Superfluid response and the neutrino emissivity of neutron matter

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We calculate the neutrino emissivity of superfluid neutron matter in the inner crust of neutron stars. We find that neutrino emission due to fluctuations resulting from the formation of Cooper pairs at finite temperature is highly suppressed in nonrelativistic systems. This suppression of the pair-breaking emissivity in a simplified model of neutron matter with interactions that conserve spin is of the order of v_F^4 for density fluctuations and v_F^2 for spin fluctuations, where v_F is the Fermi velocity of neutrons. The larger suppression of density fluctuations arises because the dipole moment of the density distribution of a single component system does not vary in time. For this reason, we find that the axial current response (spin fluctuations) dominates. In more realistic models of neutron matter that include tensor interactions where the neutron spin is not conserved, neutrino radiation from bremsstrahlung reactions occurs at order v_F^0. Consequently, even with the suppression factors due to superfluidity, this rate dominates near T_C. Present calculations of the pair-breaking emissivity are incomplete because they neglect the tensor component of the nucleon-nucleon interaction.

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I. INTRODUCTION

The long-term (10^5–10^6 yr) cooling of isolated neutron stars and the thermal evolution of accreting neutron stars in binary systems are sensitive to the weak interaction rates at high density. A recent review of the theory, modeling, and observational constraints on neutron star thermal evolution can be found in Ref. [1]. Here, we focus on a specific neutrino process that is expected to be relevant in superfluid neutron matter.

In superfluids, Cooper pairs break and recombine constantly at finite temperature. Such processes can dominate the density and spin-density fluctuations. In pioneering work, several decades earlier, Flowers, Ruderman, and Sutherland [2] recognized that these fluctuations can couple to neutrinos through the weak neutral current. They showed that when the temperature was less than but comparable to the critical temperature for superfluidity, neutrino emission due to the Cooper pair recombination processes was important [2]. Subsequently, this process, which is now commonly referred to as the pair-breaking and formation (PBF) process, was recomputed in Refs. [3] and [4], and its role in the thermal evolution of isolated neutron stars was shown to be important [5–8].

Another context in which the PBF process plays a role is in accreting neutron stars that exhibit x-ray bursts and superbursts. Current models for superbursts indicate that they arise because of unstable burning of carbon in the ocean of accreting neutron stars [9]. Agreement between theoretical models (for the light-curves and recurrence times) and observation relies on the assumption that carbon is ignited at a column depth of about 10^{12} g/cm^2. However, to ignite carbon at this depth the temperature there should be $\simeq 5 \times 10^8$ K [10,11]. Consequently the ignition condition for superbursts is sensitive to the temperature profile of the crust in accreting systems. The temperature profile in turn depends on the balance between heating in the crust due to electron captures and pycnonuclear reactions [12] and cooling due to neutrino emission [10].

The inner crust of the neutron star is expected to contain a neutron superfluid. When the matter density exceeds the neutron drip density (4×10^{12} g/cm^3) a relatively low-density neutron liquid coexists with a lattice of nuclei [13]. Here, the attractive s-wave interaction between neutrons induces superfluidity. The pairing gap rises from zero at neutron drip to a maximum of about 1 MeV when the neutron Fermi momentum $k_F \sim 200$ MeV and then decreases to zero in the vicinity of the crust-core interface. For temperatures of relevance to the accreting neutron stars, it was found that the PBF process in the neutron superfluid resulted in rapid neutrino losses and cooled the crust to temperatures below those required for carbon ignition at the favored depth [11]. Subsequently, additional heating processes in the outer crust due to electron captures on nuclei were shown to be relevant but were unable to produce the necessary heating in models that included the PBF process in the crust [14].

The preceding discussion motivates a detailed investigation of the neutrino emissivity arising because of the PBF process in neutron matter in the inner crust. Recently, this was recalculated by Leinson and Perez who found that earlier calculations violated vector current conservation [15,16]. An improved treatment that satisfies current conservation yielded a result that was suppressed by the factor v_F^4, where $v_F = k_F/M$ is the neutron Fermi velocity, k_F is the neutron Fermi momentum, and M is the neutron mass. In the neutron star crust, where $v_F \sim 0.1$, this suppression is significant. Subsequently, Sedrakian, Muther, and Schuck also calculated the PBF rate using an improved treatment based on Landau

Fermi liquid theory and found that it was suppressed by the factor $\sim T/M$, where T is the temperature [17]. This suppression is parametrically different from that obtained in Ref. [15]. The PBF rate was also recently examined in Ref. [18] where the authors also accounted for Fermi liquid effects in superfluid neutron matter using the Larkin-Migdal-Leggett formalism [19,20]. They found that the vector current response is suppressed by the factor v_F^4 in agreement with the finding of Leinson and Perez [15,16].

In this article, we reexamine the nature of density and spin-density fluctuations in superfluid neutron matter using a simplified nuclear Hamiltonian. Our main findings are

(i) The spectrum of density fluctuations is suppressed at order v_F^4 in pure neutron matter in agreement with the findings of Leinson and Perez [15,16].
(ii) The v_F^4 suppression is not generic and is *not* a consequence of vector current conservation. It is specific to simple one component systems where all particles have the same (weak) charge to mass ratio. In multicomponent systems such as the neutron star crust where neutrons coexist and interact with nuclei, the density fluctuations of the neutron superfluid occur at order v_F^2.
(iii) For the case of simple nuclear Hamiltonian with only central interactions that conserve spin, spin-density fluctuations occur at order v_F^2 and these fluctuation dominate the neutrino emissivity.
(iv) For the case of realistic nuclear interactions that contain a strong tensor component, spin is *not* conserved, and spin fluctuations arise at order v_F^0. This feature is well known in the context of neutrino emission from neutron-neutron bremsstrahlung. We find that the bremsstrahlung rate continues to be the dominant neutrino emission mechanism in nonrelativistic systems even in the superfluid state unless the temperature is much smaller than T_C.

This article is organized as follows. We begin by discussing the relation between the neutrino emissivity and the density and spin-density response functions. This is followed by a detailed investigation of the density-density response function and the role of vertex corrections in the superfluid state. Here we show that the vertex corrections required by conservation laws strongly suppress the response relative to the predictions of mean-field theory as suggested in earlier work. Subsequently we discuss the spin-density response function and show that it dominates over the density response. Finally, we discuss the various contributions to the neutrino emissivity and conclude that the neutron-neutron bremsstrahlung rate is typically larger than the PBF process even in the superfluid phase. We conclude with a critical discussion of our study here and related earlier work. We recognize that all calculations of the neutrino rates from PBF are missing a key aspect of the nuclear force, namely, the tensor interaction.

II. NEUTRINO EMISSIVITY AND RESPONSE FUNCTIONS

The neutrino emissivity is defined as the rate of energy loss per unit volume and is given by

$$\epsilon_{\nu\bar{\nu}} = -\frac{G_F^2}{4}\int \frac{d^3q_1}{(2\pi)^3 2\omega_1}\int \frac{d^3q_2}{(2\pi)^3 2\omega_2} \times \int d^4\vec{k}\quad \delta^4(\vec{q}_1+\vec{q}_2-\vec{k}) \times \frac{\omega}{\exp(\beta\omega)-1} L^{\alpha\beta}(q_1,q_2)\,\Im m\left[\Pi^R_{\alpha\beta}(k)\right], \tag{1}$$

where G_F is the Fermi weak coupling constant, $k=(\omega,\vec{k})$, $q_{i=1,2}$ are the on-mass-shell four-momenta of neutrinos, $L^{\alpha\beta}(q_1,q_2) = \mathrm{Tr}\,[\gamma^\mu(1-\gamma^5)\,\not{q}_1\gamma^\nu(1-\gamma^5)\,\not{q}_2]$, and $\Pi^R_{\alpha\beta}(q)$ is the retarded polarization tensor [17]. Using Lenard's identity [21], we can simplify Eq. (1) to obtain

$$\epsilon_{\nu\bar{\nu}} = \frac{G_F^2}{192\pi^5}\int d^3\vec{k}\int_0^\infty d\omega\,\Theta[\omega^2-|\vec{k}|^2]\,(k^\alpha k^\beta - k^2 g^{\alpha\beta}) \times \frac{\omega}{\exp(\beta\omega)-1} R_{\alpha\beta}(-\omega,|\vec{k}|), \tag{2}$$

where the superfluid response function $R_{\alpha\beta}(\omega,|\vec{k}|)$ in general contains both the vector and the axial-vector response functions and is given by

$$R_{\alpha\beta}(-\omega,|\vec{k}|) = -c_V^2\,\Im m\left[\Pi^V_{\alpha\beta}(\omega,|\vec{k}|)\right] - c_A^2\,\Im m\left[\Pi^A_{\alpha\beta}(\omega,|\vec{k}|)\right]. \tag{3}$$

In the nonrelativistic limit, we focus on density fluctuations and ignore velocity fluctuations. In this case the vector-polarization function $\Pi^V_{\alpha\beta}(\omega,|\vec{k}|) = \delta^0_\alpha\,\delta^0_\beta\,\Pi_0(\omega,|\vec{k}|)$, where $\Pi_0(\omega,\vec{k})$ is the density-density correlation function [22] given by

$$\Pi_0(\omega,|\vec{k}|) = -i\int d^4x\, e^{-i(\vec{k}\cdot\vec{x}-\omega t)}\,\mathrm{Tr}(\rho_G[\rho(x,t),\rho(0,0)]), \tag{4}$$

where ρ_G is the density matrix and $\rho(x,t)$ is the density operator. Similarly the axial response in the nonrelativistic limit is dominated by spin fluctuations and we can write $\Pi^A_{\alpha\beta}(\omega,|\vec{k}|) = \delta^i_\alpha\,\delta^j_\beta\,\Pi_{ij}(\omega,|\vec{k}|)$, where $i,j=1,2,3$ and $\Pi_{ij}(\omega,\vec{k})$ is the spin correlation function [22] given by

$$\Pi_{ij}(\omega,|\vec{k}|) = -i\int d^4x\, e^{-i(\vec{k}\cdot\vec{x}-\omega t)}\,\mathrm{Tr}\,(\rho_G[\sigma_i(x,t),\sigma_j(0,0)]). \tag{5}$$

The diagonal components in Eq. (5) are equal and are denoted by Π_σ, while the off-diagonal components of Π_{ij} do not contribute to the emissivity of an isotropic medium [4]. We can therefore write the neutrino emissivity as

$$\epsilon_{\nu\bar{\nu}} = \frac{G_F^2}{192\pi^5}\int d^3\vec{k}\,k^2\left[c_V^2 I_\rho(k) + 3c_A^2 I_\sigma(k)\right], \tag{6}$$

where

$$I_\rho(k) = -\int_k^\infty d\omega\quad \frac{\omega}{\exp(\beta\omega)-1}\Im m[\Pi_0(\omega,|\vec{k}|)], \tag{7}$$

$$I_\sigma(k) = -\int_k^\infty d\omega\quad \frac{\omega}{\exp(\beta\omega)-1} \times\left(\frac{\omega^2}{k^2}-\frac{2}{3}\right)\Im m[\Pi_\sigma(\omega,|\vec{k}|)]. \tag{8}$$

III. VECTOR RESPONSE

First, we calculate the vector current response function to verify and understand the nature of suppression factors found in Refs. [15] and [17]. Calculations of the superfluid density response have a long history in condensed matter physics and a pedagogic discussion can be found in Ref. [23]. Here we describe neutron matter at low density with the model Hamiltonian

$$\mathcal{H} = \sum_{p,\text{spin}} \xi_p a^\dagger_{k\uparrow} a_{k\uparrow} + V \sum_{p,p'} a^\dagger_{k\uparrow} a^\dagger_{-k\downarrow} a_{k\uparrow} a_{-k\downarrow}, \tag{9}$$

where V is the effective four-fermion interaction. We regulate the short-range interaction by using a momentum cutoff. The renormalization scheme is implemented by specifying the gap and requiring that $V(\Lambda)$ satisfy the gap equation at zero temperature,

$$\Delta = -V(\Lambda) \int_0^\Lambda \frac{d^3p}{(2\pi)^3} \frac{\Delta}{2E_p}. \tag{10}$$

To describe neutron matter at densities of relevance to the crust, we choose to display results at $k_F^2/(2m) = \mu = 30$ MeV and a momentum cutoff of $\Lambda = 2k_F$. We are ultimately interested in calculating the PBF rate in the temperature range $T \sim 1 - 10 \times 10^8$ K and near $k_F \lesssim 100$ MeV, where the gap $\Delta \lesssim 0.1$ MeV [24]. Although finite-range effects of the nucleon-nucleon interaction are relevant in computing the magnitude of the pairing gap in the neutron star crust, here we restrict our analysis to a simple zero-range interaction but at a strength adjusted to reproduce the pairing gap in more sophisticated calculations [25].

We define the polarization tensor in the mean-field approximation by

$$\Pi^V_{\alpha\beta}(\omega, |\vec{k}|) = -i \int \frac{d^4p}{(2\pi)^4} \text{Tr}[\gamma_\alpha G(p+k) \gamma_\beta G(p)], \tag{11}$$

where $\gamma_\alpha = (\tau_3, \hat{1}(\vec{p} + \vec{k}/2)/M)$. Here the zero-zero component of $\Pi^V_{\alpha,\beta}$ corresponds to the density-density response function defined in Eq. (4). Explicitly this is given by

$$\Pi_{\text{MF}}(\omega, |\vec{k}|) = -i \int \frac{d^4p}{(2\pi)^4} \text{Tr}\,[\tau_3 G(p+k) \tau_3 G(p)], \tag{12}$$

where the quasiparticle propagator in the ${}^1S_0$ superfluid state is given by

$$G(p) = \frac{p_0 \hat{1} + \xi_p \tau_3 + \Delta \tau_1}{p_0^2 - E_p^2 + i\epsilon} \tag{13}$$

and $\hat{1}$ is the unit matrix and $\tau_{i=1,2,3}$ are the 2×2 Pauli matrices (acting in the Nambu-Gorkov space). The quasiparticle energy is $E_p = \sqrt{\xi_p^2 + \Delta^2}$, where $\xi_p = (p^2/2m - \mu)$ and Δ is the superfluid gap [23].

The mean-field polarization tensor violates current conservation and the F-sum rule. This is a well-established finding and a lucid discussion can be found in the original articles by Anderson [26] and Nambu [27]. To restore current conservation it is necessary to replace the bare vertex function $\gamma_\alpha(p+q, p)$ by the dressed vertex $\Gamma_\mu(p+k, p)$ corresponding to the dressed quasiparticles in the superfluid.

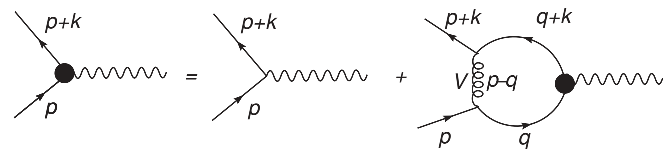

FIG. 1. Diagrammatic representation of the RPA vertex equation. The wavy line is the external weak current and the curly line represents the short-range strong interaction, which in this work is simplified to a point interaction. Solid lines represent the quasiparticles and the dark circle is the dressed vertex.

The dressed vertex is then required to satisfy the generalized Ward identity (GWI) given by

$$\begin{aligned}\omega\Gamma^0(p+k, p) - k_i \Gamma^i(p+k, p) \\ = G^{-1}(p+k)\tau_3 - \tau_3 G^{-1}(p),\end{aligned} \tag{14}$$

where $i = 1, 2, 3$ and as before the four-vector $k = (\omega, \vec{k})$. However the GWI does not uniquely determine the vertex function. It must be obtained explicitly as a solution to an integral equation that describes the modification of the vertex from the medium. This integral equation for the vertex, obtained in the random phase approximation (RPA), is known to satisfy the GWI and is diagrammatically represented in Fig. 1. Explicitly this is given by

$$\Gamma_\alpha = \gamma_\alpha + iV \int \frac{d^4q}{(2\pi)^4} \tau_3\, G(q+k) \Gamma_\alpha G(q) \tau_3. \tag{15}$$

We are interested in the zeroth component of Eq. (15) because it is this that affects the density response. In this case an approximate solution has the form

$$\Gamma_0 = \frac{1}{1+\chi}\tau_3 + \frac{2\kappa}{1+\chi} i\tau_2. \tag{16}$$

In weak coupling, $\chi \simeq VN(0) \ll 1$, where V is the four-fermion coupling and $N(0) \propto Mk_F/\pi$ is the density of state at the Fermi surface and can be neglected. However, κ has an essential singularity at $T = 0$ corresponding to the existence of a Goldstone excitation that couples to density fluctuations. At $T = 0$ and weak coupling,

$$\kappa \simeq \frac{\Delta\omega}{\omega^2 - c_s^2 k^2}, \tag{17}$$

where the speed of the Goldstone mode $c_s \simeq k_F/\sqrt{3}M$. This approximate form for the vertex was used in Ref. [16] to compute the response function. It is possible to solve the vertex equation [Eq. (15)] to obtain the exact solution in the random phase approximation [28]. In Fig. 2 we compare the exact solution (solid line) to the vertex equation to the approximate expression above (dashed line). For a better approximation to the full vertex, we can employ a shifted value of c_s by shifting it so that the approximate and full vertices diverge at exactly the same value of q_0. This "shifted" vertex is also plotted in Fig. 2 with a dotted line, but the result matches sufficiently well with the full vertex that it is not visible. We also find that this relative agreement between the real parts of the full and approximate results is not strongly modified at finite temperature.

In the region where $\omega \geqslant 2\Delta$, where we are above the threshold for producing quasiparticles, we can expect that

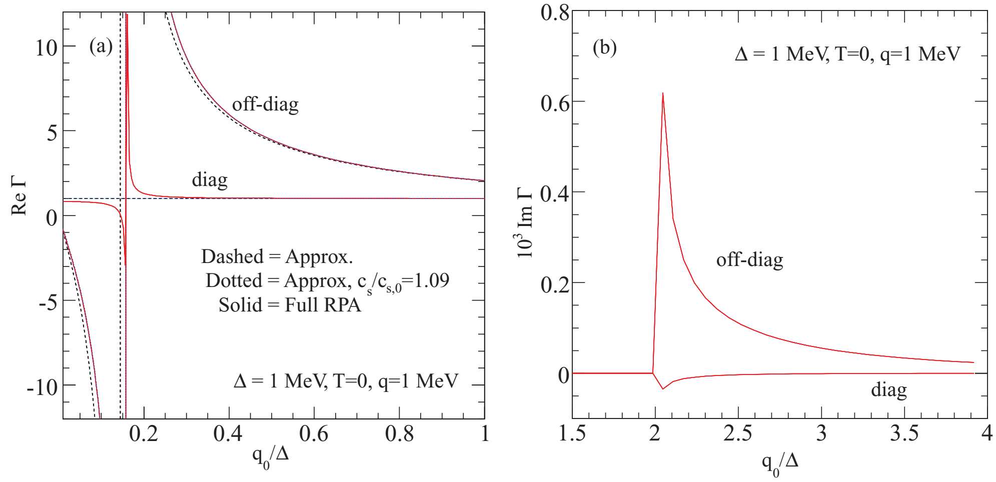

FIG. 2. (Color online) (a) Real parts of the dressed vertex function at fixed momentum transfer. The approximate solution in Eq. (17) is in excellent agreement with the "exact" RPA vertex especially when the speed of sound is shifted to match the pole structure. (b) The imaginary part of the vertex function, which is finite but small. The approximate vertex assumes that the imaginary part is zero.

the vertex equation will have a nonzero imaginary part. The approximate solution to the vertex equation in Eq. (17) neglects this contribution. The imaginary part of the full vertex is plotted in the right-hand panel of Fig. 2. As expected below $\omega = 2\Delta$, the imaginary part of the response vanishes as required from the delta function given by the imaginary part of the propagator in Eq. (18). The magnitude of the imaginary part is much smaller than the real part, but, as we discuss later, could make an be important contribution to the response.

To help make contact with earlier results obtained in Refs. [2] and [15] we discuss the response function in different approximations. First, we obtain the mean-field response at $T = 0$ by doing the p_0 integration in Eq. (12),

$$\Pi_{\rm MF}(\omega, |\vec{k}|) = \int \frac{d^3p}{2(2\pi)^3}\left(1 - \frac{\xi_p\xi_{p+k} - \Delta^2}{E_pE_{p+k}}\right) I_0$$
$$I_0 = \left(\frac{1}{\omega - E_p - E_{p+k} + i\epsilon} - \frac{1}{\omega + E_p + E_{p+k} - i\epsilon}\right). \tag{18}$$

In the long-wavelength limit ($k \to 0$) this can be simplified further and we find that

$$\Pi_{\rm MF}(\omega, |\vec{k}| \to 0) = \int \frac{d^3p}{(2\pi)^3}\frac{\Delta^2}{E_p^2} I_0. \tag{19}$$

For $\omega > 2\Delta$ and $k = 0$ the mean-field result predicts a nonvanishing response given by

$$\Im m[\Pi_{\rm MF}(\omega)] = -\frac{Mp_F}{\pi^2}\left(\frac{\Delta^2}{\omega\sqrt{\omega^2 - 4\Delta^2}} + \frac{\Delta^2}{4\mu\omega}\right). \tag{20}$$

As mentioned earlier a nonzero response at $k = 0$ violates current conservation and the related F-sum rule given by

$$\int_{-\infty}^{\infty} d\omega\, \omega\, \Im m[\Pi_{\rm MF}(\omega)] = \langle[[H, \rho(k)], \rho(k)]\rangle, \tag{21}$$

where the RHS vanishes in the $k = 0$ limit when ρ commutes with the Hamiltonian. This is simply a consequence of the well-known fact that radiation can arise only as a result of particle acceleration for conserved charges. The vertex correction discussed earlier remedies this problem. The RPA response function is defined by

$$\Pi_{\rm RPA}(\omega, |\vec{k}|) = -i\int \frac{d^4p}{(2\pi)^4}{\rm Tr}[\tau_3 G(p+k)\Gamma_0 G(p)], \tag{22}$$

where Γ_0 is the dressed vertex that satisfies Eq. (15). First we use the approximate form of the vertex given in Eq. (17) to obtain the zero-temperature density-density polarization function. As we showed earlier the approximate vertex

$$\Gamma_0 = \tau_3 + \frac{2\Delta\omega}{\omega^2 - c_s^2k^2} i\tau_2 \tag{23}$$

provides a very good description of the real part of the exact result. Substituting this into Eq. (22) and performing the integration over p_0, we obtain

$$\Pi_{\rm RPA}(\omega, k) = \int \frac{d^3p}{2(2\pi)^3}\left(1 - \frac{\xi_p\xi_{p+k} - \Delta^2 + 2\omega\kappa\Delta}{E_pE_{p+k}}\right) I_0. \tag{24}$$

The response function is related to the imaginary part, which is explicitly given by

$$\Im m[\Pi_{\rm RPA}(\omega, k)] = -\int \frac{d^3p}{2(2\pi)^3} J\delta(\omega - E_p - E_{p+k}), \quad \text{where} \tag{25}$$
$$J = \left(1 - \frac{\xi_p\xi_{p+k} - \Delta^2 + 2\omega\kappa\Delta}{E_pE_{p+k}}\right).$$

It is straightforward to verify that the imaginary part of $\Pi_{\rm RPA}$ vanishes in the limit $k \to 0$ as required by current conservation. At finite temperature and for the full vertex this continues to hold and the response vanishes at $k = 0$ because of the gap equation. The critical question then is to inquire if the order k^2 term also vanishes. To address this we expand the zero-temperature RPA response in Eq. (25) in powers of k. Expanding the integrand in Eq. (25), we obtain

$$J = \frac{c_s^2 \Delta^2}{2E_p^4}\left[\frac{p^2x^2}{m^2c_s^2} - 1\right]k^2 + \mathcal{O}\left[c_s k\right]^4, \tag{26}$$

where x is the cosine of the angle between the momenta p and k. In the long wavelength most of the support from the energy delta function is near $p = p_F$. Further, in this region we can approximately replace x^2 by its mean value given $x^2 \simeq \langle x^2\rangle = 1/3$. This implies that the quadratic term nearly vanishes because $c_s \equiv p_F/(m\sqrt{3})$. The nature of this cancellation depends on the value of c_s employed in the approximated vertex. When we use the self-consistent RPA vertex obtained by solving Eq. (15) we indeed find that this cancellation is nearly exact and Eq. (26) receives contributions at order $c_s^4 k^4$.

The imaginary parts of the response function at zero temperature obtained in different approximate schemes are shown in Fig. 3. The striking feature is that results obtained using the self-consistent RPA vertex labeled "Full RPA" in the figure are even highly suppressed because the relevant contribution occurs at order c_s^4. The result labeled "Approx." is obtained using $c_s = c_{s,0} = p_F/(\sqrt{3}M)$. Here the cancellation at order c_s^2 occurs to a high degree but is not exact. In Refs. [15,16], and [29], the authors employ this approximation for the vertex function and make a consistent approximation to the full response function to ensure that cancellation at order c_s^2 is exact. The result from Ref. [29] is included in the figure and differs from our "Full RPA" because we include the imaginary part of the vertex and do not employ an expansion in p_F.

We also note that the density response function calculated within RPA coincides with earlier calculations by Kundu and Reddy [28]. Although our results for the response function and RPA vertex derived here are essentially the same as those in Kundu and Reddy, our numerical results for the response function in the time-like region differ because of a numerical error in the plots shown in Fig. 2 of Ref. [28]. From Fig. 3 it is clear that the magnitude of the suppression obtained using the approximate vertex is sensitive to c_s. This is because of the large cancellation at order $c_s^2 k^2$. To understand the nature of this cancellation we first note that vector current conservation does not require the order $v_F^2 k^2$ contribution to vanish. This would require fine tuning the velocity of the Goldstone mode to precisely cancel the contribution at order k^2. In general, we are aware of no mechanism that can accomplish this in a multicomponent system.

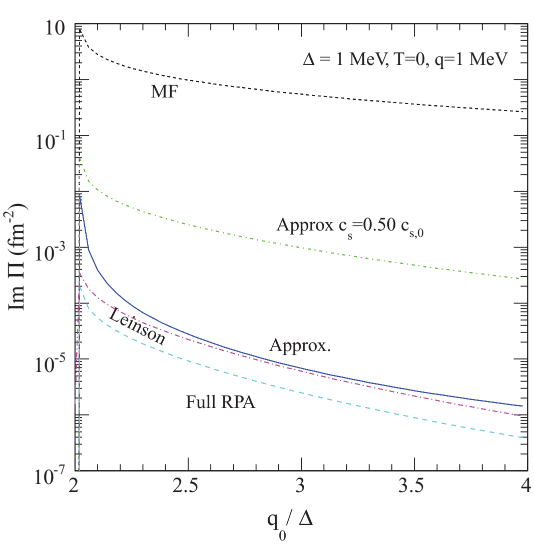

FIG. 3. (Color online) Imaginary part of the density-density polarization at $T = 0$ obtained using different approximations. The vertex corrections induce a large suppression in the response relative to the mean-field result. The result obtained using the self-consistent RPA vertex (shown with the dotted line) is suppressed by a factor $\sim 10^{-4}$. See text for a discussion relating to the response obtained using the approximate vertex.

The preceding arguments raises the following question: *what is the underlying physics responsible for the large cancellation at order $v_F^2 k^2$ occurring in the RPA calculation of the response function for pure neutron matter*? In earlier work described in Refs. [15] and [18], the authors attribute this cancellation to vector current conservation. However, as we show subsequently, even in the case of a simplified nuclear Hamiltonian where the axial current is conserved, the axial response occurs at order $v_F^2 k^2$. Hence current conservation is not adequate to explain this cancellation. To address how this arises we note that a similar cancellation at order $v_F^2 k^2$ occurs in neutrino emission due to density fluctuations in the bremsstrahlung reaction $nn \to nn\nu\bar{\nu}$ in the normal phase. Here the square of the matrix element contributes only at order $v_F^4 k^4$ [30,31]. This result is well understood, especially in the context of electromagnetic bremsstrahlung in proton-proton collisions where radiation occurs due to time variation of the quadrupole moment. In fact, the dipole moment of any system of N identical particles with equal charge e and mass m, irrespective of the underlying dynamics,

$$\vec{d} = \Sigma_i e\vec{r}_i = \frac{e}{m}\Sigma_i m\,\vec{r}_i = \frac{e}{m}\Sigma_i m\,\vec{R}_{\rm CM}, \tag{27}$$

does not vary because momentum conservation ensures that $\dot{R}_{\rm CM} = 0$ and there cannot be any dipole radiation at long wavelength. The leading order radiation is quadrapole radiation, which occurs at order k^4 in the momentum expansion. In our case because each power of k is accompanied by one factor v_F, we expect that radiation can occur only at order $v_F^4 k^4$. This is evident from the observation that in our simple model, where Δ is independent of k, k only enters through the expression for ξ_{p+k}, which at the Fermi surface is given by

$$\xi_{p+k}|_{|\vec{p}|=p_F} = \frac{1}{2m}\left(p_F^2 + k^2 - 2p_F\,kx\right) - \mu \simeq v_F\,kx, \tag{28}$$

because $k \ll p_F$. In the quantum mechanical calculation of the bremsstrahlung process this arises due to the destructive interference between the amplitudes for radiation from the two charges. This interference ensures that the square of the matrix element vanishes at order $v_F^2 k^2$ and the leading contribution is quadrupolar and occurs at order $v_F^4 k^4$.

When the (weak) charge to mass ratio of the two particles is different, the cancellation at order $v_F^2 k^2$ would be absent. For example bremsstrahlung from neutron-proton scattering occurs at order $v_F^2 k^2$. Hence we argue in multicomponent systems that there is *no* symmetry that requires the sensitive cancellation at this order. In the neutron star context, where neutrons in the crust couple to a background lattice of neutron-rich ions, we can expect that these interactions can induce a response at order $c_s^2 k^2$. In our calculation, such interactions can, for example, induce a shift in the superfluid velocity c_s by the polarization of the lattice. As we have seen in Fig. 3, even a small shift leads to a relevant contribution at order $c_s^2 k^2$. Thus we conclude that in realistic situations a nonvanishing density response at order $c_s^2 k^2$ is to expected. However, as we show below, this response is still small compared to the axial current response.

IV. AXIAL CURRENT RESPONSE

In the nonrelativistic limit, the diagonal part of the axial polarization tensor defined in Eq. (5) can be written in terms of the Nambu-Gorkov propagators [28]. To begin we ignore vertex corrections because there is no Goldstone mode associated with spin fluctuations in the case of singlet pairing and discuss the one-loop mean-field polarization tensor. In this case we can write

$$\Pi_\sigma(\omega, |\vec{k}|) = -i \int \frac{d^4 p}{(2\pi)^4} \,\mathrm{Tr}\,[\hat{1} G(p+k) \hat{1} G(p)]. \quad (29)$$

At $T = 0$ we can do the p_0 integration to obtain

$$\Pi_\sigma(\omega, |\vec{k}|) = \int \frac{d^3 p}{2(2\pi)^3} \left(1 - \frac{\xi_p \xi_{p+k} + \Delta^2}{E_p E_{p+k}}\right) I_0, \quad (30)$$

and the imaginary part in the region $\omega \geqslant 0$ is given by

$$\Im m[\Pi_\sigma(\omega, |\vec{k}|)] = -\int \frac{d^3 p}{2(2\pi)^3} \left(1 - \frac{\xi_p \xi_{p+k} + \Delta^2}{E_p E_{p+k}}\right) \times \delta(\omega - E_p - E_{p+k}). \quad (31)$$

It is easily verified that the imaginary part of Eq. (31) vanishes at $k = 0$. Expanding the integrand in Eq. (31) in powers of k we find that

$$\Im m[\Pi_\sigma(\omega, |\vec{k}|)] = -\frac{1}{32\pi^2} \int dp\, p^2 \int dx \times \left(\frac{p^2}{m^2} \frac{\Delta^2}{E_p^4} x^2 k^2 + \mathcal{O}[k^4]\right) \times \delta(\omega - E_p - E_{p+k}), \quad (32)$$

where x is the angle between $\vec{k}$ and $\vec{p}$. In the long-wavelength limit the delta function provides support only in region $p \simeq p_F$. Here we can further simply the result to obtain

$$\Im m[\Pi_\sigma(\omega, |\vec{k}|)] = -\frac{1}{48\pi^2} v_F^2 k^2 \int dp\, p^2 \frac{\Delta^2}{E_p^4} \int dx\, \delta(\omega - E_p - E_{p+k}) + \mathcal{O}[k]^4. \quad (33)$$

In Fig. 4 we plot the axial response function. The vector response obtained in different approximations discussed earlier is also shown for comparison. The results indicate the axial response is significantly larger because of the large

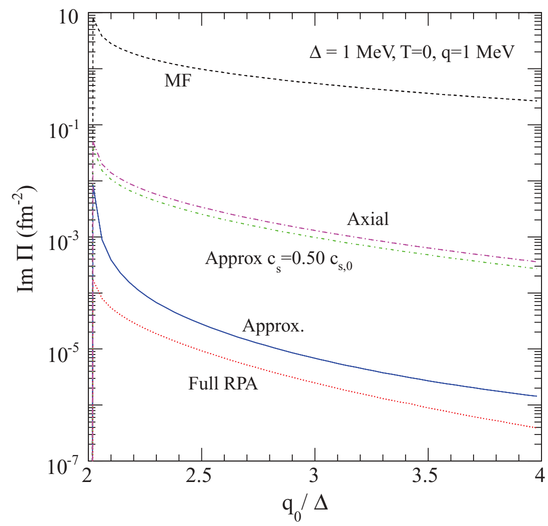

FIG. 4. (Color online) The imaginary part of the axial response function in the nonrelativistic limit.

cancellation in the vector response at order k^2. In situations where this cancellation does not occur, as in the case when we set $c_s \simeq 0.5 c_{s,0}$, the vector and axial response functions can become comparable.

Note that the argument for the cancellation in the vector response for pure neutron matter at order k^2 does not apply in the case of the axial response. Neutrons have two spin states and we can assign a plus and minus "axial" charge to the up and down spin states. The corresponding dipole moment of the axial charge will change during collisions and radiation can occur at the k^2.

The one-loop axial response in Eq. (29) vanishes in the $k = 0$ limit. This is consistent with the F-sum rule for the spin response associated the model Hamiltonian in Eq. (9). However it is well-known that the realistic nuclear Hamiltonian that contains both tensor and spin-orbit interactions does not commute with the spin operator. This implies that for realistic nuclear interactions that include pion exchange

$$\int_{-\infty}^{\infty} d\omega\, \omega\, \mathrm{lt}_{k\to 0} \Im m[\Pi_\sigma(\omega)] = \langle [[H, \sigma(k=0)], \sigma(k=0)] \rangle \neq 0. \quad (34)$$

Consequently, the correct axial response obtained using realistic interactions has to be finite in the limit $k = 0$. From Eq. (33) we see that the mean-field response function violates this expectation. Corrections to the one-loop axial response function arising due to noncentral interactions such as pion exchange therefore will be critical in the long-wavelength limit [32].

The importance of pion exchange in the axial response and neutrino emissivity was already realized in the pioneering work of Friman and Maxwell [30]. Although their calculations were only applicable to the normal phase of neutron matter, they showed that pion exchange was more important than central nuclear interactions. In particular, they found that the matrix element for neutrino emission from neutron-neutron bremsstrahlung in the axial channel was nonzero in the

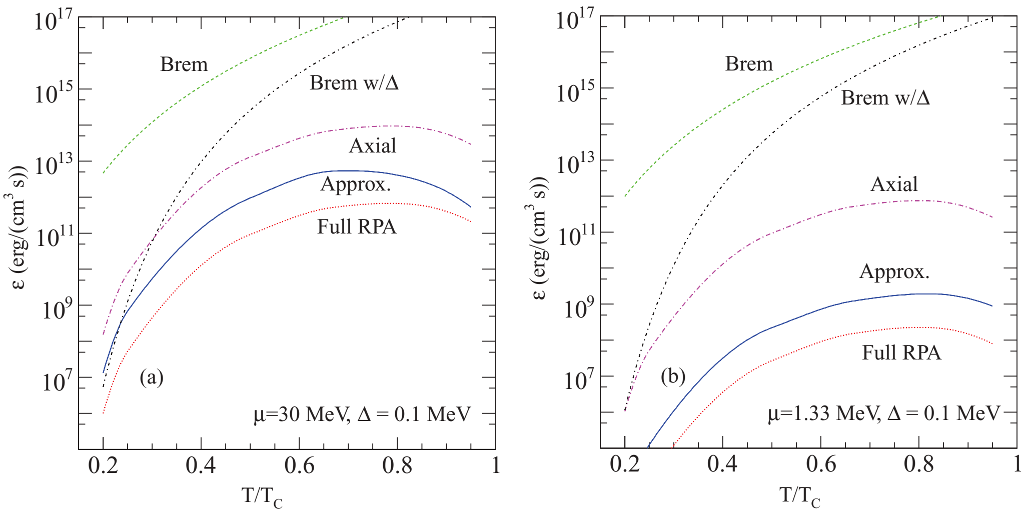

FIG. 5. (Color online) The neutrino bremsstrahlung and pair-breaking emissivities: (a) at a neutron chemical potential of 30 MeV, corresponding to a density slightly below nuclear saturation density, and (b) at a neutron chemical potential of 1.33 MeV corresponding to a density just above the neutron drip density. The curves labeled "Brem" are the bremsstrahlung emissivity in non-superfluid neutron matter and the curves labeled "Brem w/Delta" are the corresponding emissivity in superfluid neutron matter. The curves labeled "Axial" are the axial part of the pair-breaking emissivity and the curves labeled "Full RPA" are the vector part of the pair-breaking emissivity using the full RPA vertex. The curves labeled "Approx" are the vector pair-breaking emissivity using the approximate vertex. The solid lines are the vector pair-breaking emissivity using the approximate vertex.

long-wavelength limit when they included one-pion exchange. It is therefore critical to incorporate these noncentral interactions in the calculation of the axial response function of the neutron superfluid. The role of these noncentral interactions have been studied in earlier work for the case of neutron-neutron bremsstrahlung in the normal phase [33], but their role in the PBF process remains to be studied. This would require that we include both vertex corrections and two-loop effects that include 2p-2h excitations in the superfluid.

This work is in progress and will be reported elsewhere.

V. NEUTRINO EMISSIVITY

The PBF emissivity is largest when the temperature is just below the critical temperature. As neutron density increases from zero at the neutron drip density, the gap increases rapidly with the increase in the density of states at Fermi surface. Eventually above the saturation density, the neutron-neutron interaction becomes repulsive in the $^1S_0$ channel and the gap decreases with increasing density. In between these two extremes, the largest value of the gap is typically on the order of 1–3 MeV (for recent computations of the gap, see Refs. [25,34–38]). The PBF neutrino emissivity in the crust is expected to be particularly relevant for the crustal cooling of accreting neutron stars where the temperature is $\lesssim 10^9$ K. Correspondingly, we can expect two regions in a neutron star crust where $T \simeq T_c$: one just above neutron drip and one near the saturation density before the gap becomes very small due to nucleon-nucleon repulsion. In the region just above neutron drip where $k_F a$ (where a is the neutron-neutron scattering length) is large and the effective range of the neutron-neutron interaction is small, the gap can be determined quite precisely from quantum Monte Carlo simulations that have been tested in analogous systems in cold atom experiments involving ^{6}Li fermion atoms [39]. When $k_F a$ is small, the gap is given by Gorkov's celebrated result $\Delta \simeq 0.5\mu \exp(-\pi/k_F a)$. At moderate values of $k_F a \simeq 1$, the gap parameter is not yet well determined. For the purpose of obtaining numerical results choose $\Delta = 0.1$ MeV at $\mu = 1.33$ MeV, a value inspired by a simple interpolation between QMC results at large $k_F a$ and Gorkov's result at $k_F a \ll 1$. At higher densities, medium- and finite-range effects make the computation of the gap imprecise. We arbitrarily choose $\Delta = 0.1$ MeV at $\mu = 30$ MeV to enable easy comparison to Ref. [28].

The neutrino emissivity for the processes discussed above is plotted in Fig. 5, including the axial and vector contributions to the PBF emissivity and the bremsstrahlung emissivity with and without the suppression factors resulting from superfluidity as obtained in Ref. [4]. The emissivity from the axial response dominates and is larger than the full RPA vector response. As the density decreases, the much larger decrease in the vector part of the pair-breaking emissivity in comparison to a smaller decrease in the axial part reflects the larger suppression of $c_s^4 k^4$ for the vector part. A shift of the speed in sound of the Cooper pair excitations will modify the emissivity quadratically in c_s as expected from the imaginary part of the vector response. The bremsstrahlung emissivity is larger than the pair-breaking emissivity except at low temperatures when the suppression of the bremsstrahlung from superfluidity is strong.

VI. CONCLUSIONS

We have studied the one-loop vector and axial current response functions of relevance to neutrino emission in superfluid neutron matter. Through explicit calculations we have found that there is strong suppression of the vector response when vertex corrections are included. This is in close agreement with the findings of Leinson and Perez [15]. For pure neutron matter, the RPA vertex function does indeed show that the supression factor is of order v_F^4. However, we have shown that this suppression arises not only because of vector current conservation, but also because the radiation in the vector channel for pure neutron matter occurs only due to time variation of the quadrupole moment. In the realistic context where neutrons interact with the background lattice of neutron-rich ions, the suppression in the vector channel is of order v_F^2. We showed that even a small shift in the speed of the Goldstone mode due to the lattice can make a relevant contribution to the response at this order. Finally, we showed that both of these emissivities are likely smaller than the neutrino bremsstrahlung emissivity (including the suppression factors from superfluidity) unless the temperature is significantly smaller than the critical temperature.

The axial response function was shown to be numerically more important because the dominant contribution occurs at order v_F^2. Although this was the relevant contribution in our model, we demonstrated that tensor interactions arising due to pion exchange would lead to important corrections to this estimate. The F-sum rule in this case indicates that these corrections would result in a nonvanishing response at order v_F^0. In the normal phase the tensor interaction is responsible for the emissivity at long wavelength and we suspect that this will continue to be the case in superfluid matter. We anticipate that tensor interactions will also affect the PBF fluctuations in the spin channel and can modify our results and the regime in temperature where PBF can dominate over the bremsstrahlung emissivity. This issue is currently being investigated and will be reported elsewhere. Although it is now clear that the neutrino emissivity due to density fluctuations arising from PBF processes in the vicinity of the critical temperature is not significant, our present understanding of neutrino processes in the superfluid phase remains rather incomplete and warrants further study.

Finally, the vector part of the pair-breaking emissivity is unlikely to make a large contribution to the emissivity of accreting neutron star crusts. Our results show that the pair-breaking emissivity is smaller than the bremsstrahlung emissivity only for T much smaller than T_C, where the contributions from both emissivities are small. This is good news for superburst models because unlike the rapid cooling induced by the erroneous pair-breaking emissivity of Ref. [2], the bremsstrahlung rate is too small [11] to reduce the temperature in the outer crust needed to ignite carbon. However, a thorough investigation of how the nuclear tensor modifies the axial response in the superfluid phase is needed to adequately resolve this issue.

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