Quark matter under strong magnetic fields in the Nambu–Jona-Lasinio model

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In the present work we use the large-N_c approximation to investigate quark matter described by the SU(2) Nambu–Jona-Lasinio model subject to a strong magnetic field. The Landau levels are filled in such a way that usual kinks appear in the effective mass and other related quantities. β equilibrium is also considered and the macroscopic properties of a magnetar described by this quark matter is obtained. Our study shows that the magnetar masses and radii are larger if the magnetic field increases but only very large fields ($\geqslant 10^{18}$ G) affect the equation of state in a non-negligible way.

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I. INTRODUCTION

In 1979, telescopes in spacecrafts and astronomers around the world detected the emission of very intense γ and x rays. The sources of these rays were first called soft γ repeaters (SGR) and later identified as possible remnants of supernova explosions, the tragic death of very massive stars. If this remnant is a neutron star that spins very rapidly, an intense magnetic field is formed; if it spins slowlier the magnetic field is strong but not as much as in the first case. The ordinary neutron stars, also known as pulsars, bear a magnetic field of the order of 10^{12}–10^{13} G. The neutron stars with very strong magnetic fields of the order of 10^{14}–10^{15} G are known as magnetars and they are believed to be the sources of the intense γ and x rays detected in 1979. Most of the time the magnetar remains inactive, but the strong magnetic field causes the solid crust to break into small pieces. The crustquake leaves a fireball that cools down and emits x rays from its surface until it evaporates completely [1,2].

In Ref. [3] the equation of state (EOS) used to describe neutron stars in a strong magnetic field is obtained from a field theoretical approach. Two relativistic models are used: one normally called the nonlinear Walecka model (NLWM) and the other one with a derivative coupling between mesons and baryons. The importance of including anomalous magnetic moments (AMM) is discussed. A more recent work [4] analyses the importance of the scalar-isovector δ mesons in the EOS that describes magnetars. In Ref. [5] density-dependent hadronic models are used with the same purpose. In all three works the AMM of the electrons were not considered because they were shown to cancel out if properly introduced [1]. In two articles [3,4] the AMM of the muons were also taken into account. All above-mentioned articles refer to neutron stars composed of hadrons and leptons. These stars are called hadronic stars.

In the stellar modeling, the structure of the star depends on the assumed equation of state built with appropriate models. The true ground state of matter remains a source of speculation. In conventional models, hadrons are assumed to be the true ground state of the strong interaction. However, it has been argued [6–10] that *strange quark matter* (SQM) is the true ground state of all matter. This hypothesis is known as the Bodmer-Witten conjecture. Hence, the interior of neutron stars should be composed predominantly of u, d, and s quarks plus leptons to ensure charge neutrality. Pulsars described by matter composed of SQM are often called strange stars. However, as the strangeness content depends on the model used to describe the quark matter, we prefer to describe any model in which the interior involves deconfined quarks (not bound in hyperons) as *quark stars* [11]. Apart from the differences in the EOS, an important distinction between quark stars and conventional neutron stars is that the quark stars are self-bound by the strong interaction, whereas neutron stars are bound by gravity. This allows a quark star to rotate faster than would be possible for a neutron star. Moreover, some authors have argued that quark stars should be bare [12,13], in the sense that any crust would either not form or would be destroyed during the supernova explosion. The characteristics of the radiation from hot, bare strange stars have been identified [14] and the electron-positron pairs that can be emitted from bare quark stars [15] require the existence of a surface layer of electrons tied to the star by a strong electric field. Two of the most common models used to describe quark matter are the MIT bag model [16] and the Nambu–Jona-Lasinio model (NJL) [17]. In Ref. [11] it was shown that although the electron chemical potential of a quark star described by the MIT bag model is very low (less than 20 MeV), the NJL model gives much higher values, reaching 100 MeV inside the star, accounting for the necessary electric field that explains the emission of electron-positron pairs.

An important point to be investigated refers to the stability of quark matter in the interior of quark stars. Two different possibilities for the MIT bag model can be found in relation to quark matter in the interior of quark stars: the unpaired phase [16], which is widely favored in the literature on strange stars, and the color-flavor-locked phase (CFL) [18], which allows the quarks near the Fermi surface to form Cooper pairs that condense and break the color gauge symmetry [19]. At sufficiently high densities the favored phase is the CFL. In Ref. [20] the stability of quark matter described by the two-flavor NJL model and subject to an external magnetic field was investigated. It was shown that the stability depends on the strength of this field and on model parameters. In Ref. [21] the EOS for magnetized quark stars was described with the help of the MIT bag model. AMM for the quarks were properly taken into account.

The NJL model in a magnetic field was considered in Ref. [22] and it was shown that the magnetic field spontaneously breaks chiral symmetry. The formalism used in the above-mentioned study was based on a previous calculation performed in Ref. [23]. An interesting study based on a multiquark extended version of the NJL model, at zero density and in the presence of a magnetic field, can be found in Ref. [24]

The scope of the present work is to study ud quark matter in a magnetic field within the NJL model, with or without the requirement of β equilibrium. We focus our work on the SU(2) version of the model. Moreover, we remark that the present work may also be relevant regarding the physics of noncentral heavy-ion collisions such as the ones performed at RHIC and LHC-CERN that can provide a possible signature for the presence of CP-odd domains in the presumably formed quark-gluon plasma phase [25]. In this particular case one reaches magnetic fields of about 10^{19} G or $B \simeq 6m_\pi^2/e$ (m_π representing the pion mass and e the fundamental electric charge).

This work is organized as follows: In Sec. II we set up a Lagrangian density adequate to describe two-flavor quark matter, in β equilibrium, in the presence of an external magnetic field. This allows the derivations to be carried out in a uniform fashion from a common Lagrangian density making the evaluations more transparent and pedagogically more interesting to the reader. Using functional techniques we derive the effective potential, which represents Landau's free energy density, using the standard large-N_c approximation. In Sec. III the relevant expressions for quark matter EOS subject to a magnetic field are displayed. Specific cases, for the symmetric version, are discussed in Sec. IVA for finite density and no magnetic field, in Sec. IVB for zero density in a magnetic field, and in Sec. IVC for the general case of finite density matter in a magnetic field. In Sec. V, details of matter in β equilibrium are given and in Sec. VI our results are shown and discussed. In Sec. VII the more important conclusions are drawn.

II. GENERAL FORMALISM

To consider (two flavor) quark stars in β equilibrium with strong magnetic fields one may define the following Lagrangian density

$$\mathcal{L}_{\beta f} = \mathcal{L}_f + \mathcal{L}_l - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{1}$$

where the quark sector is described by the standard Nambu–Jona-Lasinio model

$$\mathcal{L}_f = \bar\psi_f[\gamma_\mu(i\partial^\mu - q_f A^\mu) - m_c]\psi_f + G[(\bar\psi_f\psi_f)^2 + (\bar\psi_f i\gamma_5\vec\tau\psi_f)^2], \tag{2}$$

where a summation over the quark flavors, $f = u, d$ is implied while q_f represents the quark electric charge. As emphasized in Refs. [26–29] a term of the form $G_V(\bar\psi_f\gamma^\mu\psi_f)^2$ is important at finite densities so that a saturation mechanism depending on the vector coupling strenght appears and makes matter stable. Within the large-N_c (or mean field) approximation, the effect of such a term in the thermodynamical potential is to produce a shift on the chemical potential. The question of how this channel influences our predictions is carefully addressed in Sec. VI which contains our results.

Note that we have used $m_c = m_u \simeq m_d$ as representing the current masses.

The leptonic sector is given by

$$\mathcal{L}_l = \bar\psi_l[\gamma_\mu(i\partial^\mu - q_l A^\mu) - m_l]\psi_l, \tag{3}$$

where $l = e, \mu$. One recognizes this sector as being represented by the usual QED type of Lagrangian density. As usual, A_μ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ are used to account for the external magnetic field. Then, because we are interested in a static and constant magnetic field in the z direction, $A_\mu = \delta_{\mu 2}x_1 B$.

Regarding the actual evaluations, $\mathcal{L}_f$ and $\mathcal{L}_l$ bear two fundamental differences. Due to the quadratic fermionic interaction, the former is nonrenormalizable in 3+1 dimensions (G has dimensions of eV^{-2}), meaning that eventual divergences cannot be eliminated by a consistent redefinition of the original model parameters (fields, masses, and couplings). The renormalizability issue arises during the evaluation of momentum integrals that represent Feynman loops and, in the process, one usually employs regularization prescriptions (e.g., dimensional regularization, sharp cutoff, etc.) that are formal ways to isolate divergences. However, the procedure introduces *arbitrary* parameters with dimensions of energy that do not appear in the *original* Lagrangian density. Later, when (unlike the 3+1 NJL model) the theory is renormalizable, one may choose any value for the arbitrary energy scale and the original parameters *run* with it as dictated by the renormalization group. Within the NJL model a sharp cutoff (Λ) is preferred and because the model is nonrenomalizable one gives up the very high energy scales fixing Λ to a value related to the physical spectrum under investigation. This strategy turns the 3+1 NJL model into an effective model while Λ is treated as a *parameter*. The experimental values of quantities such as the pion mass (m_π) and the pion decay constant (f_π) are used to fix both, G and Λ.

A second important issue regards the fact that, when $m_c \to 0$, the quark propagator brings unwanted infrared divergences, meaning that the evaluations have to be carried out in a *nonperturbative* fashion. Moreover, very often physical quantities (such as the self-energy) appear as powers of the dimensionless quantity $G\Lambda^2$ that is greater than the unity spoiling any possibility of success via standard perturbative evaluations.

As far as analytic nonperturbative evaluations are concerned, one can consider one loop contributions dressed up by a fermionic propagator whose effective mass (M) is determined in a self-consistent way. This approximation is known under different names, e.g., Hartree, large-N_c, or mean-field approximations (MFA). The leptonic sector, however, is described by a QED type of Lagrangian density that is renormalizable and that, in principle, could be treated in a perturbative fashion. Here, we treat the complete $\mathcal{L}_{\beta f}$ in a consistent way by evaluating the thermodynamical potential related to $\mathcal{L}_{\beta f}$ up to one loop. It is interesting to remark that, in practice, one does not have to consider the full $\mathcal{L}_{\beta f}$ because both sectors have similar polynomial structures, apart from the four-quark interaction term that, as discussed, makes the theory nonrenormalizable and renders perturbative calculations useless. So, concerning the evaluation of the equation of state, our strategy is the following. We use quantum field methods, in the imaginary time formalism, to evaluate the effective potential, or Landau's free energy density ($\mathcal{F}_f$), for the quark sector in the large-N_c approximation. After summing over the Matsubara's frequencies we will regularize the divergent (three) momentum integrals by using a sharp *noncovariant* cutoff, Λ. When evaluated at its minimum, the effective potential is equivalent to the thermodynamical potential, $\Omega_f = -P_f = \mathcal{E}_f - T\mathcal{S} - \mu_f \rho_f$, where P_f represents the pressure, $\mathcal{E}_f$ the energy density, T the temperature, $\mathcal{S}$ the entropy density, and μ_f the chemical potential (a sum over repeated indices is implied). The quark density, ρ_f, and the baryonic density, ρ_B, are simply related by $\rho_f = 3\rho_B$. For the present study, just the zero temperature case is important and, as a consequence, the term with the entropy vanishes. Equivalent results for the leptonic sector, within the same approximation, can be trivially obtained by performing the replacements $G \to 0, m_c \to m_l, q_f \to q_l$, and $N_c \to 1$. Finally, this procedure allows us to obtain the EOS from the full pressure for β stable dense quark matter in the presence of a magnetic field (B)

$$P_{\beta f}(\mu_f, \mu_l, B) = P_f(\mu_f, B)|_M + P_l(\mu_l, B)|_{m_l} + \frac{B^2}{2}, \quad (4)$$

where our notation means that P_f is evaluated in terms of the quark effective mass, M, which is determined in a (nonperturbative) self-consistent way while P_l is evaluated at the leptonic bare mass, m_l. The term $B^2/2$ arises due to the electromagnetic term $F_{\mu\nu}F^{\mu\nu}/4$ in the original Lagrangian density. To normalize our results we shall require that the pressure vanishes at zero chemical potentials by defining

$$P_{\beta f,\text{eff}}(\mu_f, \mu_l, B) = P_{\beta f}(\mu_f, \mu_l, B) - P_{\beta f}(0, 0, B). \quad (5)$$

Note that the above normalization prescription, which washes away the $B^2/2$ term, is not unique and one could as well require that the pressure vanishes at zero chemical potential and zero magnetic field in which case the $B^2/2$ term survives. Although we use the former prescription for most of the time we will also consider the latter in Sec. VI, when stellar matter is discussed. Throughout this article we consider the following set of parameters [28]: $\Lambda = 587.9$ MeV, $m_c = 5.6$ MeV, $m_e = 0.511$ MeV, $m_\mu = 105.66$ MeV, and $G\Lambda^2 = 2.44$.

III. EOS FOR QUARK MATTER AT FINITE DENSITY IN A MAGNETIC FIELD

To obtain the effective potential (or Landau free energy density) for the quarks, $\mathcal{F}_f$, it is convenient to consider the bosonized version of the NJL that is easily obtained by introducing auxiliary fields $(\sigma, \vec{\pi})$ through a Hubbard-Stratonovich type of transformation. Here, $\mathcal{F}_f$ is evaluated using the large-N_c approximation that is equivalent to the MFA. Then, to introduce the auxiliary bosonic fields and to render the theory more suitable to apply the large-N_c approximation it is convenient to use $G \to \lambda/(2N_c)$ formally treating N_c as a large number that is set to its relevant value, $N_c = 3$, at the end of the evaluations. To retrieve some well-known results related to chiral symmetry breaking (CSB), in the absence of a magnetic field, let us set $B = 0$ for the moment.[1] One then has

$$\mathcal{L}_f = \bar{\psi}_f (i \not\partial) \psi_f - \bar{\psi}_f(\sigma' + i\gamma_5 \vec{\tau}\vec{\pi})\psi_f - \frac{N_c}{2\lambda}(\sigma^2 + \vec{\pi}^2), \quad (6)$$

where $\sigma' = \sigma + m_c$. Note that the introduction of the auxiliary (or background) fields does not change the physics because they do not propagate and their Euler-Lagrange equations of motion trivially lead to $\sigma = -(\lambda/N_c)\bar{\psi}_f\psi_f = -2G\bar{\psi}_f\psi_f$ and $\vec{\pi} = -2Gi\bar{\psi}_f\gamma_5\vec{\tau}\psi_f$. However, their introduction simplifies the selection of the relevant contributions at any order in $1/N_c$ [30]. Basically, each closed quark loop contributes with a factor of N_c while each internal bosonic line brings a factor of $1/N_c$. The effective potential (or Landau's free energy density), $\mathcal{F}_f$, is defined as the classical potential that appears (at the tree level) in the original plus radiative (quantum) corrections. As it is well known, $\mathcal{F}_f$ is particularly useful in the study of symmetry breaking/restoration because a symmetry that is observed at the classical level may be broken by quantum corrections with the appearance of a nonvanishing order parameter. The effective potential, which is also the generating functional all 1PI contributions with zero external momenta [30], can be readily obtained by integrating over the fermionic fields. Within the large-N_c approximation this procedure results in [31]

$$\mathcal{F}_f = \frac{N_c(\sigma^2 + \pi^2)}{2\lambda} + i\int \frac{d^4p}{(2\pi)^4} \text{tr}\ln[\not p - (\sigma' + i\gamma_5\vec{\tau}\vec{\pi})]. \quad (7)$$

The first term on the right-hand side is the classical (tree) contribution while the second accounts for a radiative (loop) contribution of order N_c. Let us define some important physical quantities by quickly reviewing how CSB arises within the NJL model. For this let us take the chiral limit ($m_c = 0$). Equation (7) can be written as

$$\mathcal{F}_f(\chi) = \frac{N_c\chi^2}{2\lambda} + \frac{i}{2}\text{tr}\int \frac{d^4p}{(2\pi)^4} \ln[-p^2 + \chi^2], \quad (8)$$

where we have defined $\chi = \sqrt{\sigma^2 + \pi^2}$. Because the potential is symmetric we can set $\vec{\pi} = 0$ considering only the σ direction. Minimizing with respect to σ leads to the

[1] As we shall see in the sequel one may easily incorporate contributions for T, μ, and B by modifying some of the relevant Feynman rules.

well-known gap equation

$$\left.\frac{d\mathcal{F}_f}{d\sigma}\right|_{\sigma=\langle\sigma\rangle} = 0, \tag{9}$$

which, as expected, gives the self-consistent relation

$$\langle\sigma\rangle = -\langle\sigma\rangle \frac{\lambda}{N_c} \mathrm{tr} \int i \frac{d^4p}{(2\pi)^4} \frac{1}{[-p^2 + \langle\sigma\rangle^2]}. \tag{10}$$

Chiral symmetry is broken when the true ground state lies in $\langle\sigma\rangle \neq 0$. Therefore $\langle\sigma\rangle$ is the order parameter that signals CSB, which comes as no suprise because, by the equations of motion, $\langle\sigma\rangle = -2G\langle\bar{\psi}_f\psi_f\rangle$ and the existence of a nonvanishing quark condensate breaks CS. Equation (10) also shows that $\langle\sigma\rangle$ is identical to the quark self-energy within our approximation that allows us to write the effective quark mass as $M = m_c + \langle\sigma\rangle$. Finally, in the present work, the relevant quantity is the thermodymical potential, Ω_f, which is defined as Landau's free energy at its minimum $\Omega_f = \mathcal{F}_f(\langle\sigma\rangle)$. The equation of state can be obtained by using $\Omega_f = -P_f$. Then, Eq. (8) allows us to write the quark pressure, away from the chiral limit, as

$$P_f = -\frac{(M-m_c)^2}{4G} - \frac{i}{2}\mathrm{tr}\int \frac{d^4p}{(2\pi)^4} \ln[-p^2 + M^2]. \tag{11}$$

To obtain results valid at finite T and μ in the presence of an external magnetic field B one can use the following replacements, which come from dispersion relations for quarks [32]

$$p_0 \to i(\omega_\nu - i\mu_f), \quad \mathbf{p}^2 \to p_z^2 + (2n+1-s),$$

with $s = \pm 1, n = 0, 1, \ldots$

$$\int \frac{d^4p}{(2\pi)^4} \to i\frac{T|q_f|B}{2\pi} \sum_{\nu=-\infty}^{\infty} \sum_{n=0}^{\infty} \int \frac{dp_z}{(2\pi)}.$$

In the above relations, $\omega_\nu = (2\nu+1)\pi T$, with $\nu = 0, \pm 1, \pm 2, \ldots$ representing the Matsubara frequencies for fermions while n represents the Landau levels (LL) and s represents the spin states that, at $B \neq 0$, must be treated separately. As explained in Ref. [33] one may understand the origin of those replacements, by recalling that, quantum mechanically, the energy associated with the circular motion in the x-y plane is quantized in units of $2qB$ due to the field in the z direction while the energy associated to linear motion along z is taken as a continuous. All these levels for which the values of $p_x^2 + p_y^2$ lie between $2qBn$ and $2qB(n+1)$ now coalesce together into a single level characterized by n whose number is given by

$$\frac{S}{(2\pi)^2} \int\int dp_x\, dp_y = \frac{SqB}{2\pi}, \tag{12}$$

where S is the area in the x-y plane and q stands for $|q_f|$. Then, using those replacements, taking the trace, and summing over Matsubara's frequencies (see Appendix A) one obtains

$$\begin{aligned} P_f = &-\frac{(M-m_c)^2}{4G} + \frac{N_c}{2\pi}\sum_{s,n,f}(|q_f|B)\int \frac{dp_z}{(2\pi)} E_p(B) \\ &+ \frac{N_c}{2\pi}\sum_{s,n,f}(|q_f|B)\int \frac{dp_z}{(2\pi)} \left\{T \ln\left[1 + e^{-[E_p(B)+\mu_f]/T}\right]\right. \\ &+ \left. T\ln\left[1 + e^{-[E_p(B)-\mu_f]/T}\right]\right\}, \end{aligned} \tag{13}$$

where $E_p(B) = \sqrt{p_z^2 + (2n+1-s)|q_f|B + M^2}$. Next, by analyzing the degeneracy of the lowest Landau level (LLL) one can define $E_{p,k}(B) = \sqrt{p_z^2 + 2k|q_f|B + M^2}$, replacing n by k in the sum that appears in Eq. (13) that, now, also runs over the degeneracy label, $\alpha_k = 2 - \delta_{k0}$. Being mainly concerned with the case $T = 0$, $\mu \neq 0$, and $B \neq 0$ we can take the limit $T \to 0$ in Eq. (13) arriving at (see Appendix A)

$$\begin{aligned} P_f(\mu, B) = &-\frac{(M-m_c)^2}{4G} + P_f^{\rm med} \\ &+ \frac{N_c}{2\pi}\sum_{f=u}^{d}\sum_{k=0}^{\infty}\alpha_k(|q_f|B)\int_{-\infty}^{\infty}\frac{dp_z}{(2\pi)} E_{p,k}(B), \end{aligned} \tag{14}$$

where the contribution from the medium is

$$P_f^{\rm med} = \frac{N_c}{2\pi}\sum_{f=u}^{d}\sum_{k=0}^{\infty}\alpha_k(|q_f|B)\int_{-\infty}^{\infty}\frac{dp_z}{(2\pi)}[\mu_f - E_{p,k}(B)]. \tag{15}$$

Note that, although not explicitly written, a θ function [more specifically, $\theta(\mu_f - E_{p,k})$] must be considered as multiplying all μ_f-dependent terms, such as $P_f^{\rm med}$, appearing in our work (see Appendix A). Simple power counting reveals that the last term in Eq. (14) is (ultraviolet) divergent while the second term, which contains the in medium contributions, is finite because it has a natural cut off given by the Fermi momentum, $p_{f,F}^2 = \mu_f^2 - M^2$. In Appendix B we show how Eq. (14) can acquire a physically more appealing form by separating the (divergent) vacuum contribution from the (finite) magnetic field contribution. As a by-product, those manipulations also produce more elegant relations in which the infinite sum over Landau levels appearing in the last term of Eq. (14) are shuffled into Riemann-Hurwitz ζ functions. We finally get

$$P_f(\mu_f, B) = -\frac{(M-m_c)^2}{4G} + \left[P_f^{\rm vac} + P_f^{\rm mag} + P_f^{\rm med}\right]_M, \tag{16}$$

where the vacuum contribution reads

$$P_f^{\rm vac} = 2N_cN_f\int\frac{d^3\mathbf{p}}{(2\pi)^3}E_{\mathbf{p}}, \tag{17}$$

with $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + M^2}$ and $N_f = 2$. We are now in position to present the explicit expressions for $P_f^{\rm vac}$, $P_f^{\rm mag}$, and $P_f^{\rm med}$. Let us start with the vacuum which, on using a sharp noncovariant cutoff, Λ, can be written as

$$P_f^{\rm vac} = -\frac{N_cN_f}{8\pi^2}\left\{M^4\ln\left[\frac{(\Lambda+\epsilon_\Lambda)}{M}\right] - \epsilon_\Lambda\Lambda\left[\Lambda^2 + \epsilon_\Lambda^2\right]\right\}, \tag{18}$$

where we have defined $\epsilon_\Lambda = \sqrt{\Lambda^2 + M^2}$. The evaluations performed in Appendix B also give the following finite magnetic contribution

$$P_f^{\rm mag} = \sum_{f=u}^{d} \frac{N_c(|q_f|B)^2}{2\pi^2} \left\{ \zeta'[-1, x_f] - \frac{1}{2}\left[x_f^2 - x_f\right] \ln x_f + \frac{x_f^2}{4} \right\}, \quad (19)$$

where $x_f = M^2/(2|q_f|B)$ while $\zeta'(-1, x_f) = d\zeta(z, x_f)/dz|_{z=-1}$ where $\zeta(z, x_f)$ is the Riemann-Hurwitz ζ function [34]. Finally, after integration, the medium contribution can be written as

$$P_f^{\rm med} = \sum_{f=u}^{d} \sum_{k=0}^{k_{f,\max}} \alpha_k \frac{|q_f|BN_c}{4\pi^2} \left\{ \mu_f \sqrt{\mu_f^2 - s_f(k,B)^2} - s_f(k,B)^2 \ln\left[\frac{\mu_f + \sqrt{\mu_f^2 - s_f(k,B)^2}}{s_f(k,B)}\right] \right\}, \quad (20)$$

where $s_f(k, B) = \sqrt{M^2 + 2|q_f|Bk}$. The upper Landau level (or the nearest integer) is defined by

$$k_{f,\max} = \frac{\mu_f^2 - M^2}{2|q_f|B} = \frac{p_{f,F}^2}{2|q_f|B}. \quad (21)$$

Finally, the term M entering the quark pressure is just the effective self-consistent mass at finite density and in the presence of an external magnetic field:

$$M = m_c + \frac{MGN_cN_f}{\pi^2} \left\{ \Lambda\sqrt{\Lambda^2 + M^2} - \frac{M^2}{2} \ln\left[\frac{(\Lambda + \sqrt{\Lambda^2 + M^2})^2}{M^2}\right] \right\} + \sum_{f=u}^{d} \frac{M|q_f|BN_cG}{\pi^2} \left\{ \ln\{\Gamma[x_f]\} - \frac{1}{2}\ln(2\pi) + x_f - \frac{1}{2}[2x_f - 1]\ln[x_f] \right\} - \sum_{f=u}^{d} \sum_{k=0}^{k_{f,\max}} \alpha_k \frac{M|q_f|BN_cG}{\pi^2} \times \left\{ \ln\left[\frac{\mu_f + \sqrt{\mu_f^2 - s_f(k,B)^2}}{s_f(k,B)}\right] \right\}, \quad (22)$$

where we have used some $\zeta(z, a)$ properties given in Appendix B. Note that our Eq. (16), although obtained in a different fashion, exactly agrees[2] with the result obtained in Refs. [20,31] where the authors have used Schwinger's proper time formalism [35]. To make this article self-contained let us review, in the next section, some standard results obtained for symmetric quark matter.

[2]When one sets $\mu_u = \mu_d = \mu$.

IV. EOS FOR SYMMETRIC MATTER

In this section we use Eq. (16) to reproduce some of the most important results concerning symmetric quark matter at $\mu \neq 0$ and/or $B \neq 0$. The result for this particular case can be readily obtained by setting $\mu_u = \mu_d = \mu$. It is worth emphasizing that, when the magnetic field is turned on, the u and d quark densities are not the same, as it is discussed next. In this sense, both quark chemical potentials are identical, but the densities are not and, hence, matter is not strictly symmetric. In addition to this, we have opted to keep the nomenclature.

A. EOS for symmetric matter at $\mu \neq 0$ and $B = 0$

This case has been extensively discussed in the literature and we refer the reader to Refs. [28,36] for more details.

By setting $B = 0$ one obtains the following relation for the pressure

$$P_f(\mu, 0) = -\frac{[M_\mu - m_c]^2}{4G} - \frac{N_cN_f}{8\pi^2}\left\{M_\mu^4 \ln\left[\frac{(\Lambda + \epsilon_\Lambda)}{M_\mu}\right] - \epsilon_\Lambda\Lambda\left[\Lambda^2 + \epsilon_\Lambda^2\right]\right\} + \frac{N_cN_f}{8\pi^2}\left\{M_\mu^4 \ln\left[\frac{(\mu + p_F)}{M_\mu}\right] + \frac{5\mu}{3}p_F^3 - \mu^3 p_F\right\},$$

where $p_F^2 = \mu^2 - M_\mu^2$. The effective mass, M_μ, satisfies

$$M_\mu = m_c + \frac{M_\mu GN_cN_f}{\pi^2}\left\{\Lambda\epsilon_\Lambda - M_\mu^2 \ln\left[\frac{\Lambda + \epsilon_\Lambda}{M_\mu}\right] - \mu p_F + {M_\mu}^2 \ln\left[\frac{\mu + p_F}{M_\mu}\right]\right\}. \quad (23)$$

The normalized pressure is simply given by $P_{f,\rm eff}(\mu, 0) = P_f(\mu, 0)|_{M_\mu} - P_f(0, 0)|_{M_0}$ where

$$P_f(0, 0) = -\frac{[M_0 - m_c]^2}{4G} - \frac{N_cN_f}{8\pi^2}\left\{M_0^4 \ln\left[\frac{\Lambda + \epsilon_{\Lambda,0}}{M_0}\right] - \Lambda\epsilon_{\Lambda,0}\left(\Lambda^2 + \epsilon_{\Lambda,0}^2\right)\right\}, \quad (24)$$

with $\epsilon_{\Lambda,0} = \sqrt{M_0^2 + \Lambda^2}$. The effective quark mass, M_0, appearing in Eq. (24) is just M_μ evaluated at $\mu = 0$. Namely

$$M_0 = m_c + \frac{M_0GN_cN_f}{\pi^2}\left\{\Lambda\epsilon_{\Lambda,0} - M_0^2 \ln\left[\frac{\Lambda + \epsilon_{\Lambda,0}}{M_0}\right]\right\}. \quad (25)$$

Finally, at $B = 0$ and $\mu \neq 0$, the EOS for the quarks is given by $\mathcal{E}_f(\mu, 0) = -P_{f,\rm eff}(\mu, 0) + \mu\rho(\mu, 0)$, where $\rho(\mu, 0) = dP_f(\mu, 0)/d\mu$ is the mean-field result

$$\rho(\mu, 0) = \frac{N_cN_f}{3\pi^2}\left[\mu^2 - M_\mu^2\right]^{(3/2)} = \frac{N_cN_f}{3\pi^2}p_F^3. \quad (26)$$

We can also define the *bag constant*

$$\mathcal{B} = P_{f,\rm eff}(\mu, 0)|_{m_c} - P_{f,\rm eff}(\mu, 0)|_{M_\mu}. \quad (27)$$

As emphasized in Ref. [28] one should remark that, in the same way as in the bag model, $\mathcal{B}$ describes the pressure difference between the trivial and nontrivial vacuum, but it is not an input of the model, being a dynamical consequence of the interactions that leads to vacuum masses $M(0,0) \neq m_c$. Using our chosen values for G and Λ one obtains $\mathcal{B}$ =(181 MeV)^4. Regarding CSB, one obtains the quark effective mass, $M \simeq 400$ MeV for $\mu = 0$ and observes a first-order transition for chiral symmetric matter at $\mu_c \simeq 360$ MeV. A quantity of particular interest for the present work is the energy per baryon as a function of the density. The relations obtained above for the $B = 0$ case will allow us to discuss, at the end of this section, the influence of the magnetic field regarding the stability of quark matter.

B. EOS for symmetric matter at $\mu = 0$ and $B \neq 0$.

In this subsection we set $\mu = 0$ and concentrate in the behavior of the quark condensate under the influence of an external magnetic field. One of the most remarkable effects of this external field regards its role so as to enhance chiral symmetry breaking. This issue is related to the phenomenon known as magnetic catalysis that has been well exploited by Klimenko and collaborators among others [20,31]. Going back to our general relation for the pressure, Eq. (16), and setting $\mu_u = \mu_d = \mu = 0$, one obtains that the relevant pressure is given by

$$P_f(0,B) = -\frac{[M_B - m_c]^2}{4G} + \left[P_f^{\rm vac} + P_f^{\rm mag}\right]_{M_B}, \qquad (28)$$

with the effective quarks mass at $\mu = 0$ and $B \neq 0$, M_B, being determined by

$$\begin{aligned} M_B = m_c &+ \frac{M_B G N_c N_f}{\pi^2}\left\{\Lambda\sqrt{\Lambda^2+M_B^2} - \frac{M_B^2}{2}\ln\left[\frac{\left(\Lambda+\sqrt{\Lambda^2+M_B^2}\right)^2}{{M_B}^2}\right]\right\} \\ &+ \sum_{f=u}^{d}\frac{M_B|q_f|BN_cG}{\pi^2}\left\{\ln\Gamma[x_f] - \frac{1}{2}\ln(2\pi) + x_f - \frac{1}{2}[2x_f-1]\ln x_f\right\}, \end{aligned} \qquad (29)$$

where now $x_f = M_B^2/(2|q_f|B)$ [see Eq. (22)].

Figure 1 shows the behavior of the effective quark mass as a function of B indicating that the former increases (although slightly) with the latter that stabilizes the condensate so that the gap equation always has a nontrivial solution for finite B. As pointed out in Ref. [36] the B field facilitates the binding by antialigning the helicities of the quark and the antiquark, which are then bound by the NJL interaction while an electric field, E, has an opposite effect opposing condensate formation by polarizing the $\bar{\psi}_f \psi_f$ pairs.

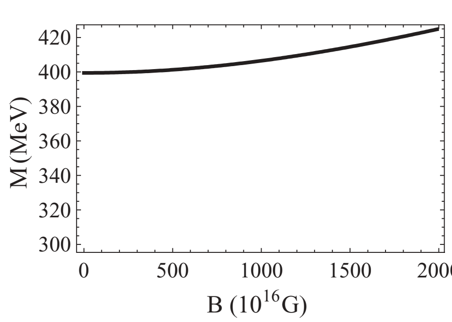

FIG. 1. Quark effective mass as a function of B showing how the latter enhances CSB (magnetic catalysis).

C. EOS for symmetric matter at $\mu \neq 0$ and $B \neq 0$

In this subsection, still considering only the symmetric case, we shall review how B stabilizes quark matter due to magnetic catalysis, according to Ref. [20]. For this we can consider the general relation, Eq. (16) with $\mu_u = \mu_d = \mu$, from which the quark density is easily extracted as

$$\rho_f(\mu,B) = \sum_{f=u}^{d}\sum_{k=0}^{k_{f,\max}} \alpha_k \frac{|q_f|BN_c}{6\pi^2} k_{F,f}(k,s_f), \qquad (30)$$

where $k_{F,f}(k,s_f) = \sqrt{\mu^2 - s_f(k,B)^2}$ while the effective pressure is defined by

$$P_{f,\rm eff}(\mu,B) = P_f(\mu,B)|_M - P_f(0,B)|_{M_B}. \qquad (31)$$

Then, the energy density follows as

$$\mathcal{E}_{\beta f}(\mu,B) = -P_{\beta f,\rm eff}(\mu,B) + \mu\rho_f(\mu,B). \qquad (32)$$

In all the above equations M is given by Eq. (22) with the obvious substitution $\mu_u = \mu_d = \mu$. In Fig. 2 we present some results which compare the energy per baryon ($\mathcal{E}/\rho_B$) as a function of the density for $B = 0$ and $B = 2 \times 10^{19}$G.

It is shown that for $B = 0$ the curve has an absolute minimum at $\rho_B \simeq 2\rho_0$ and that at $\rho_B = 0$ there is a local minimum corresponding to the real empty physical QCD vacuum. This fact indicates that, in this case, the vacuum is metastable but the situation changes drastically in the presence of an external magnetic field and the graph has a maximum at $\rho_B = 0$ so that at arbitrarily small densities stable quark droplets appear in the system. One then concludes that the external magnetic field supports the creation of stable quark matter. Later, when discussing our numerical results, in Sec. VI, we again address some issues related to the symmetric case. As one can see, in Fig. 2, the minima obtained for both cases have different values. The case of symmetric magnetized quark matter has a minimum at a density lower than the one obtained

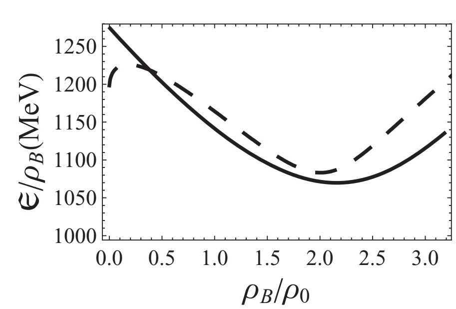

FIG. 2. Energy per baryon as a function of the baryonic density for symmetric quark matter. Continuous line $B = 2 \times 10^{19}$ G, dashed line $B = 0$.

for symmetric quark matter. The concavity of the curve in that case is also greatest.

Now, after reviewing some of the most important issues regarding how symmetric quark matter is affected by the presence of an external magnetic field we can incorporate β equilibrium in our results to consider stellar matter.

V. ASYMMETRIC QUARK MATTER WITH β EQUILIBRIUM IN A MAGNETIC FIELD

In a star with quark matter we must impose both, β equilibrium and charge neutrality [37]. Through out this article we only consider the latest stage in the star evolution, when entropy is maximum and neutrinos have already diffused out. The neutrino chemical potential is then set to zero. For β-equilibrium matter we must add the contribution of leptons (electrons and muons) in a magnetic field to the energy density and pressure. The relations between the chemical potentials of the different particles are given by

$$\mu_d = \mu_u + \mu_e, \quad \mu_e = \mu_\mu. \tag{33}$$

For charge neutrality we must impose

$$\rho_e + \rho_\mu = \tfrac{1}{3}(2\rho_u - \rho_d). \tag{34}$$

Now, Eqs. (33) and (34) will have to be satisfied together with the self-consistent relations for the quark effective mass during the numerical evaluations. The EOS for the leptonic sector is also needed. Let us now obtain this EOS by recalling that, as emphasized in the introduction, the total leptonic pressure, $P_l(\mu_l, B)$, is quickly recovered from the quark pressure, $P_f(\mu_f, B)$, on performing obvious replacements such as $f \to l$ and $N_c = 1$. Also, because the leptonic sector does not have the analog of the quartic interaction, $G \to 0$, so that when translating the results one takes $M \to m_c \to m_l$ that lead to

$$P_l(\mu_l, B) = \left[P_l^{\rm vac} + P_l^{\rm mag} + P_l^{\rm med}\right]_{m_l}. \tag{35}$$

Being evaluated at the same (one loop) approximation level all terms have exactly the same mathematical structure as those in $P_f(\mu_f, B)$ (including the divergences in the vacuum contribution). A major difference is that all terms in Eq. (14) are evaluated with the bare m_l, reflecting the fact that an undressed lepton propagator has been used. At this one loop level of approximation the leptonic contribution is that of a free gas of relativistic fermions and the divergences contained in the vacuum contribution can be properly absorbed with a zero point subtraction. However, note that according to our normalization procedure we require $P_l(\mu_l, B) = 0$ at $\mu_l = 0$ as in the quark case that leads to the following effective pressure for the leptonic sector

$$P_{l,\rm eff}(\mu_l, B) = [P_l(\mu_l, B) - P_l(0, B)]_{m_l} = P_l^{\rm med}, \tag{36}$$

because $P_l(0, B) = P_l^{\rm vac} + P_l^{\rm mag}$ and all these quantities are written in terms of the μ_l-independent mass, m_l. The result shows that, at the one loop level, only the following (finite) medium contribution has to be considered

$$P_{l,\rm eff}(\mu_l, B) = \sum_{l=e}^{\mu}\sum_{k=0}^{k_{l,\max}} \alpha_k \frac{|q_l|B}{4\pi^2}\left\{\mu_l\sqrt{\mu_l^2 - s_l(k,B)^2} - s_l(k,B)^2 \ln\left[\frac{\mu_l + \sqrt{\mu_l^2 - s_l(k,B)^2}}{s_l(k,B)}\right]\right\}. \tag{37}$$

Then, the leptonic density is easily evaluated, yielding

$$\rho_l(\mu_l, B) = \sum_{l=e}^{\mu}\sum_{k=0}^{k_{l,\max}} \alpha_k \frac{|q_l|B}{2\pi^2} k_{F,l}(k, s_l), \tag{38}$$

where $k_{F,l}(k, s_l) = \sqrt{\mu_l^2 - s_l(k,B)^2}$. Finally, the leptonic energy density reads

$$\mathcal{E}_l(\mu_l, B) = -P_{l,\rm eff}(\mu_l, B) + \mu_l \rho_l(\mu_l, B), \tag{39}$$

where, again, a sum over the repeated (l) indices is implied. Finally, the total effective pressure corresponding to the theory described by $\mathcal{L}_{\beta f}$ in the presence of a constant external magnetic field, B, is

$$P_{\beta f,\rm eff}(\mu_f, \mu_l, B) = P_{f,\rm eff}(\mu_f, B) + P_{l,\rm eff}(\mu_l, B). \tag{40}$$

We now have all the ingredients to evaluate the EOS in β equilibrium because

$$\mathcal{E}_{\beta f}(\mu_f, \mu_l, B) = -P_{\beta f,\rm eff}(\mu_f, \mu_l, B) + \mu_l \rho_l + \mu_f \rho_f, \tag{41}$$

where the quark and leptonic densities are given by Eqs. (30) and (38), respectively.

VI. RESULTS AND DISCUSSION

We start by discussing, in more detail, the effects of the magnetic field on the EOS of symmetric matter described by the usual SU(2) version of the NJL model without β equilibrium whose inclusion will be addressed afterward. Whenever mentioned in the figures, $B_0 = 10^{19}$ G. By symmetric matter we mean ud matter for which the chemical potential of both particles are equal. For a zero magnetic field this is equivalent to having symmetric matter. For finite magnetic fields due to the different electrical charge of both quarks and the appearance of Landau levels, ud is symmetric only for restricted densities.

In the next four figures, i.e., Figs. 3(a), 3(b), 4(a), and 4(b), the results for $B = 0.2 \times 10^{19}$ G are almost coincident with the results for magnetic free matter. They are kept so that minor differences can be seen.

In Figs. 3(a) and 3(b) we compare the effective quark mass and the baryonic density as a function of the chemical potential for matter subject to different values of the magnetic field. One can see that a magnetic field of the order of 0.2×10^{19} G barely affects the effective mass as compared with the results for ordinary matter (not subject to the magnetic field). Due to the Landau quantization, the increase of the strength of the magnetic field provokes a decrease of the number of the

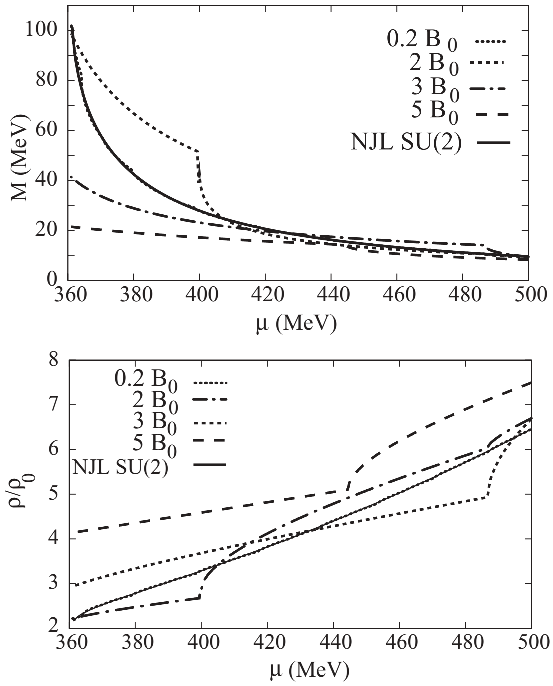

FIG. 3. (a) Effective quark mass and (b) baryonic density as a function of the chemical potential for matter subject to different values of the magnetic field. ρ_0 was chosen as 0.17 fm^{-3} and $B_0 = 10^{19}$ G.

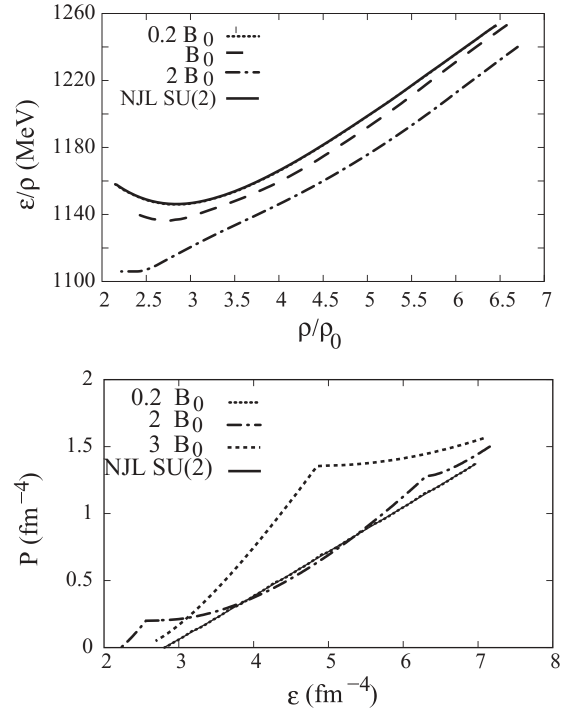

FIG. 4. (a) Binding energy and (b) EOS for different values of the magnetic field for ud matter with equal chemical potentials. $B_0 = 10^{19}$ G.

filled LL and the amplitude of the oscillations is more clear in the graphics. For each value of the magnetic field, the kink appearing at the smallest chemical potential corresponds to the case when only the first LL has been occupied. For $B = 5 \times 10^{19}$ G matter is totally polarized for chemical potentials below 490 MeV. For the small values of the magnetic fields the number of filled LL is quite large and the effects of the quantization are less visible. For the larger magnetic fields the chiral symmetry restoration occurs for smaller values of the chemical potentials that, however, correspond to larger densities as can be seen from Fig. 3(b).

In Fig. 4(a) one can see that the inclusion of the magnetic field makes matter more and more bound. The energy per baryon E/A of magnetized quark matter described by the SU(2) version of the NJL model is less bound than nuclear matter made of iron nuclei, $\frac{E}{A}|_{^{56}\mathrm{Fe}} \sim 930$ MeV, which means that quark matter is not the preferential ground-state matter even in presence of the magnetic fields under consideration [27,38].

In Fig. 4(b) we plot the EOS for different values of the magnetic field. Once again one can see that the magnetic fields modify the EOS but the modifications are more significant when the magnetic fields reach values higher than 0.2×10^{19} G; as can be seen these graphics show also oscillations every time that a different number of LL is filled. As the magnetic field increases the number of LL for a given interval of energy is reduced, increasing the gap between them and making more visible the oscillations, the so-called Haas van Alphen effect [39].

We have identified the densities at which pressure is zero, for different values of the magnetic field. This corresponds to the saturation density of symmetric quark matter when $B = 0$ or to $\mu_u = \mu_d$ otherwise. In particular, for $B = 0.2 \times 10^{19}$ G, the density is 1.44 fm^{-3} and for $B = 2 \times 10^{19}$ G, the density is 1.19 fm^{-3}, and for $B = 3 \times 10^{19}$ G, the density is around 2.7 fm^{-3}, i.e., the saturation density decreases as the magnetic field increases for magnetic fields lower than $B = 2 \times 10^{19}$ G. For larger fields the opposite may occur. The minimum of the binding energy is the result of two contributions with different behavior: on one hand, due to Landau quantization, the kinetic energy decreases with the increase of the magnetic field and, on the other, the effective bag parameter defined by the interaction terms minus the vacuum energy increases because the restoration of chiral symmetry occurs at larger densities. The saturation density depends on how fast each contribution changes with density.

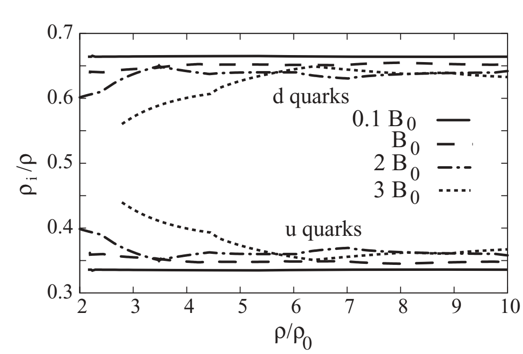

FIG. 5. Quark population (ρ_i/ρ, $i = d, u$ in terms of density for $B = 0.1 B_0$ B_0, $2B_0$, and $3B_0$, with $B_0 = 10^{19}$ G, respectively, for d quarks from top to bottom and u quarks from bottom to top.

TABLE I. Baryon densities of β-equilibrium matter for which the pressure becomes negative, for the NJL model, and for the MIT model with the bag pressure (180 MeV)^4. In both models only u and d quarks were considered.

B (G)	0	$0.1B_0$	B_0	$2B_0$	$3B_0$
$\rho_{\rm NJL}$ (fm^{-3})	0.40	0.47	0.48	0.44	0.54
$\rho_{\rm MIT}$ (fm^{-3})	0.54	0.47	0.45	0.36	0.31

We now study quark matter in β equilibrium, the original motivation of our studies aimed at understanding the constitution of magnetars. In what follows, NJL SU(3) refers to the EOS of quark matter with u, d, s quarks and parameters used in Ref. [11] and NJL SU(2) refers to ud quark matter. In both cases, matter is not subject to a magnetic field. The designation [NJL SU(2)]$_B$ refers to ud quark matter subject to a magnetic field. The reader should keep in mind that here the comparison with the SU(3) case has only a qualitative character because both the SU(2) and SU(3) versions employ different parametrization sets that prevent one from drawing any conclusion related to quantitative aspects. Just as in symmetric matter, as well as for matter in β equilibrium, the effects of Landau quantization are clearly seen in the EOS [see Fig. 6(a)].

The strange quark matter within the MIT bag has been discussed in Refs. [33,40]. Once the s quarks are removed, so that only u, d matter is considered one can see that although the overall behavior is similar, there are clear differences between both models. The zero pressure density decreases with the increase of the magnetic field for the MIT model while there is no clear trend for the NJL. These densities are given in Table I, where only u and d quarks are taken into account. For quark stars this density defines the density at the surface. We should remember, however that it is only a field above 10^{19} G that has noticeable effects and, at the surface, we expect much smaller fields.

As for hadronic stars, the density is zero for a null pressure at the surface if no magnetic field is included. The magnetic field gives rise to an increase of the density in the surface [3,5]. In this sense, the NJL model predicts a more similar behavior, i.e., an increase of the density at zero pressure as a result of the magnetic field, whereas the MIT model predicts the opposite.

The saturation density is defined by the way chiral symmetry is restored: A chiral symmetry restoration at smaller energies implies a smaller saturation density. As discussed, the saturation in the NJL model is a balance between the attractive scalar field that saturates at chiral symmetry restoration and the kinetic term that gains importance once chiral symmetry restoration has occurred.

The conditions of charge neutrality and β equilibrium give rise to different chemical potentials, and therefore different quark fractions, as seen in Fig. 5. Although for $B = 0$ β-equilibrium matter is formed essentially by two-thirds of d quarks and one-third of u quarks and only a very small fraction of electrons, the presence of the magnetic field changes the particle fractions. Matter becomes more symmetric for a finite magnetic field larger than $\sim 10^{19}$ G, and a larger fraction of electrons and muons occurs. In Fig. 5 the lepton population is

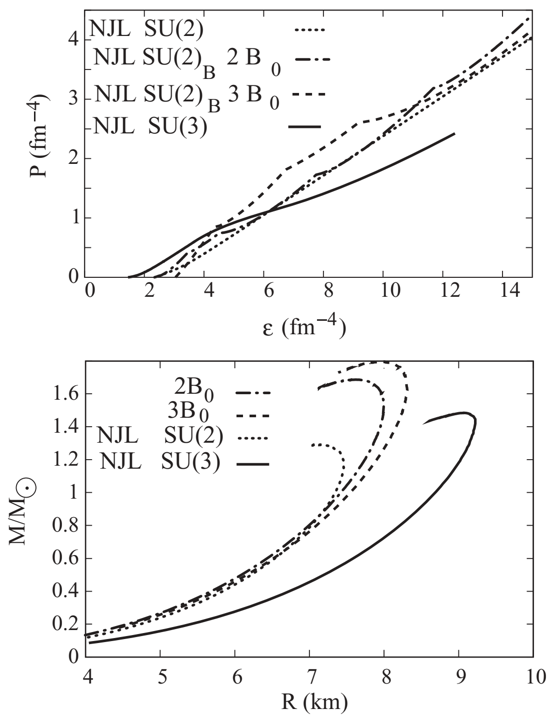

FIG. 6. (a) EoS for quark matter in β equilibrium and (b) resulting stellar properties for the SU(2) version of the NJL model both with and without magnetic field and the SU(3) version without magnetic field. In the figures, $B_0 = 10^{19}$ G.

not shown so that the effects of the magnetic field on the quark population can be better noticed.

In Fig. 6(a) we show the EOS for β-equilibrium quark stellar matter. We compare the SU(2) and SU(3) NJL EOS for magnetic free matter with the EOS for $B = 2 \times 10^{19}$ G and $B = 3 \times 10^{19}$ G. In the last two EOS the effect of the LL quantization is clearly seen. Both EOS are harder than the corresponding $B = 0$ EOS.

Notice that, contrary to hadronic matter in β-equilibrium, where the magnetic field makes the EOS softer [3,5], the NJL model predicts a slightly harder EOS, as seen in Fig. 6(a).

To obtain the properties of the stars described by these EOS, we have added the contribution of the magnetic field to the pressure and to the energy density, $B^2/2$. As discussed in Sec. II the formal consideration of this term is related to the prescription adopted to normalize the pressure. At the surface the magnetic field should be not larger than $\sim 10^{15}$ G and, therefore, have introduced a density-dependent magnetic field

$$B(\rho) = B_s + B_i[1 - \exp[-\alpha(\rho/\rho_0)^\gamma]],$$

where ρ_0 is the saturation density, $B_s = 10^{15}$ G is the magnetic field at the surface, B_i is the magnetic field at the interior for large densities, and the parameters $\alpha = 5 \times 10^{-5}$ and $\gamma = 3$ were chosen in such a way that the field increases fast with density to its central value but still describes correctly the surface, namely with a zero pressure.

The properties of compact stars were obtained from the integration of the Tolman-Oppenheimer-Volkoff equations, using the EOS obtained with the density-dependent magnetic

TABLE II. Quark star properties for the EOSs described in the text.

Type	B_i Gauss	$M_{\max}$ ($M_\odot$)	$M_{b\max}$ ($M_\odot$)	R (km)	ε_0 (fm^{-4})	B (10^{19} G)
NJL SU(3) [11]	0	1.47	1.56	9.02	7.52	0
NJL$_{Gv}$ SU(2)	0	1.60	1.44	8.17	10.7	0
NJL SU(2)	0	1.29	1.25	7.11	13.5	0
(NJL SU(2))$_B$	2×10^{19}	1.69	1.55	7.63	12.0	0.62
(NJL SU(2))$_B$	3×10^{19}	1.80	1.60	7.97	10.8	0.63

field and that includes the magnetic field contribution. For $B = 3 \times 10^{19}$ G the magnetic field contribution is as large as the contribution of stellar matter. This is seen from the properties of the star with maximum mass: the gravitational mass becomes larger than the baryonic mass (see Table II).

The SU(3) EOS is also included. It becomes softer after the onset of the s quark, which occurs at a quite high density in this model [41]. The SU(3) version of the NJL model is obviously more complete and the chiral symmetry restoration of the s quark produces a smooth change of declination in the EOS around 6 fm^{-4} as can be seen in Fig. 6. As this energy density is very high, the strangeness content of the star is small. The comparisons with the SU(3) NJL version has to be considered with care because this version of the model was fitted to a different set of variables, which, as we can see from Fig. 6, describe stellar matter (below 5 fm^{-4}) in a different way from the SU(2) version.

The results of the integration of the TOV equations are shown in Table II and in Fig. 6(b). We do not take into account the anisotropy introduced by the magnetic field [42] and have assumed a spherical configuration. Compact stars with poloidal magnetic fields were studied in Ref. [43].

One can see that the results for the masses and radii of the maximum mass stable configuration obtained with the inclusion of the magnetic field equal to 2×10^{19} G and 3×10^{19} G in the SU(2) version of the NJL model is larger than the corresponding SU(2) magnetic free star. The radius is still quite small, below 8 km, much smaller than the average neutron star value, ∼12–15 km but the mass is larger than the value 1.4 $M_\odot$ that many neutron stars have. As previously mentioned, the gravitational mass for the two cases including the magnetic field is larger than the baryonic mass due to the inclusion of the $B^2/2$ term in the EOS.

Finally, let us analyze how a vector-isoscalar term, present in Eq. (C1), influences our results. In this case one adds a term like $-G_V(\bar\psi\gamma^\mu\psi)^2$ to the standard NJL Lagrangian density, $\mathcal{L}_f$, given by Eq. (2). In the following discussion regarding the influence of this term we do not have to consider the presence of the magnetic field that we now neglect. In Appendix C we summarize how the vector term can be treated within our calculational framework. The net result of these manipulations is to modify the standard NJL quark pressure, $P_f(\mu_f)|_{M(\mu_f)}$, to

$$P_f(\tilde\mu_f)|_{M(\tilde\mu)} + \frac{(\mu_f - \tilde\mu_f)^2}{4G_V}, \tag{42}$$

where we have used the effective chemical potential

$$\tilde\mu_f = \mu_f - G_V \frac{2p_{f,F}^3}{3\pi^2}, \tag{43}$$

where $p_{f,F} = \sqrt{\tilde\mu_f^2 - M^2}$. The effective mass and the effective chemical potential have to be determined simultaneously in a self-consistent way. In Table II there is a line with the results of this extended NJL model that we refer to as NJL$_{Gv}$ SU(2). We have used $G_v = G/2$. Indeed, it gives a better result for the maximum mass of the stars than the ordinary NJL SU(2) model, as had already been shown in Ref. [29]. When $B \neq 0$ the effective chemical potential depends on it as the reader can be convinced by looking at the momentum integrals displayed in Appendix C on recalling the relevant substitutions presented in Sec. III. The modifications introduced by the magnetic field in the NJL$_{Gv}$ SU(2) model go along the same lines already discussed for the NJL SU(2).

VII. CONCLUSIONS

The role of the magnetic field effects on quark matter has an importance of its own and it has already been exploited in the literature. One of the conclusions is that the consequences of Landau quantization are not negligible for large values of the magnetic field. Quark matter is generally described by the MIT bag model. In this work we tackle the inclusion of the magnetic field in quark matter described by the SU(2) version of the NJL model, known to have more realist features.

Using the standard large-N_c technique we have evaluated the effective potential for the SU(2) version of the NJL model in 3+1 dimensions that has allowed us to obtain its thermodynamical potential at finite density and in the presence of an external magnetic field. Then, we have reviewed some of the most important well-known results related to three different situations: (a) $\mu \neq 0$ and $B = 0$ (chiral symmetry breaking/restoration), (b) $\mu = 0$ and $B \neq 0$ (magnetic catalysis), and (c) $\mu \neq 0$ and $B \neq 0$ (magnetic induced quark matter stability). The quantity E/A, for magnetized quark matter described by NJL SU(2) model, has a minimum that is lower than the one determined for magnetic free quark matter. We have also obtained that a magnetic field of the order of 0.2×10^{19} G barely affects the effective mass as compared with the results for matter not subject to the magnetic field and for $B = 5 \times 10^{19}$ G matter is totally polarized for chemical potentials below 490 MeV. For small values of the magnetic fields the number of filled LL is large and the quantization effects are washed out, whereas for large magnetic fields the chiral symmetry restoration occurs for smaller values of the chemical potentials.

To introduce β equilibrium, we have extended the thermodynamical potential so as to consider leptonic contributions. Our numerical results show that, for the SU(2) case, only very high magnetic fields ($B \geqslant 10^{18}$ G) affect the EOS in a noticeable way. Although for quark matter in β equilibrium, the densities at which the pressure is zero increase with the inclusion of the magnetic field, the opposite happens in matter not subject to β equilibrium.

Regarding the TOV equation results, we can see that the SU(2) version with magnetic field provides masses and radii for the maximum mass stable configuration larger than the analogous magnetic free configuration. The radii are quite lower than the corresponding one obtained within the magnetic free SU(3) NJL. This is mainly due to the softness of the SU(2) EOS at the lower densities. The mass-radius results were obtained with a density-dependent magnetic field, which increases from the surface to the interior of the star. The masses of the maximum mass stable configuration for fields of the order or larger than 10^{19} G are larger than 1.4 $M_\odot$ and much larger than the analogous magnetic free configuration, due to the contribution of the magnetic field. We have also discussed how the modifications introduced by the magnetic field in the presence of a vector channel follow the same trend observed when this term is turned off.

The inclusion of the magnetic field in the SU(3) version of the NJL is not trivial and is currently under way. We expect to obtain larger results for the radius. Knowing that the effects of the anomalous magnetic moments is very relevant we also intend to take them into account in the next calculations. The alternative theory, described by the Polyakov–Nambu–Jona-Lasinio [44], can also be investigated in the presence of magnetic fields.

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APPENDIX A: SUMMING MATSUBARA FREQUENCIES AND RELATED FORMULAS

In this appendix we give the results for the main integrals and Matsubara sums appearing along the text. The Matsubara sums that are relevant for the different integrals considered in our work can be derived as (see, e.g., Ref. [45]):

$$T\sum_{n=-\infty}^{+\infty}\ln\left[(\omega_n - i\mu)^2 + E_p^2\right] = E_p + T\ln\{1+e^{-[E_p+\mu]/T}\} + T\ln\{1+e^{-[E_p-\mu]/T}\}, \quad \text{(A1)}$$

which, as $T\to 0$, becomes

$$\lim_{T\to 0} T\sum_{n=-\infty}^{+\infty}\ln\left[(\omega_n - i\mu)^2 + E_p^2\right] = E_p + [\mu - E_p]\theta(\mu - E_p) = \max(E_p,\mu). \quad \text{(A2)}$$

APPENDIX B: EVALUATION OF DIVERGENT INTEGRALS

Consider the divergent term

$$\frac{N_c}{2\pi}\sum_{f=u}^{d}\sum_{k=0}^{\infty}(2-\delta_{k0})(|q_f|B)\int_{-\infty}^{\infty}\frac{dp_z}{(2\pi)}E_{p,k}(B), \quad \text{(B1)}$$

appearing in Eq. (14). By adding and subtracting a LLL term to it one gets

$$\frac{N_c}{\pi}\sum_{f=u}^{d}\sum_{k=0}^{\infty}(|q_f|B)\int_{-\infty}^{\infty}\frac{dp_z}{(2\pi)}\left[E_{p,k}(B) - \frac{E_{p,0}(B)}{2}\right]. \quad \text{(B2)}$$

Now, the integrals can be performed in the following way. We change dimensions from $1\to d = 1-\epsilon$ using the standard dimensional regularization formula [46]

$$\int_{-\infty}^{\infty}\frac{d^d q}{(2\pi)^d}[q^2+M^2]^{-A} = \frac{\Gamma[A-d/2]}{(4\pi)^{d/2}\Gamma[A](M^2)^{A-d/2}}, \quad \text{(B3)}$$

obtaining, after defining $x = M^2/(2qB)$,

$$\frac{2N_c}{\pi}\sum_{f=u}^{d}\sum_{k=0}^{\infty}(|q_f|B)^2\frac{\Gamma[-1+\epsilon/2]}{(4\pi)^{1/2-\epsilon/2}\Gamma[-1/2]} \times\left\{\frac{1}{[k+x]^{-1+\epsilon/2}} - \frac{1}{2x^{-1+\epsilon/2}}\right\}. \quad \text{(B4)}$$

Then, using the definition of the Riemann-Hurwitz ζ function one can get rid of the summation over Landau levels, obtaining

$$-\frac{N_c}{2\pi^2}\sum_{f=u}^{d}(|q_f|B)^2\Gamma[-1+\epsilon/2] \times\left[\zeta(-1+\epsilon/2,x) - \frac{1}{x^{1-\epsilon/2}}\right]. \quad \text{(B5)}$$

After expanding around $\epsilon = 0$ and canceling few terms one has the (still divergent) contribution

$$-\frac{N_c}{2\pi^2}\sum_{f=u}^{d}(|q_f|B)^2 \times\left[\frac{x^2}{\epsilon} + \frac{x^2}{2}(1-\gamma_E) - \frac{x}{2}\ln x - \zeta'(-1,x)\right]. \quad \text{(B6)}$$

One can now get rid of the above divergence by adding and subtracting a vacuum term [see Eq. (FF)]

$$P^{\text{vac}} = 2N_cN_f\int\frac{d^3\mathbf{p}}{(2\pi)^3}[\mathbf{p}^2+M^2]^{1/2}. \quad \text{(B7)}$$

The subtracted term can be conveniently treated by performing a change of variables $\mathbf{p}^2 \to [\mathbf{p}']^2/(2qB)$ and $M^2 \to x = M^2/(2qB)$. Then replacing N_f by $\sum_{f=u}^{d}$ and performing the integration in $d = 3-\epsilon$ dimensions leads to

$$-P^{\text{vac}} = \frac{N_c}{2\pi^2}\sum_{f=u}^{d}(|q_f|B)^2\left[\frac{x^2}{\epsilon} + \frac{x^2}{2}(1-\gamma_E) - \frac{x^2}{2}\ln x + \frac{x^2}{4}\right]. \quad \text{(B8)}$$

Finally, adding all contributions one may write Eq. (B1) as

$$P^{\rm vac} + \frac{N_c}{2\pi^2}\sum_{f=u}^{d}(|q_f|B)^2 \times \left[\zeta'(-1,x) - \frac{1}{2}(x^2-x)\ln x + \frac{x^2}{4}\right], \quad \text{(B9)}$$

where the added quantity, $P^{\rm vac}$, can be evaluated using a sharp cut off reproducing the results quoted in the text, which are in agreement with those of Ref. [20]. Also note that our strategy avoids possible regularization complications introduced if one first performs a proper time type of calculation and then introduces a three-momentum cutoff.

The following relation has also been used in the above derivation [34]

$$\zeta(-1,x) = -\frac{B_2(x)}{2} = -\frac{1}{2}\left[\frac{1}{6} - x + x^2\right], \quad \text{(B10)}$$

where $B_n(x)$ represents the Bernoulli polynomial. Further, to obtain the gap equation for the $B \neq 0$ case the following relations [34] are useful, $d\zeta[0,x]/dz = \ln\Gamma(x) - (1/2)\ln(2\pi)$, $\zeta[0,x] = 1/2 - x$, and $d\zeta[z,x]/dx = -z\zeta[z+1,x]$.

APPENDIX C: INTRODUCTION OF THE VECTOR-ISOSCALAR TERM

In the presence of a vector-isoscalar channel the NJL can be written as

$$\mathcal{L}_f = \bar\psi_f\left[\gamma_\mu(i\partial^\mu - q_f A^\mu) - m_c\right]\psi_f + G\left[(\bar\psi_f\psi_f)^2 + (\bar\psi_f i\gamma_5\vec\tau\psi_f)^2\right] - G_V(\bar\psi_f\gamma^\mu\psi_f)^2. \quad \text{(C1)}$$

One can implement the large-N_c approximation by defining $G_V = \lambda_V/(2N_c)$ and the model can be bosonized with an additional (vector) background field, V^μ, as

$$\mathcal{L}_f = \bar\psi_f(i\,\partial\!\!\!/)\psi_f - \bar\psi_f(\sigma' + i\gamma_5\vec\tau\vec\pi - V_\mu\gamma^\mu)\psi_f - \frac{N_c}{2\lambda}(\sigma^2+\vec\pi^2) + \frac{N_c}{2\lambda_V}V_\mu V^\mu. \quad \text{(C2)}$$

For the new vector field, the Euler-Lagrangian equation of motion yields $V^\mu = -(\lambda_V/N_c)\bar\psi_f\gamma^\mu\psi_f = -2G_V\bar\psi_f\gamma^\mu\psi_f$. Then, after integrating over the fermionic fields one obtains Landau's free energy density in the large-N_c approximation

$$\mathcal{F}_f = \frac{N_c(\sigma^2)}{2\lambda} - \frac{N_c}{2\lambda_V}V_\mu V^\mu + i\int\frac{d^4p}{(2\pi)^4}\mathrm{tr}\ln[p\!\!\!/ - (\sigma' - V_\mu\gamma^\mu)], \quad \text{(C3)}$$

where we have set $\pi = 0$ for simplicity. Now to obtain the thermodynamical potential one must extremize $\mathcal{F}_f$ with respect to the vector (and scalar) field, obtaining

$$\langle V^0\rangle = \frac{\lambda_V}{N_c}i\int\frac{d^4p}{(2\pi)^4}\mathrm{tr}\frac{\gamma^0}{[p\!\!\!/ - M + V_0\gamma^0]}, \quad \text{(C4)}$$

where we have dropped the spatial components so as to preserve the vacuum translational invariance. Restricting ourselves to the $T = 0$ and $B = 0$ case, as discussed in the text, one can take the traces and use the general relation [47]

$$\int\frac{dp_0}{2\pi}\frac{p_0 - i\xi}{(p_0 - i\xi)^2 + E^2} = \frac{i}{2}\mathrm{sgn}(\xi)\theta(|\xi| - |E|) \quad \text{(C5)}$$

to write

$$\langle V^0\rangle = -4G_V N_c N_f\int d^3\vec p(2\pi)^3\theta(p_F - \vec p), \quad \text{(C6)}$$

which is finite and yields

$$\tilde\mu = \mu - G_V\frac{2p_F^3}{3\pi^2}, \quad \text{(C7)}$$

where $p_F = \sqrt{\tilde\mu^2 - M^2}$ as quoted in the text.

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