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Further examination of prolate-shape dominance in nuclear deformation

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The observed almost complete dominance of prolate over oblate deformations in the ground states of deformed even-even nuclei is related to the splitting of high ℓ "surface" orbits in the Nilsson diagram: on the oblate side the occurrence of numerous strongly avoided crossings which reduce the fanning out of the low Λ orbits, while on the prolate side the same interactions increase the fanning out. It is further demonstrated that the prolate dominance is rather special for the restricted particle number of available nuclei and is not generic for finite systems with mean-field potentials resembling those in atomic nuclei.

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I. INTRODUCTION

The ground states of some nuclei are spherical, while others are deformed as a result of the latent anisotropy inherent in the ground state of a Fermi gas. In a very simplified view of the filling of particle orbits in the shell model based on spherical symmetry, the ground state configurations at the beginning and end of major shells are related by a particle-hole symmetry. Since the quadrupole moment of hole states has the opposite signs from that of particle states one might have expected the number of prolate and oblate shapes to be equal. In fact, almost all known deformed nuclei can be interpreted in terms of prolate axially-symmetric dominantly quadrupole deformed shape. The observed dominance of prolate over oblate shape is indeed overwhelming: Of the 98 known deformed even-even nuclei identified in Fig. 4.3 of Ref. [1], only one (^{12}C) is oblate. Additional experimental work since the review in Ref. [1] has not provided evidence for more than a few possibly oblate deformations.

Calculating one-particle spectra of the quadrupole-deformed infinite-well potential (spheroidal cavity) which simulates the potential for clusters of metallic atoms, we have obtained in Ref. [2] the result that the number of prolate systems is considerably larger than that of oblate systems. A number of publications using both quantum-mechanical and semiclassical treatments are available in which one tried to pin down the origin of the dominance of prolate nuclear shape over oblate shape. For example, when Hartree-Fock (HF) calculations with appropriate effective interactions are performed in many nuclei, the dominance of prolate shape except for very light nuclei is obtained in agreement with the experimental observations. Nevertheless, in our opinion, the nature of the parameters responsible for the prolate dominance has not yet been adequately understood. In particular we are concerned with the generality of the argument for prolate dominance: is it a universal property of saturating Fermi systems bound by short-range interactions? It is the purpose of the present article to extend the calculations of Ref. [2] with the aim of obtaining additional evidence concerning prolate dominance.

In the present work only one kind of particles, neutrons or protons, are considered. For simplicity, we neglect both spin-orbit potential and pairing correlation which are important in the understanding of nuclear ground states, since an essential element in producing more prolate systems than oblate ones can be shown without them. In the spectra of both the harmonic oscillator potential and an isolated single-j-shell (or single-ℓ-shell) model there is a particle-hole symmetry, while in both of them the surface effect is absent. The particle-hole symmetry leads to the number of prolate systems equal to that of oblate ones. However, the presence of a more sharply defined surface leads to the presence of "surface states" that break the particle-hole symmetry. Even in the case of the deformed harmonic oscillator, the deformations in mid-shell become so large that there occurs crossing between the one-particle levels in adjacent shells that violates particle-hole symmetry for the ground state occupations. We perform numerical calculations, choosing two models which can be easily solvable taking exactly into account the volume conservation when the system is quadrupole deformed; the pure harmonic oscillator potential and the infinite-well potential (cavity). The numerical results of the two models are used to elucidate the important role played by the surface of one-body potentials.

In Sec. II A we study the quadrupole deformation preferred by the pure harmonic oscillator model, while in Sec. II B the preference for prolate over oblate shape in the spheroidal cavity is examined. In Sec. III the origin of the dominance of prolate systems is explained using the numerical results presented in Sec. II. In Sec. IV we present considerably extended numerical results of calculations of ground states of independent particle motion in spheroidal cavities. In Sec. V comments on some other approaches to the present subject are briefly described, while conclusion and discussions are given in Sec. VI.

II. MODEL CALCULATION

A. Harmonic oscillator potential

We parametrize the axially-symmetric quadrupole-deformed oscillator potential by

$$\omega_\perp = \omega_0 \, e^{\alpha}, \tag{1}$$

$$\omega_z = \omega_0 \, e^{-2\alpha}, \tag{2}$$

TABLE I. Relation between parameters.

α	−0.15	−0.10	−0.05	0	0.05	0.10	0.15
$R_z/R_\perp$	0.638	0.741	0.861	1.000	1.162	1.349	1.568
δ_{osc}	−0.477	−0.313	−0.153	0	0.146	0.283	0.412

so that the volume conservation

$$\omega_\perp^2 \omega_z = \omega_0^3 \tag{3}$$

is exactly satisfied. $\alpha > 0$ ($\alpha < 0$) corresponds to prolate (oblate) shape. For reference, in Table I a numerical comparison between α, the ratio of radii $R_z/R_\perp$ (aspect ratio) and the commonly employed deformation parameter δ_{osc} [1] is tabulated, where

$$\delta_{osc} \equiv 3\frac{\omega_\perp - \omega_z}{2\omega_\perp + \omega_z} = 3\frac{R_z - R_\perp}{2R_z + R_\perp}. \tag{4}$$

The one-particle spectrum of the axially-symmetric quadrupole-deformed oscillator potential can be found in many references. See, for example, Fig. 6-48 of Ref. [1] or Fig. 1 of Ref. [3]. Choosing the energy unit $\hbar\bar{\omega}$ where

$$\bar{\omega} = \tfrac{1}{3}(2\omega_\perp + \omega_z), \tag{5}$$

one-particle energies can be drawn by straight lines as a function of δ_{osc}. We note that a given level specified by quantum numbers $(n_z, n_\perp)$ has a degeneracy $2(n_\perp + 1)$ including the nucleon spin. The magic numbers of the spherical harmonic oscillator potential are 2, 8, 20, 40, 70, 112, 168, 240, ... For a given nucleon (proton or neutron) number we numerically search for the total energy minimum on prolate and oblate sides, respectively. The total energy of the system in the present work is defined as the sum of the lowest-lying one-particle energies for a given deformation and a given particle number

$$E(\alpha) = \sum_{i=1}^{N_F} \varepsilon_\Lambda^i(\alpha), \tag{6}$$

where the conserved one-particle quantum-number Λ is the projection of the particle orbital angular-momentum onto the symmetry axis.

The one-particle spectrum originating from a given major $N = n_z + n_\perp$ shell has a symmetry between the prolate and oblate sides. Therefore, if for a given particle number we keep adiabatically the configuration consisting of the orbits occupied at a very small $|\delta_{osc}|$ value and look for the total energy minimum as $|\delta_{osc}|$ further increases, the shape of the total energy minimum is prolate for the Fermi level lying in the first half of the major shell while it is oblate for the second half of the major shell. Namely, the number of prolate systems is equal to that of oblate systems. This prolate-oblate symmetry may be broken, in the case that the energy minimum occurs at $|\alpha|$ values larger than those at which one-particle levels coming from adjacent major shells cross each other, namely in the case that the adiabatic configuration defined above is no longer the configuration at the total energy minimum. In practice, this situation occurs only in a limited number of systems, in which the major shell is almost half filled.

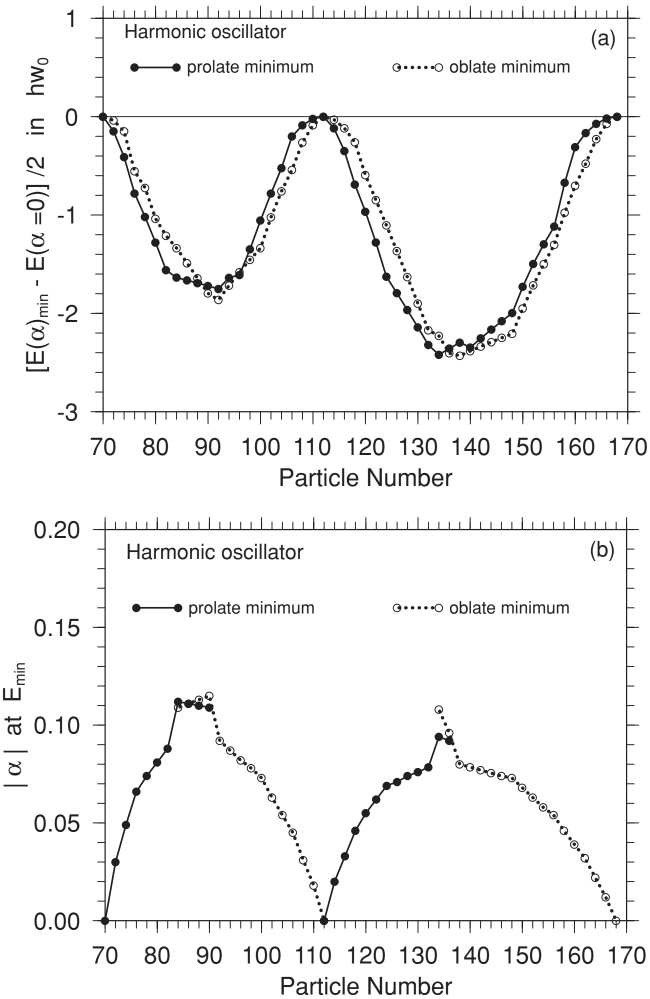

FIG. 1. (a) Total energies at the prolate and oblate minima, respectively, relative to the energy of the spherical shape as a function of particle number. The harmonic oscillator potential is used and the energy unit is $\hbar\omega_0$. (b) Absolute values of the deformation parameter α at the total energy minimum as a function of particle number. Around the middle of respective major shells the $|\alpha|$ values of both prolate and oblate minima are shown.

In Fig. 1(a) the total energy of the prolate and oblate minima, respectively, relative to the energy of the spherical shape is plotted as a function of the particle number, of which the Fermi levels lie within the $N = 5$ and 6 shells in the spherical limit. It is seen that in the beginning (end) of the respective N shells the prolate (oblate) minimum is deeper than the oblate (prolate) minimum, and as a total the number of prolate systems is approximately equal to that of oblate systems. In a few systems in which a given major shell is almost half filled, the lowest lying curves in Fig. 1(a) look somewhat irregular, however, the curves become much smoother when a deviation from axial symmetry is taken into account. Similarly, in the region of the particle number for which the curve in Fig. 1(a) locally resembles a straight line, taking into account a deviation from axial symmetry makes the curve more smoothly varying. In all these cases, the deviation from respective axially-symmetric deformations is relatively small.

In Fig. 1(b) absolute values of α at the total energy minimum for a given particle number are plotted, except around the middle of respective major shells in which the $|\alpha|$ values of both prolate and oblate minima are shown. In the latter cases, precisely speaking for the particle number 84–90 and 134–136, the energy minimum is obtained at $|\alpha|$ which is larger than that of the crossing of one-particle levels coming from adjacent major shells. The presence of those few systems provides a small deviation from the exact equality of the number of prolate and oblate shapes in the harmonic oscillator potential.

B. Infinite-well potential (cavity)

The eigenvalue of spherical cavity $\varepsilon_{n\ell}$ is obtained from the nth zero of the spherical Bessel function of order ℓ. In Fig. 2(a) eigenvalues of relatively small spherical cavities together with the total particle number A for several Fermi levels are plotted as a function of orbital angular momentum ℓ, while in Fig. 2(b) eigenvalues of a much broader region are shown.

From Fig. 2(b) two kinds of families of one-particle levels ($n\ell$) are easily identified; (a) a family with $\Delta\ell = 2$ originating from a given harmonic-oscillator major shell, of which the one-particle energies decrease as ℓ increases. This decrease of one-particle energies in realistic potentials is approximated by introducing the $\vec{\ell}^2$ term in the modified oscillator potential [4]; (b) families defined by the number of radial nodes $n = 0, 1, 2, \ldots$ and each comprising all possible ℓ-values $\ell = 0, 1, 2, \ldots$. These families are referred to as $n = 0 =$ yrast, $n = 1 =$ yrare, ...

It is seen that in the family (a) the $\Delta\ell = 2$ approximate degeneracy around $\ell = 0$ remains all the way to very large systems, corresponding to elliptical orbits in terms of closed classical orbits [1]. Since the total degeneracy of those $\Delta\ell = 2$ close-lying levels in the family is small compared with the degeneracy of all possible open shells around the Fermi levels, those levels do not play an important role in the present discussion of the prolate-oblate competition. In contrast, orbits close to the yrast line have the largest ℓ values and thus the largest degeneracies among orbits around a given Fermi level. Thus, the splitting or the shell structure of those large ℓ shells for quadrupole deformation may govern the preferred prolate or oblate shape.

As seen in Fig. 2(a), for $138 < A < 186$ almost degenerate $1j$ and $2g$ shells are considerably separated from other $n\ell$ shells. The near degeneracy of these two shells with $\Delta\ell = 3$ may lead to octupole deformations. In Ref. [2] this possibility is examined under the assumption of deformations involving only a single spherical harmonic mode. It is found that quadrupole mode dominates except for the particle number $A = 152$–156 where Y_{32} deformation provides the lowest total energy.

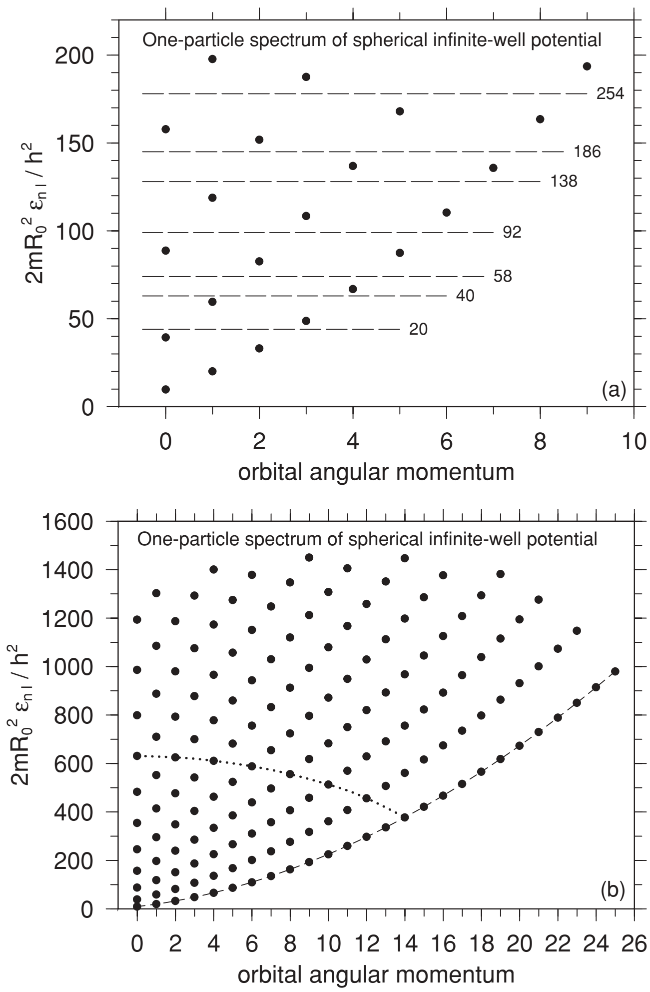

FIG. 2. (a) Eigenvalues of smaller spherical cavity in units of $\hbar^2/2mR_0^2$ as a function of one-particle orbital angular momentum. The total particle number including the factor 2 due to the nucleon spin is shown for several Fermi levels. (b) Eigenvalues of spherical cavity of a region much broader than that plotted in (a). An example of the family (a) coming from a given harmonic-oscillator major shell is denoted by a dotted curve, while the yrast $\Delta\ell = 1$ family (b) with no radial node is connected by a dashed curve.

The radii of the spheroidal cavity are parametrized as

$$R_\perp = R_0\, e^{-\alpha}, \tag{7}$$

$$R_z = R_0\, e^{2\alpha}, \tag{8}$$

so that the volume conservation is exactly satisfied under deformation. The numerical relation between α and $R_z/R_\perp$ is given in Table I. Eigenvalues of spheroidal cavity are calculated using the method described by Moszkowski [5]. The Nilsson diagram for the spheroidal cavity is shown in Figs. 3(a) and 3(b), in which one-particle levels with $\Lambda = 0$ are doubly degenerate while those with $\Lambda \neq 0$ are four-fold degenerate. It is noted that Figs. 3(a) and 3(b) cover a region of the deformation which is as much as a factor of two larger than that of possible ground states. See Fig. 4(b).

In Fig. 4(a) the total energies at the minima of the prolate and oblate shapes, respectively, relative to the energy of the spherical shape are plotted for systems with even

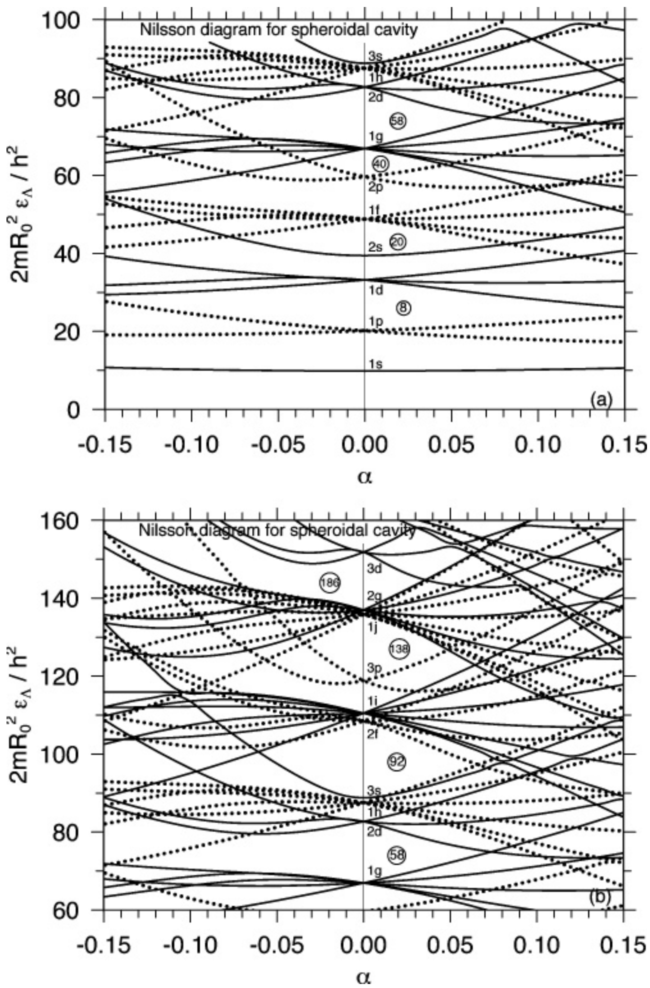

FIG. 3. (a) One-particle energies of spheroidal cavity as a function of deformation parameter. At spherical point $\alpha = 0$ the quantum numbers, $n\ell$, are written. The particle number of the system obtained by filling all lower-lying levels is written with a circle in several places. Positive-parity levels are plotted by solid curves, while negative-parity levels by dotted curves. (b) One-particle energies of spheroidal cavity as a function of deformation parameter, for the system larger than that plotted in (a).

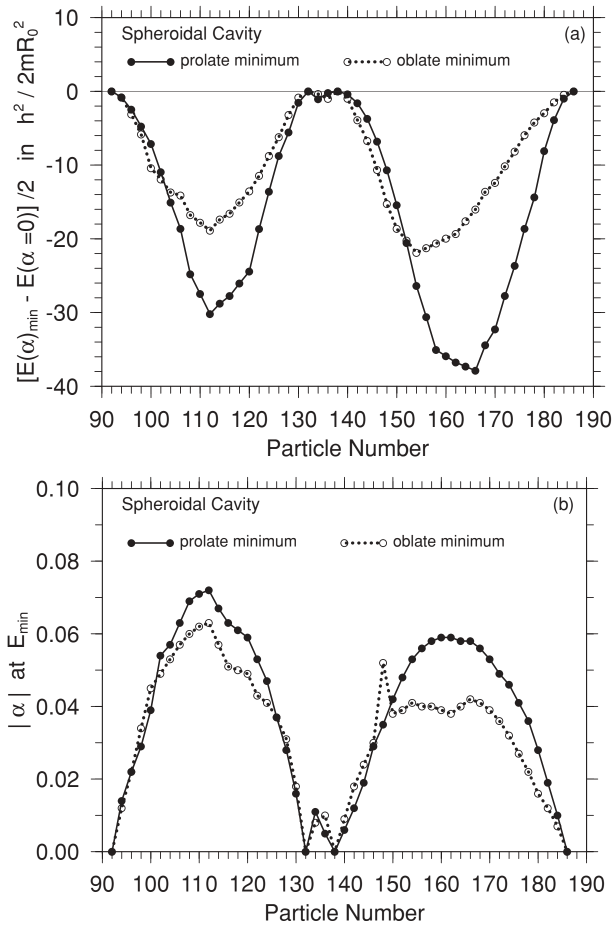

FIG. 4. (a) Total energies at the prolate and oblate minima, respectively, relative to the energy of the spherical shape as a function of particle number of the system. The infinite-well potential is used and the energy unit is $\hbar^2/2mR_0^2$. (b) Absolute values of the deformation parameter α at the energy minima of prolate and oblate shapes, respectively.

particle-number 92–186. What is shown in Fig. 4(a) is indeed identical to a part of Fig. 23 of Ref. [2]. As already stated in Ref. [2], we have found: (a) at the beginning of the two major shells (the particle number 92–138 and 138–186) the optimum oblate shape is energetically more favorable than the optimum prolate shape while at the end of the shells the optimum prolate shape is more favorable; (b) the number of systems in which a prolate shape is favorable is much larger than the number for which an oblate shape has a lower energy. For example, for the major shell with the particle number 138–186 only six systems prefer oblate while the prolate minimum is lower than the oblate minimum in seventeen systems.

Absolute values of α at the energy minima of prolate and oblate shapes, respectively, are plotted in Fig. 4(b). It is seen that in the midshell region the absolute magnitudes of the deformations are appreciably larger for the prolate than for the oblate deformations.

III. ORIGIN OF THE DOMINANCE OF PROLATE SYSTEMS

The origin of the dominance of prolate systems obtained in the spheroidal cavity comes from an asymmetry in the splitting of one-particle levels on the prolate and oblate sides, which is absent in the axially-symmetric quadrupole-deformed harmonic oscillator model. The asymmetry originates from the fact that already in the spherical shape the presence of the surface in the potential implies that the higher ℓ subshells have lower energies than the lower ℓ orbits of the same oscillator shell. When the potential is moderately deformed, around the Fermi level of the system where high ℓ shells are partially filled the local one-particle level density is considerably higher on the oblate side than on the prolate side. The origin of the different splittings of one-particle levels coming from high ℓ shells on prolate and oblate sides can be understood in terms of the asymptotic quantum numbers which have been found useful in the deformed oscillator model.

As an example of the splitting of the yrast family orbits, in Fig. 5 we reproduce the one-particle levels originating

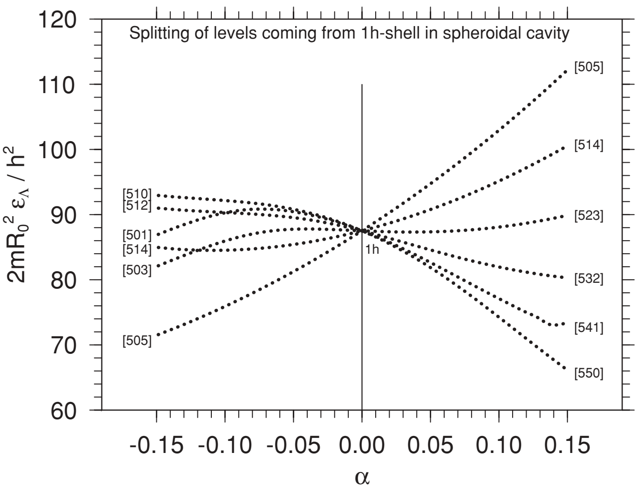

FIG. 5. Splitting of levels originating from the $1h$ shell in spheroidal cavity. The asymptotic quantum numbers $[N\ n_z\ \Lambda]$ are assigned to the levels on both prolate and oblate sides. See the text for details.

from the $1h$ shell which are taken from Fig. 3(b). It is noted that the splitting of ℓ orbits belonging to the yrast family in Fig. 2(b) is all very similar. Two characteristic features in the level splitting are noted; (a) on the oblate side strongly avoided crossings among the low Λ orbits with resulting reduction in fanning out; (b) on the prolate side increased fanning out of the low Λ levels due to the same inter-shell interactions.

In Fig. 5 the quantum numbers $[N\,n_z\,\Lambda]$, which are called the asymptotic quantum numbers in the deformed oscillator model [1], are assigned to respective one-particle levels, where Λ is a good quantum number also in the spheroidal cavity. The quantum number n_z in the deformed oscillator potential is usually known as the number of oscillator quanta in the direction of the symmetry axis (z-axis). Generally speaking, for a prolate (oblate) shape the kinetic energy is energetically cheaper (more expensive) in the direction of the symmetry axis. In the spheroidal infinite-well potential n_z can be interpreted as the number of the node of the wave function in the direction of the symmetry axis, while N represents the sum of the nodes of the wave functions in the directions of the symmetry axis and the x and y axes. Defining the meaning of the quantum numbers, N and n_z, in this way, one obtains the asymptotic behavior of one-particle levels in the spheroidal cavity as follows: (i) For a large deformation the quantum numbers $[N\,n_z\,\Lambda]$ become good quantum numbers; (ii) For a large deformation the slope of the one-particle energies in Figs. 3(a), 3(b), and 5 is determined by the quantum numbers N and n_z; (iii) Levels with larger n_z for a given N lie energetically lower (higher) in prolate (oblate) shape; (iv) The presence of the surface in the potential makes the levels with larger Λ values for given N and n_z lower, since already in the spherical shape higher ℓ shells are energetically pushed down compared with lower ℓ shells.

On the prolate side (i.e., $\alpha > 0$) of Fig. 5 the levels split for a small α value have already the internal structure which smoothly changes to respective asymptotic quantum numbers and, thus, the splitting grows smoothly and monotonically as α further increases. In contrast, on the oblate side (i.e., $\alpha < 0$) the levels except two levels with [505] and [514] have to change drastically the internal structure to approach their asymptotic quantum numbers as $|\alpha|$ increases. The drastic change of the internal structure comes from the interaction (or the avoided crossing) with the one-particle levels originating from the $2f$ and $3p$ shells. Consequently, it produces the strongly nonlinear behavior for those one-particle levels on the oblate side, in striking contrast to the prolate side, in the deformation region relevant to that of the ground states. More generally speaking, for a given $N \gg 1$ the splitting of one-particle levels coming from the highest $\ell\,(= N)$ subshell grows monotonically on the prolate side, while on the oblate side all levels except the two levels with $[N\,0\,N]$ and $[N\,1\,N-1]$ have to change drastically the internal structure very soon after $|\alpha|$ increases from zero. From Fig. 3(b) one can indeed see that the splitting structure of one-particle levels originating from the $1i$ and $1j$ shells is very similar to Fig. 5 on both prolate and oblate sides.

The similar asymmetry of the splitting of one-particle levels on the prolate and oblate sides can be identified also for one-particle levels coming from the $\ell = N - 2$ shell (the yrare family in Fig. 2(b)) if N is sufficiently large, though the asymmetry is less striking. If we take an example of the one-particle levels coming from the $N = 5$, $2f$ shell in Fig. 3(b), the one-particle levels on the prolate side have the asymptotic quantum numbers, [530], [521], [512], and [503], from the bottom to the top, and the level splitting grows monotonically as α increases from zero. In contrast, on the oblate side the asymptotic quantum numbers are [523], [521], [532], and [530], from the bottom to the top. That means, two levels starting with [512] and [503] at very small values of $|\alpha|$ on the oblate side have to change the internal structure soon after $|\alpha|$ increases from zero. Consequently, in the deformation region relevant to the ground states the local one-particle level density on the oblate side is higher than that on the prolate side. This helps further the prolate dominance produced by the level splitting of the $1i$ ($\ell = N = 6$) shell in the region of particle number 94–130.

The level splitting of the $\ell \ll N$ orbits does not play an important role in the present discussion, since the degeneracy of these subshells is small.

For reference, in Fig. 6 the level splitting of an isolated single $\ell = 5$ shell in cavity is shown, which is obtained by switching off the coupling to other shells. The volume conservation taken into account is the same as the one in Fig. 5 and leads to the curves in Fig. 6 instead of straight lines which may often be seen in the literatures as the splitting of an isolated single ℓ shell. In the case of this isolated single ℓ shell the wave functions of all one-particle levels are independent of deformation, and there is a particle-hole symmetry or a symmetry between the prolate and oblate sides. The shape of the total energy minimum is oblate for the Fermi level lying in the first half of the shell while it is prolate in the second half of the shell. The preference for prolate or oblate shape in the beginning and the end of the shell, respectively, is opposite to that in a major shell of the pure harmonic oscillator potential.

From the comparison of the splitting on the prolate side of Fig. 6 with that of Fig. 5, it is noted that the interaction with the normal-parity states in the next shell above implies

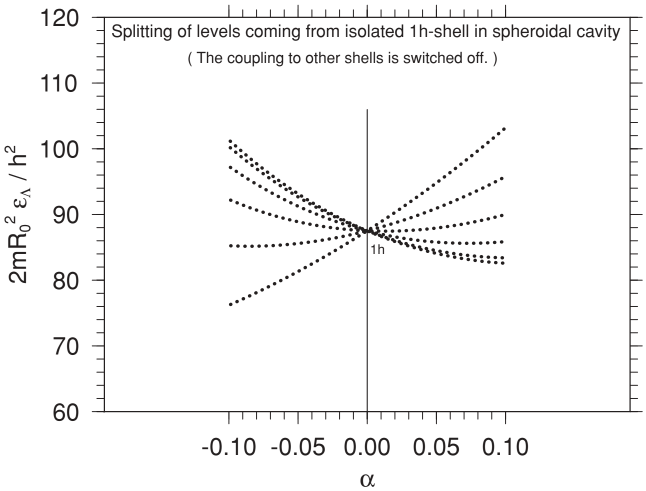

FIG. 6. Splitting of levels coming from an isolated $1h$ shell, which is obtained by switching off the coupling to other shells in spheroidal cavity.

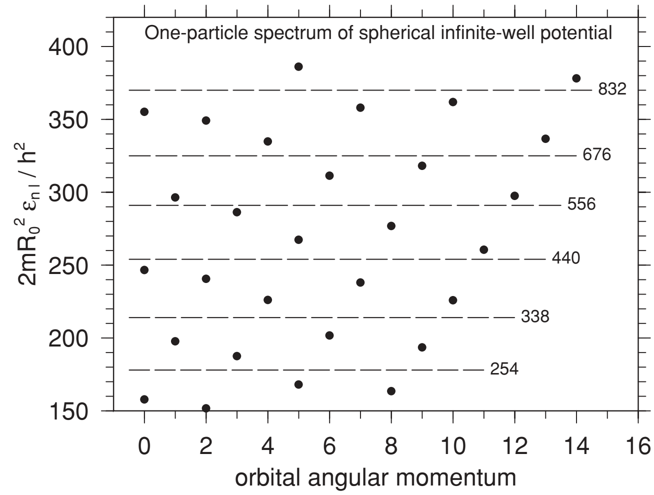

FIG. 7. Eigenvalues of spherical cavity, which are larger than those in Fig. 2(a), in units of $\hbar^2/2mR_0^2$ as a function of one-particle orbital angular momentum. The total particle number including the factor 2 due to the spin 1/2 is shown for several Fermi levels.

non-linear convexity in the lower Λ orbits in Fig. 5. This increase of fanning out is another important factor favoring prolate dominance.

So far, using the cavity potential we have explained the dominance of prolate shape in terms of the prolate-oblate difference of the bunching of high-ℓ "surface" states, which are recognized as a surface mode bound to surface by the large centrifugal potential. The characteristic feature of the bunching of those high ℓ one-particle levels, can be found in all realistic nuclear potentials. It comes from the presence of the surface in the Woods-Saxon potential and Hartree-Fock potentials and is parametrized by the ℓ^2 term in the modified oscillator potential. The bunching unique to the prolate and oblate shapes, respectively, described above can be found already in the Nilsson diagram of a small system such as the sd-shell in realistic nuclear potentials. For example, see the splitting of the levels originating from the $1d_{5/2}$ shell in Fig. 5-1 of Ref. [1], where the additional effect of the spin-orbit potential included should be also noticed.

IV. PROLATE-OBLATE COMPETITION IN SPHEROIDAL CAVITY FOR LARGER SYSTEMS

In previous sections we have shown that the different bunching of Nilsson levels for cavity on the prolate and oblate sides, which come from high ℓ shells, leads to the prolate dominance in deformed nuclei. While the number of particles which can be accommodated in the highest ℓ shell (the yrast family in Fig. 2(b)) is a large portion of the particle number accommodated in one major shell of smaller systems, the portion becomes smaller in a larger system. This is because the number of particles accommodated in the highest ℓ shell in a spherical potential is the order of $A^{1/3}$ where the total number of particles is expressed by A, while the degeneracy of one major shell is the order of $A^{2/3}(A^{1/2})$ in the harmonic oscillator potential (in potentials such as an infinite-well potential).

The calculation of Ref. [2] of equilibrium shapes of independent particles in a spheroidal cavity have been extended to particle number up to 850 with a view to examine the prolate-oblate competition in a more general context.[1]

In Fig. 7 we show eigenvalues of spherical cavity for systems larger than those in Fig. 2(a) and up to those investigated in Ref. [6]. As some numerical examples, in Figs. 8(a) and 9(a) the total energies at the energy minima relative to the energies of the spherical shape as a function of particle number are plotted for systems with the particle number 338–440 and 676–832, respectively. In Figs. 8(b) and 9(b) the absolute values of α at the total energy minima are shown. The close relation between the prolate or oblate minima and the location of high-ℓ orbits can be easily seen in the same way as that in the smaller systems described previously. In Table II we tabulate the calculated number of prolate and oblate systems, and the ratio of the number of oblate system to that of all deformed (namely oblate plus prolate) systems. From Table II it is seen that the dominance of prolate shape over oblate shape is gradually reduced as the particle number increases. The decrease of deformation with particle number shown in Table II follows rather well the expected power law $A^{-1/2}$.

V. COMMENTS ON SOME OTHER APPROACHES

Exploiting the theory of periodic orbits, Frisk [7] has suggested that the dominance of prolate systems over oblate systems originates in the landscape of the locally averaged one-particle level density considered as a function of particle

[1]Motivated by the issue of deformations in the sodium clusters, similar calculations based on the shell correction method without a detailed discussion of prolate-oblate competition have been carried out by Reimann and Brack in Ref. [6].

TABLE II. The number of the prolate- and oblate-deformed systems and the maximum deformation for an infinite-well potential (spheroidal cavity). The first column shows the region of the particle number, of which only the system with even particle-number is examined. The second column denotes the number of prolate-deformed system, while the number of oblate-deformed system is given in the third column. The ratio of the number of oblate-deformed system to the sum of the prolate- and oblate-deformed systems is shown in the fourth column, while the α value of the maximum $|\alpha|$ in respective regions of the particle number is given in the fifth column.

Particle number	Prolate	Oblate	Ratio of oblate to total	α of $|\alpha_{\max}|$
58–92	11	4	0.27	0.081
92–138	15	6	0.29	0.072
138–186	17	6	0.26	0.059
186–254	24	7	0.23	0.064
254–338	28	13	0.32	0.060
338–440	38	11	0.22	0.050
440–556	35	20	0.36	0.043
556–676	38	19	0.33	−0.029
676–832	45	29	0.39	−0.027

number and spheroidal deformation parameters in the potential. It is however difficult for us to assess the scope of the approach in Ref. [7] because it fails to provide quantitative measures for the relative number of prolate and oblate deformations.

A variety of theoretical models, some of which are fully microscopic while others are some combinations of macroscopic (or phenomenological) and microscopic approaches, have been used in the study of the prolate-oblate competition of the ground states of deformed nuclei. Here, as an example, we take the work by Tajima *et al.* in Ref. [8], which is an HF plus BCS calculation with the SIII interaction as the HF effective interaction. The ground states of even-even nuclei with the proton number $2 \leqslant Z \leqslant 114$ and the neutron number N ranging from outside the proton drip line to beyond the experimental frontier on the neutron-rich side are considered. In Ref. [8] it is found that in heavier nuclei the oblate ground states are very rare. They state that the dominance of prolate deformation for $N > 50$ may be attributed to the change of the nature of the major shells from the harmonic-oscillator shell to the Mayer-Jensen shell. This statement can be interpreted as that the spin-orbit splitting is an essential element in the dominance of prolate systems. In order to clarify the role by the spin-orbit potential in the prolate-oblate competition, let us briefly consider a model consisting of the harmonic oscillator plus spin-orbit potentials. In the spherical limit of this simple model the splitting of one-particle energies in a given major N shell comes only from the spin-orbit potential. As an example, we take the $N = 5$ major shell. Then, taking the sign of the spin-orbit potential in nuclear physics, in the spherical limit the lowest-lying shell among j shells belonging to the major shell is $1h_{11/2}$, while the highest is the $1h_{9/2}$ shell. It is easy to find that in both prolate and oblate sides the level splitting of the $1h_{11/2}$ shell is similar to that shown in Fig. 5, when

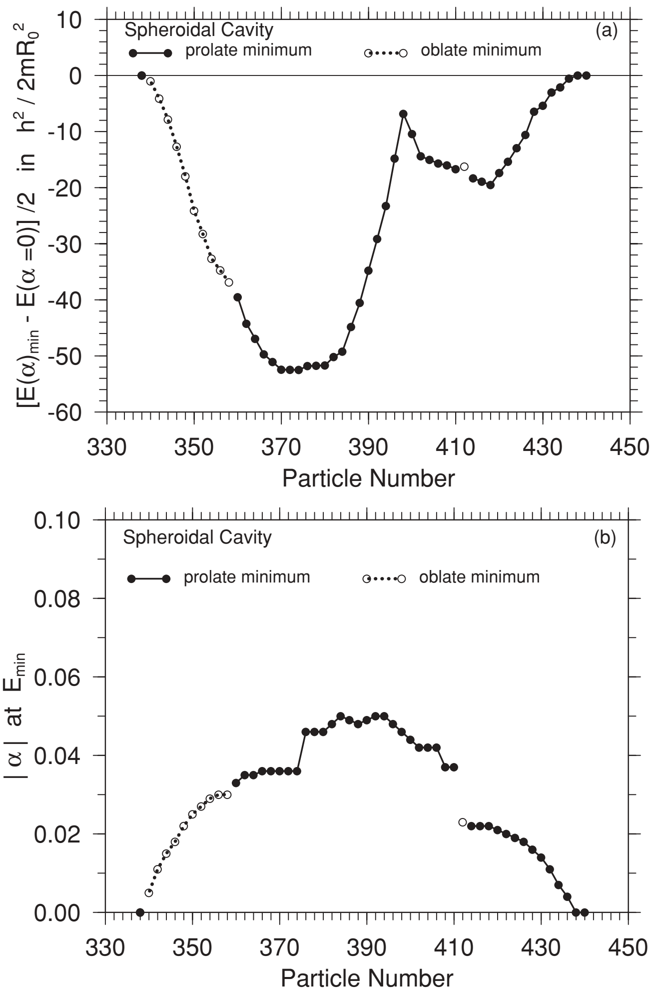

FIG. 8. (a) Total energies at the energy minima relative to the energy of the spherical shape as a function of particle number 338–440 of the system. The infinite-well potential is used and the energy unit is $\hbar^2/2mR_0^2$. (b) Absolute values of the deformation parameter α at the energy minima of (a).

Λ is replaced by $\Omega = \Lambda + s_z$ where Ω expresses the nucleon angular-momentum component along the symmetry axis.In contrast, the splitting of the levels of the highest-lying $1h_{9/2}$ shell is quite different. On the oblate side the splitting for a small $|\alpha|$ grows smoothly and monotonically as $|\alpha|$ further increases (just like the level splitting on the prolate side of Fig. 5), while on the prolate side all one-particle levels must have asymptotically $N = 5$ and $n_z = 0$ or 1. Namely, all levels originating from the $1h_{9/2}$ shell on the prolate side must be asymptotically up-sloping as α increases. That means, on the prolate side three levels other than the two levels, [505 9/2] and [514 7/2], have to change the internal structure from that for very small α values as α further increases, in order to approach the asymptotic behavior. Thus, we expect more prolate systems for the Fermi level lying at the beginning of the $N = 5$ major shell (due to the levels coming from the $1h_{11/2}$ shell), while more oblate systems may be obtained for the Fermi level lying at the end of the major shell (due to the levels coming from the $1h_{9/2}$ shell). As a total, we conclude that the spin-orbit

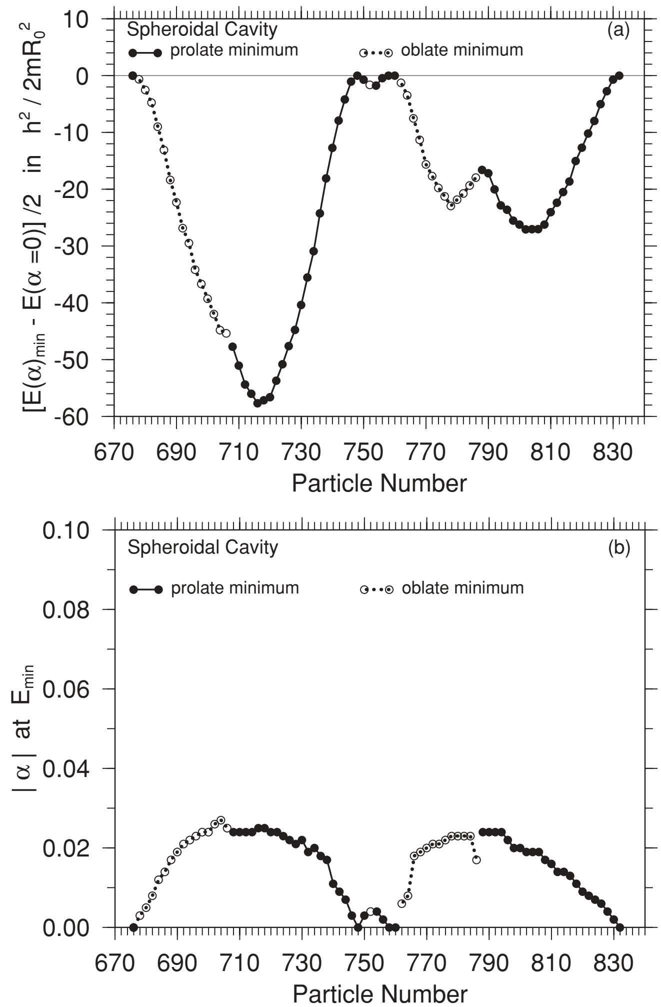

FIG. 9. (a) Total energies at the energy minima relative to the energy of the spherical shape as a function of particle number 676–832 of the system. The infinite-well potential is used and the energy unit is $\hbar^2/2mR_0^2$. (b) Absolute values of the deformation parameter α at the energy minima of (a).

potential alone may not have any strong preference for prolate shape over oblate shape.

VI. CONCLUSION AND DISCUSSIONS

This article began with the question of why almost all deformed nuclei are prolate in their ground state rather than an equal division between prolate and oblate as suggested by the particle-hole symmetry that follows if we ignore subshell structure of spherical major shells. We have only been able to identify a mechanism that leads to a significant dominance of prolate over oblate shapes in the range of particle number up to those experimentally examined. This mechanism involves the transition from harmonic oscillator mean field to systems with a much more sharply defined surface, modeled by a deformed cavity. In the latter region there occur surface states with orbital angular momentum and degeneracy appreciably greater than any other in the shell. This pattern leads to the occurrence of avoided crossings on the oblate side of the Nilsson diagram. This leads to shell filling in which the deformation is oblate in the beginning of the shell, but changes to prolate well before midshell because of the energy gain implied by the fanning out of the prolate orbits that is increased by the interaction with the normal-parity subshells in the next higher major shell. The role played by the surface becomes less important for much larger systems. Thus, the observed overwhelming dominance of prolate shape in deformed nuclei may be identified as the feature of the system with a relatively small number of particles.

In the present work we focus our attention on the prolate shape dominance in the ground states of stable or well-bound nuclei. In drip line nuclei with weakly bound nucleons, especially neutrons, the shell structure around the Fermi level as well as the role played by the nuclear surface can be different [9]. Therefore, the prolate-oblate competition in drip line nuclei has to be separately and more carefully examined.

We recognize that the origin of the prolate dominance presented in this work is the essential element but it does not immediately lead to the observed overwhelming prolate dominance in the ground states of deformed nuclei. Thus, in the following we make comments on some elements which have not been taken into account in our present work. First, we have not included the spin-orbit potential, which is important in nuclear spectroscopy. We conclude that already in the absence of the spin-orbit potential the dominance of prolate systems is obtained and the spin-orbit potential alone has no strong preference for prolate or oblate systems. However, it is possible that a particular combination of the surface effect of the nuclear potential with the spin-orbit splitting may make a further contribution to the prolate dominance. In particular, Tajima and Suzuki in Ref. [10] have identified an interesting coherence in the contributions of the ℓ^2 and the spin-orbit terms to the prolate/oblate competition. It is noted that the spin-orbit potential is closely related to the surface property of systems.

Second, we have not taken into account the pairing correlation. It is reported in Ref. [11] that the inclusion of the pairing correlation may enhance the prolate dominance, depending on the pairing strength. As seen in Figs. 4(a) and 4(b), the systems with oblate minima occur in the neighborhood of closed shells and the deformation is relatively small. Therefore, some of those oblate systems may easily become spherical when pair correlation is included. Then, the ratio of prolate systems to oblate ones may become larger, in agreement with the numerical results of Ref. [11].

Third, the Coulomb interaction between protons clearly prefers prolate shape to oblate shape as exhibited by the cubic term, α^3, in the Coulomb energy of a deformed uniformly-charged ellipsoid. However, for the moderate deformation such as that of nuclear ground states the preference is expected to play a minor role.

Fourthly, only one kind of nucleons (protons or neutrons) are considered in our present work. Different shapes may be preferred by protons and neutrons in some nuclei with $N \neq Z$, and the possible difference may make a minor modification of the degree of the dominance of prolate systems.

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