Exact three-dimensional wave function and the on-shell t matrix for the sharply cut-off Coulomb potential: Failure of the standard renormalization factor

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The three-dimensional wave function for a sharply cut-off Coulomb potential is analytically derived. The asymptotic form of the related scattering amplitude reveals a failure of the standard renormalization factor which is believed to be generally valid for any type of screening.

I. INTRODUCTION

The long range behavior of the Coulomb force causes technical problems in the scattering for more than two particles. For instance the three-body Faddeev kernel develops singularities, which deny a direct numerical approach. A way out has been searched for in the past by starting with a screened Coulomb potential, which for instance in the context of the three-body problem leads to a screened two-body Coulomb t-matrix. In the limit of an infinite screening radius it is claimed in the literature [1–3] that the on-shell two-body t-matrix approaches the physical one except for an infinitely oscillating phase factor, known analytically. Thus removing that factor, called renormalization, the physical result can be obtained.

As a basis for that approach the works by Gorshkov [4,5], Ford [6,7], and Taylor [1,2] are most often quoted. Gorshkov [4,5] regards potential scattering on a Yukawa potential in the limit of the screening radius going to infinity. He works directly in three dimensions avoiding a partial wave decomposition. He sums up the perturbation series to infinite order. As a result he finds the limit for the wave function of a Yukawa potential for an infinite screening radius. That limit function equals the standard Coulomb wave function multiplied by an infinitely oscillating phase factor. Contrary to what is quoted in Chen [8] he has not achieved the wave function for a Yukawa potential at an arbitrary screening radius but only its limiting form.

The work by Ford [6,7] relies on a partial wave decomposition. This leads to a very difficult technical task to handle the situation when the orbital angular momentum l is about pR, where p is the asymptotic wave number and R the screening radius. This task is left unsolved and the infinite sum in l is carried out without controlling the l-dependence of certain correction terms depending on R. In other words the correction terms for a given l are assumed to remain valid also for three-dimensional objects. This leaves at least doubts about the rigorousness of that approach. The same is true for the investigations of Taylor [1,2], where again a partial wave decomposition is the basis and the infinite sum over l is carried through without control of its validity for the correction terms.

In such a situation we felt that a rigorous analytical approach for a sharply cut-off Coulomb potential carried through directly in three dimensions is in order. This paper delivers an analytical solution for an arbitrary cut-off radius. Further we also provide an exact expression for the corresponding scattering amplitude (equivalent to the on-shell t-matrix). The paper is organized as follows. In Sec. II the wave function is derived. In Sec. III the scattering amplitude and its limit for vanishing screening is given. These purely analytical results are confirmed by numerical studies presented in Sec. IV. In the Appendix we regard the much simpler case for s-wave scattering. We summarize in Sec. V.

II. THE WAVE FUNCTION FOR A SHARPLY CUT-OFF COULOMB POTENTIAL

Let us regard two equally charged particles with mass m. Then the two-body Schrödinger equation reads

$$\left(-\nabla^2 - p^2 + \frac{me^2}{r}\right)\Psi^{(+)}(\vec{r}) = 0. \qquad (1)$$

It is well known that in parabolic coordinates

$$u = r - z, \qquad (2)$$
$$v = r + z, \qquad (3)$$
$$\phi = \tan^{-1}\frac{y}{x}, \qquad (4)$$

the partial differential equation factorizes and yields the solution

$$\Psi^{(+)}(\vec{r}) = const \; e^{i\vec{p}\cdot\vec{r}} \,_1F_1(-i\eta, 1, i(pr - \vec{p}\cdot\vec{r})) \qquad (5)$$

with Somerfeld parameter $\eta = \frac{me^2}{2p}$.

Now we switch to a sharply screened Coulomb potential

$$V(r) = \Theta(R - r)\frac{e^2}{r} \qquad (6)$$

and rewrite Eq. (1) into the form of the Lippmann-Schwinger equation

$$\Psi_R^{(+)}(\vec{r}) = \frac{1}{(2\pi)^{3/2}}e^{i\vec{p}\cdot\vec{r}} - \frac{m}{4\pi}\int d^3r' \frac{e^{ip|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}$$
$$\times \Theta(R - r')\frac{e^2}{r'}\Psi_R^{(+)}(\vec{r}'). \qquad (7)$$

This defines uniquely the wave function $\Psi_R^{(+)}(\vec{r})$ for a given cut-off radius R. Acting on Eq. (7) with $(-\nabla^2 - p^2)$ and using the well-known property of the free Greens function in the

integral kernel one obtains the Schrödinger equation

$$(-\nabla^2 - p^2)\Psi_R^{(+)}(\vec{r}) = -m\Theta(R-r)\frac{e^2}{r}\Psi_R^{(+)}(\vec{r}). \quad (8)$$

Thus for $r < R$ one has to have

$$\Psi_R^{(+)}(\vec{r}) = Ae^{i\vec{p}\cdot\vec{r}}{}_1F_1(-i\eta, 1, i(pr - \vec{p}\cdot\vec{r})) \quad (9)$$

with some to be determined constant A. The idea is, therefore, to insert that knowledge into the Lippmann-Schwinger equation (7) leading to

$$\Psi_R^{(+)}(\vec{r}) = \frac{1}{(2\pi)^{3/2}}e^{i\vec{p}\cdot\vec{r}} - \frac{m}{4\pi}\int d^3r'\frac{e^{ip|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}$$
$$\times \Theta(R-r')\frac{e^2}{r'}Ae^{i\vec{p}\cdot\vec{r}'}{}_1F_1(-i\eta, 1, i(pr' - \vec{p}\cdot\vec{r}')). \quad (10)$$

If we choose $r < R$ then also the left hand side is known and one obtains the following identity:

$$Ae^{i\vec{p}\cdot\vec{r}}{}_1F_1(-i\eta, 1, i(pr - \vec{p}\cdot\vec{r}))$$
$$= \frac{1}{(2\pi)^{3/2}}e^{i\vec{p}\cdot\vec{r}} - \frac{m}{4\pi}\int d^3r'\frac{e^{ip|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}$$
$$\times \Theta(R-r')\frac{e^2}{r'}Ae^{i\vec{p}\cdot\vec{r}'}{}_1F_1(-i\eta, 1, i(pr' - \vec{p}\cdot\vec{r}')). \quad (11)$$

This provides the factor A. If A is known one can determine the scattering amplitude f_R defined for $r \to \infty$ by

$$\Psi_R^{(+)}(\vec{r}) \to \frac{1}{(2\pi)^{3/2}}e^{i\vec{p}\cdot\vec{r}} + \frac{e^{ipr}}{r}A\left(-\frac{m}{4\pi}\right)\int d^3r'e^{-ip\hat{r}\cdot\vec{r}'}$$
$$\times \Theta(R-r')\frac{e^2}{r'}e^{i\vec{p}\cdot\vec{r}'}{}_1F_1(-i\eta, 1, i(pr' - \vec{p}\cdot\vec{r}'))$$
$$\equiv \frac{1}{(2\pi)^{3/2}}e^{i\vec{p}\cdot\vec{r}} + \frac{e^{ipr}}{r}f_R. \quad (12)$$

It is not difficult using properties of the confluent hypergeometric function to show that the corresponding LS equation, for instance for an s-wave, is identically fulfilled as it should be. Doing that one can read off the corresponding analytical expression for A. That calculation is deferred to the Appendix.

The three-dimensional case is much harder. Let us choose $\hat{p} = \hat{z}$ and work with the parabolic coordinates. Then Eq. (11) turns into

$$Ae^{i\frac{p}{2}(v-u)}{}_1F_1(-i\eta, 1, ipu)$$
$$= \frac{1}{(2\pi)^{3/2}}e^{i\frac{p}{2}(v-u)} + A\frac{e^2}{2}\int_0^{2R}du'e^{-i\frac{p}{2}u'}{}_1F_1(-i\eta, 1, ipu')$$
$$\times \int_0^{2R-u'}dv'e^{i\frac{p}{2}v'}\left(-\frac{m}{4\pi}\right)\int_0^{2\pi}d\phi'\frac{e^{ip|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}. \quad (13)$$

Since we want to determine just one factor A one value of u and v is sufficient and we choose the simplest case $u = v = 0$. Then the ϕ' integration is trivial and one obtains

$$A = \frac{1}{(2\pi)^{3/2}} - A\frac{e^2m}{2}\int_0^{2R}du'{}_1F_1(-i\eta, 1, ipu')$$
$$\times \int_0^{2R-u'}dv'e^{ipv'}\frac{1}{u'+v'}, \quad (14)$$

where we used ${}_1F_1(-i\eta, 1, 0) = 1$. Substituting $u' = 2Rx$, $v' = 2Ry$ and defining $A \equiv \tilde{A}\frac{1}{(2\pi)^{3/2}}$ one obtains

$$\tilde{A} = 1 - \tilde{A}\eta T\int_0^1 dx{}_1F_1(-i\eta, 1, iTx)\int_0^{1-x}dye^{iTy}\frac{1}{x+y} \quad (15)$$

with $T \equiv 2pR$.

Introducing $z \equiv iT$ let us define

$$\tilde{F}(z) = 1 + \frac{\eta z}{i}\int_0^1 dx{}_1F_1(-i\eta, 1, zx)\int_0^{1-x}dye^{zy}\frac{1}{x+y}. \quad (16)$$

Substituting $zx = \tau$, $zy = \tau'$ we get

$$\tilde{F}(z) = 1 - i\eta\int_0^z d\tau{}_1F_1(-i\eta, 1, \tau)\int_0^{z-\tau}d\tau'e^{\tau'}\frac{1}{\tau+\tau'}. \quad (17)$$

Then it follows that

$$\frac{d\tilde{F}(z)}{dz} = -\frac{i\eta}{z}e^z\int_0^z d\tau{}_1F_1(-i\eta, 1, \tau)e^{-\tau}, \quad (18)$$

$$\frac{d^2\tilde{F}(z)}{dz^2} = \frac{i\eta}{z^2}e^z\int_0^z d\tau{}_1F_1(-i\eta, 1, \tau)e^{-\tau}$$
$$- \frac{i\eta}{z}e^z\int_0^z d\tau{}_1F_1(-i\eta, 1, \tau)e^{-\tau}$$
$$- \frac{i\eta}{z}e^z\int_0^z d\tau{}_1F_1(-i\eta, 1, \tau)e^{-z}$$
$$= -\frac{1}{z}\frac{d\tilde{F}(z)}{dz} + \frac{d\tilde{F}(z)}{dz} - \frac{i\eta}{z}{}_1F_1(-i\eta, 1, z). \quad (19)$$

Consequently

$$z\frac{d^2\tilde{F}(z)}{dz^2} + (1-z)\frac{d\tilde{F}(z)}{dz} = -i\eta{}_1F_1(-i\eta, 1, z). \quad (20)$$

We add $i\eta\tilde{F}(z)$ on both sides

$$z\frac{d^2\tilde{F}(z)}{dz^2} + (1-z)\frac{d\tilde{F}(z)}{dz} + i\eta\tilde{F}(z)$$
$$= i\eta(\tilde{F}(z) - {}_1F_1(-i\eta, 1, z)). \quad (21)$$

The left side put to zero is the defining differential equation for ${}_1F_1(-i\eta, 1, z)$. Thus Eq. (21) is fulfilled for

$$\tilde{F}(z) = {}_1F_1(-i\eta, 1, z) \quad (22)$$

which also fixes the normalization.

A rather lengthy sequence of analytical steps (not given) using an integral representation, recurrence relations, and further properties of the confluent hypergeometric function yields the same result.

Thus we obtain based on Eq. (15)

$$\tilde{A} = 1 - \tilde{A}({}_1F_1(-i\eta, 1, iT) - 1). \quad (23)$$

The cancellation of $\tilde{A}$ on the left against $\tilde{A}$ on the right is a verification that the LS equation (14) at $r = 0$ is fulfilled, as it

should and we end up with the exact relation

$$\tilde{A} = \frac{1}{{}_1F_1(-i\eta, 1, iT)}. \tag{24}$$

This is valid for any $T = 2pR$ and therefore

$$\Psi_R^{(+)}(\vec{r}) = \frac{1}{(2\pi)^{3/2}} \frac{1}{{}_1F_1(-i\eta, 1, iT)}$$
$$\times e^{i\vec{p}\cdot\vec{r}} {}_1F_1(-i\eta, 1, i(pr - \vec{p}\cdot\vec{r})) \tag{25}$$

is exactly fulfilled for $r < R$, inside the range of the potential. To the best of our knowledge this is the first time that this has been achieved.

At the same time it provides, due to Eq. (12), the exact expression for the scattering amplitude f_R or the on-shell t-matrix element for a sharply cut-off Coulomb potential. This will be dealt with in the next section.

III. THE SCATTERING AMPLITUDE

The starting point due to Eq. (12) with $\tilde{f}_R = \frac{1}{(2\pi)^{3/2}} f_R$ is

$$\tilde{f}_R = \tilde{A}\left(-\frac{m}{4\pi}\right) \int^R d^3r' e^{-ip\hat{r}\cdot\vec{r}'} \frac{e^2}{r'} e^{i\vec{p}\cdot\vec{r}'}$$
$$\times F(-i\eta, 1, (pr' - \vec{p}\cdot\vec{r}')). \tag{26}$$

We use the general integral representation of $F(\alpha, \beta, z)$

$${}_1F_1(\alpha, \beta, z) = C(\alpha, \beta) \int_\Gamma dt\, e^{zt} t^{\alpha-1}(1-t)^{\beta-\alpha-1}, \tag{27}$$

where the path Γ encircles the logarithmic cut between $t = 0$ and $t = 1$ in the positive sense and the prefactor is

$$C(\alpha, \beta) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \frac{1}{1 - e^{2\pi i(\beta-\alpha)}}. \tag{28}$$

Inserting Eq. (27) into Eq. (26) yields

$$\tilde{f}_R = \tilde{A}\left(-\frac{me^2}{4\pi}\right) C(-i\eta, 1) \int_\Gamma dt \left(\frac{1-t}{t}\right)^{i\eta} \frac{1}{t}$$
$$\times \int^R d^3r' e^{-ip\hat{r}\cdot\vec{r}'} \frac{1}{r'} e^{i\vec{p}\cdot\vec{r}'} e^{i(pr'-\vec{p}\cdot\vec{r}')t}. \tag{29}$$

The $\vec{r}'$ integral is straightforward and one obtains

$$\tilde{f}_R = \tilde{A}\left(-\frac{me^2}{2p^2\alpha}\right) C(-i\eta, 1) \int_\Gamma dt \left(\frac{1-t}{t}\right)^{i\eta} \frac{1}{t(1-t)}$$
$$\left(1 + e^{i\tilde{R}t}\left(it \frac{\sin\tilde{R}\sqrt{t^2 + 2(1-t)\alpha}}{\sqrt{t^2 + 2(1-t)\alpha}}\right.\right.$$
$$\left.\left. - \cos\tilde{R}\sqrt{t^2 + 2(1-t)\alpha}\right)\right), \tag{30}$$

where α contains the dependence on the scattering angle θ

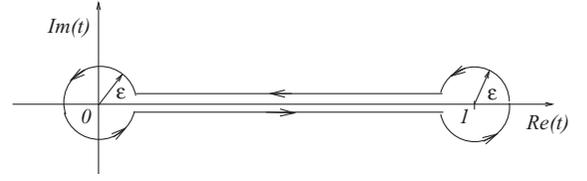$$\alpha = 1 - \hat{p}\cdot\hat{r} = 2\sin^2\frac{\theta}{2}. \tag{31}$$

FIG. 1. The path of integration Γ in Eq. (33).

$C(-i\eta, 1) = \frac{-i}{2\pi} e^{\pi\eta}$, and $\tilde{R} \equiv pR$. The "1" in the bracket does not contribute since

$$\int_\Gamma dt \left(\frac{1-t}{t}\right)^{i\eta} \frac{1}{t(1-t)} = 0. \tag{32}$$

Thus we obtain the intermediate result

$$\tilde{f}_R = -\tilde{A}\frac{\eta}{\alpha p} C(-i\eta, 1)\left[i \int_\Gamma dt \left(\frac{1-t}{t}\right)^{i\eta} \frac{1}{1-t} e^{i\tilde{R}t}\right.$$
$$\times \frac{\sin\tilde{R}\sqrt{t^2 + 2(1-t)\alpha}}{\sqrt{t^2 + 2(1-t)\alpha}} - \int_\Gamma dt \left(\frac{1-t}{t}\right)^{i\eta}$$
$$\left.\times \frac{1}{t(1-t)} e^{i\tilde{R}t} \cos\tilde{R}\sqrt{t^2 + 2(1-t)\alpha}\right]. \tag{33}$$

In the following we choose the path of integration Γ as depicted in Fig. 1 with small circles around $t = 1$ and $t = 0$ of vanishingly small radius ϵ and two straight integration lines between $t = \epsilon$ and $t = 1 - \epsilon$ above and below the logarithmic cut. The phases are $arg(t) = 0$ and $arg(1-t) = \pi$ for $t = 1 + \epsilon$. The rest follows by continuity: $arg(1-t) = 2\pi$ along the upper rim of the cut, $arg(t) = 2\pi$ along the lower rim, and $arg(1-t) = 3\pi$ back again at $t = 1 + \epsilon$. The phase of $\frac{1-t}{t}$ does not change after a full sweep of Γ, of course.

In this manner the integrals in Eq. (33) can be split into four pieces. Let us define

$$B \equiv \int_\Gamma dt \left(\frac{1-t}{t}\right)^{i\eta} \frac{1}{1-t} e^{i\tilde{R}t} \left(i \frac{\sin\tilde{R}\sqrt{t^2 + 2(1-t)\alpha}}{\sqrt{t^2 + 2(1-t)\alpha}}\right.$$
$$\left. - \frac{\cos\tilde{R}\sqrt{t^2 + 2(1-t)\alpha}}{t}\right). \tag{34}$$

Then

$$B = \int_{\text{zero}} + \int_\epsilon^{1-\epsilon} + \int_{\text{one}} + \int_{1-\epsilon}^\epsilon. \tag{35}$$

It simply follows that

$$\int_{1-\epsilon}^\epsilon dt + \int_\epsilon^{1-\epsilon} dt = (1 - e^{-2\pi\eta}) \int_\epsilon^{1-\epsilon} dt \left(\frac{1-t}{t}\right)^{i\eta}$$
$$\times \frac{1}{1-t} e^{i\tilde{R}t} \left(i \frac{\sin\tilde{R}\sqrt{t^2 + 2(1-t)\alpha}}{\sqrt{t^2 + 2(1-t)\alpha}}\right.$$
$$\left. - \frac{\cos\tilde{R}\sqrt{t^2 + 2(1-t)\alpha}}{t}\right). \tag{36}$$

In order to remove the pole singularities at $t = 0$ and $t = 1$ we split the integration interval into two parts

$$\int_\epsilon^{1-\epsilon} dt = \int_\epsilon^{1/2} dt + \int_{1/2}^{1-\epsilon} dt. \qquad (37)$$

Of course the value $1/2$ could be replaced by any number a between $t = \epsilon$ and $t = 1 - \epsilon$ without changing the result.

Thus

$$\int_\epsilon^{1-\epsilon} dt$$
$$= i \int_\epsilon^{1/2} dt (1-t)^{i\eta-1} t^{-i\eta} e^{i\tilde{R}t} \frac{\sin \tilde{R}\sqrt{t^2+2(1-t)\alpha}}{\sqrt{t^2+2(1-t)\alpha}}$$
$$+ i \int_{1/2}^{1-\epsilon} dt (1-t)^{i\eta-1} t^{-i\eta} e^{i\tilde{R}t} \frac{\sin \tilde{R}\sqrt{t^2+2(1-t)\alpha}}{\sqrt{t^2+2(1-t)\alpha}}$$
$$- \int_\epsilon^{1/2} dt\, t^{-i\eta-1}(1-t)^{i\eta-1} e^{i\tilde{R}t} \cos \tilde{R}\sqrt{t^2+2(1-t)\alpha}$$
$$- \int_{1/2}^{1-\epsilon} dt (1-t)^{i\eta-1} t^{-i\eta-1} e^{i\tilde{R}t} \cos \tilde{R}\sqrt{t^2+2(1-t)\alpha}.$$
$$(38)$$

Now we perform partial integrations such that $\epsilon \to 0$ can be taken:

$$\int_\epsilon^{1-\epsilon} dt$$
$$= i \int_0^{1/2} dt (1-t)^{i\eta-1} t^{-i\eta} e^{i\tilde{R}t} \frac{\sin \tilde{R}\sqrt{t^2+2(1-t)\alpha}}{\sqrt{t^2+2(1-t)\alpha}}$$
$$+ i \left[\frac{-1}{i\eta}(1-t)^{i\eta} t^{-i\eta} e^{i\tilde{R}t} \frac{\sin \tilde{R}\sqrt{t^2+2(1-t)\alpha}}{\sqrt{t^2+2(1-t)\alpha}} \Big|_{1/2}^{1-\epsilon} \right.$$
$$+ \frac{1}{i\eta} \int_{1/2}^{1-\epsilon} dt (1-t)^{i\eta}$$
$$\left. \times \frac{d}{dt}\left(t^{-i\eta} e^{i\tilde{R}t} \frac{\sin \tilde{R}\sqrt{t^2+2(1-t)\alpha}}{\sqrt{t^2+2(1-t)\alpha}} \right) \right]$$
$$- \left[\frac{1}{-i\eta} t^{-i\eta}(1-t)^{i\eta-1} e^{i\tilde{R}t} \cos \tilde{R}\sqrt{t^2+2(1-t)\alpha} \Big|_\epsilon^{1/2} \right.$$
$$+ \frac{1}{i\eta} \int_0^{1/2} dt\, t^{-i\eta} \frac{d}{dt}((1-t)^{i\eta-1} e^{i\tilde{R}t}$$
$$\left. \times \cos \tilde{R}\sqrt{t^2+2(1-t)\alpha}) \right] - \left[\frac{1}{-i\eta}(1-t)^{i\eta} t^{-i\eta-1} e^{i\tilde{R}t} \right.$$
$$\left. \times \cos \tilde{R}\sqrt{t^2+2(1-t)\alpha} \Big|_{1/2}^{1-\epsilon} \right.$$
$$+ \frac{1}{i\eta} \int_{1/2}^1 dt (1-t)^{i\eta} \frac{d}{dt}(t^{-i\eta-1} e^{i\tilde{R}t}$$
$$\left. \times \cos \tilde{R}\sqrt{t^2+2(1-t)\alpha}) \right]. \qquad (39)$$

After some lengthy algebra one obtains

$$\int_\epsilon^{1-\epsilon} dt = \frac{1}{i\eta}\epsilon^{i\eta} - \frac{1}{i\eta}\epsilon^{-i\eta}\cos \tilde{R}\sqrt{2\alpha}$$
$$+ \frac{1}{\eta} e^{i\frac{\tilde{R}}{2}} \frac{\sin \tilde{R}\sqrt{1/4+\alpha}}{\sqrt{1/4+\alpha}}$$
$$+ i \int_0^{1/2} dt (1-t)^{i\eta-1} t^{-i\eta} e^{i\tilde{R}t} \frac{\sin \tilde{R}\gamma}{\gamma}$$
$$+ \frac{i\eta-1}{i\eta} \int_0^{1/2} dt\, t^{-i\eta}(1-t)^{i\eta-2} e^{i\tilde{R}t} \cos \tilde{R}\gamma$$
$$- \frac{1}{\eta} \int_{1/2}^1 dt (1-t)^{i\eta} t^{-i\eta} e^{i\tilde{R}t} \frac{\sin \tilde{R}\gamma}{\gamma^3}(t-\alpha) - i$$
$$\times \int_{1/2}^1 dt (1-t)^{i\eta} t^{-i\eta-1} e^{i\tilde{R}t} \frac{\sin \tilde{R}\gamma}{\gamma}$$
$$+ \frac{i\eta+1}{i\eta} \int_{1/2}^1 dt (1-t)^{i\eta} t^{-i\eta-2} e^{i\tilde{R}t} \cos \tilde{R}\gamma$$
$$+ \frac{\tilde{R}}{\eta}\left[- \int_0^{1/2} dt\, t^{-i\eta}(1-t)^{i\eta-1} e^{i\tilde{R}t} \cos \tilde{R}\gamma \right.$$
$$+ \frac{1}{i} \int_0^{1/2} dt\, t^{-i\eta}(1-t)^{i\eta-1} e^{i\tilde{R}t} \sin \tilde{R}\gamma \frac{t-\alpha}{\gamma}$$
$$+ i \int_{1/2}^1 dt (1-t)^{i\eta} t^{-i\eta} e^{i\tilde{R}t} \frac{\sin \tilde{R}\gamma}{\gamma}$$
$$+ \int_{1/2}^1 dt (1-t)^{i\eta} t^{-i\eta} e^{i\tilde{R}t} \frac{\cos \tilde{R}\gamma}{\gamma} \frac{t-\alpha}{\gamma}$$
$$- \int_{1/2}^1 dt (1-t)^{i\eta} t^{-i\eta-1} e^{i\tilde{R}t} \cos \tilde{R}\gamma$$
$$\left. + \frac{1}{i} \int_{1/2}^1 dt (1-t)^{i\eta} t^{-i\eta-1} e^{i\tilde{R}t} \sin \tilde{R}\gamma \frac{t-\alpha}{\gamma} \right]$$
$$(40)$$

with $\gamma = \sqrt{t^2+2(1-t)\alpha}$.

It is straightforward to evaluate the two integrals around $t = 0$ and $t = 1$:

$$\int_{\text{zero}} dt \left(\frac{1-t}{t}\right)^{i\eta} \frac{1}{1-t} e^{i\tilde{R}t} \left(i \frac{\sin \tilde{R}\sqrt{t^2+2(1-t)\alpha}}{\sqrt{t^2+2(1-t)\alpha}} \right.$$
$$\left. - \frac{\cos \tilde{R}\sqrt{t^2+2(1-t)\alpha}}{t} \right)$$
$$= -i \cos \tilde{R}\sqrt{2\alpha}\, \epsilon^{-i\eta} \frac{1}{\eta}(1-e^{-2\pi\eta}), \qquad (41)$$

$$\int_{\text{one}} dt \left(\frac{1-t}{t}\right)^{i\eta} \frac{1}{1-t} e^{i\tilde{R}t} \left(i\frac{\sin \tilde{R}\sqrt{t^2 + 2(1-t)\alpha}}{\sqrt{t^2 + 2(1-t)\alpha}}\right.$$
$$\left. - \frac{\cos \tilde{R}\sqrt{t^2 + 2(1-t)\alpha}}{t}\right)$$
$$= \frac{i}{\eta} \epsilon^{i\eta} (1 - e^{-2\pi\eta}). \qquad (42)$$

The ϵ-dependent terms cancel in Eq. (40) [multiplied by $(1 - e^{-2\pi\eta})$], Eqs. (41) and (42) as they should and one obtains the finite result

$$B = (1 - e^{-2\pi\eta})\left[\frac{1}{\eta} e^{i\frac{\tilde{R}}{2}} \frac{\sin \tilde{R}\sqrt{1/4 + \alpha}}{\sqrt{1/4 + \alpha}}\right.$$
$$+ i \int_0^{1/2} dt(1-t)^{i\eta-1} t^{-i\eta} e^{i\tilde{R}t} \frac{\sin \tilde{R}\gamma}{\gamma}$$
$$+ \frac{i\eta - 1}{i\eta} \int_0^{1/2} dt\, t^{-i\eta}(1-t)^{i\eta-2} e^{i\tilde{R}t} \cos \tilde{R}\gamma$$
$$- \frac{1}{\eta} \int_{1/2}^1 dt(1-t)^{i\eta} t^{-i\eta} e^{i\tilde{R}t} \frac{\sin \tilde{R}\gamma}{\gamma^3}(t - \alpha)$$
$$- i \int_{1/2}^1 dt(1-t)^{i\eta} t^{-i\eta-1} e^{i\tilde{R}t} \frac{\sin \tilde{R}\gamma}{\gamma}$$
$$+ \frac{i\eta + 1}{i\eta} \int_{1/2}^1 dt(1-t)^{i\eta} t^{-i\eta-2} e^{i\tilde{R}t} \cos \tilde{R}\gamma$$
$$+ \frac{\tilde{R}}{\eta}\left[- \int_0^{1/2} dt\, t^{-i\eta}(1-t)^{i\eta-1} e^{i\tilde{R}t} \cos \tilde{R}\gamma\right.$$
$$+ \frac{1}{i} \int_0^{1/2} dt\, t^{-i\eta}(1-t)^{i\eta-1} e^{i\tilde{R}t} \sin \tilde{R}\gamma \frac{t-\alpha}{\gamma}$$
$$+ i \int_{1/2}^1 dt(1-t)^{i\eta} t^{-i\eta} e^{i\tilde{R}t} \frac{\sin \tilde{R}\gamma}{\gamma}$$
$$+ \int_{1/2}^1 dt(1-t)^{i\eta} t^{-i\eta} e^{i\tilde{R}t} \frac{\cos \tilde{R}\gamma}{\gamma} \frac{t-\alpha}{\gamma}$$
$$- \int_{1/2}^1 dt(1-t)^{i\eta} t^{-i\eta-1} e^{i\tilde{R}t} \cos \tilde{R}\gamma$$
$$\left.\left.+ \frac{1}{i} \int_{1/2}^1 dt(1-t)^{i\eta} t^{-i\eta-1} e^{i\tilde{R}t} \sin \tilde{R}\gamma \frac{t-\alpha}{\gamma}\right]\right].$$
$$(43)$$

This together with Eqs. (33)–(36) is an exact expression for the scattering amplitude for an arbitrary cut-off radius R.

But of course we are interested only in its asymptotic limit $R \to \infty$.

It is advisable to introduce $e_{\pm} = e^{i\tilde{R}(t\pm\gamma)}$ and to rearrange Eq. (43). We regard first the pieces explicitly proportional to $\tilde{R}$ in Eq. (43):

$$\frac{\tilde{R}}{2\eta}\left[- \int_0^{1/2} dt \left(\frac{1-t}{t}\right)^{i\eta} \frac{1}{1-t}(e_+ + e_-)\right.$$
$$- \int_0^{1/2} dt \left(\frac{1-t}{t}\right)^{i\eta} \frac{1}{1-t}(e_+ - e_-)\frac{t-\alpha}{\gamma}$$

$$+ \int_{1/2}^1 dt \left(\frac{1-t}{t}\right)^{i\eta} \frac{e_+ - e_-}{\gamma}$$
$$+ \int_{1/2}^1 dt \left(\frac{1-t}{t}\right)^{i\eta} \frac{e_+ + e_-}{\gamma} \frac{t-\alpha}{\gamma}$$
$$- \int_{1/2}^1 dt \left(\frac{1-t}{t}\right)^{i\eta} \frac{1}{t}(e_+ + e_-)$$
$$\left.- \int_{1/2}^1 dt \left(\frac{1-t}{t}\right)^{i\eta} \frac{1}{t}(e_+ - e_-)\frac{t-\alpha}{\gamma}\right]. \qquad (44)$$

Leading terms will arise from the boundaries of integration $t = 0$, $t = 1/2$, and $t = 1$, where the $t = 1/2$ contributions have to cancel in the total expression. We use the standard method of steepest descent [9] and expand around boundaries of integration. For example at $t = 0$

$$e_{\pm} = e^{\pm i\tilde{R}\sqrt{2\alpha}} e^{i\tilde{R}t(1\mp\sqrt{\frac{\alpha}{2}})}(1 + O(t)) \qquad (45)$$

and corresponding expressions for the remaining parts of the integrand. One obtains

$$\frac{\tilde{R}}{2\eta}\left[- \int_0 dt \left(\frac{1-t}{t}\right)^{i\eta} \frac{1}{1-t}(e_+ + e_-)\right.$$
$$\left.- \int_0 dt \left(\frac{1-t}{t}\right)^{i\eta} \frac{1}{1-t}(e_+ - e_-)\frac{t-\alpha}{\gamma}\right]$$
$$\to \frac{-i}{2\eta} e^{\frac{\pi}{2}\eta} \Gamma(1 - i\eta) \tilde{R}^{i\eta} \left(e^{2i\tilde{R}\sin\frac{\theta}{2}} \left(1 - \sin\frac{\theta}{2}\right)^{i\eta}\right.$$
$$\left.+ e^{-2i\tilde{R}\sin\frac{\theta}{2}} \left(1 + \sin\frac{\theta}{2}\right)^{i\eta}\right). \qquad (46)$$

Correspondingly we proceed at the upper limit of integration $t = 1$ and it turns out that the e_+ part decreases as $O(\frac{1}{\tilde{R}})$ and only the e_- part survives as

$$\frac{\tilde{R}}{2\eta}\left[\int^1 dt \left(\frac{1-t}{t}\right)^{i\eta} \frac{e_+ - e_-}{\gamma}\right.$$
$$+ \int^1 dt \left(\frac{1-t}{t}\right)^{i\eta} \frac{e_+ + e_-}{\gamma} \frac{t-\alpha}{\gamma}$$
$$- \int^1 dt \left(\frac{1-t}{t}\right)^{i\eta} \frac{1}{t}(e_+ + e_-)$$
$$\left.- \int^1 dt \left(\frac{1-t}{t}\right)^{i\eta} \frac{1}{t}(e_+ - e_-)\frac{t-\alpha}{\gamma}\right]$$
$$\to \frac{i}{\eta}(2\tilde{R})^{-i\eta} \left(\sin^2\frac{\theta}{2}\right)^{-i\eta} e^{\frac{\pi}{2}\eta} \Gamma(1 + i\eta). \qquad (47)$$

The remaining pieces resulting from the integration limits $t = 1/2$ yield

$$\frac{\tilde{R}}{2\eta}\left[- \int^{1/2} dt\, t^{-i\eta}(1-t)^{i\eta-1}(e_+ + e_-)\right.$$
$$- \int^{1/2} dt\, t^{-i\eta}(1-t)^{i\eta-1}(e_+ - e_-)\frac{t-\alpha}{\gamma}$$
$$+ \int_{1/2} dt(1-t)^{i\eta} t^{-i\eta} \frac{e_+ - e_-}{\gamma}$$

$$+ \int_{1/2} dt (1-t)^{i\eta} t^{-i\eta} \frac{e_+ + e_-}{\gamma} \frac{t-\alpha}{\gamma}$$

$$- \int_{1/2} dt (1-t)^{i\eta} t^{-i\eta-1} e^{i\tilde{R}t} (e_+ + e_-)$$

$$- \int_{1/2} dt (1-t)^{i\eta} t^{-i\eta-1} (e_+ - e_-) \frac{t-\alpha}{\gamma} \Bigg]$$

$$\rightarrow -\frac{1}{\eta} \frac{1}{\sqrt{1/4+\alpha}} e^{i\frac{\tilde{R}}{2}} \sin \tilde{R} \sqrt{1/4+\alpha}. \qquad (48)$$

This cancels exactly against the first term in Eq. (43) after multiplication by $(1 - e^{-2\pi\eta})$, as it should.

The terms in Eq. (43) not directly proportional to $\tilde{R}$ decrease like $O(\frac{1}{\tilde{R}})$. Finally the contributions from the interior of the integration intervals decay faster as can be seen by deforming the path of integration into the upper half plane, where $e^\pm$ is exponentially damped.

Thus we are left with the leading asymptotic expression

$$B \rightarrow (1 - e^{-2\pi\eta}) \frac{i}{2\eta} e^{\frac{\pi}{2}\eta} \left(- \Gamma(1-i\eta) \tilde{R}^{i\eta} \left(e^{2i\tilde{R}\sin\frac{\theta}{2}} \right. \right.$$

$$\times \left(1 - \sin\frac{\theta}{2} \right)^{i\eta} + e^{-2i\tilde{R}\sin\frac{\theta}{2}} \left(1 + \sin\frac{\theta}{2} \right)^{i\eta} \Bigg)$$

$$+ 2(2\tilde{R})^{-i\eta} \left(\sin^2\frac{\theta}{2} \right)^{-i\eta} \Gamma(1+i\eta) \Bigg). \qquad (49)$$

This is now to be combined with Eq. (33). Using Eq. (28),

$$\frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)} \equiv e^{2i\sigma_0} \qquad (50)$$

and the asymptotic form of $\tilde{A}$

$$\tilde{A} \rightarrow e^{-\frac{\pi}{2}\eta} T^{-i\eta} \Gamma(1+i\eta) \qquad (51)$$

based on the asymptotic form [10]

$${}_1F_1(\alpha, \beta, z) \rightarrow \frac{e^{\pm i\pi\alpha} z^{-\alpha}}{\Gamma(\beta-\alpha)} + \frac{e^z z^{\alpha-\beta}}{\Gamma(\alpha)} + O\left(\frac{1}{|z|}\right), \quad (52)$$

we get

$$\tilde{f}_R = -(2\tilde{R})^{-2i\eta} e^{2i\sigma_0} \frac{\eta}{2p} \frac{\left(\sin^2\frac{\theta}{2}\right)^{-i\eta}}{\sin^2\frac{\theta}{2}}$$

$$+ \frac{\eta}{2\alpha p} \left(e^{2i\tilde{R}\sin\frac{\theta}{2}} \left(\frac{(1-\sin\frac{\theta}{2})}{2} \right)^{i\eta} \right.$$

$$+ e^{-2i\tilde{R}\sin\frac{\theta}{2}} \left(\frac{(1+\sin\frac{\theta}{2})}{2} \right)^{i\eta} \Bigg). \qquad (53)$$

Now the physical Coulomb scattering amplitude is

$$A_c(\theta) = -\frac{\eta}{2p} \frac{\left(\sin^2\frac{\theta}{2}\right)^{-i\eta}}{\sin^2\frac{\theta}{2}} e^{2i\sigma_0} \qquad (54)$$

and we end up with

$$\tilde{f}_R = (2\tilde{R})^{-2i\eta} A_c(\theta) + \frac{\eta}{4p\sin^2\frac{\theta}{2}} \left(e^{2i\tilde{R}\sin\frac{\theta}{2}} \left(\frac{(1-\sin\frac{\theta}{2})}{2} \right)^{i\eta} \right.$$

$$\left. + e^{-2i\tilde{R}\sin\frac{\theta}{2}} \left(\frac{(1+\sin\frac{\theta}{2})}{2} \right)^{i\eta} \right)$$

$$= \left[e^{-2i\eta\ln(2\tilde{R})} - \frac{1}{2} e^{i\eta\ln\sin^2\frac{\theta}{2} - 2i\sigma_0} \left(e^{2i\tilde{R}\sin\frac{\theta}{2} + i\eta\ln\frac{1-\sin\frac{\theta}{2}}{2}} \right. \right.$$

$$\left. \left. + e^{-2i\tilde{R}\sin\frac{\theta}{2} + i\eta\ln\frac{1+\sin\frac{\theta}{2}}{2}} \right) \right] A_c(\theta). \qquad (55)$$

The first term is the result expected from the literature [1,3] and references therein. As [11] has shown, the diverging phase factor $e^{-2i\Phi_R(p)}$ in case of an often used form of screening the Coulomb potential

$$V_R(r) = \frac{e^2}{r} e^{-\left(\frac{r}{R}\right)^n} \qquad (56)$$

using the prescription of [1] turns out to be

$$\Phi_R(p) = \eta[\ln(2pR) - C/n] \qquad (57)$$

with the Euler number C. For $n \rightarrow \infty$ one recovers the sharp cutoff, which we consider in this paper. This expectation for the screening limit agrees with the first term in Eq. (55) but not with the necessity of adding a second term. Therefore the derivations in the literature based on partial wave decomposition must be incomplete. Whether this is also true for a finite value n in Eq. (56) remains to be seen.

IV. NUMERICAL RESULTS

We performed a number of numerical tests to check the basic points in the derivation of the sharp cut-off Coulomb wave function (25) and the asymptotic scattering amplitude (55).

First we checked numerically how well the solution (24) fulfills Eq. (15). In Table I the left and right sides of Eq. (15) are shown for a number of cut-off radii R for pp scattering with $E_p^{\text{lab}} = 13$ MeV. The right side was obtained by a direct two-dimensional numerical integration over x and y. A very good agreement up to four significant digits is seen.

We also compared at the same energy the exact expression for $\tilde{A}$ as given in Eq. (24) with its asymptotic form (51) at a number of screening radii. The results are shown in Fig. 2 and Table II. The oscillating behavior seen in real and imaginary parts of exact $\tilde{A}$ (solid lines in Fig. 2) gradually diminishes with increasing cut-off radius R. These oscillations are absent in the asymptotic form for $\tilde{A}$ (dashed lines in Fig. 2). The asymptotic form for $\tilde{A}$ approaches its exact value at $R \approx 50$ fm as can be seen in Fig. 2 and in the third column of Table II where the ratio of $\tilde{A}/\tilde{A}_{\text{approx}}$ is given.

To check the quality of our renormalization factor (55) we applied it directly to the numerical solutions of the Lippmann-Schwinger equation for the sharp cut-off Coulomb potential with different cut-off radii.

In the case of a short-ranged potential V two-body scattering is described by the solution of the Lippmann-Schwinger

TABLE I. The left and right sides of Eq. (15) at $E_p^{\text{lab}} = 13$ MeV ($\eta = 0.0439$, $p = 0.3959$ fm^{-1}) and different screening radii R.

R [fm]	$\tilde{A}$	$1 - \tilde{A}\eta T \int_0^1 dx_1 F_1(-i\eta, 1, iTx) \int_0^{1-x} dy e^{iTy} \frac{1}{x+y}$
0.5	$(0.98301, -0.00166)$	$(0.98301, -0.00166)$
1	$(0.96724, -0.00634)$	$(0.96724, -0.00634)$
5	$(0.91933, -0.08246)$	$(0.91933, -0.08246)$
10	$(0.92770, -0.10294)$	$(0.92770, -0.10294)$
20	$(0.91961, -0.13491)$	$(0.91961, -0.13491)$
50	$(0.91606, -0.17185)$	$(0.91606, -0.17185)$
100	$(0.90960, -0.20061)$	$(0.90960, -0.20061)$
500	$(0.89376, -0.26439)$	$(0.89377, -0.26439)$
1000	$(0.88528, -0.29140)$	$(0.88528, -0.29140)$
5000	$(0.86250, -0.35307)$	$(0.86252, -0.35307)$

equation

$$T(z) = V + V \frac{1}{z - H_0} T(z), \tag{58}$$

where V is the two-body potential, H_0 is the free Hamiltonian, and $T(z)$ the transition operator. In momentum space Eq. (58) takes the form of an integral equation for the matrix elements of the transition operator $\langle \vec{q}' \mid T(z) \mid \vec{q} \rangle \equiv T(\vec{q}', \vec{q})$. In this equation matrix elements of the potential V are used $\langle \vec{q}' \mid V \mid \vec{q} \rangle \equiv V(\vec{q}', \vec{q})$. In our case both $V(\vec{q}', \vec{q})$ and $T(\vec{q}', \vec{q})$ depend only on the magnitudes $q' \equiv \mid \vec{q}' \mid, q \equiv \mid \vec{q} \mid$ and the cosine of the angle between $\vec{q}$ and $\vec{q}'$, $\hat{q}' \cdot \hat{q}$:

$$V(\vec{q}', \vec{q}) = V(q', q, \hat{q}' \cdot \hat{q}), \tag{59}$$

$$T(\vec{q}', \vec{q}) = T(q', q, \hat{q}' \cdot \hat{q}). \tag{60}$$

(Note we dropped the dependence on the parameter z.) As a consequence the Lippmann-Schwinger equation can be written as a two-dimensional integral equation [12]

$$T(q', q, x') = \frac{1}{2\pi} v(q', q, x', 1)$$
$$+ \int_0^\infty dq'' q''^2 \int_{-1}^1 dx'' v(q', q'', x', x'')$$
$$\times \frac{1}{z - \frac{q''^2}{m}} T(q'', q, x''), \tag{61}$$

where

$$v(q', q, x', x)$$
$$= \int_0^{2\pi} d\varphi V(q', q, x'x + \sqrt{1 - x'^2}\sqrt{1 - x^2} \cos \varphi) \tag{62}$$

and m is the reduced mass of the system.

For the sharply screened Coulomb potential of the range R considered in this paper

$$V(q', q, x) = \frac{e^2}{2\pi^2} \frac{1 - \cos(QR)}{Q^2}, \tag{63}$$

where $Q \equiv \sqrt{q'^2 + q^2 - 2q'qx}$. However, the integral over φ in Eq. (62) cannot be carried out analytically.

It is clear that $V(q', q, x)$ shows a highly oscillatory behavior, especially for large R. Thus solving the two-dimensional equation (61) is a difficult numerical problem. We were interested in solutions for positive energies where

$$z = E_{\text{c.m.}} + i\epsilon \equiv \frac{q_0^2}{m} + i\epsilon. \tag{64}$$

We solved Eq. (61) by generating the corresponding Neumann series and summing it up by Pade which is a very reliable and accurate method. Usually six iterations were fully sufficient. In

TABLE II. The exact value of $\tilde{A}$ as in Eq. (24) (left column), asymptotic form given by Eq. (51) (middle column), and their ratio (right column) at $E_p^{\text{lab}} = 13$ MeV for different screening radii R.

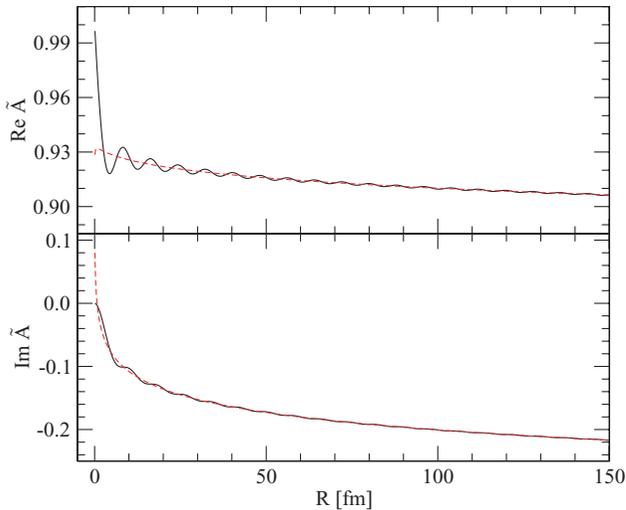| R [fm]	$\tilde{A}$	$\tilde{A}_{\text{approx}}$	$\tilde{A}/\tilde{A}_{\text{approx}}$
0.1	$(0.99653, -0.00007)$	$(0.92852, 0.07999)$	$(1.06534, -0.09185)$
1.0	$(0.96724, -0.00634)$	$(0.93186, -0.01402)$	$(1.03784, 0.00881)$
2.0	$(0.94156, -0.02249)$	$(0.93100, -0.04233)$	$(1.01035, 0.02178)$
3.0	$(0.92536, -0.04353)$	$(0.93010, -0.05888)$	$(0.99389, 0.01611)$
5.0	$(0.91933, -0.08246)$	$(0.92855, -0.07970)$	$(0.99040, -0.00380)$
10.0	$(0.92770, -0.10294)$	$(0.92570, -0.10788)$	$(1.00152, 0.00551)$
20.0	$(0.91961, -0.13491)$	$(0.92199, -0.13596)$	$(0.99731, 0.00074)$
50.0	$(0.91606, -0.17185)$	$(0.91578, -0.17289)$	$(1.00009, 0.00115)$
100.0	$(0.90960, -0.20061)$	$(0.91011, -0.20064)$	$(0.99946, -0.00008)$

FIG. 2. (Color online) The real (top) and imaginary (bottom) part of $\tilde{A}$ as a function of the screening radius R at $E_p^{\text{lab}} = 13$ MeV. The solid (black) line represents the exact expression given in Eq. (24) and the dashed (red) line shows the asymptotic form as given in Eq. (51).

each iteration the Cauchy singularity was split into a principal-value integral (treated by subtraction) and a δ-function piece. We used 120 or 140 q-points and 150 or 190 x-points. The q-integral points are chosen in the definite interval $(0, \bar{q})$, where typically $\bar{q} = 50$ fm^{-1}. In order to obtain directly the on-shell t-matrix element $T(q_0, q_0, x, E_{\text{c.m.}})$ we added $q = q_0$ to the set of q-points. To better control the behavior of the transition matrix element for small scattering angles also $x = 1$ was
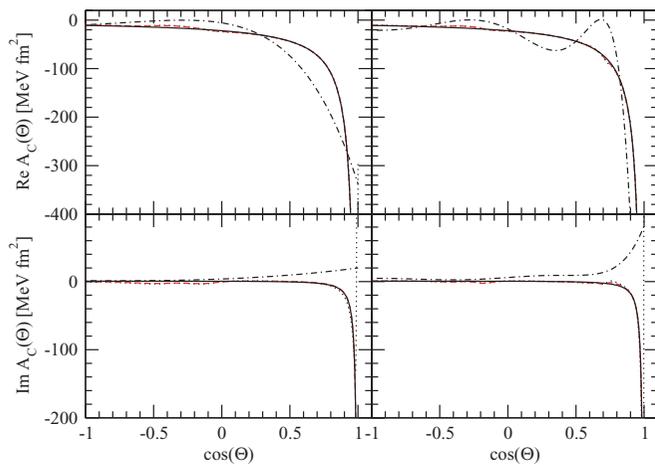

FIG. 3. (Color online) The real (top) and imaginary (bottom) part of $A_C(\theta) \equiv -2\pi^2 m T(q_0, q_0, \cos\theta)$ as a function of $\cos\theta$ for $R = 10$ fm (left panel) and 20 fm (right panel) at $E_p^{\text{lab}} = 13$ MeV. The dash-dotted line represents a direct numerical prediction (without any renormalization). The dotted line shows $A_C(\theta)$ with inclusion of the asymptotic renormalization factor given in Eq. (55) and the dashed (red) line is for $A_C(\theta)$ with inclusion of the exact renormalization factor obtained from Eqs. (33), (34), and (43) (see text). The solid line represents the pure Coulomb amplitude given in Eq. (54). Note that the dashed, dotted, and solid lines practically overlap with the exception of very forward angles for the imaginary part.
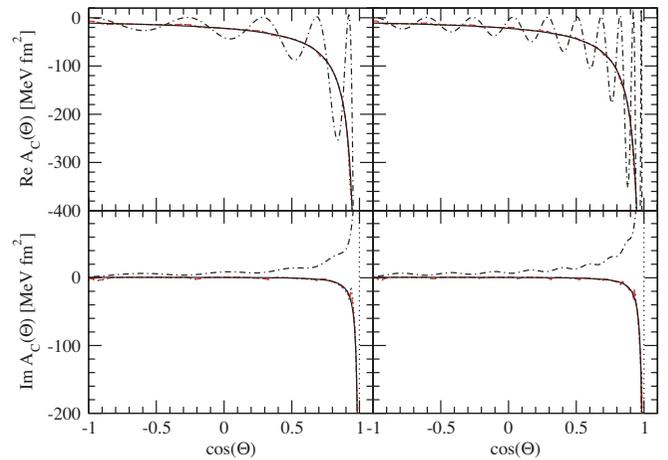

FIG. 4. (Color online) The same as in Fig. 3 but for $R = 40$ fm (left panel) and 80 fm (right panel).

added to the set of x-points. A typical run required less than 9 min on 256 nodes (1024 processors) on the IBM Blue Gene/P parallel computer at the Jülich Supercomputing Centre.

In Figs. 3–5 we show with dash-dotted line the real and imaginary parts of the transition amplitudes $A_C(\theta) \equiv -2\pi^2 m T(q_0, q_0, \cos\theta)$ for sharp cut-off Coulomb potential pp scattering at $E_p^{\text{lab}} = 13$ MeV and a number of cut-off radii $R = 10$ and 20 fm (Fig. 3), $R = 40$ and 80 fm (Fig. 4), and $R = 100$ and 120 fm (Fig. 5). With increasing cut-off radius a development of strong oscillations in the scattering angle dependence for the real parts of the numerical solutions is clearly seen. These oscillations follow on average the real part of the pure Coulomb amplitude given by Eq. (54) and shown by the solid line. The imaginary parts of the numerical solutions are totally off from the imaginary part of the pure Coulomb amplitude and have even an opposite sign. Now applying to the numerical solutions the asymptotic renormalization factor from Eq. (55) dramatically improves the agreement (dotted lines in Figs. 3–5). Not only are the oscillations in the real parts practically removed and the pure Coulomb and renormalized amplitudes practically overlapping but the renormalization
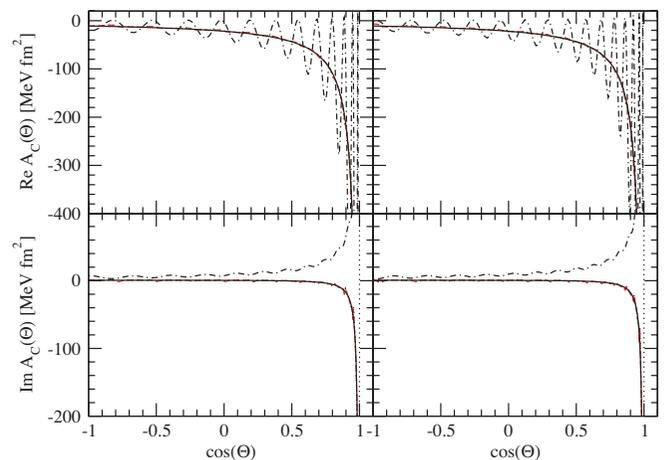

FIG. 5. (Color online) The same as in Fig. 3 but for $R = 100$ fm (left panel) and 120 fm (right panel).
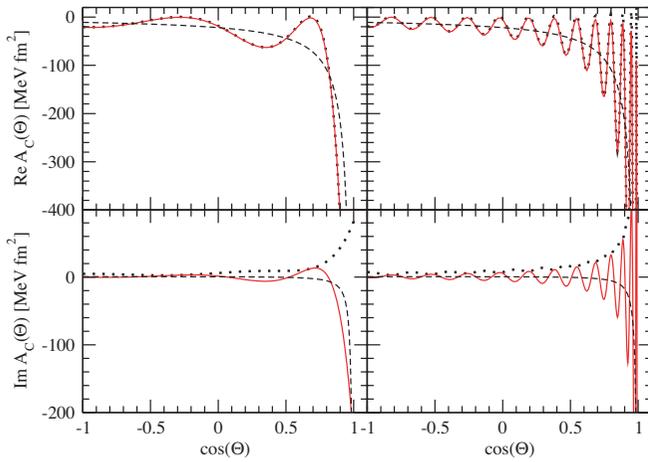
FIG. 6. (Color online) The real (top) and imaginary (bottom) part of $A_C(\theta) \equiv -2\pi^2 mT(q_0, q_0, \cos\theta)$ as a function of $\cos\theta$ for $R = 20$ fm (left panel) and 100 fm (right panel) at $E_p^{\text{lab}} = 13$ MeV. The dotted line represents a direct numerical prediction (without any renormalization). The solid (red) line shows $A_C(\theta)$ with renormalization factor $e^{-2i\eta\ln(2pR)}$ and the dashed line represents the pure Coulomb amplitude given in Eq. (54).

brings also imaginary parts into agreement with the exception of very forward angles. When one desists to use the asymptotic expansion for $\tilde{f}_R$ and instead calculates it exactly according to Eqs. (33), (34), and (43) then the ratio $\frac{\tilde{f}_R}{A_C(\theta)}$ provides the exact renormalization factor. Performing exact renormalization of the numerical solutions provides very good agreement between imaginary parts of the numerical and pure Coulomb amplitudes also at the very forward angles (dashed line in Figs. 3–5).

We also checked how important the two additional terms in the renormalization factor of Eq. (55) are. To this aim we renormalized the numerical solutions with the standard form of the renormalization factor, given by the first term in Eq. (55). In Fig. 6 solid (red) lines show the amplitude renormalized in this way. It is clearly seen, that keeping only the standard form of the renormalization factor, it is not possible to reach the physical amplitude. Standard renormalization reduces slightly oscillations in the real part of the numerical solution, and changing the sign of the imaginary part invokes in it large oscillations. So after standard renormalization strong oscillations are present both in the real and imaginary parts and fail totally.

V. SUMMARY

The renormalization method for a screened on-shell Coulomb t-matrix enjoys a widespread use; see for instance [13,14]. As pointed out in the Introduction the underlying mathematical considerations leave room for doubt. To shed light on that issue we regarded potential scattering on a sharply cut-off Coulomb potential directly in three dimensions, avoiding obstacles in the infinite sum of angular momenta. The idea was to use the Lippmann-Schwinger equation which uniquely defines the wave function including its boundary conditions. Inside the range of the potential it is the standard Coulomb

wave function multiplied by an unknown normalization factor. Using that form also on the left side of the Lippmann-Schwinger equation for radii smaller than the cut-off radius determines that normalization factor uniquely. Based on that we succeeded analytically to determine the normalization factor and thus obtained in this manner the exact analytic result for the wave function. This also allowed us to derive the analytical expression for the scattering amplitude in the limit of infinite cut-off radius. The connection to the standard Coulomb scattering amplitude $A_c(\theta)$ turned out, however, to be different from the standard form used widely in the literature and is given in Eq. (55). Our form consists of two terms, one of which is the standard one, $e^{-2i\eta\ln 2pr} A_c(\theta)$. To that, however, is added a new expression which is singular at $\theta = 0$ and $\theta = \pi$. These analytical results are fully backed up by accompanying numerical investigations. Our renormalization factor brings in a very good agreement between the strongly deviating and oscillating numerical solution of the Lippmann-Schwinger equation with the sharp cut-off Coulomb potential and the exact Coulomb amplitude. The standard renormalization factor fails completely.

APPENDIX: S-WAVE POTENTIAL SCATTERING FOR A SHARPLY CUT-OFF COULOMB POTENTIAL

The (reduced) wave function for s-wave scattering obeys the Lippmann-Schwinger equation

$$\phi^{(+)}(r) = \sin(pr) - \frac{m}{p} \int_0^R dr' e^{ipr_>} \sin(pr_<) \frac{e^2}{r'} \phi^{(+)}(r')$$

(A1)

with $r_{<(>)}$ the smaller (greater) of r, r'. Inside the potential range $\phi^{(+)}(r)$ has to have the form

$$\phi^{(+)}(r) = A F_0(pr),$$ (A2)

where $F_0(pr)$ is proportional to the standard Coulomb wave function

$$F_0(pr) = pr e^{ipr} F(1 + i\eta, 2, -2ipr).$$ (A3)

Inserting Eq. (A2) into Eq. (A1) yields

$$\phi^{(+)}(r) = \sin(pr) - 2\eta A \left(e^{ipr} \int_0^r dr' \sin(pr') \frac{1}{r'} F_0(pr') \right.$$

$$+ \left. \sin(pr) \int_r^R dr' e^{ipr'} \frac{1}{r'} F_0(pr') \right)$$

$$= \sin(pr) - \frac{2\eta p A}{2i} \left(e^{ipr} \int_0^R dr' e^{2ipr'} \right.$$

$$\times F(1 + i\eta, 2, -2ipr')$$

$$- e^{-ipr} \int_r^R dr' e^{2ipr'} F(1 + i\eta, 2, -2ipr')$$

$$- \left. e^{ipr} \int_0^r dr' F(1 + i\eta, 2, -2ipr') \right). \quad (A4)$$

One faces two types of integrals, which can be solved using the following properties of the confluent hypergeometric function:

$$F(1 + i\eta, 2, -2ipr) = \frac{1}{2p\eta} \frac{d}{dr} F(i\eta, 1, -2ipr), \quad (A5)$$

$$F(1 + i\eta, 2, -2ipr) = -\frac{e^{-\pi\eta}}{2\pi\eta} \int_\Gamma dt e^{-2iprt} \left(\frac{t}{1-t} \right)^{i\eta} \quad (A6)$$

with the path Γ given in Sec. III, and

$$F(i\eta, 1, -2ipr) - F(1 + i\eta, 1, -2ipr)$$
$$= 2ipr F(1 + i\eta, 2, -2ipr). \quad (A7)$$

One obtains

$$\int_0^r dr' F(1 + i\eta, 2, -2ipr')$$
$$= \frac{1}{2p\eta}(F(i\eta, 1, -2ipr) - 1), \quad (A8)$$

$$\int_0^r dr' e^{2ipr'} F(1 + i\eta, 2, -2ipr')$$
$$= -\frac{1}{2\eta p}(1 - e^{2ipr} F(1 + i\eta, 1, -2ipr)). \quad (A9)$$

Therefore

$$\phi^{(+)}(r) = \sin(pr) - \frac{2\eta p A}{2i} \left(\frac{2i}{2\eta p} \sin(pr) e^{2ipR} \right.$$

$$\times F(1 + i\eta, 1, -2ipR)$$

$$- \left. \frac{2ipr}{2\eta p} e^{ipr} F(1 + i\eta, 2, -2ipr) \right)$$

$$= \sin(pr)(1 - Ae^{2ipR} F(1 + i\eta, 1, -2ipR))$$

$$+ Apr e^{ipr} F(1 + i\eta, 2, -2ipr)$$

$$= \phi^{(+)}(r) + \sin(pr)(1 - Ae^{2ipR}$$

$$\times F(1 + i\eta, 1, -2ipR)). \quad (A10)$$

Consequently the LS equation (A1) is identically fulfilled, as it should be and one obtains an explicit condition for the constant

A:

$$1 - Ae^{2ipR} F(1 + i\eta, 1, -2ipR) = 0 \quad (A11)$$

or

$$A = \frac{e^{-2ipR}}{F(1 + i\eta, 1, -2ipR)}. \quad (A12)$$

Inserting this result into Eq. (A2) the exact s-wave function for a sharply cut-off Coulomb is obtained

$$\phi^{(+)}(r) = \frac{e^{-2ipR}}{F(1 + i\eta, 1, -2ipR)} pr e^{ipr} F(1 + i\eta, 2, -2ipr). \quad (A13)$$

It obeys the LS equation (A1).

The asymptotic behavior $r \to \infty$, which provides the scattering phase shift $\delta_R(p)$, is given through the LS equation and we read off from Eq. (A4)

$$\phi^{(+)}(r) \to \sin(pr) - e^{ipr} A' \quad (A14)$$

with

$$A' = 2\eta p A \int_0^R dr' \sin(pr') e^{ipr'} F(1 + i\eta, 2, -2ipr'). \quad (A15)$$

At the same time this yields

$$e^{2i\delta_R(p)} = 1 - 2iA'. \quad (A16)$$

Using Eqs. (A8) and (A9) again gives

$$A' = \frac{A}{2i}(e^{2ipR} F(1 + i\eta, 1, -2ipR) - F(i\eta, 1, -2ipR)) \quad (A17)$$

and consequently

$$e^{2i\delta_R(p)} = 1 - A(e^{2ipR} F(1 + i\eta, 1, -2ipR)$$
$$- F(i\eta, 1, -2ipR)). \quad (A18)$$

The interest lies now in the limit $R \to \infty$. We use Eq. (A12) and the asymptotic form (52) of F and obtain

$$e^{2i\delta_R(p)} \to e^{2i\sigma_0 - 2i\eta \ln(2pr)} \quad (A19)$$

or

$$\delta_R(p) \to \sigma_0 - \eta \ln(2pr). \quad (A20)$$

Of course this result is well known and can be trivially obtained by matching the interior Coulomb wave function to the free one containing $\delta_R(p)$.

We performed this exercise to explicitly demonstrate that the LS equation (A1) is indeed identically fulfilled for arbitrary r below the cut-off radius R. In the three-dimensional case we succeeded analytically to do this only for the special value $r = 0$, though it is valid for any $r < R$, and were forced to verify the general case numerically.

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