Mass tensor in the Bohr Hamiltonian from the nondiagonal energy weighted sum rules

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Relations are derived in the framework of the Bohr Hamiltonian that express the matrix elements of the deformation-dependent components of the mass tensor through the experimental data on the energies and the $E2$ transitions relating the low-lying collective states. These relations extend the previously obtained results for the intrinsic mass coefficients of the well-deformed axially symmetric nuclei on nuclei of arbitrary shape. The expression for the mass tensor is suggested, which is sufficient to satisfy the existing experimental data on the energy weighted sum rules for the $E2$ transitions for the low-lying collective quadrupole excitations. The mass tensor is determined for ^{106,108}Pd, $^{108-112}$Cd, ^{134}Ba, ^{150}Nd, $^{150-154}$Sm, $^{154-160}$Gd, ^{164}Dy, ^{172}Yb, ^{178}Hf, $^{188-192}$Os, and $^{194-196}$Pt.

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I. INTRODUCTION

In our previous publications [1,2], on the basis of the experimental data, we have shown that in the case of the well-deformed axially symmetric nuclei the mass coefficients for β and γ vibrations are several times larger than the mass coefficient for the rotational motion. However, if we start with the Bohr Hamiltonian written in the laboratory frame and if we have, as it is usually assumed, a constant mass coefficient in the expression for the kinetic energy term, then the Hamiltonian transformed into the intrinsic frame will have the same mass coefficients for the rotational motion and for the β and γ vibrations. This means that in the Bohr Hamiltonian written in the laboratory frame the mass tensor cannot be reduced to a constant mass coefficient if we want to have different inertia coefficients for the rotational and the vibrational modes.

The kinetic energy term is a vector product of the two operators of the collective momentum and the mass tensor. The operators of the collective momentum $-\imath\hbar\frac{\partial}{\partial\alpha_{2\mu}}$, each having the angular momentum equal to $2\hbar$ units, can be coupled to the total momentum L, taking the values 0, 2, and 4 ($L=1$ and 3 are absent because of the symmetrization). Because the kinetic energy term is a scalar, the mass tensor will also have components with $L=0, 2$, and 4. The component of the mass tensor with $L=0$ can be a constant, but the component with $L=2$ contains at least one degree of the collective coordinate $\alpha_{2\mu}$. The component with $L=4$ can be at least of the second order in $\alpha_{2\mu}$. If the mass tensor is restricted to be a constant, then it can be used as a scaling parameter to fit the energy of the first 2^+ state. However, if the mass tensor contains the deformation-dependent components with $L=2$ and 4, we suggest a procedure for extracting the information about the mass tensor from the experimental data.

The aim of the present article is to extend the analysis of the mass tensor performed by us earlier for the well-deformed axially symmetric nuclei to the spherical and transitional nuclei. We consider the Bohr collective Hamiltonian in the laboratory frame with the mass tensor of the most general form. By using the experimental data for different nuclei we show that the mass tensor definitely contains nonvanishing quadrupole and hexadecupole components and therefore cannot be reduced to a scalar mass as it is assumed in many recent publications [3–6]. We should mention, however, that a deformation-dependent mass tensor having not only a monopole component has been considered in Refs. [7–12].

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II. THE RELATIONS FOR THE COMPONENTS OF THE MASS TENSOR

We consider the collective quadrupole Bohr Hamiltonian

$$H = T + V(\alpha_{2\mu}), \tag{1}$$

with the most general form of the kinetic energy term

$$T = -\frac{\hbar^2}{2}\sum_{\mu,\mu'}\frac{\partial}{\partial\alpha_{2\mu}}(B^{-1})^{\rm lab}_{\mu,\mu'}\frac{\partial}{\partial\alpha_{2\mu'}}, \tag{2}$$

where $(B^{-1})^{\rm lab}_{\mu,\mu'}$ is an inverted mass tensor. This Hamiltonian generates all eigenstates that will be considered. Because the angular momentum L is a good quantum number that is used to characterize the excited states, it is convenient to express the mass tensor through the components having fixed values of L,

$$(B^{-1})^{\rm lab}_{\mu,\mu'} = \sqrt{5}\sum_{LM}C^{LM}_{2\mu2\mu'}(B^{-1})^{\rm lab}_{LM}, \tag{3}$$

where $C^{LM}_{2\mu2\mu'}$ is a Clebsch Gordan coefficient. In the special case when the kinetic energy term is characterized by only one constant mass coefficient, one finds from Eq. (3) that in this case the mass tensor does not contain components with $L=2$ and 4. Our task is to check on the basis of the experimental data for different nuclei whether these components are present in the mass tensor or whether they are equal to zero in real nuclei. We determine, however, only two matrix elements of the mass tensor and do not describe the mass tensor as a function of the collective variables. To get more information about the mass tensor we need a larger number of its matrix elements but there is not enough experimental data to determine them.

To obtain information about the quadrupole and the hexadecupole components of the mass tensor we calculate a double commutator $[[H, Q_{2\mu}], Q_{2\mu'}]$ using Bohr's form of the quadrupole moment operator

$$Q_{2\mu} = q\alpha_{2\mu}, \tag{4}$$

where $q = \frac{3}{4\pi}eZr_0^2A^{2/3}$. The double commutator $[[H, Q_{2\mu}], Q_{2\mu'}]$ is suitable to obtain information about the mass tensor because of the crucial fact that the potential energy does not contribute to it (see Appendix A) [13,14]. Indeed, both the potential energy and the quadrupole operator depend only on the collective coordinate $\alpha_{2\mu}$, whose components with different values of μ commute with each other. We note that, as shown in Eq. (A1) in Appendix A, the potential energy $V(\alpha_2)$ does not already contribute to the single commutator. Of course both parts of the collective Hamiltonian, the mass tensor and the potential energy, are equally important for the description of the collective properties of nuclei but the matrix elements of the double commutator considered above give us an opportunity to obtain those relations between the excitation energies and the reduced $E2$ transition probabilities that are sensitive only to the mass tensor.

It is convenient to express the coefficient q in terms of the Weisskopf units for $E2$ transitions because the experimental data for $B(E2)$ are given frequently in Weisskopf units (Wu) [13],

$$q = \frac{5}{\sqrt{4\pi}} Z\sqrt{B(E2)_{\rm Wu}}. \tag{5}$$

The result for the double commutator is [see Appendix A, Eq. (A2)]

$$[[H, Q_{2\mu}], Q_{2\mu'}] = -\hbar^2 q^2 \sqrt{5} \sum_{L=0,2,4} \sum_M C^{LM}_{2\mu 2\mu'} (B^{-1})^{\rm lab}_{LM}. \tag{6}$$

We stress once more that the double commutator (6) is independent of the value of the potential V.

Taking the matrix elements of both sides of Eq. (6) between some eigenstates of a nucleus, we get on the left an expression containing only the measurable quantities, namely, the excitation energies and the matrix elements of the quadrupole operator, and on the right we get the matrix elements of the mass tensor.

Let us introduce for convenience a short notation for the matrix elements of the double commutator,

$$S^{(L)} \equiv \sqrt{2L+1}\langle LM| \sum_{\mu,\mu'} C^{LM}_{2\mu\,2\mu'}[[H, Q_{2\mu}], Q_{2\mu'}]|0^+_{\rm gs}\rangle, \tag{7}$$

where $|LM\rangle$ is the lowest state with angular momentum L, and express the matrix element of the mass tensor through the reduced matrix elements

$$\langle LM|(B^{-1})^{\rm lab}_{LM}|0^+_{\rm gs}\rangle = \frac{1}{\sqrt{2L+1}}\langle L\|(B^{-1})^{\rm lab}_L\|0^+_{\rm gs}\rangle. \tag{8}$$

Then taking the matrix element of Eq. (6) between the ground state and the first excited state with angular momentum L ($|LM\rangle$) and using definitions (7) and (8), we obtain [see Appendix B, Eq. (B1)]

$$\begin{aligned} S^{(L)} &= \frac{1}{\sqrt{5}} \sum_i (E(L) - 2E(2^+_i))\langle L\| Q_2\|2^+_i\rangle\langle 2^+_i\| Q_2\|0^+_{\rm gs}\rangle \\ &= -\hbar^2 q^2\sqrt{5}\langle L\|(B^{-1})^{\rm lab}_L\|0^+_{\rm gs}\rangle. \end{aligned} \tag{9}$$

Equation (9) is the main result of this article. It shows that one can obtain information on the mass tensor from excitation energies and $B(E2)$'s. In the case of $L = 2$ or 4, $S^{(L)}$ is equal to zero if the mass tensor is approximated by the one constant mass coefficient.

In most cases we can keep with good accuracy in the sum in Eq. (9) only the three lowest 2^+ states. In the case of the well-deformed axially symmetric nuclei, these three states are the 2^+ states of the ground, β, and γ bands. In the case of the spherical nuclei they are usually one-, two-, and three-phonon 2^+ states.

Below we use the notation 2^+_1 for the first 2^+ state. However, it is inconvenient to use the numeration $i = 2$ and 3 for the higher lying 2^+ states because in the well-deformed nuclei in some cases the second 2^+ state is 2^+_γ but in some cases, like ^{154}Sm, it is 2^+_β. So, everywhere in the text below we use the notations 2^+_γ and 2^+_β for the higher lying 2^+ states that will be taken into account in our formulas. This will be done not only for the well-deformed nuclei but also for the transitional nuclei, where it would be more correct to call these states quasi-γ and quasi-β. In order not to use different notations for different types of nuclei, we use these notations also for the spherical nuclei. In this case the notation 2^+_γ is used for the two-phonon state and 2^+_β for the three-phonon state. It has some meaning because both the three-phonon 2^+ state in spherical nuclei and the 2^+_β state in the well-deformed nuclei have a node in the β-dependent part of the wave function. Nevertheless the wave functions for these states are completely different.

The matrix elements in Eq. (9) can be expressed through the excitation energies and the reduced $E2$ transition probabilities. To do this we use some phase relations for the reduced matrix elements of the quadrupole operator. The details are considered in Appendix B. The final results are

$$\begin{aligned} S^{(4)} &\equiv \sum_{i=1,\beta,\gamma} S^{(4)}_i = \sum_{i=1,\gamma,\beta} (E(4^+_1) - 2E(2^+_i)) \\ &\quad\times \sqrt{B(E2; 0^+_{\rm gs} \to 2^+_i)\cdot B(E2; 2^+_i \to 4^+_1)}, \end{aligned} \tag{10}$$

$$\begin{aligned} S^{(2)} &= \sum_{i=1,\beta,\gamma} S^{(2)}_i = -E(2^+_1)\sqrt{\frac{35}{32\pi}}|Q(2^+_1)| \\ &\quad\cdot\sqrt{B(E2; 0^+_{\rm gs} \to 2^+_1)} + (2E(2^+_\gamma) - E(2^+_1)) \\ &\quad\times\sqrt{B(E2; 2^+_\gamma \to 2^+_1)\cdot B(E2; 0^+_{\rm gs} \to 2^+_\gamma)} \\ &\quad- (2E(2^+_\beta) - E(2^+_1)) \\ &\quad\times\sqrt{B(E2; 2^+_\beta \to 2^+_1)\cdot B(E2; 0^+_{\rm gs} \to 2^+_\beta)}, \end{aligned} \tag{11}$$

where the notation $S_i^{(L)}$ is introduced for the separate terms in the sums. For $S^{(0)}$ we get

$$S^{(0)} = -2\sqrt{5} \sum_{i=1,\beta,\gamma} E(2_i^+)B(E2; 2_i^+ \to 0_{gs}^+). \quad (12)$$

Using the existing experimental data for the excitation energies, $B(E2)$ values, and spectroscopic quadrupole moments, we can calculate the values of $S^{(4)}$, $S^{(2)}$, and $S^{(0)}$. Deviations of $S^{(4)}$ and $S^{(2)}$ from zero indicate the presence of the nonzero components with $L = 4$ and 2 in the mass tensor. As a measure of deviations we can compare the values of $S^{(4)}$ and $S^{(2)}$ with the values of the first terms in the sums (10) and (11), namely, with $S_1^{(4)}$ and $S_1^{(2)}$, which are the largest ones among $S_i^{(4)}$ and $S_i^{(2)}$, respectively.

For the aim of presentation it is convenient to introduce the nondimensional quantities

$$\tilde{S}^{(L)} = \frac{S^{(L)}}{E(2_1^+)B(E2; 2_1^+ \to 0_{gs}^+)}, \quad (13)$$

which fluctuate from nucleus to nucleus less than $S^{(L)}$.

The results of calculations of $\tilde{S}^{(L)}$ are presented in Table I. In the cases of the Os and Pt isotopes and also in ^{110}Pd, ^{134}Ba, ^{160}Gd, and ^{164}Dy there is no data on $E2$ transitions from the 2_β^+ to the 4_1^+ and 2_1^+ states. However, in the case of $\tilde{S}^{(4)}$ the expected corrections are small because of the usually small values of the corresponding transition probabilities. In the case of $\tilde{S}^{(2)}$ the corresponding terms will only increase the absolute value of $\tilde{S}^{(2)}$. As is discussed in detail in Appendix B, the signs of $S^{(4)}$ and $S^{(2)}$ depend on the agreement about the phases of the states that appear only once in Eq. (9). However, for our conclusion only the fact of deviation of the absolute values of $S^{(4)}$ and $S^{(2)}$ from zero is important. Of course the mass tensor is independent of the choices of phases.

In the Table I the results for nuclei with different collective properties (spherical, transitional, and deformed) are shown. As is seen from these results, the values of $\tilde{S}^{(4)}$ are large in all considered nuclei. The nonzero values of $\tilde{S}^{(4)}$ are the consequence of the absence of a full compensation of the positive and negative terms in the sum. Because in many cases $\tilde{S}^{(4)}$ is comparable with $\tilde{S}_1^{(4)}$, the negative terms in the sum for $\tilde{S}^{(4)}$ are rather small. It means that the component of the mass tensor with $L = 4$ is strongly different from zero and should be taken into account in all these nuclei. The negative value of $\tilde{S}^{(4)}$ in the case of ^{150}Sm is a consequence of the unusually large value of the $B(E2; 2_\beta^+ \to 4_1^+)$ given in Ref. [15].

For the calculations of $\tilde{S}^{(2)}$ in the case of transitional and deformed nuclei it is important to determine correctly what 2^+ state is considered as 2_β^+ (quasi-β) and what state is considered as 2_γ^+ (quasi-γ). For that we have followed the indications given in Ref. [15]. The values of $\tilde{S}^{(2)}$ are also large in the majority of the considered nuclei excluding ^{110}Cd and ^{190,192}Os

TABLE I. The calculated values of $\tilde{S}^{(L)}$ and $\tilde{S}_i^{(L)}$. They are determined in the text by Eqs. (10)–(13). The values of $\tilde{S}^{(4)}$ and $\tilde{S}_{1,2}^{(4)}$ are given assuming a positive sign of the product $\langle 4_1^+ \| Q_2 \| 2_1^+ \rangle \langle 2_1^+ \| Q_2 \| 0_{gs}^+ \rangle$. The values of $\tilde{S}^{(2)}$ and $\tilde{S}_{1,2}^{(2)}$ are given assuming a negative sign of the product $\langle 2_1^+ \| Q_2 \| 2_1^+ \rangle \langle 2_1^+ \| Q_2 \| 0_{gs}^+ \rangle$. The experimental data were taken from Ref. [15].

Nucleus	$\tilde{S}^{(4)}$	$\tilde{S}_1^{(4)}$	$\tilde{S}_2^{(4)}$	$\tilde{S}^{(2)}$	$\tilde{S}_1^{(2)}$	$\tilde{S}_2^{(2)}$	$\tilde{S}^{(0)}$
^{106}Pd	1.1(2)	1.5	−0.1	−0.9(4)	−2.0	1.2	−4.78
^{108}Pd	1.1(1)	1.5	−0.2	−1.1(3)	−2.1	1.1	−4.66
^{110}Pd	–	–	–	−1.4(4)	−2.3	0.9	−4.61
^{110}Cd	–	–	–	0.0(8)	−1.8	1.8	−5.08
^{112}Cd	–	–	–	−1.0(3)	−1.6	0.7	−4.78
^{114}Cd	0.6(2)	1.3	–	−0.8(2)	−1.5	0.8	−4.74
^{134}Ba	1.1(1)	1.2	−0.1	0.4(7)	−0.7	1.1	−4.77
^{150}Nd	2.1(4)	3.4	−0.6	–	–	–	−5.56
^{150}Sm	−0.9(5)	1.3	−0.8	–	–	–	−5.25
^{152}Sm	2.6(3)	3.6	−0.7	−1.6(2)	−2.7	1.5	−5.66
^{154}Sm	3.2(4)	3.5	−0.7	–	–	–	−6.26
^{154}Gd	2.6(2)	3.8	−0.6	−1.7(2)	−2.7	1.5	−5.95
^{156}Gd	3.5(2)	4.4	−0.4	−1.3(2)	−2.6	1.7	−6.11
^{158}Gd	3.9(5)	4.7	−0.5	−1.2(2)	−2.6	1.5	−5.73
^{160}Gd	–	–	–	−1.2(1)	−2.7	1.5	−5.58
^{164}Dy	–	–	–	−1.4(3)	−2.6	1.2	−5.37
^{172}Yb	4.3(3)	4.6	−0.2	–	–	–	−5.06
^{178}Hf	4.2(2)	4.7	−0.3	–	–	–	−6.00
^{188}Os	2.3(4)	4.2	−1.9	−0.8(4)	−2.6	1.8	−5.63
^{190}Os	–	–	–	−0.1(6)	−2.3	2.2	−5.57
^{192}Os	–	–	–	0.1(3)	−2.0	2.1	−5.40
^{194}Pt	–	–	–	−0.8(3)	−1.5	0.6	−4.52
^{196}Pt	–	–	–	−1.7(2)	−1.7	0.0	−4.47

where $\tilde{S}^{(2)}$ is practically zero. Therefore, in many nuclei the mass tensor should contain the component $(B^{-1})^{\rm lab}_{2M}$ in the expression for the kinetic energy term in the Bohr collective Hamiltonian.

In Table I, not only the summed quantities $\tilde{S}^{(L)}$ but also separate terms, namely, $\tilde{S}_1^{(L)}$ and $\tilde{S}_2^{(L)}$, are shown.

Some lines for the values of $\tilde{S}^{(4)}$ in Table I are empty because of the absence of data for the $4_1^+ \to 2_{\gamma,\beta}^+$ transitions in the corresponding nuclei. The absence of experimental information about the spectroscopic quadrupole moment is the reason for the appearance of the empty lines for the values of $\tilde{S}^{(2)}$ in some nuclei.

The main contribution to $S^{(0)}$ comes from the product $E(2_1^+)B(E2;2_1^+ \to 0_{\rm gs}^+)$. For this reason the nondimensional quantity $\tilde{S}^{(0)}$ contains a large constant term equal to $-2\sqrt{5}$. This explains the approximate constancy of the values of $\tilde{S}^{(0)}$ given in the last column of Table I.

What are the consequences of the fact that the values of $S^{(4)}$ and $S^{(2)}$ are not equal to zero?

Equation (9) together with the nonzero values of $S^{(2)}$ and $S^{(4)}$ that follow from the experimental data tells us that the mass tensor cannot be reduced to a constant. We see from Eqs. (6) and (9) that we obtain from the energies and $B(E2)$'s the matrix elements of the mass tensor and not the mass tensor directly. However, if we make the plausible ansatz for the mass tensor as being a quadratic function of the α's then we can determine the three constant parameters A_0, A_2, and A_4 from the three constants $S^{(0)}$, $S^{(2)}$, and $S^{(4)}$. And in this way we can determine the mass tensor. The simplest form of the mass tensor sufficient to satisfy Eq. (9) is

$$\hbar^2(B^{-1})^{\rm lab}_{LM} = \frac{\hbar^2}{A_0}\delta_{L0} + \frac{\hbar^2}{A_2}\alpha_{2M}\cdot\delta_{L2} + \frac{\hbar^2}{A_4}(\alpha\alpha)_{4M}\delta_{L4}, \quad (14)$$

where A_0, A_2, and A_4 are numbers. Instead of the quadrupole tensor α_{2M} we can introduce $(\alpha\alpha)_{2M}$ or their linear combination. However, in the last case there will not be sufficient experimental information to determine all parameters using Eqs. (10) and (11). The same is true for the $(\alpha\alpha)_{4M}$ tensor where the other possibilities are $(\alpha(\alpha\alpha)_2)_{4M}$ and $((\alpha\alpha)_2(\alpha\alpha)_2)_{4M}$.

The expressions for the parameters of the mass tensor A_4, A_2, and A_0 in terms of $S^{(L)}$ and some reduced $E2$ transition probabilities are derived in Appendix C. They are

$$\frac{\hbar^2}{A_0} = -\frac{S^{(0)}}{q^2} \quad (15)$$

$$\frac{\hbar^2}{A_2} = \frac{S^{(2)}}{q\sqrt{B(E2;0_{\rm gs}^+ \to 2_1^+)}} \quad (16)$$

$$\frac{\hbar^2}{A_4} = -\frac{S^{(4)}}{\sqrt{B(E2;0_{\rm gs}^+ \to 2_1^+)B(E2;2_1^+ \to 4_1^+)}}. \quad (17)$$

Equations (15), (16), and (17) determine completely the simplest form of the mass tensor (14) that is still compatible with the existing experimental data and therefore determines the anharmonic terms in the Hamiltonian coming from the kinetic energy term. Substituting Eqs. (10) and (11) into Eqs. (16) and (17), we obtain the expressions for $\hbar^2/A_2$ and $\hbar^2/A_4$ in terms of the measurable quantities. Because the quantities $\hbar^2/A_2$ and $\hbar^2/A_4$ have a dimension of energy it is convenient to give them in units of $E(2_1^+)$.

$$\frac{\hbar^2}{A_2E(2_1^+)} = -\sqrt{\frac{35}{32\pi}}\frac{|Q(2_1^+)|}{q} + \left(2\frac{E(2_\gamma^+)}{E(2_1^+)} - 1\right) \times\sqrt{\frac{B(E2;2_\gamma^+ \to 2_1^+)B(E2;0_{\rm gs}^+ \to 2_\gamma^+)}{q^2B(E2;0_{\rm gs}^+ \to 2_1^+)}} - \left(2\frac{E(2_\beta^+)}{E(2_1^+)} - 1\right) \times\sqrt{\frac{B(E2;2_\beta^+ \to 2_1^+)B(E2;0_{\rm gs}^+ \to 2_\beta^+)}{q^2B(E2;0_{\rm gs}^+ \to 2_1^+)}} \quad (18)$$

$$-\frac{\hbar^2}{A_4E(2_1^+)} = \left(\frac{E(4_1^+)}{E(2_1^+)} - 2\right) - \sum_{i=2,3}\left(2\frac{E(2_i^+)}{E(2_1^+)} - \frac{E(4_1^+)}{E(2_1^+)}\right) \times\sqrt{\frac{B(E2;4_1^+ \to 2_i^+)B(E2;2_i^+ \to 0_{\rm gs}^+)}{B(E2;4_1^+ \to 2_1^+)B(E2;2_1^+ \to 0_{\rm gs}^+)}} \quad (19)$$

The values of the parameters A_0, A_2, A_4 are given in Table II. They are independent of the choices of the phases of the wave functions. Looking at the results presented in Table II we see that the values of the parameters $\hbar^2/A_L (L = 0, 2, 4)$ are quite similar for the well-deformed nuclei. Although they can deviate much more from these values in the case of spherical and γ-unstable nuclei.

The parameter $\hbar^2/A_0$ is approximately proportional to the Grodzins product for the first 2^+ state. For this product it was shown in Ref. [16] that its dependence on charge Z and the mass number A is described by the function $Z^2/A^{2/3}$. Together with Eq. (15) it indicates that $\hbar^2/A_0$ is proportional to A^{-2}. This fact is illustrated by the values of $\hbar^2/A_0 \cdot (A/100)^2$ shown in the last column of Table II.

The results presented above tell us that the mass tensor in the collective Hamiltonian cannot be taken as a constant and the deformation-dependent quadrupole and hexadecupole components of the mass tensor should be taken into account. However, it is very useful to know not only this fact but also the most characteristic qualitative effect of an inclusion of these components of the mass tensor into the expression for the kinetic energy. This point is clarified below.

Equations (10) and (11) are given by the sums of terms having different signs. The first term in these sums represents a contribution of the matrix elements characterizing $E2$ transitions inside the ground state band and have the largest values as compared to the other terms in the sums. The next two terms in Eq. (10) include the matrix elements describing the interband $E2$ transitions to the 4_1^+ state. These terms are negative. From this we can see the effect of an increase in the value of the hexadecupole component of the mass tensor that is proportional to $S^{(4)}$ [see Eq. (9)]. The large absolute value of $\langle 4_1\|(B^{-1})^{\rm lab}_4\|0_{\rm gs}^+\rangle$ means the large value of $S^{(4)}$. However, the first term in Eq. (10), i.e., $S_1^{(4)}$, being measured in units of $E(2_1^+)B(E2;2_1^+ \to 0_{\rm gs}^+)$ is restricted in value. Therefore, the

TABLE II. The values of $R_{4/2} = E(4_1^+)/E(2_1^+)$ and the calculated values of $\hbar^2/A_4$, $\hbar^2/A_2$, and $\hbar^2/A_0$ determined in the text by Eqs. (15)–(17). The last three quantities are given in units of 10^{-1} MeV. In the last column, A is the nuclear mass number. The experimental data are taken from Ref. [15].

Nucleus	$R_{4/2}$	$\hbar^2/A_4$	$\hbar^2/A_2$	$\hbar^2/A_0$	$\hbar^2/A_0(A/100)^2$
^{106}Pd	2.40	−1.52(19)	−0.22(8)	0.26	0.29
^{108}Pd	2.41	−1.35(17)	−0.22(9)	0.24	0.28
^{110}Pd	2.46	–	−0.27(7)	0.24	0.29
^{110}Cd	2.34	–	0.01(9)	0.20	0.24
^{112}Cd	2.29	–	−0.21(6)	0.19	0.24
^{114}Cd	2.30	−0.74(14)	−0.16(5)	0.18	0.23
^{134}Ba	2.32	−1.85(4)	0.07(11)	0.16	0.29
^{150}Nd	2.93	−0.78(13)	–	0.12	0.27
^{150}Sm	2.31	0.71(47)	–	0.13	0.29
^{152}Sm	3.00	−0.87(4)	−0.12(1)	0.13	0.30
^{154}Sm	3.26	−0.77(7)	–	0.12	0.28
^{154}Gd	3.02	−0.86(4)	−0.13(2)	0.14	0.33
^{156}Gd	3.24	−0.87(2)	−0.08(1)	0.125	0.30
^{158}Gd	3.29	−0.86(2)	−0.08(1)	0.11	0.27
^{160}Gd	3.32	–	−0.070(5)	0.10	0.26
^{164}Dy	3.32	–	−0.07(1)	0.095	0.26
^{172}Yb	3.29	−0.94(1)	–	0.09	0.27
^{178}Hf	3.30	−1.09(2)	–	0.09	0.29
^{188}Os	3.08	−0.93(13)	−0.06(2)	0.06	0.21
^{190}Os	2.93	–	−0.01(2)	0.065	0.23
^{192}Os	2.82	–	−0.01(1)	0.06	0.22
^{194}Pt	2.47	–	−0.080(15)	0.06	0.23
^{196}Pt	2.46	–	−0.15(1)	0.05	0.19

large value of $S^{(4)}$ can be obtained only due to the small values of the relative interband transitions to the 4_1^+ state. Thus, in the Hamiltonian the component of the mass tensor with $L = 4$ can be used to regulate the strength of the interband transitions decreasing them.

In the case of Eq. (11), which determines the component of the mass tensor with $L = 2$ through Eq. (9), two terms, namely, the first one containing the spectroscopic quadrupole moment and the matrix element for the $E2$ transition inside the ground band and the third one related to the transitions between the β (quasi-β) and the ground bands, are negative. But the second term with the matrix elements connecting the γ (quasi-γ) and the ground bands is positive. Thus, increasing the absolute value of the component of the mass tensor with $L = 2$ we can also decrease the relative intensity of the $E2$ transitions between the γ (quasi-γ) and the ground bands.

Equations (10) and (11) include the products of the energies and $B(E2)$'s. But the energies of the states of the β (quasi-β) and the γ (quasi-γ) bands are restricted from below by the energies of the states of the ground state band with the same angular momentum. Therefore, decreasing the values of the products of the energies of the collective state and the $B(E2)$'s we can decrease down to zero only the values of the $E2$ interband transition probabilities.

Regulating the values of both components of the mass tensor with $L = 4$ and 2 we can regulate independently the strength of the transitions between the β (quasi-β) and γ (quasi-γ) bands on one side and the ground state band on the other side.

The effect described above is similar to that observed in our previous article [2] where it was shown that by increasing the mass coefficients for the β modes and the γ modes we decrease the strength of the $E2$ transitions between the ground and the collective excited bands, keeping unchanged the energies of the band head states.

As an illustration of the idea discussed above let us consider how the variations of the parameter $\hbar^2/A_4$ can influence the strength of the $E2$ transitions from the excited bands to the ground band. Consider Eq. (19). The largest possible value of $E(4_1^+)/E(2_1^+)$ is 10/3. The smallest possible value of $E(2_{i=\gamma,\beta}^+)/E(2_1^+)$ is 2. Therefore,

$$-\frac{\hbar^2}{A_4 E(2_1^+)} \leqslant \frac{4}{3} - \frac{2}{3}\sum_{i=\gamma,\beta}\sqrt{\frac{B(E2;4_1^+\to 2_i^+)B(E2;2_i^+\to 0_{gs}^+)}{B(E2;4_1^+\to 2_1^+)B(E2;2_1^+\to 0_{gs}^+)}}. \quad (20)$$

We see that if the value of $-\frac{\hbar^2}{A_4 E(2_1^+)}$ approaches 1.33, then the product $B(E2;4_1^+\to 2_{\gamma,\beta}^+)B(E2;2_{\gamma,\beta}^+\to 0_{gs}^+)$ should go to zero.

Let us consider concrete examples. In the spherical nuclei ^{106,108}Pd the parameter $-\hbar^2/(A_4 E(2_1^+))$ takes the values 0.30 and 0.31, which are quite far from the critical value 1.33 when the interband transitions disappear. In the transitional nuclei ^{150}Nd and ^{152}Sm, $-\hbar^2/(A_4 E(2_1^+))$ is equal to 0.60 and 0.71, respectively. In the well-deformed axially symmetric nuclei

^{154}Sm and ^{158}Gd, $-\hbar^2/(A_4 E(2_1^+))$ takes the values 0.94 and 1.08. Thus, this parameter approaches the critical value when we go from spherical to deformed nuclei.

The fact that the mass tensor can be a reason for very weak interband transitions is very interesting because it indicates a possibility that in nuclei some excited bands can have a collective nature and nevertheless their $E2$ transitions into the ground state band are weak. A similar picture is observed in the well-deformed nuclei where $E2$ transitions from the β band to the ground band can be very small.

In our previous publications [1,2,17] we have shown that to describe the experimental data on the interband $E2$ transitions in the well-deformed axially symmetric nuclei, it is necessary to use different mass coefficients for the rotational and β- and γ-vibrational modes in the Bohr Hamiltonian presented in the intrinsic frame. This result can be obtained using the more general relations derived above. This is done in the Appendix D where the more exact expressions for $B_{\rm rot}$, B_γ, and B_β are derived. However, using for $E(4_1^+)/E(2_1^+)$ the value 10/3 and neglecting the ratios $E(2_1^+)/E(2_{\beta,\gamma}^+)$ and $E(4_1^+)/E(2_{\beta,\gamma}^+)$, which are small in the well-deformed nuclei, we come to the results obtained in Ref. [1], namely,

$$\frac{\hbar^2}{B_{\rm rot}} = \frac{E(2_1^+)B(E2;0_{\rm gs}^+ \to 2_1^+)}{q^2} \qquad (21)$$

$$\frac{\hbar^2}{B_\gamma} = \frac{E(2_\gamma^+)B(E2;0_{\rm gs}^+ \to 2_\gamma^+)}{q^2} \qquad (22)$$

$$\frac{\hbar^2}{B_\beta} = \frac{2E(2_\beta^+)B(E2;0_{\rm gs}^+ \to 2_\beta^+)}{q^2}. \qquad (23)$$

The terms neglected in Eqs. (21) and (22) introduce the errors of the order of (2–4)%. The terms neglected in Eq. (23) can decrease the value of B_β up to 30%.

III. SUMMARY

We have shown in this article on the basis of the experimental data that the mass tensor in the Bohr collective quadrupole Hamiltonian given in the laboratory frame and written in terms of Bohr's collective variables $\alpha_{2\mu}$ cannot be reduced to one constant mass coefficient. The mass tensor contains not only scalar but also quadrupole and hexadecupole components and therefore is a function of the collective coordinates.

We have shown that the matrix elements of the mass tensor can be expressed through the nondiagonal energy weighted sum rules. We have suggested also the simplest form of the mass tensor that satisfies the relations derived in this article [Eq. (14)]. The parameters of this mass tensor are determined completely for many nuclei by the existing experimental data (Table II). The values of these parameters derived from the experimental data for the well-deformed axially symmetric nuclei are quite close to each other, indicating the possibility to use an approximately universal mass tensor for the description of the well-deformed nuclei. There is not enough data to reach a similar conclusion about a mass tensor for the spherical and the γ-unstable nuclei. The second term in Eq. (14) was already used in Refs. [10] and [12]. However, to our knowledge there were no publications where the third term in Eq. (14) was used. But as it follows from our analysis this term is very important.

In our previous publications we have derived the expressions for the mass coefficients $B_{\rm rot}$, B_γ, and B_β for the Bohr Hamiltonian written in the intrinsic frame for the well-deformed axially symmetric nuclei. Above, these expressions are obtained using the general form of the Bohr Hamiltonian given in the laboratory frame.

The relations derived in this article show that the quadrupole and the hexadecupole components of the mass tensor can decrease the strength of the $E2$ transitions between the ground and the excited collective bands if they approach sufficiently large absolute values. This result is a generalization of the result obtained in Ref. [2] for the well-deformed nuclei where it was shown that an increase of the mass coefficients for the β- and γ-vibrational modes decreases the strength of the $E2$ transitions, although the energies of the vibrational states can be kept unchanged if the stiffness coefficient is varied in a corresponding way.

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APPENDIX A

Let us calculate a double commutator $[[H, Q_{2\mu}], Q_{2\mu'}]$ with $Q_{2\mu} = q\alpha_{2\mu}$ and Eq. (2) for the kinetic energy term. First of all let us calculate the commutator $[H, Q_{2\mu}]$,

$$\begin{aligned}[H, Q_{2\mu}] &= [T + V(\alpha_2), q\alpha_{2\mu}] = q[T, \alpha_{2\mu}]\\ &= -\frac{\hbar^2}{2}q\left[\sum_{\nu,\nu'}\frac{\partial}{\partial\alpha_{2\nu}}(B^{-1}(\alpha_2))^{\rm lab}_{\nu,\nu'}\frac{\partial}{\partial\alpha_{2\nu'}}, \alpha_{2\mu}\right]\\ &= -\frac{\hbar^2}{2}q\sum_{\nu}\left(\frac{\partial}{\partial\alpha_{2\nu}}(B^{-1}(\alpha_2))^{\rm lab}_{\nu,\mu}\right.\\ &\qquad\left. + (B^{-1}(\alpha_2))^{\rm lab}_{\mu,\nu}\frac{\partial}{\partial\alpha_{2\nu}}\right). \qquad (A1)\end{aligned}$$

Using Eq. (A1), for the double commutator we get

$$\begin{aligned}[[H, Q_{2\mu}], Q_{2\mu'}] &= -\frac{\hbar^2q^2}{2}\left((B^{-1}(\alpha_2))^{\rm lab}_{\mu',\mu} + (B^{-1}(\alpha_2))^{\rm lab}_{\mu,\mu'}\right)\\ &= -\hbar^2q^2\sqrt{5}\sum_{LM}C^{LM}_{2\mu2\mu'}(B^{-1})^{\rm lab}_{LM}. \qquad (A2)\end{aligned}$$

APPENDIX B

In this appendix we derive the expression for $S^{(L)}$ in terms of the excitation energies and the reduced matrix elements of the quadrupole operator. From a definition of $S^{(L)}$ we obtain

$$\begin{aligned} S^{(L)} &= \sqrt{2L+1}\sum_{\mu,\mu'} C^{LM}_{2\mu 2\mu'}\langle LM|HQ_{2\mu}Q_{2\mu'} \\ &\quad - Q_{2\mu}HQ_{2\mu'} - Q_{2\mu'}HQ_{2\mu} + Q_{2\mu'}Q_{2\mu}H|0^+_{\rm gs}\rangle \\ &= \sqrt{2L+1}\sum_{\mu,\mu'} C^{LM}_{2\mu 2\mu'} \\ &\quad \times \sum_i (\langle LM|HQ_{2\mu}|2^+_i\mu'\rangle\langle 2^+_i\mu'|Q_{2\mu'}|0^+_{\rm gs}\rangle \\ &\quad - \langle LM|Q_{2\mu}|2^+_i\mu'\rangle\langle 2^+_i\mu'|HQ_{2\mu'}|0^+_{\rm gs}\rangle \\ &\quad - \langle LM|Q_{2\mu'}|2^+_i\mu\rangle\langle 2^+_i\mu|HQ_{2\mu}|0^+_{\rm gs}\rangle) \\ &= \frac{1}{\sqrt{5}}\sum_i (E(L)-2E(2^+_i))\langle L\|Q_2\|2^+_i\rangle\langle 2^+_i\|Q_2\|0^+_{\rm gs}\rangle, \end{aligned} \quad \text{(B1)}$$

where $|LM\rangle$ is the lowest state with angular momentum L. Now we use some phase relations for the reduced matrix elements of the quadrupole operator. Because in Eq. (B1) for $L = 2$ and 4 some eigenstates appear only once in the nondiagonal matrix elements of $Q_{2\mu}$, the signs of $S^{(4)}$ and $S^{(2)}$ depend on the agreement about the phases of these states. However, for our conclusion only the fact of deviation of the absolute values of $S^{(4)}$ and $S^{(2)}$ from zero is important. Of course a mass tensor is independent of the choices of phases.

In the case of $L = 4$ it can be shown for the limits of the well-deformed axially symmetric nuclei and of the spherical nuclei with small anharmonicities that

$$\begin{aligned} &\text{sign}(\langle 4^+_1\|Q_2\|2^+_1\rangle\langle 2^+_1\|Q_2\|0^+_{\rm gs}\rangle) \\ &\quad = \text{sign}(\langle 4^+_1\|Q_2\|2^+_i\rangle\langle 2^+_i\|Q_2\|0^+_{\rm gs}\rangle), \end{aligned} \quad \text{(B2)}$$

at least for $i = \gamma$ and $i = \beta$. Then

$$\begin{aligned} S^{(4)} &= \text{sign}(\langle 4^+_1\|Q_2\|2^+_1\rangle\langle 2^+_1\|Q_2\|0^+_{\rm gs}\rangle) \\ &\quad \times \sum_{i=1,\beta,\gamma}(E(4^+_1)-2E(2^+_i)) \\ &\quad \times \sqrt{B(E2;0^+_{\rm gs}\to 2^+_i)\cdot B(E2;2^+_i\to 4^+_1)}. \end{aligned} \quad \text{(B3)}$$

For definiteness we assume in the text and below in Appendices C and D that the product $\langle 4^+_1\|Q_2\|2^+_1\rangle\langle 2^+_1\|Q_2\|0^+_{\rm gs}\rangle$ is positive.

In the case of $L = 2$ there is the well-known relation between the signs of the reduced matrix elements of the quadrupole operator [18,19]

$$\begin{aligned} &\text{sign}(\langle 2^+_1\|Q_2\|2^+_1\rangle\langle 2^+_1\|Q_2\|0^+_{\rm gs}\rangle) \\ &\quad = -\text{sign}(\langle 2^+_1\|Q_2\|2^+_\gamma\rangle\langle 2^+_\gamma\|Q_2\|0^+_{\rm gs}\rangle), \end{aligned} \quad \text{(B4)}$$

where in the spherical limit the 2^+_γ state is the two-phonon state. The last relation was obtained in the limits of the well-deformed axially symmetric nuclei and of the spherical nuclei with small anharmonicity. It was also checked in the IBA with the consistent Q Hamiltonian [20] for the wide variations of the parameters [21]. In the limits of the well-deformed (see expressions for the matrix elements in Appendix D) and spherical nuclei we can obtain also that

$$\begin{aligned} &\text{sign}(\langle 2^+_1\|Q_2\|2^+_1\rangle\langle 2^+_1\|Q_2\|0^+_{\rm gs}\rangle) \\ &\quad = \text{sign}(\langle 2^+_1\|Q_2\|2^+_\beta\rangle\langle 2^+_\beta\|Q_2\|0^+_{\rm gs}\rangle), \end{aligned} \quad \text{(B5)}$$

where the 2^+_β state is the three-phonon 2^+ state in the spherical limit. Using these phase relations we can write

$$\begin{aligned} S^{(2)} &= \text{sign}(\langle 2^+_1\|Q_2\|2^+_1\rangle\langle 2^+_1\|Q_2\|0^+_{\rm gs}\rangle) \\ &\quad \times \Bigg(E(2^+_1)\sqrt{\frac{35}{32\pi}}|Q(2^+_1)|\cdot\sqrt{B(E2;0^+_{\rm gs}\to 2^+_1)} \\ &\quad - (2E(2^+_\gamma)-E(2^+_1)) \\ &\quad \times \sqrt{B(E2;2^+_\gamma\to 2^+_1)\cdot B(E2;0^+_{\rm gs}\to 2^+_\gamma)} \\ &\quad + (2E(2^+_\beta)-E(2^+_1)) \\ &\quad \times \sqrt{B(E2;2^+_\beta\to 2^+_1)\cdot B(E2;0^+_{\rm gs}\to 2^+_\beta)}\Bigg). \end{aligned} \quad \text{(B6)}$$

Note again that in Eq. (B6) we assume that in the case of the spherical nuclei the 2^+_γ state is the two-phonon state. We see from Eq. (B6) that for the correct calculations of $S^{(2)}$ it is important to do a correct assignment for the states. In our calculations we have followed the assignments of Ref. [15]. For definiteness, we assume in the text and in Appendices C and D that the product $(\langle 2^+_1\|Q_2\|2^+_1\rangle\langle 2^+_1\|Q_2\|0^+_{\rm gs}\rangle$ is negative.

The expression for $S^{(0)}$ follows directly from Eq. (B1) and the expression for $B(E2;2^+_i\to 0^+_{\rm gs})$ through the reduced matrix elements of Q_2,

$$S^{(0)} = -2\sqrt{5}\sum_{i=1,\beta,\gamma}E(2^+_i)B(E2;2^+_i\to 0^+_{\rm gs}). \quad \text{(B7)}$$

APPENDIX C

Equation (9) expresses the components of the mass tensor through $S^{(L)}$. Substituting Eq. (14) into Eq. (9) and taking into account the phase conventions from Appendix B we have

$$\begin{aligned} \frac{\hbar^2}{A_0} &= -\frac{S^{(0)}}{q^2} \\ \frac{\hbar^2}{A_2}\langle 2^+_1M|\alpha_{2M}|0^+_{\rm gs}\rangle &= \frac{S^{(2)}}{\sqrt{5}q^2} \\ \frac{\hbar^2}{A_4}\langle 4^+_1M|(\alpha\alpha)_{4M}|0^+_{\rm gs}\rangle &= -\frac{S^{(4)}}{3q^2}. \end{aligned} \quad \text{(C1)}$$

The matrix element $\langle 2^+_1M|\alpha_{2M}|0^+_{\rm gs}\rangle$ can be expressed through the $B(E2)$,

$$\langle 2^+_1M|\alpha_{2M}|0^+_{\rm gs}\rangle = \frac{1}{q}\sqrt{B(E2;2^+_1\to 0^+_{\rm gs})}. \quad \text{(C2)}$$

The same can be done for the matrix element $\langle 4^+_1M|(\alpha\alpha)_{4M}|0^+_{\rm gs}\rangle$, however, not exactly but in good approximation. The result is

$$\begin{aligned} &\langle 4^+_1M|(\alpha\alpha)_{4M}|0^+_{\rm gs}\rangle \\ &\quad = \frac{1}{q^2}\sqrt{B(E2;4^+_1\to 2^+_1)\cdot B(E2;2^+_1\to 0^+_{\rm gs})}. \end{aligned} \quad \text{(C3)}$$

Using Eqs. (C1), (C2), and (C3), we obtain

$$\frac{\hbar^2}{A_0} = -\frac{S^{(0)}}{q^2} \tag{C4}$$

$$\frac{\hbar^2}{A_2} = \frac{S^{(2)}}{q\sqrt{B(E2;0^+_{\rm gs}\to 2^+_1)}} \tag{C5}$$

$$\frac{\hbar^2}{A_4} = -\frac{S^{(4)}}{\sqrt{B(E2;0^+_{\rm gs}\to 2^+_1)B(E2;2^+_1\to 4^+_1)}}. \tag{C6}$$

APPENDIX D

In this appendix we derive the expressions for the mass coefficients $B_{\rm rot}$, B_γ, and B_β considered in our previous publications. From Eq. (9) we have

$$\sqrt{2L+1}\langle LM|(B^{-1})^{\rm lab}_{LM}|0^+_{\rm gs}\rangle = -\frac{S^{(L)}}{\sqrt{5}\hbar^2 q^2}. \tag{D1}$$

In the case of the well-deformed axially symmetric nuclei,

$$|LM\rangle = \sqrt{\frac{2L+1}{8\pi^2}}D^L_{M0}\left|0^{+({\rm int})}_{\rm gs}\right\rangle \tag{D2}$$

$$|0^+_{\rm gs}\rangle = \sqrt{\frac{1}{8\pi^2}}\left|0^{+({\rm int})}_{\rm gs}\right\rangle, \tag{D3}$$

and the intrinsic components of the mass tensor are introduced by the standard relation

$$(B^{-1})^{\rm lab}_{LM} = \frac{1}{\sqrt{2(1+\delta_{K0})}} \times \left(D^L_{MK}+D^L_{M-K}\right)(B^{-1})^{\rm int}_{LK}. \tag{D4}$$

Using Eqs. (D2), (D3), and (D4), we obtain

$$\langle LM|(B^{-1})^{\rm lab}_{LM}|0^+_{\rm gs}\rangle = \frac{1}{\sqrt{2L+1}}(B^{-1})^{\rm int}_{L0}, \tag{D5}$$

and from Eq. (9) we get

$$\hbar^2(B^{-1})^{\rm int}_{L0} = -\frac{S^{(L)}}{\sqrt{5}q^2}. \tag{D6}$$

In the case of the well-deformed axially symmetric nuclei we can substitute into Eq. (D6) for the reduced matrix elements of the quadrupole operator the expressions that follow from the Alaga rules:

$$\langle 2^+_1\|Q_2\|0^+_{\rm gs}\rangle = \sqrt{B(E2;0^+_{\rm gs}\to 2^+_1)} \tag{D7}$$

$$\langle 2^+_\beta\|Q_2\|0^+_{\rm gs}\rangle = \sqrt{B(E2;0^+_{\rm gs}\to 2^+_\beta)} \tag{D8}$$

$$\langle 2^+_\gamma\|Q_2\|0^+_{\rm gs}\rangle = \sqrt{B(E2;0^+_{\rm gs}\to 2^+_\gamma)} \tag{D9}$$

$$\langle 2^+_1\|Q_2\|2^+_1\rangle = -\sqrt{\frac{10}{7}}\sqrt{B(E2;0^+_{\rm gs}\to 2^+_1)} \tag{D10}$$

$$\langle 2^+_1\|Q_2\|2^+_\beta\rangle = -\sqrt{\frac{10}{7}}\sqrt{B(E2;0^+_{gs}\to 2^+_\beta)} \tag{D11}$$

$$\langle 2^+_1\|Q_2\|2^+_\gamma\rangle = \sqrt{\frac{10}{7}}\sqrt{B(E2;0^+_{\rm gs}\to 2^+_\gamma)} \tag{D12}$$

$$\langle 4^+_1\|Q_2\|2^+_1\rangle = 3\sqrt{\frac{2}{7}}\sqrt{B(E2;0^+_{\rm gs}\to 2^+_1)} \tag{D13}$$

$$\langle 4^+_1\|Q_2\|2^+_\beta\rangle = 3\sqrt{\frac{2}{7}}\sqrt{B(E2;0^+_{\rm gs}\to 2^+_\beta)} \tag{D14}$$

$$\langle 4^+_1\|Q_2\|2^+_\gamma\rangle = \frac{1}{2}\sqrt{\frac{2}{7}}\sqrt{B(E2;0^+_{\rm gs}\to 2^+_\gamma)}. \tag{D15}$$

Above we have used the standard expressions for the eigenvectors of the well-deformed nuclei [7].

In Ref. [1] we derived the following relations valid for the case of the well-deformed axially symmetric nuclei:

$$\frac{1}{B_{\rm rot}} = (B^{-1})^{\rm int}_{00} - \sqrt{\frac{5}{56}}(B^{-1})^{\rm int}_{20} - \sqrt{\frac{8}{7}}(B^{-1})^{\rm int}_{40} \tag{D16}$$

$$\frac{1}{B_\gamma} = (B^{-1})^{\rm int}_{00} + \sqrt{\frac{10}{7}}(B^{-1})^{\rm int}_{20} + \sqrt{\frac{1}{14}}(B^{-1})^{\rm int}_{40} \tag{D17}$$

$$\frac{1}{B_\beta} = (B^{-1})^{\rm int}_{00} - \sqrt{\frac{10}{7}}(B^{-1})^{\rm int}_{20} + \sqrt{\frac{18}{7}}(B^{-1})^{\rm int}_{40}. \tag{D18}$$

Substituting Eq. (D6) into Eqs. (D16)–(D18) and using Eqs. (D7)–(D15), we obtain

$$\begin{aligned}\frac{\hbar^2}{B_{\rm rot}} &= \frac{E(2^+_1)B(E2;0^+_{\rm gs}\to 2^+_1)}{q^2}\\ &\quad -\frac{12}{35}\left(\frac{10}{3}-\frac{E(4^+_1)}{E(2^+_1)}\right)\cdot\frac{E(2^+_1)B(E2;0^+_{\rm gs}\to 2^+_1)}{q^2}\\ &\quad +\frac{(12E(4^+_1)-5E(2^+_1))}{35E(2^+_\beta)}\cdot\frac{E(2^+_\beta)B(E2;0^+_{\rm gs}\to 2^+_\beta)}{q^2}\\ &\quad +\frac{(2E(4^+_1)+5E(2^+_1))}{35E(2^+_\gamma)}\cdot\frac{E(2^+_\gamma)B(E2;0^+_{\rm gs}\to 2^+_\gamma)}{q^2}\end{aligned} \tag{D19}$$

$$\begin{aligned}\frac{\hbar^2}{B_\gamma} &= \frac{E(2^+_\gamma)B(E2;0^+_{\rm gs}\to 2^+_\gamma)}{q^2}\\ &\quad -\frac{(20E(2^+_1)+E(4^+_1))}{70E(2^+_\gamma)}\cdot\frac{E(2^+_\gamma)B(E2;0^+_{\rm gs}\to 2^+_\gamma)}{q^2}\\ &\quad +\frac{3}{35}\left(\frac{10}{3}-\frac{E(4^+_1)}{E(2^+_1)}\right)\cdot\frac{E(2^+_1)B(E2;0^+_{\rm gs}\to 2^+_1)}{q^2}\\ &\quad +\frac{(10E(2^+_1)-3E(4^+_1))}{35E(2^+_\beta)}\cdot\frac{E(2^+_\beta)B(E2;0^+_{\rm gs}\to 2^+_\beta)}{q^2}\end{aligned} \tag{D20}$$

$$\begin{aligned}\frac{\hbar^2}{B_\beta} &= \frac{2E(2^+_\beta)B(E2;0^+_{\rm gs}\to 2^+_\beta)}{q^2}\\ &\quad -\frac{(5E(2^+_1)+9E(4^+_1))}{35E(2^+_\beta)}\cdot\frac{2E(2^+_\beta)B(E2;0^+_{\rm gs}\to 2^+_\beta)}{q^2}\end{aligned}$$

$$+\frac{18}{35}\left(\frac{10}{3}-\frac{E(4_1^+)}{E(2_1^+)}\right)\cdot\frac{E(2_1^+)B(E2;0_{gs}^+\to 2_1^+)}{q^2}$$
$$+\frac{(10E(2_1^+)-3E(4_1^+))}{35E(2_\gamma^+)}\cdot\frac{E(2_\gamma^+)B(E2;0_{gs}^+\to 2_\gamma^+)}{q^2}. \quad \text{(D21)}$$

In Eqs. (D19)–(D21) the main contribution gives the first terms. Only these terms were derived in Ref. [1]. In Eqs. (D19) and (D20), the rest are of the order of (2–4)% of the first term. In Eq. (D21) the contribution of the second and the third terms is about 1/3 that of the first one. Using $E(4_1^+)/E(2_1^+) = 10/3$ and neglecting the ratios $E(2_1^+)/E(2_{\beta,\gamma}^+)$ and $E(4_1^+)/E(2_{\beta,\gamma}^+)$, which are relatively small in the well-deformed nuclei, we come to the results obtained in Ref. [1], namely,

$$\frac{\hbar^2}{B_{rot}}=\frac{E(2_1^+)B(E2;0_{gs}^+\to 2_1^+)}{q^2} \quad \text{(D22)}$$

$$\frac{\hbar^2}{B_\gamma}=\frac{E(2_\gamma^+)B(E2;0_{gs}^+\to 2_\gamma^+)}{q^2} \quad \text{(D23)}$$

$$\frac{\hbar^2}{B_\beta}=\frac{2E(2_\beta^+)B(E2;0_{gs}^+\to 2_\beta^+)}{q^2}. \quad \text{(D24)}$$

[1] R. V. Jolos and P. von Brentano, Phys. Rev. C **76**, 024309 (2007).
[2] R. V. Jolos and P. von Brentano, Phys. Rev. C **77**, 064317 (2008).
[3] D. Bonatsos, D. Lenis, N. Minkov, P. P. Raychev, and P. A. Terziev, Phys. Rev. C **69**, 014302 (2004).
[4] L. Fortunato, Phys. Rev. C **70**, 011302(R) (2004).
[5] D. Bonatsos, E. M. McCutchan, N. Minkov *et al.*, Phys. Rev. C **76**, 064312 (2007).
[6] M. A. Caprio, Phys. Rev. C **72**, 054323 (2005).
[7] A. Bohr and B. Mottelson, *Nuclear Structure* (Benjamin, Elmsford, NY 1975), Vol. 2, p. 407.
[8] J. M. Eisenberg and W. Greiner, *Nuclear Models* (North-Holland, Amsterdam, 1970), Vol. 1.
[9] W. Greiner and J. Maruhn, *Kern–Modelle* (Verlag Harri Deutsch, Frankfurt, 1995).
[10] G. Gneuss and W. Greiner, Nucl. Phys. **171**, 449 (1971).
[11] K. Kumar, Nucl. Phys. **A231**, 189 (1974).
[12] P. Petkov, A. Dewald, and W. Andrejtscheff, Phys. Rev. C **51**, 2511 (1995).
[13] P. Ring and P. Schuck, *The Nuclear Many Body Problem* (Springer, New York, 1980).
[14] A. de Shalit and H. Feshbach, *Theoretical Nuclear Physics* (Wiley, New York, 1974).
[15] http://www.nndc.bnl.gov/nndc/ensdf/.
[16] S. Raman, C. W. Nestor, Jr., and P. Tikkanen, At. Data Nucl. Data Tables **78**, 1 (2001).
[17] R. V. Jolos and P. von Brentano, Phys. Rev. C **74**, 064307 (2006).
[18] T. Tamura, Phys. Lett. **B28**, 90 (1968).
[19] K. Kumar, Phys. Lett. **B29**, 25 (1969).
[20] R. F. Casten and D. D. Warner, Rev. Mod. Phys. **60**, 389 (1988).
[21] R. V. Jolos and P. von Brentano, Phys. Lett. **B381**, 7 (1996).