Coulomb corrections to superallowed β decay in nuclei

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(Received 2 December 2008; published 10 March 2009)

Corrections to the superallowed β decay matrix elements are evaluated in perturbation theory using the notion of the isovector monopole resonance. The calculation avoids the separation into different contributions and thus presents a consistent, systematic, and more transparent approach. Explicit expressions for δ_c as a function of the mass number A are given.

DOI: 10.1103/PhysRevC.79.035502 PACS number(s): 21.10.Hw, 21.10.Sf, 23.40.−s, 24.80.+y

I. INTRODUCTION

One of the recent activities in nuclear structure is the attempt to determine the corrections one has to introduce in the evaluation of the β-decay matrix elements for superallowed transitions in $T = 1$, $T_z = +1$ (or $T_z = -1$) nuclei [1,2]. This is considered to be an important issue because using the measured ft values one can relate these to the u-quark to d-quark transition matrix element V_{ud} in the Cabibbo-Kobayashi-Maskawa (CKM) matrix. In the standard model (SM) this matrix satisfies the unitarity condition, that is, the sum of the squares of the matrix elements in each row (column) is equal to one:

$$V_{ud}^2 + V_{us}^2 + V_{ub}^2 = 1. \tag{1}$$

Departures from 1 may indicate physics not described by the SM.

In order to use the experimental ft values to determine V_{ud} one has to introduce corrections [1–3]. There is a class of important radiative corrections which we will not treat here. Discussions of these can be found abundantly in the literature [1–3]. The second type of correction, that is usually termed as the isospin symmetry breaking term, denoted as δ_c and defined by the following equation:

$$|M_F|^2 = \left|M_F^0\right|^2(1 - \delta_c), \tag{2}$$

where M_F is the physical Fermi matrix element:

$$M_F = \langle\Psi_1|T_+|\Psi_2\rangle. \tag{3}$$

$|\Psi_1\rangle$and $|\Psi_2\rangle$ are the parent and daughter physical states. The symbol M_F^0 stands for the Fermi matrix element obtained in the limit when in the Hamiltonian all the charge-dependent parts are put to zero, and the wave functions are eigenstates of the charge-independent Hamiltonian.

The calculation of δ_c is usually done by breaking it up into several contributions that result from the inclusion of charge-dependent terms in the Hamiltonian of the nucleus. This separation into different types of contributions is model dependent. As pointed out recently [4] the approach taken in a number of studies [1,2] used the notion of analog spin [5], (also called the W-spin), instead of isospin, and this complicates matters because in Eq. (2) the isospin raising operator appears and not the W_+.

In the present approach we start from a charge-independent Hamiltonian so that the matrix element in Eq. (1) is exactly $\sqrt{2T}$ and we then treat the Coulomb force in perturbation theory. In the way we approach the problem there is no need to break up the contribution of the Coulomb interaction into various separate components. All the effects of Coulomb mixing (such as isospin mixing, the change in the radial part of the wave functions, etc.) are taken into account in a single term. (Some aspects of this approach have already been presented in the past [6,7].)

II. COULOMB MIXING

We start by introducing a nuclear charge independent Hamiltonian which does not contain any charge dependent parts H_0. The eigenstates of this Hamiltonian with isospin $T$and T_z will be denoted as $|T, T_z\rangle$ and

$$H_0|T, T_z\rangle = E_T|T, T_z\rangle. \tag{4}$$

The $2T + 1$, components with different T_z values are degenerate.

The action of the isospin lowering and raising operators, T_-, T_+, gives

$$\begin{aligned} T_-|T, T\rangle &= \sqrt{2T}|T, T-1\rangle; \\ T_+|T, T-1\rangle &= \sqrt{2T}|T, T\rangle. \end{aligned} \tag{5}$$

We now add to the charge independent Hamiltonian a charge dependent part $V_{\rm CD}$.

The dominant part in the charge dependent interaction is the charge asymmetric Coulomb force V_C. (While the charge-dependent components of the two-body nuclear force might be important in changing the relative spacing of levels in the analog nucleus its influence on isospin mixing is expected to be small.) In what follows we will deal only with off-diagonal matrix elements of the Coulomb interaction. Because of the long range nature of the Coulomb force, the prevailing part will be in such cases the one-body part.

As a good approximation we take the potential of a uniformly charged sphere.

Inside the sphere ($r \leqslant R$)

$$V_C(r) = -\frac{Ze^2}{R^3}\sum_i\left(\frac{1}{2}r_i^2 - \frac{3}{2}R^2\right)\left(\frac{1}{2} - t_z(i)\right). \tag{6}$$

Of interest to us here is the isovector part of the potential. Any off-diagonal matrix element between two states of the isovector part is

$$\langle 0|V_C|n\rangle = \frac{Ze^2}{2R^3}\langle 0|\sum_i r_i^2 t_z(i)|n\rangle \equiv \frac{Ze^2}{2R^3}\langle 0|M_0^{(1)}|n\rangle, \quad (7)$$

where $M_0^{(1)}$ denotes the z-component of the isovector monopole operator.

It is obvious that if the state $|n\rangle$ is the giant isovector monopole state (IVMS) [8], that is the state obtained by acting with the $M_0^{(1)}$ operator on the ground state $|0\rangle$ and normalizing, then the above matrix element will be proportional to $\langle 0|M_0^{(1)}M_0^{(1)}|0\rangle$ and thus will exhaust the isovector part ($r^2 t_z$) of the Coulomb sum rule [8].

We will now find in perturbation theory the effect of the charge-dependent part on the wave functions of the two members of the isomultiplet, $|T, T\rangle$ and $|T, T-1\rangle$,

$$\Psi_1 = (|T,T\rangle + \varepsilon_T|M_{T,T}\rangle + \varepsilon_{T+1}|M_{T+1,T}\rangle)N_1^{-1}, \quad (8a)$$

$$\Psi_2 = (|T,T-1\rangle + \eta_{T-1}|M_{T-1,T-1}\rangle + \eta_T|M_{T,T-1}\rangle + \eta_{T+1}|M_{T+1,T-1}\rangle)N_2^{-1}, \quad (8b)$$

where $|M_{T',T'_z}\rangle$, are the T', T'_z components of the isovector monopole, and where

$$N_1 = \sqrt{1+\varepsilon_T^2+\varepsilon_{T+1}^2} \quad \text{and}$$
$$N_2 = \sqrt{1+\eta_{T-1}^2+\eta_T^2+\eta_{T+1}^2}. \quad (9)$$

The admixtures are given in perturbation theory by the equations

$$\varepsilon_i = \frac{\langle T,T|V_C^{(1)}|M_{T+i,T}\rangle}{E_{M_{T+i,T}} - E_0}, \quad i = 0, 1, \quad (10)$$

where E_0 is g.s. energy in this nucleus,

$$\eta_i = \frac{\langle T,T-1|V_C^{(1)}|M_{T+i,T-1}\rangle}{E_{M_{T+i,T-1}} - E_1}, \quad i = -1, 0, 1. \quad (11)$$

Here E_1 is the energy of the analog state.

One can write these as

$$\varepsilon_i = \langle T,T,1,0T+i,T\rangle\langle T+i\|V_C^{(1)}\|T\rangle\big/\big(E_{M_{T+i,T}} - E_0\big), \quad (12)$$

$$\eta_i = \langle T,T,1,0T+i,T-1\rangle\langle T+i\|V_C^{(1)}\|T\rangle\big/\big(E_{M_{T+i,T-1}} - E_1\big). \quad (13)$$

The first bracketed expression is the Clebsch-Gordan (CG) coefficient, while the second bracket is the reduced matrix element. The differences between the reduced matrix elements for different isospin components are of the order of $(N-Z)/A$. (See [8] and references therein.) The reduced matrix elements for large excess neutron (proton) nuclei are such that the components with lower isospin have larger values [8]. However we will deal here with two-nucleon excess nuclei ($T = 1$ states) and in this case the differences are very small (especially for the heavier nuclei) by only a few percent, so we will assume that the various reduced matrix elements are equal. The energy denominators are the excitations of the isovector monopole components in the parent and daughter nuclei either with respect to the ground state or the analog state. We will denote these energies as ΔE_M^i.

The $T + i$ components are split by the symmetry potential:

$$V_s = \frac{V_1}{A}(\vec{t} \bullet \vec{T}), \quad (14)$$

where $\vec{t}$ is the isospin operator of the isovector excitation.

For the various components of the monopole excitation,

$$E_{T+i}^s = \frac{V_1}{A}[(T+i)(T+i+1) - T(T+1) - 2] \quad (15)$$

and

$$\Delta E_M^i = \xi\hbar\omega + E_{T+i}^s, \quad (16)$$

where $\hbar\omega = 41A^{-1/3}$ MeV and ξ is a numerical factor which depends on the model used to describe the isovector monopole. The range of values for this parameter is between 3 and 4 [8].

Introducing the values of the CG coefficients and the notation u for the reduced matrix element, denoting $\kappa = \frac{2V_1}{\xi\hbar\omega A}$ we can write

$$\varepsilon_0 = -\sqrt{\frac{T}{T+1}}\frac{u}{\xi\hbar\omega}\frac{1}{(1-\kappa)}, \quad (17)$$

$$\varepsilon_1 = \sqrt{\frac{1}{T+1}}\frac{u}{\xi\hbar\omega}\frac{1}{(1+T\kappa)}, \quad (18)$$

$$\eta_{-1} = -\sqrt{\frac{2T-1}{T(2T+1)}}\frac{u}{\xi\hbar\omega}\frac{1}{[1-(T+1)\kappa]}, \quad (19)$$

$$\eta_0 = -\frac{T-1}{\sqrt{T(T+1)}}\frac{u}{\xi\hbar\omega}\frac{1}{(1-\kappa)}, \quad (20)$$

$$\eta_1 = \sqrt{\frac{4T}{(2T+1)(T+1)}}\frac{u}{\xi\hbar\omega}\frac{1}{(1+T\kappa)}. \quad (21)$$

The matrix element of interest here is

$$\langle\Psi_1|T_+|\Psi_2\rangle = \sqrt{2T}\left(1+\varepsilon_0\eta_0+\varepsilon_1\eta_1\sqrt{\frac{2T+1}{T}}\right)N_1^{-1}N_2^{-1}, \quad (22)$$

where

$$N_1 = \left(1+\varepsilon_0^2+\varepsilon_1^2\right)^{1/2}, \quad (23a)$$

$$N_2 = \left(1+\eta_{-1}^2+\eta_0^2+\eta_1^2\right)^{1/2}. \quad (23b)$$

It is worthwhile at this point to note that in case of complete degeneracy of the $T + i$ components of the isovector monopole state (that is when $V_1 = 0$) the result for the physical matrix element in Eq. (1) will be $\sqrt{2T}$, the same as in the case when there is no charge dependent perturbation. This occurs actually when isospin mixing is not zero, and not necessarily negligible. It happens because the parent and daughter states can be related to each other via the action of the T_+ operator in spite of the fact that the Hamiltonian is not charge-independent.

We will apply this formalism to the $T_z = \pm 1$ nuclei for which experimental results exist for a number of nuclei. The value of $V_1 \approx 100$ MeV. The value of κ is smaller than 1 even in nuclei as light as ^{10}C and it becomes very small for $A = 50$.

We can express all the above coefficients in terms of one, and we choose ε_1. Neglecting terms quadratic (or higher powers) in κ, we calculate the matrix element in Eqs. (17)–(23) and arrive, after some algebra, at the expression

$$\langle\Psi_1|T_+|\Psi_2\rangle^2 = 2T\left(1-2(T+1)\frac{V_1}{\xi\hbar\omega A}\varepsilon_1^2\right)^2. \quad (24)$$

Therefore to order ε_1^2, using the definition in Eq. (2) we find

$$\delta_c = 4(T+1)\frac{V_1}{\xi\hbar\omega A}\varepsilon_1^2 \quad (25)$$

and using the expression for $\hbar\omega$ in MeV,

$$\delta_c = 4(T+1)\frac{V_1}{41\xi A^{2/3}}\varepsilon_1^2. \quad (26)$$

ε_1^2 is the isospin admixture of the $T+1$ state into the T, $T_z = T$ ground state. This is commonly defined as isospin impurity. Note that $\delta_c < \varepsilon_1^2$ because the coefficient in front of ε_1^2 for the nuclei considered is less than 1. For large T there is the factor $T+1$ that enhances the value but it is cancelled by the factor $(T+1)^{-1}$ in ε_1^2 [8].

For $T = 1$ nuclei

$$\delta_c = 8\frac{V_1}{41\xi A^{2/3}}\varepsilon_1^2. \quad (27)$$

We will now use different models for ε_1^2 introduced in the past and presented in Ref. [8]. We will use the value $\xi \approx 3$ and take $V_1 = 100$ MeV. Of course as we mentioned some models predict $\xi > 3$ and the value of V_1 is not well determined and values lower than 100 MeV are also used sometimes. However both uncertainties will not change our main conclusions.

The IVMS is not an eigenstate of the full Hamiltonian, and in nature it has a spreading width. The isovector monopole strength is spread, and we consider the IVMS to be a doorway state that determines Coulomb mixing. To account for this one usually introduces a width in the denominator of the mixing coefficients in Eqs. (10) and (11). This correction has a small effect on the final result and therefore not included.

III. RESULTS

The isospin impurities in the ground state of a nucleus (or in its isobaric analog) are computed often using the one-body part of the Coulomb potential. The two-body Coulomb interaction because of its long range is dominated by the monopole part in the multipole expansion. When considering off-diagonal Coulomb matrix elements, the most important part is the one-body matrix element involving the monopole.

In fact, the isovector monopole matrix element between the ground state and the isovector monopole can be approximately written as [8]

$$\langle 0|V_C^{(1)}|M\rangle = \tfrac{1}{7}Z \text{ MeV} \quad (28)$$

which for a nucleus like ^{40}Ca is 3 MeV.

The other parts of the two-body Coulomb force cannot contribute much to Coulomb mixing, unless there is an accidental degeneracy between levels that mix. This of course is not the case for the ground state. Coulomb mixing (including isospin mixing) is determined by the distribution of the isovector monopole strength.

In Ref. [8] studies of Coulomb mixing were presented in which the notion of the giant isovector monopole was extensively used. The reader is referred to this reference, where various models of isospin mixing via the isovector monopole are described.

Here we use the results in [8] in order to calculate expressions for δ_c.

As we deal with $T_z = \pm 1$ nuclei we will take $Z \approx \frac{A}{2}$ and express the dependence on A only. Below are presented the results for δ_c using the four models, for the $T_z = \pm 1$ nuclei.

The first one we employ is the hydrodynamical model of Bohr and Mottelson [9] in which the IVMS is the result of radial oscillations of the proton fluid against the neutron fluid. In this model the energy of the IVMS is very high, ξ being more than 4.

A. The hydrodynamical model:

$$\delta_c = 6.0\times 10^{-7}A^2. \quad (29)$$

The next model we apply here is based on the nonenergy weighted sum rule (NEWSR) for the isovector monopole strength. For the approximate derivation of this sum rule see Ref. [8]. The result for δ_c is given below.

B. NEWSR:

$$\delta_c = 0.67\times 10^{-7}A^{7/3}. \quad (30)$$

We will also use here the results for the isospin impurity obtained in [8] using the energy weighted sum rule (EWSR) for the isovector monopole resonance. Using the expression for ε_1^2 one finds

C. EWSR:

$$\delta_c = 5.7\times 10^{-7}A^2. \quad (31)$$

In all three of the above models the IVMS is treated as a doorway state and one allows for a spreading width of this resonance. (Some small fraction of this strength can reach relatively low energies.)

Finally we will use the results of some simple microscopic RPA calculations of the IVMS in which schematic p-h interactions were used. In $N > Z$ nuclei the separation of the two isospin components with $T$and $T+1$ for the IVMS was taken into account. To obtain good isospin states it is necessary to include certain 2p-2h configurations in the wave function of the IVMS [8]. This was explicitly done and the $T+1$ components of the IVMS were determined. Using these components the isospin admixtures were evaluated for a series of nuclei. A phenomenological formula for the isospin

TABLE I. Values of δ_c in % for several mass numbers A for the four models discussed in the text.

δ_c	A		
	10	40	80
Hydrodynamical	0.006%	0.1%	0.4%
NEWSR	0.001%	0.04%	0.20%
EWSR	0.005%	0.09%	0.4%
Microscopic	0.009%	0.08%	0.25%

impurity was obtained by fitting these results for several nuclei with different masses [8].

D. Microscopic:

$$\delta_c = 18.0 \times 10^{-7} A^{5/3}. \quad (32)$$

Numerical results for several masses A and the four models are presented in Table I.

IV. DISCUSSION

The spread of values for δ_c in Table I is within a factor of 2 for the various models considered. Comparing these to the results for δ_c in [1,2] one sees that our calculation predicts considerably lower values, by factors of 2–4. When comparing the calculated δ_c to the ones obtained in the shell-model [3] one observes that the numbers for $\delta_{\rm IM}$ (isospin mixing) from [3] are roughly in agreement with the δ_c in the present work. However, in [3] the $\delta_{\rm RO}$ correction (termed the "radial overlap") is large and as stated in [1–3] should be added to $\delta_{\rm IM}$ in order to get δ_c. For example in ^{46}V Ormand and Brown found $\delta_{\rm IM} = 0.04\%$ and $\delta_{\rm RO} = 0.29\%$ for Hartree-Fock wave functions. Adding the two one gets $\delta_c = 0.33\%$. As already emphasized the δ_c calculated here includes both contributions. Our result for mass $A = 40$ is $\delta_c = 0.04$–0.1% depending on the isospin mixing model. Our results for δ_c are clearly smaller than those in [3].

Why is there this difference between the results of our approach and the ones discussed above? It is difficult to pinpoint exactly the reasons; one possible reason is that in the other works collective effects are not included. In the present work on the contrary, the mixing with IVMS takes into account effects of collectivity. The IVMS is a collective excitation and because of the repulsive nature of the particle-hole interaction in the isovector mode it is shifted to higher energies and its strength is reduced. This leads to reduced Coulomb mixing both in the proton wave function and in the isopin impurity of the isospin quantum number.

One should however note that when the self-consistent Hartree-Fock (HF) theory is used to define the single-particle orbits, the Coulomb force induces a one-body isovector potential that tends to counter the Coulomb repulsion and thus reduce the $\delta_{\rm RO}$ correction [3,8,10]. This is consistent with our discussion of the role of the coupling to the IVMS. When a HF calculation is performed in the odd-even nucleus it produces the coupling of the odd nucleon to excited states of the even nucleus [8]. In a spherical HF this means the coupling to monopole states.

The question of using the analog spin W versus the use of isospin T in some calculations of δ_c [1,2] was raised recently in Ref. [4]. In the analog spin formalism the W_- operator, for example, changes a neutron in a neutron orbit into a proton occupying the corresponding proton orbit which is distorted by the Coulomb potential. The operator T_-, on the other hand, changes the neutron charge but does not change the orbit (analogously for the W_+ and T_+ operators). As mentioned the operator used in Eq. (2) is the T_+. In our approach we employ consistently the isospin formalism. It is worth noticing the Coulomb admixtures of the IVMS introduce distortions into the proton single particle wave functions and, as has been demonstrated, this mixing in the isobaric analog state is equivalent to the formation of a W-analog state [11]. In our approach this is a result of the calculation and not the starting point. The so-called "radial overlap correction" is included in a consistent manner, avoiding double counting, without the need to use the W-spin.

It is clearly exhibited in our approach that the correction δ_c depends explicitly on two quantities: the isospin impurity and the strength of the symmetry potential. Smaller symmetry potential leads to a smaller δ_c correction.

It is important to assess the uncertainties in our treatment of δ_c. As already mentioned the symmetry potential strength is not well determined. This parameter determines the splitting between the various isospin components. There is another factor that influences this splitting, namely the different degree of collectivity of these isospin components. In large neutron excess nuclei this might alter considerably the spacing [8], but in $T_z = \pm 1$ nuclei the collectivity of the various components of the IVMS is similar and the effect on the spacing is small. The assumption of equal reduced matrix elements for the transitions to the various isospin components of the IVMS in expressions (12) and (13) has a very small effect for the nuclei considered. The IVMS has a spreading width and this could bring some fraction of strength to lower energies and influence the result. Also the centroid energy represented by the factor ξ contains some degree of uncertainty. The use of a simplified charge distribution (homogenous sphere) and the neglect of short-range non-Coulomb charge-dependent interactions might affect the results somewhat. The possibility that the Thomas–Ehrman effect [8] might have some influence on the orbital and binding-energy dependence should not be forgotten. There are possibly a number of other small uncertainties. If we rely on an intuitive estimate that the maximal uncertainty is 50%, still our results for δ_c will be considerably lower than the ones found in previous studies.

Our model, that uses the notion of the IVMS, provides an alternative approach for the isospin impurity and radial overlap corrections calculated in Refs. [1–3].

In our approach the collective nature of Coulomb mixing is taken into account, however it does not have all the details regarding the orbital or binding energy dependence present in Refs. [1–3].

More work is needed to understand the differences between the results presented here and those of Refs. [1–3] in order to reduce the theoretical error on δ_c.

ACKNOWLEDGMENT

We wish to thank Byron Jennings for his hospitality at TRIUMF where this work was performed.

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