda'\mu;\lambda\nu} \left(\frac{d}{d\beta} - \frac{v+2}{\beta} \right) \sqrt{\frac{v+1}{2v+5}}, \tag{64}$$

$$\langle \lambda' \mu; v-1 ||| \hat{\pi} ||| \lambda \nu; v \rangle = -i\hbar F_{\lambda'\mu;\lambda\nu} \left(\frac{d}{d\beta} + \frac{v+1}{\beta} \right) \sqrt{\frac{v+2}{2v+1}}. \tag{65}$$

The first of these equations follows directly from the expressions for the β matrix elements, given by Eqs. (39) and (40), and the matrix elements of $\hat{\mathcal{Q}}$, given by Eq. (55). The matrix elements of the momentum operators are obtained by expressing their β matrix elements in terms of those of the $A(k)$ and $A^\dagger(k)$ operators of Eq. (34).

Matrix elements of the Laplacian

$$\nabla^2 = \sum_m \frac{\partial^2}{\partial q_m \partial q_m^*} \tag{66}$$

for $\mathbb{R}^5$ are obtained as follows. It is known [30] that ∇^2 is expressible in the form

$$\nabla^2 = \frac{1}{\beta^4} \frac{\partial}{\partial \beta} \beta^4 \frac{\partial}{\partial \beta} - \frac{1}{\beta^2} \hat{\Lambda}^2, \tag{67}$$

where $\hat{\Lambda}^2$ is the SO(5) Casimir invariant whose eigenfunctions are the SO(5) spherical harmonics with eigenvalues given in terms of the SO(5) angular momentum v, by

$$\hat{\Lambda}^2 \Psi_{v\alpha LM} = v(v+3) \Psi_{v\alpha LM}. \tag{68}$$

Thus, the action of the Laplacian on the wave functions of Eq. (5) is given by

$$\nabla^2 \Phi_{\lambda\nu;v\alpha LM} = \frac{1}{\beta^2} \Psi_{v\alpha LM} \left[\frac{d^2}{d\beta^2} - \frac{v(v+3)+2}{\beta^2} \right] \mathcal{R}^\lambda_\nu. \tag{69}$$

It follows that the matrix elements of ∇^2 are given by

$$\langle \lambda'\mu; v'\alpha' L' M'|\nabla^2|\lambda\nu; v\alpha LM\rangle = F_{\lambda'\mu;\lambda\nu}\left(\frac{d^2}{d\beta^2} - \frac{v(v+3)+2}{\beta^2}\right)\delta_{vv'}\delta_{\alpha\alpha'}\delta_{LL'}\delta_{MM'}. \quad (70)$$

VII. THE COMPUTATIONAL PROCEDURE

The preceding sections give analytical expressions for the matrix elements of $\beta, 1/\beta, d/d\beta, \beta^2, 1/\beta^2, d^2/d\beta^2, \hat{q}_m, \hat{\pi}_m, \nabla^2$, and the SO(5)-reduced matrix elements of the SO(5) spherical harmonics that are needed for collective model calculations within the framework of the ACM.

A. Hamiltonian matrices for the ACM

Most frequently, one will want to consider Hamiltonians that are time-reversal and SO(3) invariant. For example, some Hamiltonians of interest are those of the form

$$\hat{H} = -k(\beta)\nabla^2 + f_0(\beta) + f_1(\beta)\beta\cos 3\gamma + f_2(\beta)\cos^2 3\gamma + \cdots, \quad (71)$$

where $k, f_0, f_1, f_2, \ldots,$ are low-order polynomials in β^2, $1/\beta^2$, and $\beta d/d\beta$.

The class of Hamiltonians that we consider includes polynomials in $(\hat{\pi}\times\hat{\pi})^{(0)} \propto \nabla^2$, $(\hat{q}\times\hat{q})^{(0)} \propto \beta^2$, and $(\hat{q}\times\hat{q}\times\hat{q})^{(0)} \propto \beta^3\cos 3\gamma$. By summing over intermediate states, one can also include in the Hamiltonian such terms as $(\hat{\pi}\times\hat{q}\times\hat{\pi})^{(0)}$. Thus, it includes all Hamiltonians that have customarily been considered in the Frankfurt version of the collective model [10,11]. In addition, we are able to include polynomials with terms in $1/\beta^2$, which enables us to use potential energy functions of the Davidson type. What are excluded are terms that are irrational functions of the coordinates such as a lone function of $\beta = \sqrt{\hat{q}\cdot\hat{q}}$. However, polynomials in $\beta\cos 3\gamma$, for example, are included because $\beta\cos 3\gamma \propto (\hat{q}\times\hat{q}\times\hat{q})^{(0)}/(\hat{q}\times\hat{q})^{(0)}$ is rational. Restriction to such rational Hamiltonians means that their matrix elements can be computed by purely algebraic means. However, as we discuss in the Appendix, this is not an essential constraint and can be relaxed to allow polynomials in $\beta = \sqrt{\sum_m |q_m|^2}$ as well.

From the expression for a general $L = 0$, SO(5) spherical harmonic, given by Eq. (58), the Hamiltonian (71) can also be expanded in terms of SO(5) spherical harmonics, i.e.,

$$\hat{H} = -k(\beta)\nabla^2 + g_0(\beta) + g_1(\beta)\beta\hat{\Psi}_{3100} + g_2(\beta)\hat{\Psi}_{6100} + \cdots. \quad (72)$$

Thus, being SO(3) invariant, its matrix elements are M independent, diagonal in L, and given, in a $\{|\lambda_v\nu; v\alpha LM\rangle\}$ basis [see Eqs. (18) and (50)], by

$$\begin{aligned}\langle \lambda_{v'}\mu; v'\alpha' LM|\hat{H}|\lambda_v\nu; v\alpha LM\rangle &= F_{\lambda_v\mu;\lambda_v\nu}\\ &\times\left(-k(\beta)\left[\frac{d^2}{d\beta^2} - \frac{v(v+3)+2}{\beta^2}\right] + g_0(\beta)\right)\delta_{vv'}\delta_{\alpha\alpha'}\\ &+\sqrt{2L+1}F_{\lambda_{v'}\mu;\lambda_v\nu}\big(g_1(\beta)\beta\big)(v\alpha L, 310\|v'\alpha' L)\\ &\times\langle v'|||\hat{\Psi}_{310}|||v\rangle + \sqrt{2L+1}F_{\lambda_{v'}\mu;\lambda_v\nu}(g_2(\beta))\\ &\times(v\alpha L, 610\|v'\alpha' L)\langle v'|||\hat{\Psi}_{610}|||v\rangle + \cdots. \end{aligned} \quad (73)$$

B. Choice of basis states

A characteristic feature of the expressions for the β matrix elements $F_{\lambda'\mu;\lambda\nu}(\beta)$, $F_{\lambda'\mu;\lambda\nu}(1/\beta)$, and $F_{\lambda'\mu;\lambda\nu}(d/d\beta)$, given in Sec. IV, is that they are available in analytical form only for $\lambda' = \lambda \pm 1$. However, it turns out that for the evaluation of observables that are rational functions of the q_m coordinates, this is precisely what is needed. It is found that observables that are rational functions of the q_m coordinates and are odd (even) functions of β are always accompanied by odd (even) $\mathbb{R}^5$-parity functions of γ, respectively. For example, the quadrupole moments, needed for the evaluation of electromagnetic $E2$ transitions, are linear functions of $\beta\cos\gamma$ and $\beta\sin\gamma$.

Thus, for a rational Hamiltonian, as defined above, it is convenient to choose basis states $\{|\lambda_v\nu; v\alpha LM\rangle\}$ for the ACM which are those of the harmonic spherical vibrator Hamiltonian, for which $\lambda_v = v + 5/2$, for spherical and near spherical nuclei, and by modified oscillator wave functions, with

$$\lambda_v = \begin{cases}\lambda_0 & \text{for } v \text{ even},\\ \lambda_0 \pm 1 & \text{for } v \text{ odd},\end{cases} \quad (74)$$

for deformed nuclei. In this paper, we set $\lambda = \lambda_0$ for states of even $\mathbb{R}^5$ parity and $\lambda = \lambda_0 + 1$ for states of odd $\mathbb{R}^5$ parity.

Good choices of λ_0 and the rigidity parameter a (see Sec. II) are ones that minimized the expectation value of the chosen Hamiltonian in the $\nu = v = 0$ basis state, $|\lambda_0 0; 0100\rangle$. For Hamiltonians of the form

$$\hat{H}(B, c_1, c_2, \chi, \kappa) = -\frac{\nabla^2}{2B} + \frac{1}{2}B(c_1\beta^2 + c_2\beta^4) - \chi\beta\cos 3\gamma + \kappa\cos^2 3\gamma, \quad (75)$$

used in the following examples, this expectation is given by

$$\langle\hat{H}(B, c_1, c_2, \chi, \kappa)\rangle = \frac{a^2}{2B}\frac{4\lambda_0+5}{8(\lambda_0-1)} + \frac{B}{2a^2}c_1\lambda_0 + \frac{B}{2a^4}c_2\lambda_0(\lambda_0+1) + \frac{1}{2}\kappa. \quad (76)$$

A simpler but essentially equivalent and physically more meaningful choice of λ is obtained by observing that when $\lambda = 1 + \sqrt{(a\beta_0)^4 + 9/4}$, the realization of $\hat{\mathcal{S}}_0^{(\lambda)}$ given by Eq. (16) takes the form

$$2\hat{\mathcal{S}}_0^{(\lambda)} = -\frac{1}{2a^2}\left(\frac{d^2}{d\beta^2} - \frac{2}{\beta^2}\right) + V(\beta), \quad (77)$$

where $V(\beta)$ is the Davidson potential

$$V(\beta) = \frac{1}{2}a^2\left(\beta^2 + \frac{\beta_0^4}{\beta^2}\right). \quad (78)$$

In this case, it follows from Eqs. (69) and (14) that the $v = 0$ product wave functions, $\Phi_{\lambda\nu;0\alpha LM}$, defined in Eq. (5), are eigenfunctions of the so-called modified oscillator Hamiltonian

$$H = -\frac{\nabla^2}{2a^2} + \frac{1}{2}a^2\left(\beta^2 + \frac{\beta_0^4}{\beta^2}\right). \quad (79)$$

Thus, they are eigenfunctions of a Hamiltonian whose potential energy has a minimum at $\beta = \beta_0$. Correspondingly, for a

Hamiltonian with potential energy

$$W(\beta) = \tfrac{1}{2}B(c_1\beta^2 + c_2\beta^4), \tag{80}$$

which has a minimum at $\beta = \beta_0$ given by

$$\beta_0 = \begin{cases} -\tfrac{1}{2}c_1/c_2 & \text{if } c_1 < 0, \\ 0 & \text{if } c_1 \geqslant 0, \end{cases} \tag{81}$$

it is appropriate to make the choice

$$\lambda_0 = 1 + \sqrt{(a\beta_0)^4 + 9/4} \tag{82}$$

for the value (81) of β_0. Having fixed λ_0 as a function of the rigidity parameter a, the parameter a is determined to be the value for which $\langle \hat{H}(B, c_1, c_2, \chi, \kappa)\rangle$, as given by Eq. (76), is minimal. The parameter values determined in either of these two ways turn out to be very similar and lead to equally rapidly convergent eigenfunctions. The latter method is used in deriving the results reported in this paper.

C. Reduced $E2$ transition rates

Reduced $E2$ transition rates between sets of levels have a standard definition in nuclear physics as the transition rates obtained by summing the squared matrix elements of the $E2$ transition operator $\hat{Q}^{(E)}_m = (Ze/A)\hat{q}_m$ over the final set of states and averaging over the initial states. With this definition, reduced $E2$ transition rates acquire simple expressions in terms of reduced $E2$ matrix elements when the sets of initial and final states, as well as the components of the $E2$ transition operator, transform according to an irrep of some symmetry group. For example, if the states of initial and final energy levels span SO(3) irreps, the reduced $E2$ transition rates between them are the standard SO(3)-reduced $E2$ transition rates for a transition from an initial set of states of angular momentum L_i to a final set of angular momentum L_f given by

$$B(E2; L_i \to L_f) = \left(\frac{Ze}{A}\right)^2 \frac{|\langle L_f \|\hat{q}\| L_i\rangle|^2}{2L_i + 1}. \tag{83}$$

Similarly, reduced $E2$ transition rates between SO(5) multiplets of states, labeled by quantum numbers $(v\alpha LM)$, are given by

$$\begin{aligned} \bar{B}(E2; v_i \to v_f) &= \sum_{\substack{\alpha_i L_i M_i \\ m\alpha_f L_f M_f}} [\dim(v_i)]^{-1} \\ &\quad \times |\langle v_i \alpha_i L_i M_i | \hat{Q}^{(E)}_m | v_f \alpha_f L_f M_f\rangle|^2 \\ &= \left(\frac{Ze}{A}\right)^2 |\langle v_i \| \hat{q} \| v_f\rangle|^2, \end{aligned} \tag{84}$$

where $\langle v_i \| \hat{q} \| v_f\rangle$ is an SO(5)-reduced matrix element. Application of the relationship, Eq. (51), between SO(5)- and SO(3)-reduced matrix elements then gives the relationship between these SO(5)-reduced $E2$ transition rates and the standard SO(3)-reduced transition rates:

$$\begin{aligned} &B(E2; v_i\alpha_i L_i \to v_f\alpha_f L_f) \\ &\quad = (2L_i + 1)^{-1} |\langle v_i\alpha_i L_i \| \hat{Q}^{(E)} \| v_f\alpha_f L_f\rangle|^2 \\ &\quad = (v_f\alpha_f L_f; 112 \| v_i\alpha_i L_i)^2 \bar{B}(E2; v_i \to v_f). \end{aligned} \tag{85}$$

Note that, in deriving the last relationship we have used the symmetry relationship

$$|\langle L_i \| \hat{q} \| L_f\rangle|^2 = |\langle L_f \| \hat{q} \| L_i\rangle|^2. \tag{86}$$

VIII. APPROACHES TO SOLVABLE LIMITS AND SUBMODELS

The Bohr model has several solvable limits which provide useful test cases for their ACM counterparts. Submodels of the Bohr model, which are solvable because the Hamiltonian has a dynamical symmetry, naturally remain exactly solvable in the ACM. Thus, we are primarily concerned with solvable models that emerge as the result of an adiabatic approximation [2], which become submodels of the Bohr model in the limit in which this approximation is valid. This enables us to assess the extent to which the adiabatic approximation is valid in practical situations. In diagonalizing a general Bohr model Hamiltonian, it is also important to recognize that the Hilbert space is infinite and that calculations can only be carried out in a finite truncated subspace. Thus, the practical utility of the ACM depends on how large this subspace needs to be to obtain results to within an acceptable level of accuracy.

We consider Bohr model Hamiltonians of the form

$$\begin{aligned} \hat{H} = &-\frac{\hbar^2}{2B_0}\nabla^2 + \frac{1}{2}B_0\omega^2\left[(1-2\alpha)\beta^2 + \alpha\frac{\beta^4}{b^2}\right] \\ &- \chi\frac{\beta}{b}\cos 3\gamma + \kappa\cos^2 3\gamma, \end{aligned} \tag{87}$$

where b is a dimensional unit in which quadrupole moments (and hence β) are usefully expressed. This parametrization is useful, as will become clear in the following applications, because the various limits of the Bohr model that we wish to explore correspond to simple parameter sets. However, $\hat{H}$ can be simplified by first writing it as

$$\begin{aligned} \hat{H} = &\frac{1}{2}\hbar\omega\left(-\frac{\hbar}{B_0\omega}\nabla^2 + \frac{B_0\omega}{\hbar}\left[(1-2\alpha)\beta^2 + \alpha\frac{\beta^4}{b^2}\right]\right) \\ &- \chi\frac{\beta}{b}\cos 3\gamma + \kappa\cos^2 3\gamma. \end{aligned} \tag{88}$$

Then, by choosing $\hbar\omega$ as the unit of energy and expressing β in units of b (which corresponds to setting $\hbar\omega$ and b equal to unity), $\hat{H}$ simplifies to

$$\begin{aligned} \hat{H}(B, \alpha, \chi, \kappa) = &-\frac{\nabla^2}{2B} + \frac{1}{2}B[(1-2\alpha)\beta^2 + \alpha\beta^4] \\ &- \chi\beta\cos 3\gamma + \kappa\cos^2 3\gamma, \end{aligned} \tag{89}$$

with $B = B_0\omega/\hbar$.

This Hamiltonian, with the β potential

$$V_\alpha(\beta) = \tfrac{1}{2}B[(1-2\alpha)\beta^2 + \alpha\beta^4], \tag{90}$$

is convenient because its parameters are readily assigned physically relevant values for the range of collective spectra of interest. A set of $V_\alpha(\beta)$ potentials are shown, for different values of α, in Fig. 2. Such a potential was used in Ref. [31] to study the second-order phase transition of a model nucleus, from a spherical to a deformed phase, with α as a control

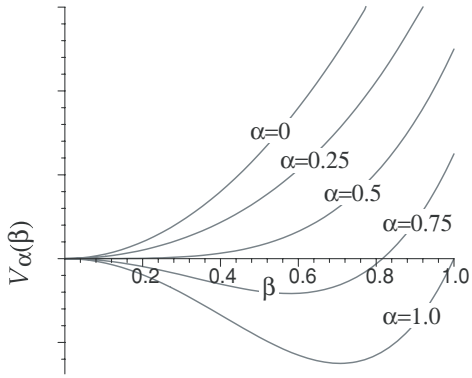

FIG. 2. Potential energy V_α as a function of β for different values of α.

parameter. For $\alpha = 0$, the potential is that of a spherical harmonic oscillator, $\frac{1}{2}B\beta^2$; and for $\alpha > 0.5$, it has a minimum at a nonzero value of β, which increases as α increases. Moreover, as the mass parameter B of the Hamiltonian increases, the strength of the potential increases, the kinetic energy decreases, and the result is a decrease in the vibrational β fluctuations of the model about its equilibrium deformation. Thus, the value of α controls the β deformation of the model, and B controls its rigidity. Moreover, because the unit b, in which β is expressed, is adjustable, one can choose values of α and B to construct a model with any equilibrium value of β and any degree of rigidity desired. As determined in Ref. [31], parameter values in the range $0 \leqslant \alpha \leqslant 2.0$ and $10 \leqslant B \leqslant 100$ are sufficient to describe the β deformations and rigidities of the range of observed nuclear collective states. Useful alternative β potentials are given by the Davidson potentials of Eq. (78) which, likewise, enable the deformation and rigidity to be adjusted independently.

For fully converged results, to a much greater precision than can be seen in the following figures, it was found to be sufficient to include basis states of seniority up to values of $v_{\max} \lesssim 20$ and with β quantum number $\nu \lesssim \nu_{\max} \sim 5$. (Somewhat larger values of $v_{\max}$ are needed when α is close to the critical value of 0.5 and the β fluctuations are particularly large.) The largest matrices to be diagonalized for $L \leqslant 12$ were then of size 150×150. Such matrices can be set up and diagonalized in a fraction of a second on a modern personal computer. The calculations reported here were repeated with $v_{\max} \sim 30$ and $\nu_{\max} \sim 10$, and it was ascertained that the results remained the same to within the required level of accuracy.

A. The harmonic spherical vibrator limit

The Bohr model Hamiltonian for the harmonic spherical vibrator is the special case

$$\hat{H}(a^2, 0, 0, 0) = -\frac{1}{2B}\nabla^2 + \frac{1}{2}B\beta^2 = -\frac{\nabla^2}{2a^2} + \frac{1}{2}a^2\beta^2. \quad (91)$$

This Hamiltonian has a complete set of eigenstates, given in the SU(1, 1) × SU(5)-coupled basis of the ACM, by the states $\{|\lambda_v \nu; v\alpha LM\rangle\}$ with $\lambda_v = v + \frac{5}{2}$. The corresponding energy eigenvalues are given by

$$E_{vv} = \lambda_v + 2v. \quad (92)$$

These are well-known results. The new result that emerges from the ACM perspective of this limit is an explicit expression for the reduced $E2$ matrix elements given by Eqs. (84) and (85) with

$$\langle \mu, v+1 ||| a\hat{q} ||| \nu v \rangle = \delta_{\mu,\nu}\sqrt{\frac{(v+1)(2v+2\nu+5)}{2(2v+5)}} + \delta_{\mu,\nu-1}\sqrt{\frac{(v+1)\nu}{2v+5}}, \quad (93)$$

$$\langle \mu, v-1 ||| a\hat{q} ||| \nu v \rangle = \delta_{\mu,\nu}\sqrt{\frac{(v+2)(2v+2\nu+3)}{2(2v+1)}} + \delta_{\mu,\nu+1}\sqrt{\frac{(v+2)(\nu+1)}{2v+1}}. \quad (94)$$

These expressions follow directly from Eqs. (39), (40), and (55).

B. Approach to the rigid-β Wilets-Jean limit

In the rigid-β Wilets-Jean model [30], the radial coordinate is frozen at a fixed value, β_0, and the potential is assumed to be independent of γ. In accord with Eq. (67), the Laplacian, ∇^2, then reduces to $-\hat{\Lambda}^2/\beta_0^2$, where $\hat{\Lambda}^2$ is the SO(5) Casimir operator, and the Hamiltonian $\hat{H} = -\frac{1}{2B}\nabla^2$ reduces to the Wilets-Jean Hamiltonian

$$\hat{H}_{\rm WJ} = \frac{1}{2B\beta_0^2}\hat{\Lambda}^2. \quad (95)$$

Thus, according to Eq. (68), the eigenfunctions of $\hat{H}_{\rm WJ}$ are SO(5) spherical harmonics with energies given by

$$E_v = \frac{1}{2B\beta_0^2}v(v+3). \quad (96)$$

From Eq. (84), the SO(5)-reduced $E2$ transition rates between SO(5) multiplets of states are given by

$$\bar{B}(E2; \mu v \to \nu, v-1) = \delta_{\mu,\nu}\left(\frac{Ze}{A}\right)^2\frac{v}{2v+3}\beta_0^2, \quad (97)$$

$$\bar{B}(E2; \mu v \to \nu, v+1) = \delta_{\mu,\nu}\left(\frac{Ze}{A}\right)^2\frac{v+3}{2v+3}\beta_0^2, \quad (98)$$

and the corresponding SO(3)-reduced $E2$ transition rates are obtained from these by means of Eq. (85). These results are shown in Fig. 3.

It is known [31] that when the Hamiltonian $\hat{H}(B, \alpha, \chi, \kappa)$ of Eq. (89) has parameter values with $\alpha \gtrsim 2$, a large value of B, and $\chi = \kappa = 0$, the ACM accurately reproduces the results of the rigid-β Wilets-Jean limit with β wave functions that are sharply peaked about a mean value, β_0, but which are nevertheless normalizable. Such results are shown for $\alpha = 5.0$ in Fig. 4(a). For smaller values of α, the SO(5) multiplets of the Wilets-Jean model are preserved, but SO(5) centrifugal stretching occurs as shown for $\alpha = 1.0$ in Fig. 4(b). In particular, it can be seen that the energy spacings

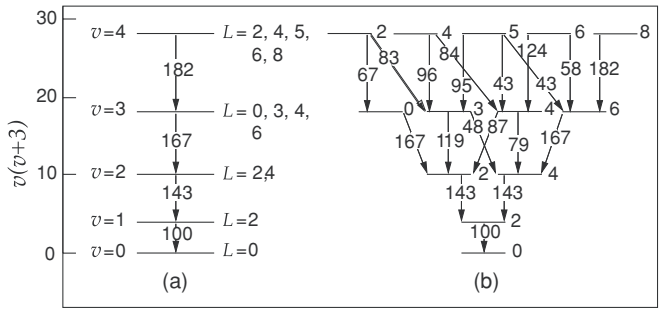

FIG. 3. Energy levels of the rigid-beta Wilets-Jean model in units of $1/(2B\beta_0^2)$. (a) SO(5)-reduced $E2$ transition rates in units such that the $v = 1$ to $v = 0$ reduced transition rate is 100. (b) More detailed SO(3)-reduced $E2$ transition rates.

are decreased and the $\bar{B}(E2)$ reduced transition rates increase with increasing v. Also a one-phonon, $L = v = 0$, β vibrational state appears. As the rigid-β Wilets-Jean limit is approached, this state rises in energy and disappears from the observable energy region. Figure 4(c) shows results for the Hamiltonian (90) with $B = 40$, $\alpha = 1.0$, $\chi = \pm 0.5$, and $\kappa = 0$. These parameters correspond to a potential energy function that has a weak γ dependence and radial wave functions that are moderately peaked about a mean value of β.

Figure 4(c) shows many characteristics of the Wilets-Jean limit. It also shows that the dominant effect of the small $\cos 3\gamma$ interaction is the beginning of an alignment of the energy levels into rotational bands. This is most pronounced at lower energies where there are signs of an approach to a rotor-vibrator spectrum (cf. Fig. 5), which is most clearly observed in the yrast band. However, with increasing energy, the energy levels rapidly revert to near degenerate SO(5) multiplets. Note also that the first excited $L = 2$ state acquires a quadrupole moment which is zero in the Wilets-Jean limit. These results are consistent with the interpretation that the low-energy states tend to be trapped in the shallow well around $\gamma = 0$ (or $\gamma = \pi/3$ for $\chi = -0.5$), whereas higher energy states are much less affected by this well. The situation is similar to the examples considered in Ref. [32], where structurally dissimilar sets of states are found below and above the energy of an essentially quadratic barrier between two wells, and where an excited state quantum phase transition (involving a singularity in the level density) occurs at this energy.

C. Approach to adiabatically decoupled rotor-vibrator limits

For large near-rigid deformations, the Bohr model exhibits weakly coupled rotations and vibrations about its deformed equilibrium [2]. In such a limit, it is common to approximate the potential energy function by the leading terms in the expansion

$$V(\beta,\gamma) = V(\beta_0,\gamma_0) + \tfrac{1}{2}B\omega_\beta^2(\beta-\beta_0)^2 + \tfrac{1}{2}B\beta_0^2\omega_\gamma^2(\gamma-\gamma_0)^2, \quad (99)$$

where β_0 and γ_0 are the equilibrium values of the deformation parameters. Moreover, if the rotational motions are relatively slow in comparison to high-frequency β and γ vibrational modes, there is an adiabatic decoupling of the rotational and

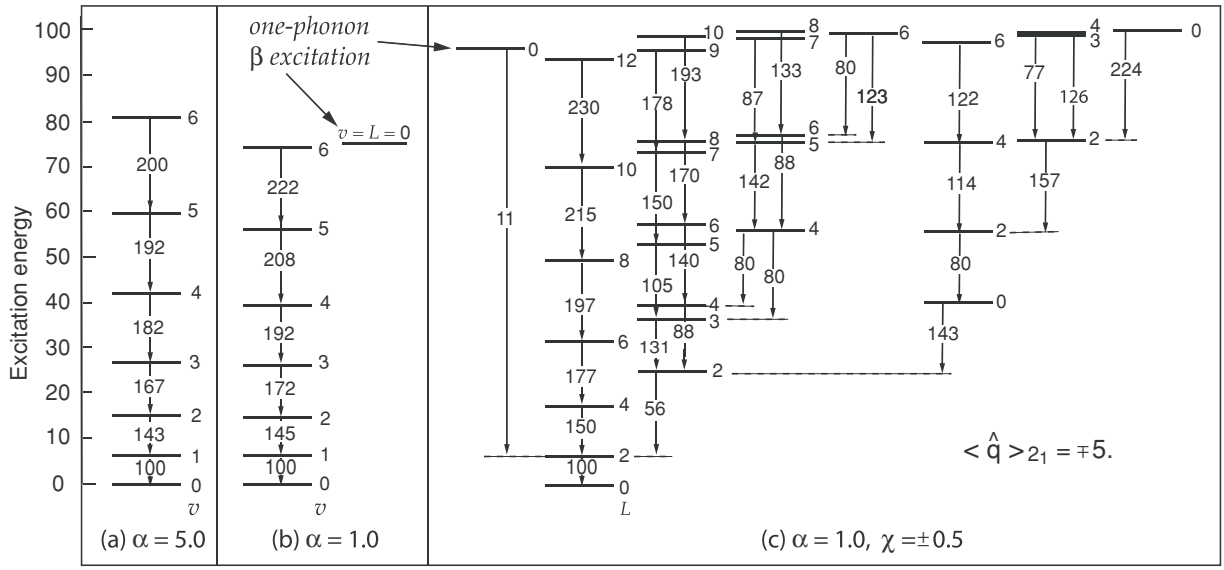

FIG. 4. Energy levels and reduced $E2$ transition rates for the Hamiltonian $\hat{H}(B,\alpha,\chi,\kappa=0)$ of Eq. (89). (a) Near rigid-β Wilets-Jean Hamiltonian with $B = 40, \alpha = 5.0, \chi = 0$. (b) A relatively soft Wilets-Jean Hamiltonian with $B = 40, \alpha = 1.0, \chi = 0$. (c) Hamiltonian of Eq. (89) with $B = 40, \alpha = 1.0, \chi = \pm 0.5$. Energy levels are given in each figure in units such that the lowest $L = 2$ state has excitation energy of $L(L+1) = 6$. The energy levels shown in (a) and (b) are degenerate SO(5) multiplets, and the $E2$ transition rates shown are SO(5)-reduced as in Fig. 3(a); they are identical in value to the standard $B(E2; L_i = 2v \to L_f = 2v-1)$ transition rates between the states of maximum L of each multiplet. The transition rates shown in (c) are standard SO(3)-reduced $B(E2; L_i \to L_f)$ transition rates. All transition rates are given in units such that $B(E2; 2_1 \to 0_1) = 100$. Note that 0_1 and 2_1 refer to the lowest energy states of $L = 0$ and 2, respectively.

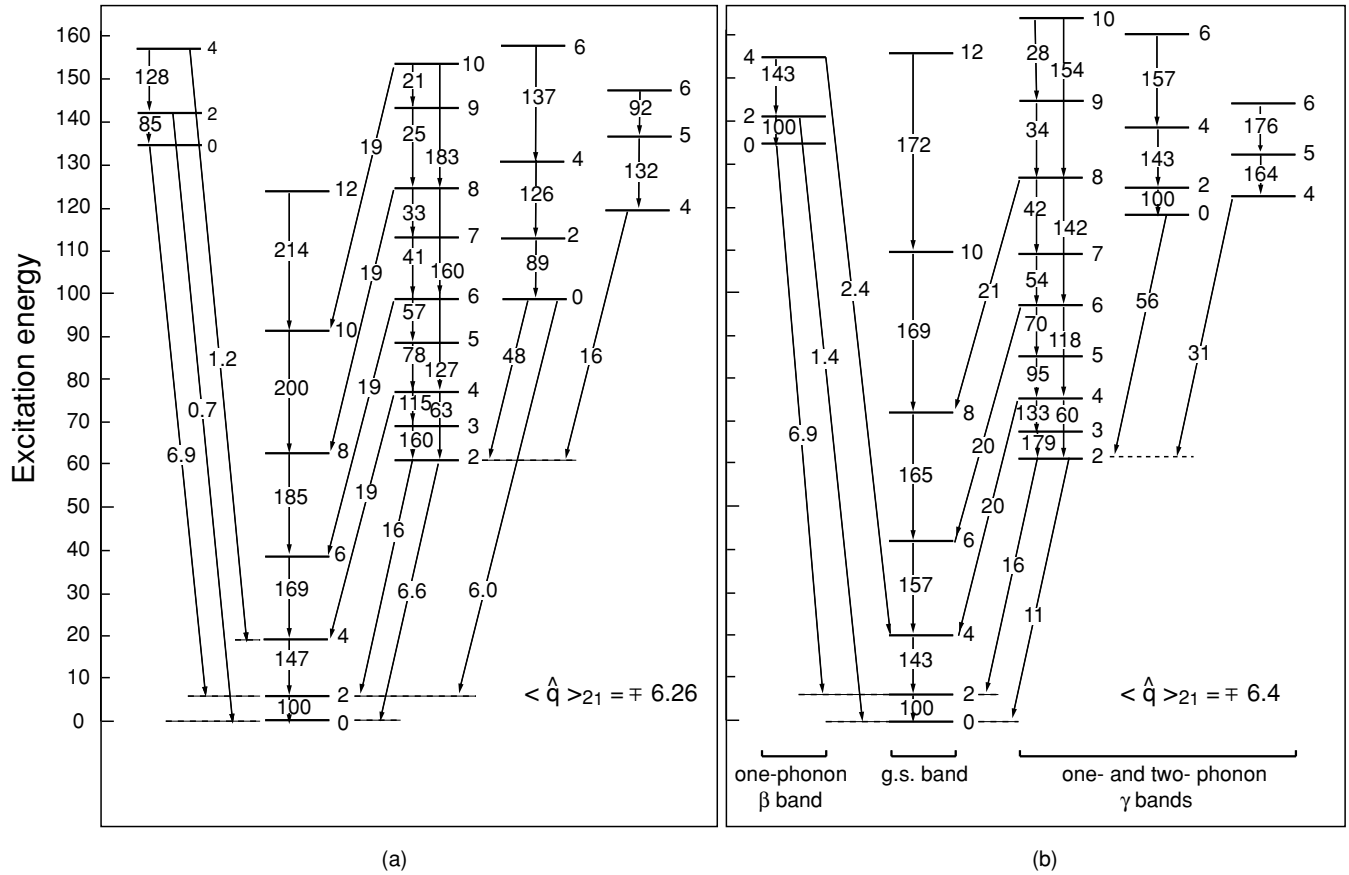

FIG. 5. (a) Low-energy spectrum of the Hamiltonian $\hat{H}(B, \alpha, \chi, \kappa = 0)$ of Eq. (89) for $B = 20, \alpha = 1.5$, and $\chi = \pm 2.0$. Reduced $E2$ transition rates are shown in units for which $B(E2; 2_1 \rightarrow 0_1) = 100$. Energy levels are given in units such that the lowest $L = 2$ state has energy $E_{2_1} = 6$. (b) Corresponding results for the adiabatic Bohr model in the axially symmetric limit showing a ground-state band, a one-phonon β vibrational band, and one- and two-phonon γ vibrational bands.

vibrational degrees of freedom, and the expression for ∇^2 simplifies. The adiabatic limit of the Bohr model is then exactly solvable when two of its three moments of inertia are equal at the equilibrium deformation, i.e., when $\gamma_0 = 0, \pi/6$, and $\pi/3$ [33].

1. *Rotations and vibrations about axial symmetry*

When $\gamma_0 = 0$ or $\pi/3$, the model exhibits rotations and adiabatically decoupled harmonic vibrations about an axially symmetric prolate or oblate spheroidal shape, respectively. In considering the approach to this limit, we again consider the ACM Hamiltonian $\hat{H}(B, \alpha, \chi, \kappa = 0)$ of Eq. (89), albeit with larger values of α and χ so that the potential has well-defined minima about β_0 and γ_0.

Figure 5(a) shows the spectrum of $\hat{H}(B, \alpha, \chi, \kappa = 0)$ for $B = 20, \alpha = 1.5$, and $\chi = \pm 2.0$. When $\chi = \pm 2$, the potential has a minimum at $\gamma_0 = 0$ or $\gamma_0 = \pi/3$, respectively. The value of B was adjusted and given the small value $B = 20$ so that the β and γ vibrational bands would appear in the low-energy domain. The quadrupole moment of the first excited $L = 2$ state, $\langle \hat{q} \rangle_{2_1}$, is a measure of the equilibrium deformation of the model; it is positive or negative according to whether the deformation is, respectively, oblate or prolate. Thus, as expected, $\langle \hat{q} \rangle_{2_1}$ is negative (positive) as χ is positive (negative). However, the energy-level spectrum is independent of the sign of χ.

The corresponding results obtained with the adiabatic model are shown in Fig. 5(b) with parameters chosen such that the excitation energy of the first-excited $L = 2$ state, the lowest energy states of the one-phonon β and γ bands, and the $B(E2; 0_{2_\beta} \rightarrow 2_{\text{g.s.}})$ and $B(E2; 0_{2_\gamma} \rightarrow 2_{\text{g.s.}})$ reduced $E2$ transition rates are the same as those of Fig. 5(a).

Although most of the results of Fig. 5(a) are close to those of the adiabatic model shown in Fig. 5(b), there are significant differences. The dominant result of choosing a small B parameter, for which the β and γ bands appear in the low-energy region, is a large degree of centrifugal stretching, which is neglected in the adiabatic approximation. This is seen, for example, in the comparison of energy levels of excited states of a band, which fall increasingly below those of the adiabatic approximation as the angular momentum is increased. The increasing $E2$ transition rates over those of the adiabatic model with increasing angular momentum is also evidence of such coupling. Note also the substantial two-γ-phonon to ground-band $0_2 \rightarrow 2_1$ transition, which is forbidden in the adiabatic approximation. Although the results are not shown here, it is determined, as expected, that the β

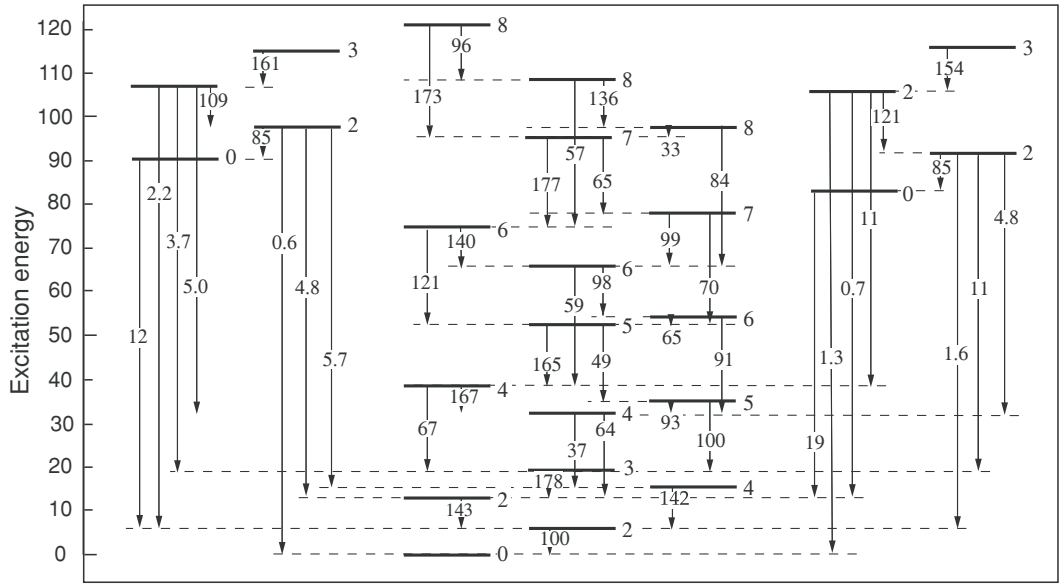

FIG. 6. Low-energy spectrum of the Hamiltonian $\hat{H}(B, \alpha, \chi = 0, \kappa)$ for $B = 22, \alpha = 1.5$, and $\kappa = 4.0$. The larger reduced $E2$ transition decay rates are shown in units for which $B(E2; 2_1 \rightarrow 0_1) = 100$. Energy levels are given in units such that the lowest $L = 2$ state has energy $E_{2_1} = 6$.

and γ excitation energies increase with increasing α and χ, and for large values of these parameters the adiabatic limit of the Bohr model is approached.

The implications of these results are considered further in the discussion section.

2. *Rotations and vibrations of a triaxial symmetry top*

A rotor is described as a symmetric top (particularly in molecular physics) if it has two equal moments of inertia. As pointed out by Meyer-ter-Vehn [33], the Bohr model with equilibrium deformation parameter $\gamma_0 = \pi/6$ is a symmetric top even though its quadrupole moments are not those of an axially symmetric rotor. Thus, in spite of its axial asymmetry, the adiabatic Bohr model is analytically solvable when $\gamma_0 = \pi/6$. Moreover, although this is a very singular case and probably not characteristic of observable nuclear states, it is nevertheless of interest as an analytically solvable prototype of more general triaxial models. For present purposes, however, our primary concern is to ascertain that the ACM provides the means to cover the range of possible Bohr model situations.

Figure 6 shows the spectrum for the Hamiltonian $\hat{H}(B, \alpha, \chi = 0, \kappa)$ of Eq. (89) with $B = 22, \alpha = 1.5$, and $\kappa = 4.0$. In addition to a "ground-state family" of levels, grouped into $K = 0, 2, 4, \ldots$ bands in the figure, there are also excited families of levels, each similarly grouped into $K = 0, 2, 4, \ldots$ bands. These families have the characteristics of β and γ vibrational excitations. The corresponding results for the adiabatic model with the same β and γ vibrational energies are shown in Fig. 7.

A common characteristic of these two sets of results is the absence of $E2$ transitions within any K band as occurs for the rigid Meyer-ter-Vehn rotor [33]. In the adiabatic model limit, this is a consequence of a selection rule that forbids $\Delta K = 0$, $E2$ transitions within all bands and between bands of the same K other than those with a γ phonon number differing by unity. From the ACM perspective, it is seen that the Hamiltonian $\hat{H}(B, \alpha, \chi = 0, \kappa)$ does not mix states of different $\mathbb{R}^5$ parity, $(-1)^v$. Thus, the quadrupole operator, which has odd $\mathbb{R}^5$ parity, has vanishing matrix elements, between states of the same $\mathbb{R}^5$ parity. An implication is that the states of the triaxial symmetric top rotor-vibrator model have $\mathbb{R}^5$ parity given by $(-1)^{K/2+n_\gamma}$, where n_γ is the γ vibrational phonon number. Another implication is that all quadrupole moments of energy eigenstates of this model are zero. Details of this model are given in Ref. [34].

A notable difference in the results of the two figures, as in the previous example, is the large amount of centrifugal stretching exhibited by the ACM results, which is neglected in the adiabatic approximation. Although the results are not shown here, it is determined, as expected, that the β and γ excitation energies respectively increase with increasing α and κ, and for large values of these parameters the adiabatic limit of the Bohr model is approached.

The implications of these results are considered further in the discussion section.

IX. GENERIC MODEL HAMILTONIANS

An important concern is the extent to which it is possible to construct collective model potentials for the ACM that have minima at arbitrary values of β_0 and γ_0 and corresponding arbitrary rigidity parameters. For the β component of the potential, the functions $V(\beta) = -a\beta^2 + b\beta^4$ and $V(\beta) = a\beta^2 + b/\beta^2$ can have a minimum at any β_0. However, for the γ component of the potential, a more interesting situation

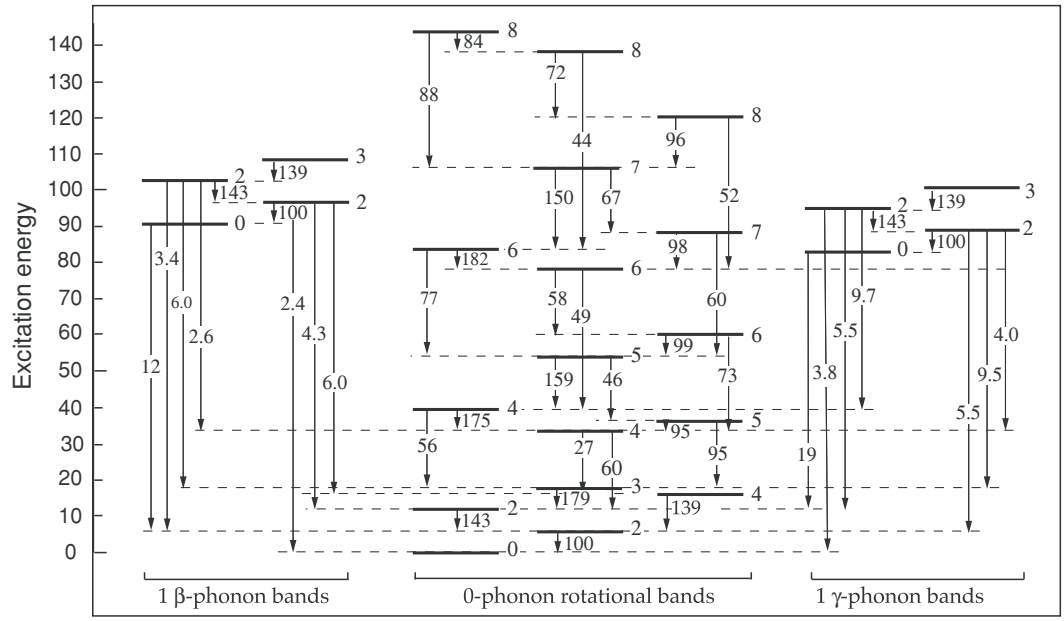

FIG. 7. Typical spectrum of the adiabatic Bohr model in the triaxial symmetric top limit in which $\gamma_0 = \pi/6$. It shows the beginnings of a zero-vibrational-phonon $K = 0, 2, 4, \ldots$ rotational band sequence and parallel excited one-β-phonon and one-γ-phonon bands. The parameters of this model are chosen such that the excitation energy of the first-excited $L = 2$ state, the lowest energy states of the one-phonon β and γ bands, and the $B(E2; 0_\beta \to 2_1)$ and $B(E2; 0_\gamma \to 2_2)$ reduced $E2$ transition rates are the same as those of Fig. 5.

is found. For example, a set of potentials of the type

$$V_{c_1c_2}(\gamma) = c_1\big(1 - \beta_0 \cos 3\gamma\big) - c_2\big(1 - \cos^2 3\gamma\big) \quad (100)$$

is shown in Fig. 8, for $\beta_0 = 1$ and different values of (c_1, c_2). It is seen that for $c_1 > 0$, the minimum value, γ_0, of $V(\gamma)$ stays at $\gamma = 0$ for $c_2 < c_1/2$ after which it progresses rapidly at first and then slowly, with increasing c_2, up to a maximum value of $\gamma = \pi/6$ as $c_2/c_1 \to \infty$. [It was suggested by Iachello [35] that this behavior of $V_{c_1c_2}(\gamma)$ can be exploited in the study of a phase transition between axial and triaxial deformation. This problem is explored using ACM methods by Caprio [36].] However, it is also seen that unless the potential is multiplied

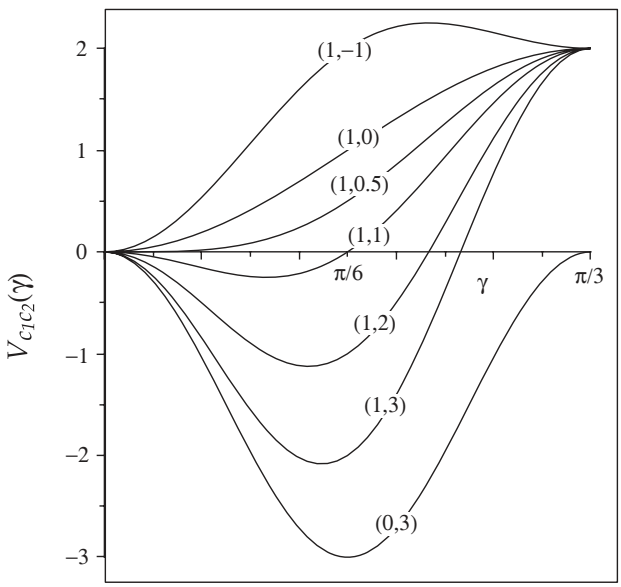

FIG. 8. γ potentials of the form given by Eq. (100) with $\beta_0 = 1$ and for different values of (c_1, c_2).

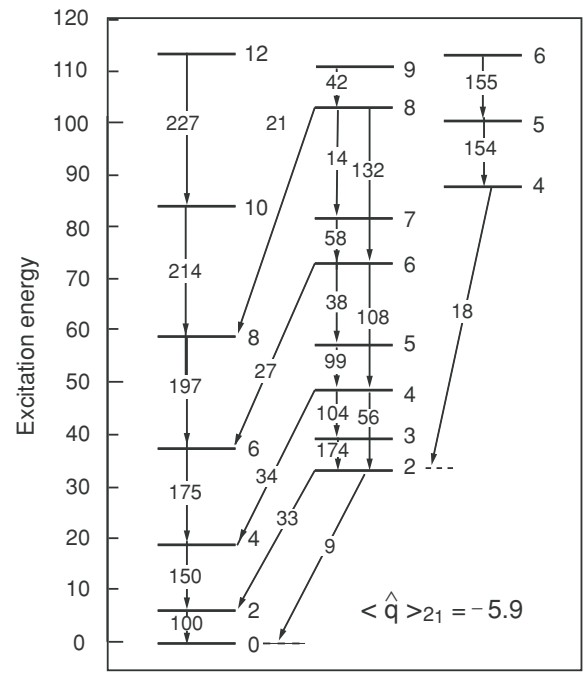

FIG. 9. Low-energy spectrum of the Hamiltonian $\hat{H}(B, \alpha, \chi, \kappa)$ of Eq. (89) for $B = 22, \alpha = 1.5, \chi = \kappa = 4.0$. The reduced $E2$ transition decay rates are in units for which $B(E2; 2_1 \to 0_1) = 100$. Energy levels are given in units such that the lowest $L = 2$ state has energy $E_{2_1} = 6$.

by a large constant factor, it is very shallow for small γ_0. For negative values of c_1, the range of potentials that can be generated for both positive and negative values of c_2 can be inferred from the observation that $V_{-c_1,-c_2}(\gamma) = -V_{c_1c_2}(\gamma)$. In particular, for $c_1 < 0$, γ_0 approaches $\pi/6$, for large values of the ratio $-c_2/c_1$, from $\gamma = \pi/3$ at $c_2 = 0$.

Figure 9 shows the spectrum for the Hamiltonian $\hat{H}(B, \alpha, \chi, \kappa)$ of Eq. (89) with $B = 22$, $\alpha = 1.5$, and $\chi = \kappa = 4.0$. The relatively small interband $E2$ transition rates, seen in this figure, as compared to the intraband rates, suggests that K is a relatively good quantum number, although there are evidently major rotational-vibrational coupling perturbations as one might expect for a shallow γ potential. Particularly notable is the appearance of what resembles a sequence of approximate $K = 0, 2, 4, \ldots$, bands similar to those of a symmetric top with one small, but nonzero, moment of inertia, i.e., the counterpart of the Meyer-ter-Vehn symmetric top for which the distinct moment of inertia is larger than the other two. The quadrupole moment $\langle \hat{q} \rangle_{2_1} = -5.9$, of the first-excited $L = 2$ state, which is close to that of an axially symmetric prolate rotor, is consistent with this perspective.

X. DISCUSSION

These results show that with the ready availability of SO(5) Clebsch-Gordan coefficients for the SO(5) irreps that appear in the space of the five-dimensional harmonic oscillator, it is easy to carry out collective model calculations for a diverse range of Hamiltonians of the form given by Eq. (71) within the framework of the ACM.

A significant result that emerges from these calculations is the large amount of centrifugal stretching that occurs, e.g., in the ground-state rotational band, when there are β and γ vibrational bands in the low-energy domain. If this result were shown to persist for all reasonable ACM Hamiltonians that give low-energy β or low-energy γ vibrational excitations, it would cause one to question the use of the standard adiabatic approximation to the Bohr model, which neglects such coupling effects. In particular, it would call into question the consistency of interpreting low-lying excited bands as β or γ vibrational bands when the corresponding centrifugal stretching effects are observed to be small. However, until the various spectra for the many possible ACM Hamiltonians have been explored, we consider it premature to come to any hard conclusions.

As emphasized in the Introduction, a primary objective of analyzing data in terms of a model is to identify the model's limitations and ways in which it can be improved. An obvious limitation of the Bohr model is that, at best, it can only describe a small subset of the large variety of rotational bands seen in deformed nuclei. A natural development to counter this limitation is to extend the Hilbert space of the model by tensoring it with an intrinsic Hilbert space spanned by a finite number of intrinsic states. This is essentially what is done in the standard Bohr-Mottelson unified model [2] albeit within the framework of a solvable adiabatic approximation to the Bohr model component. It can now be done without the adiabatic approximation.

Another limitation of the Bohr model is its inability to predict observed moments of inertia. The ratios of moments of inertia are fixed by the SO(5) Casimir operator [1] to be those of an irrotational-flow quantum fluid, and their magnitudes are fixed by the mass parameter B. In the original formulation of his model, Bohr proposed that the mass parameter should also be given its irrotational-flow value. However, this choice is at odds with experiment, which requires much larger values [37]. One can regard the mass parameter of the model as adjustable on the grounds that the model is not committed to any particular collective flow. A concern remains, however, because it is known that in the many-nucleon Hilbert space, the quadrupole moment operators are the infinitesimal generators of irrotational flows; this is shown explicitly in Ref. [34]. Moreover, there is evidence to suggest that the moment-of-inertia ratios that best fit nuclear data may also differ significantly from those of the Bohr model [38].

From a liquid-drop model perspective, the moments of inertia observed for nuclear rotations suggest that a better description of collective dynamics might be given by the inclusion of vorticity degrees of freedom in the collective model as done in superfluid hydrodynamics [39]. Thus, in addition to moment-of-inertia considerations, it would be instructive to have direct experimental evidence of the effects of vorticity degrees of freedom in nuclei. An obvious place to look for such evidence is in transverse electron scattering form factors which depend on the nuclear current flows [40,41]. Unfortunately, measurements of the needed form factors appear to be difficult.

Another concern, which turns out to be related, is that the quantization procedure used for the Bohr model may not be the most appropriate. Equivalently, one may question if the algebraic structure of the Bohr model is the most appropriate. A vital step in facilitating a study of these questions has already been achieved by replacing the surface shape parameters of Bohr's original model by quadrupole moments. Thus, if the Cartesian quadrupole moments for a nucleus with density $\rho(\mathbf{r})$ at a point $\mathbf{r} = (x_1, x_2, x_3) \in \mathbb{R}^3$, given by

$$Q_{ij} = \int x_i x_j \rho(\mathbf{r})\, dv, \tag{101}$$

are replaced by the quadrupole moments for a discrete distribution of nucleons with position coordinates $\{\mathbf{r}_n; n = 1, \ldots, A\}$, so that

$$Q_{ij} \to \sum_n x_{ni} x_{nj}, \tag{102}$$

then the time derivative of these quadrupole moments becomes

$$\dot{Q}_{ij} \to \sum_n (x_{ni}\dot{x}_{nj} + \dot{x}_{ni} x_{nj}). \tag{103}$$

Thus, they correspond to moments of momenta given by

$$P_{ij} \to \sum_n (x_{ni} p_{nj} + p_{ni} x_{nj}), \tag{104}$$

where $p_{ni} = M\dot{x}_{ni}$ is the momentum of the nth nucleon. This suggests that in a quantum theory of nuclear collective quadrupole dynamics, these observables should define the

operators

$$\hat{Q}_{ij} = \sum_n \hat{x}_{ni}\hat{x}_{nj}, \quad \hat{P}_{ij} = \sum_n (\hat{x}_{ni}\hat{p}_{nj} + \hat{p}_{ni}\hat{x}_{nj}), \qquad (105)$$

where $\hat{p}_{ni} = -i\hbar\partial/\partial x_{ni}$. Moreover, the kinetic energy for a system of nucleons, given in the usual way by $T = \frac{1}{2M}\sum_{ni} p_{ni}^2$, has the standard quantization

$$\hat{T} = \frac{1}{2M}\sum_{ni} \hat{p}_{ni}^2. \qquad (106)$$

With all these replacements, the collective model becomes the so-called sp(3, $\mathbb{R}$) symplectic model [42]. It turns out that this model does include vorticity degrees of freedom and gives moments of inertia close to those observed [43,44]. Moreover, it has the advantage that it is a submodel of the nuclear shell model. As a microscopic collective model, reviewed in Ref. [45] and in a broader context in Ref. [46], it provides a powerful tool for capitalizing on what is learned from the phenomenological interpretation of collective states in the design of shell-model calculations [34]. However, as a phenomenological algebraic collective model, the symplectic model is not as quick and easy to use in the direct interpretation of experimental data as a model with a simpler algebraic structure. This suggests consideration of a contracted form of the symplectic model [47,48] to supply what is missing in the ACM, along the lines proposed in Refs. [49,50].

ACKNOWLEDGMENTS

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APPENDIX: IRRATIONAL HAMILTONIANS

The analytical expressions given in Secs. IV–VI for basic matrix elements can be used to derive matrix elements of rational functions of the fundamental $\{\hat{q}_m, \hat{\pi}^m\}$ observables of the Bohr model in a basis in which all β wave functions associated with SO(5) angular states of seniority v belong to a common su(1, 1) irrep λ_v and are subject to the constraint that $\lambda_{v+1} = \lambda_v \pm 1$.

This constraint is much less restrictive than the conventional harmonic spherical vibrator basis, which requires that $\lambda_v = v + \frac{5}{2}$. Nevertheless, because the β wave functions of any single su(1, 1) irrep, with $\lambda > 0$, define an orthonormal basis for the Hilbert space, $L^2(\mathbb{R}_+)$ of β wave functions, it is, in principle, possible to carry out ACM calculations using only a single su(1, 1) irrep. Indeed, the only impediment to working with a single, seniority-independent su(1, 1) irrep is a need for the matrix elements of β within this single irrep. Thus, for example, if one has a Hamiltonian that contains only even powers of β and/or their inverses, then it can be diagonalized in an ACM basis with just a single su(1, 1) irrep. However, to calculate $E2$ transition rates, one still needs matrix elements of β between states of the chosen irrep.

A general algebraic expression for matrix elements such as $F_{\lambda\mu;\lambda\nu}(\beta)$, $F_{\lambda\mu;\lambda\nu}(1/\beta)$, or $F_{\lambda\mu;\lambda\nu}(d/d\beta)$ appears to be difficult to derive. However, given that it is usually sufficient to restrict to a basis of five β wave functions in a typical ACM calculation with optimally chosen values of λ and the rigidity parameter a, only a small number of such β matrix elements are required in any given situation. Moreover, it is easy to derive them by explicit case-by-case integration or by numerical methods. For example, one can make use of the fact that because β is positive definite, the infinite matrix $F(\beta)$, with elements $F_{\lambda\mu;\lambda\nu}(\beta)$, is the unique positive-definite square root of the matrix $F(\beta^2)$. Thus accurate approximations to a finite submatrix of $F(\beta)$ can be determined by taking the unique positive-definite square root of a larger $F(\beta^2)$ matrix.

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