Resonance parameters from K-matrix and T-matrix poles

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We extract K matrix poles from our fits to elastic pion-nucleon scattering and eta-nucleon production data in order to test a recently proposed method for the determination of resonance properties, based on the trace of the K matrix. We have considered issues associated with the separation of background and resonance contributions, the correspondence between K matrix and T matrix poles, and the complicated behavior of eigenphases.

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In a study by Ceci and Collaborators [1], a method for resonance parameter extraction was proposed, based on properties of the trace of the T matrix and the associated K matrix, from a multichannel fit to scattering data. The relevant relations are [1]

$$\mathrm{Tr}(K) = \frac{\tilde{\Gamma}_R/2}{E_R - E} + \sum_{j\neq R}^{N} \tan\delta_j, \tag{1}$$

and

$$\mathrm{Tr}(T) = \frac{\tilde{\Gamma}_R/2}{E_R - E - i\tilde{\Gamma}_R/2} + \sum_{j\neq R}^{N} e^{i\delta_j}\sin\delta_j, \tag{2}$$

where

$$\tilde{\Gamma}_R/2 \;=\; \Gamma_R/2 \;+ (E_R - E)\tan\delta_B, \tag{3}$$

E is the center of mass energy, E_R is the resonance energy, Γ_R is the full width, δ_B is a background phase, and N is the number of included channels. The index R labels the $j = R$ element of the diagonal K matrix, and δ_j is an eigenphase. These expressions follow directly from the expressions for the K and T matrices in terms of the eigenphases:

$$\mathrm{Tr}(K) = \sum_{j=1}^{N} \tan\delta_j,\ \mathrm{Tr}(T) = \sum_{j=1}^{N} e^{i\delta_j}\sin\delta_j, \tag{4}$$

assuming a single diagonal element of the K matrix has the resonant form

$$\tan\delta_R = \frac{\Gamma_R/2}{E_R - E} + \tan\delta_B, \tag{5}$$

with a K matrix pole at $E = E_R$ and the nonpole behavior collectively described by the background phase. The quantities Γ_R, E_R and δ_B are considered as functions of the energy, E.

The importance of Eqs. (1) and (2) for Ref. [1] is that while the position of the T-matrix pole of the first term in Eq. (2), $E_R - i\tilde{\Gamma}_R/2$, and its residue, $\tilde{\Gamma}_R/2$, depend on δ_B, the position of the K-matrix pole in Eq.(1), E_R, and its residue, Γ_R, do not. This is the model independence [2] cited in Ref. [1]. In light of this, the authors of Ref. [1] suggest the use of K-matrix pole positions and residues to give a model-independent characterization of resonance structure. Their method involves the determination of the K matrix, from a given T matrix, from which the pole positions, E_R, and their residues, Γ_R, are extracted.

We have explicitly tested this method with a set of amplitudes determined in recent fits to pion-nucleon scattering and eta-nucleon production data. Using the T matrix in a given partial wave, determined in fits to the observed data [3], we determine the K matrix, from which we extract the pole positions and residues for real energies. This analysis yields at least two results which undermine the utility of using the positions and residues of K-matrix poles as model-independent characterizations of resonance structure. First, we show that assuming a different form for Eq. (5) alters the finding of Ref. [1] that the K-matrix pole and residue are independent of the background, $\tan\delta_B$. Second, using the T matrix determined in Ref. [3], we numerically calculate the related K matrix [see Eq. (6)] and find that there are poles in the T matrix which have no nearby poles in the K matrix for real energies. This would seem to obviate the use of K-matrix poles in characterizing resonance structures observed in scattering experiments, since the structures present in the T matrix do not necessarily appear in the K matrix.

Prior to describing our numerical results, we revisit the derivation of Eqs. (1) and (2) in relation to the assumptions of Eq. (5). We then compare the result of Ref. [1] with the result obtained with a different assumption (following Dalitz [4]) for the parametrization [5] of the resonant eigenphase of Eq. (5).

The K matrix we use is real for energies above all thresholds considered in the problem, and is related to the T matrix by

$$T \;=\; K\,(1 - iK)^{-1}. \tag{6}$$

The real symmetric K matrix is diagonalized by an orthogonal transformation, U as

$$K_D = U^T\ K\ U. \tag{7}$$

This matrix also diagonalizes the T matrix, and therefore the S matrix, defined as $S = 1 + 2iT$. Since S is a unitary matrix,

$$(S_D)_{ij} \;=\; U^T S U \;=\; \delta_{ij}\, e^{2i\delta_i}, \tag{8}$$

where the δ_i are eigenphases. Using the relation between the K and T matrices above, we have

$$(K_D)_{ij} \;=\; i(1 - S_D)(1 + S_D)^{-1} \;=\; \delta_{ij}\tan\delta_i. \tag{9}$$

Having determined the eigenphases, we can reconstruct the physical T matrix

$$T_{if} \;=\; (U T_D U^T)_{if} = \sum_{\alpha} U_{i\alpha} U_{f\alpha} e^{i\delta_\alpha}\sin\delta_\alpha. \tag{10}$$

Taking the trace of Eqs. (9) and (10) gives the result in Eq. (4).

We first examine the use of these relations in a simple scenario including a single resonant eigenphase and neglect any background effects. Assuming only one eigenphase (δ_R) passes through $\pi/2$ at energy E_R and neglecting others, the resonant eigenphase may be written in the form

$$\tan\delta_R = \frac{\Gamma_R/2}{E_R - E}, \tag{11}$$

which leads to

$$T_{if} = \frac{1}{2}\frac{\Gamma_i^{1/2}\Gamma_f^{1/2}}{E_R - E - i\Gamma_R/2} \tag{12}$$

with $\Gamma_i = U_{iR}^2\Gamma_R$ and $\sum_i U_{iR}^2 = 1$ (orthogonality) giving $\sum_i \Gamma_i = \Gamma_R$. The result, Eq. (12), is consistent with Eqs. (1) and (2) for $\delta_B = 0$. Next we consider how background can be added, and whether a single dominant eigenphase is appropriate. These questions have been addressed in the works of Dalitz [4], Goebel and McVoy [6], and Weidenmüller [7].

Consider first the addition of a background phase to Eq. (11). One way of doing this is the *ansatz* of Eq. (5) employed in Ref. [1] (see also Ref. [8]) and used to obtain Eqs. (1) and (2). This leads to the model independence of Ref. [1] described above. An alternative parametrization of the resonant eigenphase is considered in Refs. [4,6,7]. The *ansatz* used there also assumes a single dominant eigenphase, which rises through $\pi/2$, but posits a phase-addition rule: the resonant eigenphase, $\tilde{\delta}_R$ has the form

$$\tilde{\delta}_R = \tilde{\delta}_B + \delta_P, \tag{13}$$

where $\tilde{\delta}_B$ is the background phase which determines the eigenphase far from the resonance energy, and δ_P is the resonant (pole) contribution. This form of resonance and background separation modifies the above conclusion of model independence. We consider the phase-addition rule in some detail to clarify this point.

As a function with a simple pole, the resonant contribution, δ_P may be written in general as

$$\tan\delta_P = \frac{\gamma(E)/2}{E^*(E) - E}, \tag{14}$$

where the position of the pole is given by $E_P^*(E^*(E_P^*) - E_P^* = 0)$ and the function $\gamma(E)$ goes to a nonzero constant at the pole. Note that, far from the pole, the eigenphase shift $\tilde{\delta}_R$ reduces to the nonpole part, $\tilde{\delta}_B$. Using Eqs. (13) and (14) we compute the resonant element of the diagonal K matrix as

$$\tan\tilde{\delta}_R = \frac{\frac{1}{2}\gamma + (E^* - E)\tan\tilde{\delta}_B}{(E^* - E) - \frac{1}{2}\gamma\tan\tilde{\delta}_B}, \tag{15}$$

which leads to a K matrix with the trace

$$\mathrm{Tr}(K) = \frac{\overline{\Gamma}(E)/2}{\overline{E}_P^*(E) - E} + \sum_{j\neq R}^{N}\tan\delta_j, \tag{16}$$

where $\overline{\Gamma}(E)/2 = \gamma/2 + (E^* - E)\tan\tilde{\delta}_B$ and the location of the pole in $\mathrm{Tr}(K)$ is $\overline{E}_P^*$, where

$$[(E^*(E) - E) - \frac{\gamma(E)}{2}\tan\tilde{\delta}_B(E)]\Big|_{E=\overline{E}_P^*} = 0. \tag{17}$$

In general, $E_P^* \neq \overline{E}_P^*$ and the pole position of the K matrix, $\overline{E}_P^*$ depends on the background, $\tan\tilde{\delta}_B$. The residue also depends on $\tilde{\delta}_B$ since $\overline{\Gamma}(\overline{E}_P^*) = \gamma/\cos^2\tilde{\delta}_B$.

We could anticipate this result by comparing the forms Eqs. (5) and (13). In Eq. (5), used in Ref. [1], the location of the K-matrix pole, E_R is independent of δ_B. The resonant structure, $\Gamma/[2(E_R - E)]$, and the nonresonant contribution are *assumed* to be decoupled. That is, if $\tan\delta_B$ is a bounded function of the energy, its value cannot affect the energy where the resonant eigenphase, δ_R is $\pi/2$. In the "Dalitz form," Eq. (13), the location of the pole in the K matrix, determined by the energy E_R where the phase $\delta_R \to \pi/2$, is affected by the "background phase," δ_B. Since the true form of the K matrix is unknown, the existence of alternative forms complicates the extraction of pole positions. In fact, in dynamical models of scattering amplitudes, the location of the K-matrix pole is expected to depend, perhaps strongly, on the nonresonant (or background) contribution to the amplitude [9].

Turning to the T matrix, in place of Eq. (12), the result is

$$T_{if} = \frac{1}{2}e^{2i\tilde{\delta}_B}\frac{\Gamma_i'^{1/2}\Gamma_j'^{1/2}}{E_R' - E - i\Gamma'/2} + U_{iR}U_{jR}e^{i\tilde{\delta}_B}\sin\tilde{\delta}_B, \tag{18}$$

for the corresponding T-matrix element with resonance 'mass' and 'width' shifted from the K-matrix pole parameters. Thus, a different scheme for the addition of resonance and background contributions can alter the relationship between K-matrix and T-matrix resonance masses.

As another example of the model dependence of K- and T-matrix poles, and to address the question of whether a single resonant eigenphase is appropriate, we consider the following simple S matrix from Refs. [6,7]:

$$S_{ij} = e^{i(\phi_i+\phi_j)}\left[\delta_{ij} + i\frac{\Gamma_i^{1/2}\Gamma_j^{1/2}}{E_R - E - i\Gamma/2}\right], \tag{19}$$

to show the effect of background on eigenphases. This S matrix is symmetric and far from the resonance energy is diagonal (the elastic background approximation) with eigenphases ϕ_i[1]. Applying the unitary transformation diagonalizing Eq. (19) and taking the determinant, yields

$$e^{2i\sum_i\delta_i} = e^{2i\sum_i\phi_i}\frac{E_R - E + i\Gamma/2}{E_R - E - i\Gamma/2}, \tag{20}$$

where δ_i is an eigenphase and the last factor has the phase behavior of an elastic resonance at E_R. From Eq. (20) we see the above phase-addition rule, Eq. (13), if only a single eigenphase is significant. In general, however, it is the sum of eigenphases that displays resonance behavior.

A few further comments on the parametrization of eigenphases may be useful. Weidenmüller [7] has shown that

[1]Note that a product form of background and resonant S matrices $S = S_B S_R$, generally yields a nonsymmetric S matrix. This violates time-reversal invariance, which is required to obtain a real K matrix; The 2×2 S matrix in Eq. (19) is sufficiently simple to allow an explicit calculation of eigenphases [6], $\sigma_\pm$, using

$$\sigma_\pm = \tfrac{1}{2}(S_{11} + S_{22}) \pm \tfrac{1}{2}\left[(S_{11} - S_{22})^2 + 4S_{12}^2\right]^{1/2}. \tag{24}$$

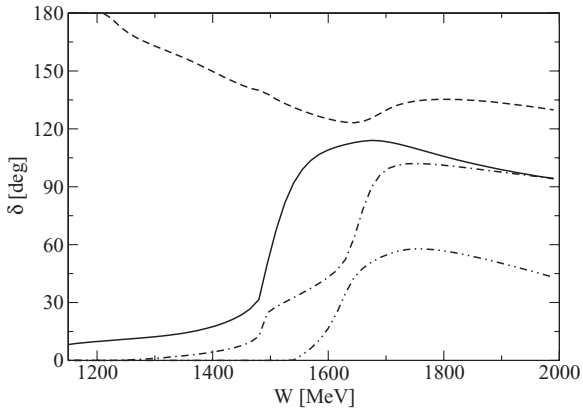

FIG. 1. The eigenphases in a four-channel fit to the S_{11} partial wave from the SP06 solution from SAID.

individual eigenphases have an energy dependence determined largely by the background. Through an application of Wigner's no-crossing theorem, he finds no single eigenphase increasing by π, except for special values of the background phases. As a result, the eigenphases 'repel' rather than crossing, the N eigenphases individually increasing only by an average of π/N over the width of the resonance in some cases. An example of this behavior is given in Fig. 1, which shows the eigenphases calculated from SAID [3] for the S_{11} partial wave, containing two resonances.

Goebel and McVoy have applied the eigenvalue method to resonant d-α scattering [6] data to explicitly study this behavior. Eigenvalues for this two-channel scattering matrix were also given, showing the appearance of square-root branch points which complicate the energy dependence [10]. There is a cancellation occurring when the sum of eigenvalues or eigenphases is taken, and this supports the basic idea of using a trace, as proposed in Ref. [1]. A direct relation between resonance energy and the sum of eigenphases is given by the equation [11,12]

$$\text{tr } Q \;=\; 2\hbar \sum_i \frac{d\delta_i}{dE}, \tag{21}$$

relating the trace of Smith's time-delay matrix to the energy derivative of the sum of eigenphases, δ_i. One diagonal element of the Q matrix has recently been shown to correlate precisely with the T-matrix pole positions of resonances [12].

To explicitly test the method of Ref. [1], we have taken amplitudes determined from our fits to pion-nucleon elastic scattering data [3], and the reaction $\pi N \to \eta N$. The parametrization we use is based on the Chew-Mandelstam (CM) K matrix, which builds in cuts associated with the opening of $\eta N, \pi\Delta$, and ρN channels. The CM form is analytic and generates a T matrix which is unitary and can be continued into the complex plane to find poles on the various sheets associated with the $\eta N, \pi\Delta$, and ρN channels. The fits can, in principle, include couplings to any of the channels, though the $\pi\Delta$ and ρN channels are not constrained by data. However, the amplitude associated with each channel has, by construction, the proper threshold behavior, two-particle channel cuts, and pole positions. Amplitudes in the elastic channel are further constrained by forward and fixed-t dispersion relations.

Thus far, we have implicitly assumed that there is a direct correspondence between K-matrix and T-matrix poles. It is known [13] that this is not true in general. For example, in the CM approach it is possible to generate T-matrix poles for the resonances without explicitly adding a pole to the CM K matrix [14]. If, however, a pure CM K-matrix pole representation is used

$$K_{ij} \;=\; \frac{\gamma_i\gamma_j(\rho_i\rho_j)^{1/2}}{E_K - E}, \tag{22}$$

the resulting T matrix is

$$T_{ij} \;=\; \frac{\gamma_i\gamma_j(\rho_i\rho_j)^{1/2}}{E_K - E - \sum_n \gamma_n^2 C_n - i\sum_n \gamma_n^2\rho_n}, \tag{23}$$

where ρ_n is the phase space factor for the ith channel, and C_n is the real part of the Chew-Mandelstam function, obtained by integrating phase space factors over appropriate unitarity cuts.

The fit under consideration uses a parametrization of a CM K matrix [14], from which the unitary T matrix is calculated. The K matrix, defined by Eq. (6), is computed from the calculated T matrix as $K = T(1 + iT)^{-1}$. The resulting K matrix was checked for consistency by reproduction of the T matrix from Eq. (6), and checked for unitarity at each stage. The K matrix was then searched for poles at energies associated with well-known resonances. When poles did appear in a given amplitude, we confirmed that they appeared in each associated amplitude at the same energy. However, we did not generally find poles closely associated with (T-matrix) resonance energies, nor did we find that each resonance produced a K-matrix pole, as shown in Tables I and II. If an explicit pole was inserted into the CM K matrix, then this approach generated a corresponding K-matrix pole. This was the case for the $\Delta(1232)$ resonance, where we found a K-matrix pole at 1232 MeV. K-matrix poles also appeared near the $N(1535)$, $N(1650)$, and $\Delta(1620)$ resonance masses, in the πN S_{11} and S_{31} partial waves, though no explicit CM K-matrix poles were used in the fit. For the P_{11} and D_{13} partial waves, however, no CM K-matrix poles were used in the fit, and no K-matrix poles were found near

TABLE I. Pole positions in complex energy plane of T and K matrix for the $\pi N \to \pi N$ reaction from SAID [3] for isospin $T = \frac{1}{2}$ partial waves. Each T-pole position is expressed in terms of its real and imaginary parts $(M_R, -\Gamma_R/2)$ in MeV. Only K-matrix pole positions which satisfy 1.1 GeV $< W <$ 2.0 GeV are considered.

ℓ_{TJ}	T poles		K poles	
S_{11}	(1500, 50)	(1650, 40)	1535	1675
P_{11}	(1360, 80)	(1390, 80)[a]	–	–
P_{13}	(1665, 175)		–	
D_{13}	(1515, 55)		–	
D_{15}	(1655, 70)		1760	
F_{15}	(1675, 60)	(1780, 130)	–	–

[a]This pole is located on the second Riemann sheet.

TABLE II. Pole positions in complex energy plane as in Table I for isospin $T = \frac{3}{2}$ partial waves.

ℓ_{TJ}	T poles		K poles	
S_{31}	(1595, 70)		1660	
P_{31}	(1770, 240)		–	
P_{33}	(1210, 50)	(1460, 200)	1232	–
D_{33}	(1630, 125)		–	
D_{35}	(2000, 195)		–	
F_{35}	(1820, 125)		–	

the resonance masses. In all of these cases, resonance poles did appear in the corresponding T matrices.

In conclusion, we have not found a simple association between K-matrix and T-matrix poles for use in the extraction of resonance properties. We have argued that: (i) K-matrix poles are not generally independent of background contributions, (ii) a pole in the T matrix does not necessarily imply a pole in the K matrix. Therefore, K-matrix poles do not appear to be useful candidates for characterizing resonance parameters obtained from scattering amplitudes. Applied to a particular S matrix obtained from a fit to pion-nucleon and eta-nucleon scattering data [3] we find no one-to-one association between K-matrix and T-matrix poles. We have also noted that the separation of background and resonance contributions is not unique and that eigenphase behavior may be more complicated than the form chosen in Ref. [1]. We have noted an explicit counterexample for the parametrization of the resonant eigenphase, specifically Eq. (13), which violates the model independence of Ref. [1]. We are currently exploring the behavior of eigenphases using S matrices from scattering amplitudes in order to determine whether eigenphase repulsion is as common as suggested in Ref. [6].

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