PHYSICAL REVIEW C **78**, 064909 (2008)

Entropy flow of a perfect fluid in (1+1) hydrodynamics

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(Received 5 September 2008; published 31 December 2008)

Using the formalism of the Khalatnikov potential, we derive exact general formulas for the entropy flow dS/dy, where y is the rapidity, as a function of temperature for the (1+1) relativistic hydrodynamics of a perfect fluid. We study in particular flows dominated by a sufficiently long hydrodynamic evolution and provide an explicit analytical solution for dS/dy. We discuss the theoretical implications of our general formulas and some phenomenological applications for heavy-ion collisions.

DOI: 10.1103/PhysRevC.78.064909 PACS number(s): 12.38.Mh, 24.10.Nz, 25.75.Ld, 25.75.Nq

I. INTRODUCTION

There is accumulating evidence that hydrodynamics may be relevant for the description of the medium created in high-energy heavy-ion collisions [1]. Indeed, experimental measurements of, for example, elliptic flow [2] show the existence of a collective effect on the produced particles which can be described in terms of a motion of the fluid. More precisely, numerical simulations of the hydrodynamic equations describe quite well the distribution of low-$p_\perp$ particles [1], with an equation of state close to that of a "perfect fluid" with a rather low viscosity. On the other hand, it seems useful to discuss a simplified picture [3,4] that can be qualitatively understood in physical terms, namely, the idea that the evolution of the system before freeze-out is dominated by the longitudinal motion. Thus, the hydrodynamic transverse motion can be neglected or at least factorized out to study only the longitudinal flow.

Indeed, the two seminal applications of relativistic hydrodynamics to particle and heavy-ion collisions, by Landau [3] and by Bjorken [4], start with this (1+1) approximation, valid in the determinative stage of the reaction. The longitudinal hydrodynamic approach has found many applications. It has been used in the literature [5,6] to discuss aspects of hydrodynamic flow that are relevant to the physical understanding of high-energy particle scattering and, more recently [7], of heavy-ion collisions.

Soon after the first proposal by Landau and its derivation of a large-time approximation [3], Khalatnikov [8] showed that (1+1) hydrodynamics derive from a potential verifying a linear equation. The Khalatnikov potential has been used in the literature in an initial period [9,10], but it has not been recently considered, to our knowledge. Very recently, the interest in looking for exact solutions of (1+1) hydrodynamics has been revived, and one finds new examples and applications of exact solutions, e.g., in Refs. [11,12]. For instance, in a recent paper [11], a unified description of Bjorken and Landau (1+1) flows has been proposed as a class of exact solutions of (1+1) hydrodynamics based on harmonic flows. (1+1) hydrodynamics appears also quite recently in the application of string-theory ideas to the formation of a strongly interacting quark-gluon plasma [13]. These exact solutions allow one to find explicit analytical solutions for relevant observables. Among these, the entropy flow dS/dy, where y is the rapidity, is quite interesting, since it may be related to the multiplicity distribution of particles. Our goal is to go beyond the particular cases and obtain a general expression of entropy flow as a function of temperature for a generic solution of (1+1) hydrodynamics, i.e., for a generic solution of the Khalatnikov equation.

We are interested in the distribution dS/dy of entropy density per unit of rapidity y which is related to the flow velocities $u_\pm = e^{\pm y}$ in the (1+1) approximation. This "hydrodynamic observable" depends in an essential way on the assumed hypersurface[1] through which we want to compute the flow (and eventually relate it to physical observables in collisions). For given entropy s and energy ϵ densities, dS/dy depends on the hypersurface one considers to follow the hydrodynamic evolution. It is particularly interesting to consider hypersurfaces corresponding to a fixed temperature. Indeed, it allows one to follow the cooling of the hydrodynamic flow, from an initial stage characterized by a high temperature, toward a final stage often associated with a freeze-out temperature. Hence our aim is to derive an expression of dS/dy as a function of temperature for a given Khalatnikov potential and to investigate its properties, from both theoretical and phenomenological points of view.

Our main new result, i.e., the general expressions for entropy flow as a function of the Khalatnikov potential, can be found in three different versions [Eqs. (49), (50), and (53)]; one or the other expression can be more suitable for a given explicit problem.

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[1] We keep the term *hypersurface* in the (1+1) case to keep track, even formally, of the transverse motion.

The plan of our paper is as follows. First, in Sec. II, we group, for completion, all the necessary material including the hydrodynamic equations and the Khalatnikov potential, its equation and solutions, recast and derived in a modern framework using light-cone variables. In Sec. III, we formulate and derive the general expression of the entropy flow in rapidity as a function of temperature evolution. In Sec. IV, we derive and study a family of exact solutions, namely, the ones in which the final entropy distribution is dominated by the hydrodynamic evolution and not by the initial conditions. They generalize the Landau flow and give the asymptotic behavior of physical flows in the limit of long hydrodynamic evolution. We provide in particular the exact analytic expression of the final entropy distribution corresponding to the Belenkij-Landau [9] solution. Then, in Sec. V, we compare the profile of the entropy distributions, as well as their energy dependence, with the relevant experimental data. The final section is devoted, as usual, to the conclusions and outlook.

II. (1+1) RELATIVISTIC HYDRODYNAMICS OF A PERFECT FLUID

A. Hydrodynamic equations

We consider a perfect fluid whose energy-momentum tensor is

$$T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu - p\eta^{\mu\nu}, \tag{1}$$

where ϵ is the energy density, p is the pressure, and u^μ ($\mu = \{0, 1, 2, 3\}$) is the four-velocity in the Minkowski metric $\eta^{\mu\nu}$. It obeys the equation

$$\partial_\mu T^{\mu\nu} = 0. \tag{2}$$

Using the standard thermodynamic identities (where we have assumed for simplicity vanishing chemical potential)

$$p + \epsilon = Tsl, \quad d\epsilon = T ds, \quad dp = s\, dT, \tag{3}$$

the system of hydrodynamic equations closes by relating energy density and pressure through the general equation of state

$$\frac{dp}{d\epsilon} = \frac{s\, dT}{T\, ds} = c_s^2(T). \tag{4}$$

We consider now the (1+1) approximation of the hydrodynamic flow, restricting it only to the longitudinal direction. Within such an approximation, the effect of the transverse dimensions is only reflected through the equation of state (4). Note that we do not *a priori* assume the traceless condition $T^{\mu\mu} = 0$, and thus the fluid is considered as "perfect" (null viscosity) but not necessarily "conformal" (null trace).

Let us introduce the light-cone coordinates

$$z^\pm = t \pm z = z^0 \pm z^1 = \tau e^{\pm\eta} \;\Rightarrow\; \left(\frac{\partial}{\partial z^0} \pm \frac{\partial}{\partial z^1}\right) = 2\frac{\partial}{\partial z^\pm} \equiv 2\partial_\pm, \tag{5}$$

where $\tau = \sqrt{z^+ z^-}$ is the proper time, and $\eta = \frac{1}{2}\ln(z^+/z^-)$ is the space-time rapidity of the fluid. We also introduce for further use the light-cone components of the fluid velocity

$$u^\pm \equiv u^0 \pm u^1 = e^{\pm y}, \tag{6}$$

where y is the usual rapidity variable (in the energy-momentum space).

The hydrodynamic equations (2) take the form

$$\begin{aligned} &(\partial_+ + \partial_-)T^{00} + (\partial_+ - \partial_-)T^{01} = 0,\\ &(\partial_+ + \partial_-)T^{01} + (\partial_+ - \partial_-)T^{11} = 0. \end{aligned} \tag{7}$$

B. Khalatnikov potential

It is known [8,9] that one can replace the nonlinear problem of (1+1) hydrodynamic evolution with a linear equation for a suitably defined potential. In this section, we follow the method of Ref. [8], recasting the calculations in the light-cone variables.

Inserting in Eq. (7) the known relations of Eq. (1) for $T^{\mu\nu}$ and expressing everything in light-cone coordinates using Eqs. (5) and (6), one obtains the following two equations:

$$\begin{aligned} &\left(\frac{e^{2y}-1}{2}\right)\partial_+(\epsilon+p) + e^{2y}(\epsilon+p)\partial_+ y\\ &\quad+\left(\frac{1-e^{-2y}}{2}\right)\partial_-(\epsilon+p) + e^{-2y}(\epsilon+p)\partial_- y\\ &\quad+\partial_+ p - \partial_- p = 0,\\ &\left(\frac{e^{2y}+1}{2}\right)\partial_+(\epsilon+p) + e^{2y}(\epsilon+p)\partial_+ y\\ &\quad+\left(\frac{1+e^{-2y}}{2}\right)\partial_-(\epsilon+p) - e^{-2y}(\epsilon+p)\partial_- y\\ &\quad-\partial_+ p - \partial_- p = 0. \end{aligned} \tag{8}$$

In Eq. (8), the energy density ϵ and pressure p are considered as functions of the kinematic light-cone variables (z^+, z^-). One key ingredient of the potential method [8] is to express the hydrodynamic equations in terms of the hydrodynamic variables $y = \ln u^+ = -\ln u^-$ and $\theta = \ln\,[T/T_0]$, where T_0 is an arbitrary temperature scale.

Relations (8) can be further transformed by inserting the differentials of the thermodynamic relations (3), namely,

$$\partial_\pm(\epsilon + p) = T_0\partial_\pm(se^\theta), \quad \partial_\pm p = T_0 s\,\partial_\pm e^\theta. \tag{9}$$

Multiplying the first equation of Eq. (8) by $(e^{-2y} + 1)$ and the second by $(e^{-2y} - 1)$, adding them, and using Eq. (9), one obtains

$$\partial_+(e^{\theta+y}) = \partial_-(e^{\theta-y}). \tag{10}$$

Equation (10) proves the existence of a potential[2] $\Phi(z^+, z^-)$ verifying that

$$\partial_\mp\Phi(z^+, z^-) \equiv u^\pm T = T_0\, e^{\theta\pm y}. \tag{11}$$

In this way, Eq. (10) is automatically satisfied.

[2]The function Φ has some degree of arbitrariness, since we could define $\partial_\mp\Phi \equiv T_0 e^{\theta\pm y} + \varphi_\mp(z^\mp)$, with $\varphi_-(z^-)$ and $\varphi_+(z^+)$ arbitrary one-variable functions. This freedom, analogous to a gauge choice, does not modify the final results.

To transform the system of equations (8) from the kinematic variables (z^+, z^-) to the dynamical ones (θ, y), one introduces [8] the Khalatnikov potential χ, considered as a function of (u^+T, u^-T) through a Legendre transform:

$$\chi(u^+T, u^-T) \equiv \Phi(z^+, z^-) - z^-u^+T - z^+u^-T, \quad (12)$$

where $z^\pm$ are functions of (u^+T, u^-T) implicitly defined by Eq. (11). Hence, we get

$$\frac{\partial\chi}{\partial(u^\mp T)} = -z^\pm + [\partial_+\Phi - u^-T]\frac{\partial z^+}{\partial(u^\mp T)} + [\partial_-\Phi - u^+T]\frac{\partial z^-}{\partial(u^\mp T)} \equiv -z^\pm, \quad (13)$$

where, due to the relations (11), the terms between brackets are zero. Knowing the Khalatnikov potential χ, which is a function of the thermodynamic variables, one can find the kinematic variables of the flow by derivation.

In the following, we will always consider the Khalatnikov potential χ as function of θ and y, keeping the same notation for χ. That change of variables corresponds for the differential operators to

$$\frac{\partial}{\partial(u^\pm T)} = \frac{1}{2T_0}\, e^{-\theta\mp y}(\partial_\theta \pm \partial_y). \quad (14)$$

In those variables, relation (13) is written as

$$z^\pm(\theta, y) = \frac{1}{2T_0}e^{-\theta\pm y}(-\partial_\theta\chi \pm \partial_y\chi). \quad (15)$$

From Eq. (15), one also gets the expressions for the proper time τ and the space-time rapidity η [defined as in Eq. (5)], that is,

$$\begin{aligned}\tau(\theta, y) &= \frac{e^{-\theta}}{2T_0}\sqrt{(\partial_\theta\chi)^2 - (\partial_y\chi)^2},\\ \eta(\theta, y) &= y + \frac{1}{2}\ln\left(\frac{-\partial_\theta\chi + \partial_y\chi}{-\partial_\theta\chi - \partial_y\chi}\right)\\ &= y - \tanh^{-1}\left(\frac{\partial_y\chi}{\partial_\theta\chi}\right).\end{aligned} \quad (16)$$

C. Khalatnikov equation

Coming back to the system of equations (8), another independent combination can be obtained. Multiplying the first equation by $(e^{-2y} - 1)$, the second by $(e^{-2y} + 1)$ and adding, we obtain $\partial_+(e^y s) + \partial_-(e^{-y}s) = 0$, or equivalently,

$$\partial_+(u^+s) + \partial_-(u^-s) = 0. \quad (17)$$

This relation corresponds physically to the conservation of the entropy along the flow. It is a property of the perfect fluid that the motion of the pieces of the fluid along the velocity lines is isentropic.

Following the logics of the Legendre transform, we transform relation (17) using the (θ, y)-base. For this sake, we write down the following partial derivatives:

$$\begin{aligned}\frac{\partial(u^\pm s)}{\partial\theta} &\equiv u^\pm\frac{ds}{d\theta} = \frac{\partial(u^\pm s)}{\partial z^+}\frac{\partial z^+}{\partial\theta} + \frac{\partial(u^\pm s)}{\partial z^-}\frac{\partial z^-}{\partial\theta}\\ \frac{\partial(u^\pm s)}{\partial y} &\equiv \pm u^\pm s = \frac{\partial(u^\pm s)}{\partial z^+}\frac{\partial z^+}{\partial y} + \frac{\partial(u^\pm s)}{\partial z^-}\frac{\partial z^-}{\partial y}.\end{aligned} \quad (18)$$

Solving this system of linear equations, we obtain

$$\begin{aligned}\frac{\partial(u^+s)}{\partial z^+} &= \frac{1}{D}\left[u^+\frac{\partial z^-}{\partial y}\frac{ds}{d\theta} - u^+s\frac{\partial z^-}{\partial\theta}\right]\\ \frac{\partial(u^-s)}{\partial z^-} &= -\frac{1}{D}\left[u^-\frac{\partial z^+}{\partial y}\frac{ds}{d\theta} + u^-s\frac{\partial z^+}{\partial\theta}\right],\end{aligned} \quad (19)$$

where[3]

$$D = \frac{\partial z^+}{\partial\theta}\frac{\partial z^-}{\partial y} - \frac{\partial z^+}{\partial y}\frac{\partial z^-}{\partial\theta}. \quad (20)$$

Inserting Eq. (19) into the entropy-flow conservation relation (17), we acquire

$$\frac{ds}{d\theta}\left[u^+\frac{\partial z^-}{\partial y} - u^-\frac{\partial z^+}{\partial y}\right] - s\left[u^+\frac{\partial z^-}{\partial\theta} + u^-\frac{\partial z^+}{\partial\theta}\right] = 0. \quad (21)$$

Obtaining the expression of the $z^\pm$ derivatives from Eq. (15), Eq. (21) leads to

$$\frac{1}{s}\frac{ds}{d\theta}\left[\partial_\theta\chi - \partial_y^2\chi\right] - \partial_\theta\chi + \partial_\theta^2\chi = 0. \quad (22)$$

Making use of the sound velocity relation (4), we finally arrive at the Khalatnikov equation [8,9]:

$$c_s^2\,\partial_\theta^2\chi(\theta, y) + \left[1 - c_s^2\right]\partial_\theta\chi(\theta, y) - \partial_y^2\chi(\theta, y) = 0. \quad (23)$$

Hence, the nonlinear system of equations that governs the (1+1) hydrodynamic flow has been converted into a linear, second-order, hyperbolic partial differential equation. Note that the Khalatnikov equation is valid independently from the specific form of the sound velocity.

D. Application: solutions of the Khalatnikov equation for fixed c_s

In this section, for our purpose, we present the solutions of the Khalatnikov equation with a constant speed of sound:

$$c_s^2 \equiv \frac{p}{\epsilon} = g^{-1}, \quad (24)$$

where g will be considered as a parameter in the Khalatnikov equation (23). Note that in this case, the general relations (3) are written as

$$\epsilon = gp = \epsilon_0\left(\frac{T}{T_0}\right)^{g+1} = \epsilon_0\, e^{(g+1)\theta} \quad (25)$$

for the energy density and

$$s = s_0\left(\frac{T}{T_0}\right)^{g} = s_0\, e^{g\theta} \quad (26)$$

[3]We assume that the determinant D is nonzero, which is the case except for exceptional lines [9].

for the entropy density.

Writing

$$\chi(\theta, y) = e^{-(\frac{g-1}{2})\theta} Z(\theta, y) \tag{27}$$

and inserting it into Eq. (23), we acquire

$$\partial_\theta^2 Z - g\partial_y^2 Z - \left(\frac{g-1}{2}\right)^2 Z = 0, \tag{28}$$

where we use a compact notation for the partial derivatives.

It is convenient to replace the variables θ and y by α and β, defined by

$$\alpha \equiv -\theta + \frac{y}{\sqrt{g}} \quad \text{and} \quad \beta \equiv -\theta - \frac{y}{\sqrt{g}}, \tag{29}$$

such that Eq. (28) takes the form

$$\partial_\alpha\partial_\beta \bar{Z}(\alpha, \beta) - \frac{(g-1)^2}{16}\bar{Z}(\alpha, \beta) = 0. \tag{30}$$

We solve this equation following the Green's function formalism, i.e., we look for distributions $\bar{G}(\alpha, \beta)$ such that

$$\partial_\alpha\partial_\beta \bar{G}(\alpha, \beta) - \frac{(g-1)^2}{16}\bar{G}(\alpha, \beta) = \delta(\alpha)\delta(\beta). \tag{31}$$

The relevant solution of Eq. (31) is[4]

$$\bar{G}(\alpha, \beta) = \Theta(\alpha)\Theta(\beta)\, I_0\left(\frac{g-1}{2}\sqrt{\alpha\beta}\right), \tag{32}$$

with I_0 the modified Bessel function of the first kind and Θ the Heaviside function. Using the relation

$$\begin{aligned}\delta(\alpha)\,\delta(\beta) &\equiv \delta\left(-\theta + \frac{y}{\sqrt{g}}\right)\,\delta\left(-\theta - \frac{y}{\sqrt{g}}\right)\\ &= \sqrt{g}\;\delta(\theta)\delta(y),\end{aligned} \tag{33}$$

we deduce from Eq. (32) the relevant Green's function of Eq. (28), i.e.,

$$\begin{aligned}G(\theta, y) &= \frac{1}{4\sqrt{g}}\bar{G}(\alpha, \beta)\\ &= \frac{1}{4\sqrt{g}}\,\Theta\left(-\theta + \frac{y}{\sqrt{g}}\right)\,\Theta\left(-\theta - \frac{y}{\sqrt{g}}\right)\\ &\quad\times I_0\left(\frac{g-1}{2}\sqrt{\theta^2 - \frac{y^2}{g}}\right),\end{aligned} \tag{34}$$

which verifies

$$\partial_\theta^2 G - g\,\partial_y^2 G - \left(\frac{g-1}{2}\right)^2 G = \delta(\theta)\delta(y). \tag{35}$$

[4]There exist other Green's functions of Eq. (31), with, e.g., $\Theta(-\alpha)$ instead of $\Theta(\alpha)$, or $\Theta(-\beta)$ instead of $\Theta(\beta)$. Assuming that the fluid naturally expands and cools down during the evolution, and taking the arbitrary temperature scale T_0 to be the maximal temperature of the sources, $-\theta$ increases with time. Thus Eq. (32) gives the only physical solution of Eq. (31), analogous to the retarded propagator of the d'Alembert equation. Finally, note that for the obvious physical requirement of finite behavior at $\alpha, \beta \to \infty$, we reject the solutions of Eq. (31) containing the Bessel-K_0 function instead of I_0.

Thus, we can construct the general solution of Khalatnikov equation (23), inserting a distribution of sources $F(\hat{\theta}, \hat{y})$, as

$$\begin{aligned}\chi(\theta, y) &= e^{-\left(\frac{g-1}{2}\right)\theta}\int d\hat{y}\int d\hat{\theta} G(\theta - \hat{\theta}, y - \hat{y})\, F(\hat{\theta}, \hat{y})\\ &= \frac{e^{-\left(\frac{g-1}{2}\right)\theta}}{4\sqrt{g}}\int d\hat{y}\int_{\theta+|y-\hat{y}|/\sqrt{g}}^{+\infty} d\hat{\theta} F(\hat{\theta}, \hat{y})\\ &\quad\times I_0\left(\frac{g-1}{2}\sqrt{(\theta-\hat{\theta})^2 - \frac{(y-\hat{y})^2}{g}}\right).\end{aligned} \tag{36}$$

Equation (36) gives the most general solution for any distribution of sources of hydrodynamic flow. In the context of heavy-ion collisions, we are mostly interested in solutions that correspond to the evolution of a flow starting from initial conditions on a curve of the (θ, y) plane. Therefore, one should impose constraints on $F(\hat{\theta}, \hat{y})$, in order to describe the initial conditions. In Sec. IV we will consider a physically interesting subclass of solutions.

III. DERIVATION OF ENTROPY FLOW

Coming back to the general formalism, let us now derive the exact formula for the entropy flow dS/dy at a given fixed temperature $T_F = T_0 e^{\theta_F}$, as a function of rapidity y. For a general (1+1) hydrodynamic expansion, we consider the solution formulated in terms of the general Khalatnikov potential $\chi(\theta, y)$, given by Eq. (36). The entropy distribution at fixed temperature is expressed through the amount of entropy flowing through the hypersurface of fixed temperature T_F, in an infinitesimal rapidity interval. It is given by (see, e.g., Ref. [11])

$$\frac{dS}{dy} \equiv s_F\frac{u^\mu d\lambda_\mu}{dy} = s_F u^\mu n_\mu \frac{d\lambda}{dy}, \tag{37}$$

where $d\lambda$ is the infinitesimal (space-like) length element along the hypersurface of fixed temperature T_F, and n^μ is the normal to the hypersurface. The entropy density depends only on the temperature and not on y. Hence it is constant along the fixed-temperature hypersurface, namely, $s_F \equiv s(T_F) \propto T_F^g$.

A. Flow through the fixed-temperature hypersurface

As we have mentioned, we concentrate on hypersurfaces at fixed temperature T_F (or equivalently at $\theta_F = \ln\,[T_F/T_0]$). It is convenient to use as kinematic functions the proper time $\tau = \sqrt{z^+z^-}$ and the space-time rapidity $\eta = \frac{1}{2}\ln(z^+/z^-)$, considered as functions of θ and y. In this (θ, y) base, the fixed-temperature hypersurface is parametrized by

$$\begin{aligned}\tau_F(y) &= \tau(\theta_F, y),\\ \eta_F(y) &= \eta(\theta_F, y),\end{aligned} \tag{38}$$

considered as functions of y at θ_F fixed. The tangent vector to the hypersurface reads

$$\begin{aligned}V^+(y) &\equiv z_F^{+'}(y) = (\tau_F' + \eta_F'\tau_F)\, e^{\eta_F},\\ V^-(y) &\equiv z_F^{-'}(y) = (\tau_F' - \eta_F'\tau_F)\, e^{-\eta_F},\end{aligned} \tag{39}$$

where the primes denote derivatives with respect to y. Hence, we can construct the normalized perpendicular vector to the fixed-temperature curve $[n^+(y), n^-(y)]$ defined by

$$\begin{aligned} n^+(y)n^-(y) &= 1, \\ \tfrac{1}{2}[n^+(y)V^-(y) + n^-(y)V^+(y)] &= 0. \end{aligned} \tag{40}$$

Using Eq. (39), the second equation translates into

$$n^+(y)e^{-\eta_F}(\eta_F'\tau_F - \tau_F') = n^-(y)\, e^{\eta_F}(\eta_F'\tau_F + \tau_F'). \tag{41}$$

Provided $|\eta_F'(y)| > |\frac{\tau_F'(y)}{\tau_F(y)}|$ for all y, we find

$$\begin{aligned} n^+(y) &= \sqrt{\frac{\eta_F'\tau_F + \tau_F'}{\eta_F'\tau_F - \tau_F'}}\, e^{\eta_F}, \\ n^-(y) &= \sqrt{\frac{\eta_F'\tau_F - \tau_F'}{\eta_F'\tau_F + \tau_F'}}\, e^{-\eta_F}. \end{aligned} \tag{42}$$

Following Ref. [11], $d\lambda^\mu \equiv d\lambda\, n^\mu$ is defined such that

$$(d\lambda)^2 = d\lambda^\mu d\lambda_\mu = -dz_F^+\, dz_F^- = -\left(\tau_F'^2 - \tau_F^2\, \eta_F'^2\right)(dy)^2, \tag{43}$$

where the minus sign comes from the fact that the hypersurface is a space-like curve. Thus, we have

$$d\lambda = \sqrt{\tau_F^2\eta_F'^2 - \tau_F'^2}\, dy. \tag{44}$$

So, inserting Eqs. (42) and (44) into Eq. (37), we finally find

$$\begin{aligned} \frac{dS}{dy}(y) = s_F[&\tau_F(y)\eta_F'(y)\cosh(\eta_F(y) - y) \\ &+ \tau_F'(y)\sinh(\eta_F(y) - y)]. \end{aligned} \tag{45}$$

B. Expression of entropy flow

Let us now introduce the expression of the entropy flow in terms of the Khalatnikov potential. Starting from Eq. (15), we obtain

$$\begin{aligned} \cosh(\eta - y) &= -\frac{1}{2\tau T_0 e^\theta}\partial_\theta\chi(\theta, y), \\ \sinh(\eta - y) &= \frac{1}{2\tau T_0 e^\theta}\partial_y\chi(\theta, y)\,. \end{aligned} \tag{46}$$

Now, inserting Eq. (46) into Eq. (45), we can eliminate the hyperbolic trigonometric functions, thereby acquiring

$$\frac{dS}{dy}(y) = s_F\, \frac{[-\tau_F(y)\,\eta_F'(y)\,\partial_\theta\chi(\theta, y) + \tau_F'(y)\partial_y\chi(\theta, y)]}{2T_0\, e^{\theta_F}\, \tau_F(y)}\bigg|_{\theta=\theta_F}. \tag{47}$$

Furthermore, by differentiation of the relations (16) with respect to y, at $\theta = \theta_F$, we find

$$\begin{aligned} \tau_F'(y) &= \frac{e^{-\theta}}{2}\, \frac{\left[(\partial_\theta\chi)(\partial_y\partial_\theta\chi) - (\partial_y\chi)\left(\partial_y^2\chi\right)\right]}{\sqrt{(\partial_\theta\chi)^2 - (\partial_y\chi)^2}}\bigg|_{\theta=\theta_F} \\ \eta_F'(y) &= \frac{\left[(\partial_\theta\chi)^2 - (\partial_y\chi)^2 + (\partial_y\chi)(\partial_y\partial_\theta\chi) - (\partial_\theta\chi)\left(\partial_y^2\chi\right)\right]}{(\partial_\theta\chi)^2 - (\partial_y\chi)^2}\bigg|_{\theta=\theta_F}. \end{aligned} \tag{48}$$

Then, inserting the relations (48) and (16) into Eq. (47), we obtain a remarkably simple expression, namely,

$$\frac{dS}{dy}(y) = \frac{s_F}{2T_F}\, \left[\partial_y^2\chi(\theta, y) - \partial_\theta\chi(\theta, y)\right]\big|_{\theta=\theta_F}, \tag{49}$$

which possesses a full generality, as long as the Khalatnikov potential $\chi(\theta, y)$ exists. In addition, using the Khalatnikov equation (23), Eq. (49) can also be written as

$$\frac{dS}{dy}(y) = \frac{s_F c_s^2(T_F)}{2T_F}\, \left[\partial_\theta^2\chi(\theta, y) - \partial_\theta\chi(\theta, y)\right]\big|_{\theta=\theta_F}. \tag{50}$$

There is an interesting third version of Eqs. (49) and (50), featuring the potential Φ instead of χ. The definition (12) of χ can be written alternatively as

$$\chi = \Phi - T\tau e^{\eta-y} - T\tau e^{-\eta+y} \equiv \Phi - 2T\tau\cosh(\eta - y). \tag{51}$$

Inserting into Eq. (51) the first relation of Eq. (46), one obtains

$$\Phi = \chi(\theta, y) - \partial_\theta\chi(\theta, y). \tag{52}$$

Inserting that last relation into Eq. (50), and considering the potential Φ [originally defined in Eq. (11) as a function of z^+ and z^-] now as a function of θ and y, one gets a third equivalent formula for the entropy flow through fixed-temperature hypersurfaces, namely,

$$\frac{dS}{dy}(y) = -\frac{s_F\, c_s^2(T_F)}{2T_F}\, \partial_\theta\Phi(\theta, y)|_{\theta=\theta_F}. \tag{53}$$

The set of expressions (49), (50), and (53) form our main formal result. They provide the exact form of the entropy flow along fixed-temperature hypersurfaces for a general (1+1) hydrodynamic evolution. We also mention that beyond the derivation of the Khalatnikov potential at fixed sound velocity, formulas (49), (50), and (53) still hold for a general speed of sound, once the solution of the general Khalatnikov equation (23) is known. It is important to note that relations (49) and (52) are valid as long as there exist χ or Φ potentials, even if there is no reduction to a linear equation, i.e., no entropy conservation in the (1+1) projection of the flow; whereas relations (50) and (53) are valid when the Khalatnikov equation holds, i.e., with entropy conservation in the (1+1) projection of the flow.

C. Examples

Let us check the general formulas for the entropy flow considering exact hydrodynamic solutions known in the literature, namely, the Bjorken flow [4] and the harmonic flows [11].

1. Bjorken flow

The Bjorken flow corresponds to boost invariance, i.e., $\partial_y \chi \equiv 0$. In this case, the Khalatnikov equation (23) reduces to

$$\chi''(\theta) + (g-1)\chi'(\theta) = 0, \tag{54}$$

which has the generic solution

$$\chi(\theta) = C e^{-(g-1)\theta}, \tag{55}$$

C being an integration constant. Let us choose C and the arbitrary temperature scale T_0 such that at the proper time $\tau = \tau_0$, the temperature of the fluid is $T = T_0$. Inserting Eq. (55) into relations (16), one finds $C = 2T_0\tau_0/(g-1)$ and the known expressions for the Bjorken flow, namely,

$$\tau(\theta, y) = \tau_0\, e^{-g\theta} \quad \text{and} \quad \eta \equiv y, \tag{56}$$

i.e., the equality of rapidity with space-time rapidity. Finally, the Khalatnikov potential for the Bjorken solution is written as

$$\chi(\theta) = \frac{2T_0\tau_0}{(g-1)} e^{-(g-1)\theta} = \frac{2T\tau}{(g-1)}. \tag{57}$$

Inserting Eq. (57) into Eq. (49), one obtains the entropy flow

$$\frac{dS}{dy}(y) = s_F \tau_F = s_0\tau_0 = \text{const.} \tag{58}$$

Hence, as expected from boost invariance of the Bjorken flow, not only the total entropy but also the entropy flow is conserved.

2. Harmonic flows

Following Ref. [11], one is led to introduce new auxiliary variables $l^+(z^+)$ and $l^-(z^-)$ satisfying

$$\frac{dl^\pm}{dz^\pm} = \lambda e^{-l^{\pm 2}}, \tag{59}$$

where $\lambda = $ const. The thermodynamic variables can be explicitly written [11] as[5]

$$\begin{aligned} \theta &= -\frac{g+1}{4g}(l^{+2} + l^{-2}) + \frac{g-1}{2g} l^+ l^-, \\ y &= \frac{1}{2}(l^{+2} - l^{-2}). \end{aligned} \tag{60}$$

Using the property (11) of the potential Φ, one writes

$$\frac{\partial \Phi}{\partial l^\pm} = \frac{dz^\pm}{dl^\pm}\partial_\pm \Phi = \frac{dz^\pm}{dl^\pm} T_0 e^{\theta \mp y}. \tag{61}$$

Now, inserting Eq. (59) and the expressions (60), one obtains

$$\frac{\partial \Phi}{\partial l^\pm} = \lambda T_0 e^{l^{\pm 2}} e^{\theta \mp y} = \lambda T_0 e^{\frac{g-1}{4g}(l^+ + l^-)^2}. \tag{62}$$

The expressions (62) are symmetric in l^+ and l^-; thus, by mere integration, one gets

$$\Phi(l^+, l^-) = \lambda T_0 \int^{l^+ + l^-} dv\, e^{\frac{g-1}{4g}v^2}, \tag{63}$$

where the potential can be expressed in terms of θ and y through

$$l^+ + l^- = \sqrt{2}\,|y| \left(-\theta - \sqrt{\theta^2 - \frac{y^2}{g}}\right)^{-1/2}. \tag{64}$$

Now, using our relation (53) and the relation

$$\partial_\theta \Phi = \lambda T_0 e^{\frac{g-1}{4g}(l^+ + l^-)^2}\, \partial_\theta(l^+ + l^-), \tag{65}$$

one gets the result for the entropy flow

$$\begin{aligned} \frac{dS}{dy}(y) = &\frac{\sqrt{2}\lambda T_0 s_F}{g T_F} e^{\frac{g-1}{2}(\theta + \sqrt{\theta^2 - y^2/g})} \\ &\times \frac{|y|}{\sqrt{\theta^2 - y^2/g}}(-\theta - \sqrt{\theta^2 - y^2/g})^{-1/2}. \end{aligned} \tag{66}$$

Using our general formalism, we thus recover the nontrivial result obtained by direct calculation [see Ref. [11], formula (58)]. Interestingly enough, we note that for the family of harmonic flows as an example, it appears to be much simpler to use formula (53) for the potential Φ than to use the Khalatnikov potential χ itself.

Note that a specific discussion is needed of the limiting case when $g = 1$, that is, when the speed of sound equals the speed of light. In fact, in this case, the harmonic flow cannot be obtained as above, and the solution for the flow acquires a more general form. Returning to Eq. (23), one finds that the Khalatnikov potential itself is harmonic, namely, $\chi(\theta, y) \equiv h_+(y + \theta\sqrt{g}) + h_-(y - \theta\sqrt{g})$, where h_+, h_- are arbitrary functions. We thus recover the results noted in Ref. [15].

IV. EVOLUTION-DOMINATED SOLUTIONS

In general, a longitudinal flow in the final state follows from a longitudinal pressure gradient and/or from a longitudinal flow in the initial state. Let us consider the subclass of solutions where the effect of the initial flow is negligible compared to the one of the initial pressure gradient. This subclass corresponds to the dominance of the hydrodynamic evolution over the influence of the initial conditions. A typical example of such a solution is the Belenkij-Landau solution [9], where the fluid is initially at rest (the so-called full stopping initial conditions) and then expands into the vacuum.

A. Khalatnikov potential and entropy flow

To model an evolution-dominated flow, let us consider all the sources at rest, i.e., $F(\hat{\theta}, \hat{y}) \propto \delta(\hat{y})$. Let us also take the arbitrary temperature scale T_0 to be the maximal temperature of the sources [hence, $\theta \equiv \ln(T/T_0) \leqslant 0$], i.e., $F(\hat{\theta}, \hat{y}) \propto \Theta(-\hat{\theta})$. All in all, we write

$$F(\hat{\theta}, \hat{y}) = 4\sqrt{g} K(\hat{\theta})\, \Theta(-\hat{\theta})\, \delta(\hat{y}). \tag{67}$$

[5] Here we use our convention $\theta = \ln(T/T_0)$ with opposite sign as in Ref. [11].

Inserting Eq. (67) into Eq. (36), and replacing the variable $\hat{\theta}$ by $\theta' \equiv \theta - \hat{\theta}$, one gets

$$\chi(\theta, y) = e^{-(\frac{g-1}{2})\theta} \int_{\theta}^{-\frac{|y|}{\sqrt{g}}} I_0\left(\frac{g-1}{2}\sqrt{\theta'^2 - y^2/g}\right) \times K(\theta - \theta')\, d\theta', \tag{68}$$

where the function $K(\theta - \theta')$ carries the information on the initial conditions. Note that θ' is also negative.

In the following, it is convenient to use a Laplace representation of Eq. (68). Since $\theta \leqslant 0$, we introduce the Laplace transform, and its inverse, with respect to $-\theta$ as

$$\begin{aligned} \tilde{f}(\gamma) &= \int_{-\infty}^{0} d\theta\, e^{\gamma\theta} f(\theta), \\ f(\theta) &= \int_{\gamma_0 - i\infty}^{\gamma_0 + i\infty} \frac{d\gamma}{2\pi i} e^{-\gamma\theta} \tilde{f}(\gamma), \end{aligned} \tag{69}$$

where γ_0 is a real constant that exceeds the real part of all the singularities of the integrand, i.e., the integral is calculated on an imaginary contour that lies on the right of all singularities. Following Ref. [9], the Khalatnikov potential (68) can be written as a convolution of the two functions

$$\begin{aligned} &\Theta(-\theta)\, K(\theta) \\ &\quad = \int_{\gamma_0 - i\infty}^{\gamma_0 + i\infty} \frac{d\gamma}{2\pi i}\, e^{-\gamma\theta} \tilde{K}(\gamma), \\ &\Theta\left(-\theta - |y|/\sqrt{g}\right) I_0\left(\frac{g-1}{2}\sqrt{\theta^2 - y^2/g}\right) \\ &\quad = \int_{\gamma_0 - i\infty}^{\gamma_0 + i\infty} \frac{d\gamma}{2\pi i} \frac{1}{\sqrt{\gamma^2 - \frac{(g-1)^2}{4}}} \\ &\qquad \times \exp\left[-\gamma\theta - \frac{|y|}{\sqrt{g}}\sqrt{\gamma^2 - \frac{(g-1)^2}{4}}\right]. \end{aligned} \tag{70}$$

As the Laplace transform changes convolutions into ordinary products, one gets the Laplace representation

$$\chi(\theta, y) = \int_{\gamma_0 - i\infty}^{\gamma_0 + i\infty} \frac{d\gamma}{2\pi i} \exp\left[-\left(\gamma + \frac{g-1}{2}\right)\theta - \frac{|y|}{\sqrt{g}}\sqrt{\gamma^2 - \frac{(g-1)^2}{4}}\right] \frac{\tilde{K}(\gamma)}{\sqrt{\gamma^2 - \frac{(g-1)^2}{4}}}. \tag{71}$$

Notice that while the expression of solution (68) restricts the phase-space domain in the interval $|y| \leqslant -\sqrt{g}\,\theta$, Eq. (71) may allow an analytic continuation of the solution of the Khalatnikov potential outside this region. However, the outside region may be different (e.g., with $\chi \equiv 0$, as in Ref. [9]).

Let us now investigate the properties of the entropy flow given by the solutions (71) of the Khalatnikov equation. Inserting the Khalatnikov potential (71) into the expression of the entropy distribution (50), one is led to the formula

$$\frac{dS}{dy}(y) = \frac{s_F}{2gT_F} \int_{\gamma_0 - i\infty}^{\gamma_0 + i\infty} \frac{d\gamma}{2\pi i} e^{-\theta_F(\gamma + \frac{g-1}{2})} [(\gamma + g/2)^2 - 1/4] \times \tilde{K}(\gamma) \frac{e^{-\frac{|y|}{\sqrt{g}}\sqrt{\gamma^2 - \frac{(g-1)^2}{4}}}}{\sqrt{\gamma^2 - \frac{(g-1)^2}{4}}}. \tag{72}$$

In formula (72), one may distinguish the *kernel*

$$\mathcal{Q}(\gamma, y) \equiv \frac{\exp\left[-\frac{|y|}{\sqrt{g}}\sqrt{\gamma^2 - \frac{(g-1)^2}{4}}\right]}{\sqrt{\gamma^2 - \frac{(g-1)^2}{4}}}, \tag{73}$$

driving the dynamical hydrodynamic evolution as expressed on the entropy flow, and the *coefficient function*

$$\tilde{C}_f(\gamma) = [(\gamma + g/2)^2 - 1/4]\tilde{K}(\gamma), \tag{74}$$

which encodes the initial conditions of the entropy flow.

B. Total entropy

Since we have a well-defined relation (72) for the entropy distribution, it is easy to perform the integration over y and obtain the total entropy flux through the hypersurface with fixed temperature $T = T_F$.

Formally, Eq. (72) leads to

$$\begin{aligned} S_{\text{tot}}\big|_{\theta=\theta_F} &= 2\int_0^{-\theta_F\sqrt{g}} dy\, \frac{s_F}{2gT_F} \int_{\gamma_0 - i\infty}^{\gamma_0 + i\infty} \frac{d\gamma}{2\pi i} \left(\gamma + \frac{g-1}{2}\right) \\ &\quad \times \left(\gamma + \frac{g+1}{2}\right) e^{-\frac{|y|}{\sqrt{g}}\sqrt{\gamma^2 - \frac{(g-1)^2}{4}}} \\ &\quad \times \frac{\tilde{K}(\gamma)}{\sqrt{\gamma^2 - \frac{(g-1)^2}{4}}} e^{-\theta_F(\gamma + \frac{g-1}{2})}, \end{aligned} \tag{75}$$

where we took into account the $\Theta(-\theta - |y|/\sqrt{g})$ function present in Eq. (70). Indeed, the hydrodynamic flow is limited in the region inside this domain, with possible contributions on the boundary $\theta = -|y|/\sqrt{g}$ (Riemann waves, see, e.g., Ref. [9]).

We know that, by construction, the flow is isentropic, and thus the total entropy S_{tot} is conserved. In fact, it is possible to show that the dominant part of the total conserved entropy results from the kernel (73) more than from other sources such as the coefficient function (74) or the boundary Riemann waves. Hence, the hydrodynamic dynamics dominate. For this reason, let us release for simplicity the boundary limitations of the integral over y. One then writes

$$\begin{aligned} S_{\text{tot}} &\approx \frac{s_F}{T_F\sqrt{g}} \int_{\gamma_0 - i\infty}^{\gamma_0 + i\infty} \frac{d\gamma}{2\pi i} \frac{\gamma + \left(\frac{g+1}{2}\right)}{\gamma - \left(\frac{g-1}{2}\right)} \tilde{K}(\gamma)\, e^{-\theta_F(\gamma + \frac{g-1}{2})} \\ &= \frac{s_F\sqrt{g}}{T_F}\, \tilde{K}\,[(g-1)/2]\, e^{-(g-1)\theta_F}. \end{aligned} \tag{76}$$

Indeed, the complex integral is obtained through the singularities of the integrand, which can be due to either the initial conditions [through singularities of $\tilde{K}(\gamma)$] or the hydrodynamic dynamics [through the pole at $\gamma = (g-1)/2$], or both. If the singularities of $\tilde{K}(\gamma)$ are situated at the left (right) of the pole, they will be subdominant (dominant) in the total entropy. Assuming a dominance of the hydrodynamic flow, we get the final result of Eq. (76). The physical meaning of Eq. (76) becomes clear when using the thermodynamic relation (26) and the entropy density s_0 at temperature T_0. The total entropy is written as[6]

$$S_{\rm tot} \approx \frac{s_0\sqrt{g}}{T_0}\,\tilde{K}\,[(g-1)/2]\,, \tag{77}$$

and thus does not depend on the features of the flow at $T = T_F$. In conformity with the isentropic property of the flow, the total hydrodynamic entropy of the perfect fluid should be conserved, as the expression (77) is independent of T_F. This provides a self-consistency check for an evolution-dominated flow. In more general cases, one should also take into account the other contributions.

A final comment is in order. *A priori*, the domain of integration $|y| \leqslant Y/2$ comes from energy-momentum conservation. However, formula (68) for the Khalatnikov potential is only valid in the domain $|y| \leqslant -\sqrt{g}\,\theta_F$. In the flow-dominated approximation, $-\sqrt{g}\,\theta_F$ and $Y/2$ are considered large enough that the integration domain can extend to infinity and only the kernel contributes.[7]

C. Relation to the Belenkij-Landau solution

We have studied the dependence of our results on the coefficient function (74) by imposing various relevant analytic forms for $\tilde{K}(\gamma)$. We observed that typical meromorphic functions, bounded by a constant[8] at infinity and with poles at the left of $\gamma = \frac{g-1}{2}$, give smooth and similar entropy flow distributions, which are almost identical at large enough θ_F. Hence, we conjecture that all physical evolution-dominated solutions are almost identical, at least for a sufficiently large value of $\theta_F = \ln\,(T_F/T_0)$. To provide an analytic expression for the entropy flow characteristic of the family of solutions, we remark that the following choice of the coefficient function (74)

$$\tilde{C}_f(\gamma) = C\left(\gamma + \frac{g-1}{2}\right) \Leftrightarrow \tilde{K}(\gamma) = \frac{C}{\gamma + \frac{g+1}{2}}, \tag{78}$$

where C is a dimensionless constant, corresponds to hydrodynamic flow with an initial full stopping condition [9].

The Belenkij-Landau solution [9] describes the evolution of a slice of fluid of width $2L$ initially at rest and expanding in the vacuum. It consists in a hydrodynamic flow bounded by Riemann waves. The matching conditions between the flow and the waves in space-time translated in terms of temperature and rapidity variables are realized by imposing zero boundary conditions on the Khalatnikov potential χ on the characteristics $\theta = \pm y/\sqrt{g}$. Another condition on the potential is that the center of the slice remains by symmetry at rest ($y = 0$) during the evolution.

We have checked that the energy flow resulting from modifications of the ansatz (78) satisfying the dominance of the kernel singularity is not sensibly modified from the one given by inserting the coefficient function (78) into Eq. (72).

Inserting now Eq. (78) into Eq. (71), the Khalatnikov potential between the characteristics $-\theta \geqslant |y|/\sqrt{g}$ acquires the analytic form [9,10]

$$\chi(\theta, y) = C\int_{\theta}^{-\frac{|y|}{\sqrt{g}}} I_0\left(\frac{g-1}{2}\sqrt{\theta'^2 - \frac{y^2}{g}}\right) e^{\theta-(\frac{g+1}{2})\theta'}\,d\theta'. \tag{79}$$

The potential is identically zero in the region $-\theta \leqslant |y|/\sqrt{g}$. Note that the constant in Eqs. (78) and (79) is such that $C \propto LT_0$ with our notations.

Let us now insert this specific solution to our general formula (50) for the entropy distribution. Calculating the derivatives, we find

$$\frac{dS}{dy}(y) = s_F\frac{(g-1)C}{4gT_F}\,e^{-\frac{(g-1)}{2}\theta_F}\left[I_0\left(\frac{g-1}{2}\sqrt{\theta_F^2 - y^2/g}\right)\right.$$
$$\left. - I_1\left(\frac{g-1}{2}\sqrt{\theta_F^2 - y^2/g}\right)\frac{\theta_F}{\sqrt{\theta_F^2 - y^2/g}}\right]. \tag{80}$$

Since θ_F is negative, this expression is always positive, at least in the region $\theta_F^2 - y^2/g \geqslant 0$. Hence the positivity of the entropy flow is ensured. Finally, the expression (80) is divergence-free, since it is finite for $\theta_F^2 = y^2/g$. We also note that it is still real in the analytic continuation of solution (80) for $\theta_F^2 < y^2/g$, since in this case both the numerator and the denominator of the second term are purely imaginary.[9]

For the total entropy, inserting Eq. (78) into the general formula (75), one gets, using the thermodynamic relations (26),

$$S_{\rm tot} = \frac{C\,s_F}{\sqrt{g}\,T_F}e^{-(g-1)\theta_F} = \frac{C s_0}{\sqrt{g}T_0}. \tag{81}$$

A comment is in order at this point. Condition (78) has been considered to describe the so-called full stopping conditions. In the original papers [9], it consists of the assumptions that (i) there is a specific plane where the medium is at rest for all times, and (ii) on the vacuum-boundary we have just a simple (Riemann) wave. In fact, the resulting entropy flow distribution is expected to be more general and is characteristic of the evolution-dominated hydrodynamic flows. Hence the Khalatnikov potential (79) (already obtained in Ref. [9]) and the entropy flow of Eq. (80) may serve as an analytic

[6]Note that $\tilde{K}$ is dimensionless as the potential χ.

[7]We have performed numerical checks that show that thanks to the decreasing exponential behavior, the boundary term contributes negligibly to the total entropy.

[8]Indeed, choosing $\tilde{K}(\gamma)$ of strictly positive degree leads to an unphysical angular point at $y = 0$ and to a function $K(\theta)$, see Eq. (70), containing derivatives of Dirac distributions, i.e., structures that are too singular to describe physical flows.

[9]Positivity may also extend but is not ensured because of the appearance of Bessel function zeros.

formulation for the class of evolution-dominated flows. In fact, their features are essentially determined by the evolution kernel $\mathcal{Q}(\gamma, y)$ [see Eq. (73)].

Finally, it is interesting to note that Eq. (72) gives the possibility of comparing the hydrodynamic predictions with those of other existing models of heavy-ion (and eventually hadron-hadron, soft scattering) reactions. This relies on the possibility of relating thermodynamic quantities, such as temperature and entropy, to observed properties of the particle multiplicities. In this scheme, the rapidity of particles is defined by the corresponding value of $y \equiv \ln u^+$ obtained from the fluid velocity of the lump of fluid giving rise locally to the hadrons. In the same context, the overall temperature gradient θ_F will be related to the total available rapidity and the entropy to the multiplicity up to phenomenological factors. We will discuss in the next section the phenomenological issues of our derivation, but the general theoretical idea is that Eq. (72) can be compared with the one-particle inclusive hadronic cross section which is related to the scattering amplitudes. In this respect, the generic form [Eq. (73)] of the hydrodynamic evolution kernel may serve as a means of comparing hydrodynamic properties with conventional models of scattering amplitudes.

V. PHENOMENOLOGICAL APPLICATIONS

Motivated by the seminal works of Landau [3] and Bjorken [4], comparisons of their predictions for (1+1) hydrodynamics with some features of the data have been made (see e.g., Refs. [5–7,9,10]). Even if such a rough approximation, ignoring the details of the transverse motion or the hadronization, cannot replace the numerical simulations, it has given some useful information on the dynamics of the quark-gluon plasma. For instance, the order of magnitude estimates made using the Bjorken flow in the central region [4] and the comparison of the multiplicity distributions with the predictions of the Landau flow [6,7] have indicated that the proper-time region during which the hydrodynamic flow is approximately (1+1) dimensional has a deep impact on the whole process. Our aim is to take advantage of the explicit form (80) representative of the entropy of evolution-dominated flows, based on the Khalatnikov potential (79), to revisit the discussion in the light of recent experimental results.

For the phenomenological application, we will concentrate on the entropy flow corresponding to the Belenkij-Landau solution. From Eqs. (80) and (81), one obtains the formula

$$\frac{dS}{dy}(y) = S_{\rm tot}\frac{(g-1)}{4\sqrt{g}}\, e^{\frac{(g-1)}{2}\theta_F}\left[I_0\left(\frac{g-1}{2}\sqrt{\theta_F^2 - y^2/g}\right) - I_1\left(\frac{g-1}{2}\sqrt{\theta_F^2 - y^2/g}\right)\frac{\theta_F}{\sqrt{\theta_F^2 - y^2/g}}\right], \tag{82}$$

where we have used the normalization by the total entropy $S_{\rm tot}$. Formula (82) still depends on two hydrodynamic parameters θ_F, the logarithmic temperature evolution, and the speed of sound $c_s = g^{-1/2}$.

A. Multiplicity distribution at fixed energy

To investigate the phenomenological validity of formula (82), let us consider the experimental BRAHMS data for the charged multiplicity distribution in the most central collisions as a function of the rapidity measured recently at the BNL Relativistic Heavy Ion Collider (RHIC) [16]. For sake of simplicity, in accordance with the (1+1) dimensional approximation of the dynamics that we consider, we will make the following assumptions. We will identify the rapidity if the fluid elements $y_f \equiv 1/2\ \ln\,(u^+/u^-)$ with the rapidity of the particles $y_p \equiv 1/2\ \ln\,(p^+/p^-)$. We thus keep the same notation y. In the same way, we assume that the multiplicity distribution of produced particles dN/dy in rapidity can be considered to be equal, up to a constant factor, to the entropy distribution[10] dS/dy. One expects that the end of the hydrodynamic behavior appears at a typical temperature T_F, related to a hadronization or freeze-out temperature, and independent of the total c.m. energy of the collision. On the other hand, the initial temperature T_0 is expected to depend on the total c.m. energy (or equivalently on the total rapidity Y) and on the centrality of the collision, through the energy density $\epsilon(T_0)$ of the medium produced by the prehydrodynamic stage of the collision. Thus, $\theta_F = \ln\,(T_F/T_0)$ should be a function of Y and of the centrality. Our formalism, based on the (1+1) dimensional approximation of the flow, is not appropriate for a precise description of the freeze-out. Note, however, that some improvement could be obtained by using, e.g., the Cooper-Frye formalism [17] in the derivation of the entropy flow. We postpone this to future studies.

Using then formula (82) for dN/dy and fitting BRAHMS data by adjusting the parameter θ_F, we obtain a good description for different values of g. In Fig. 1, as an example,

[10]We also assume that the multiplicity distribution of charged particles is proportional to the total one.

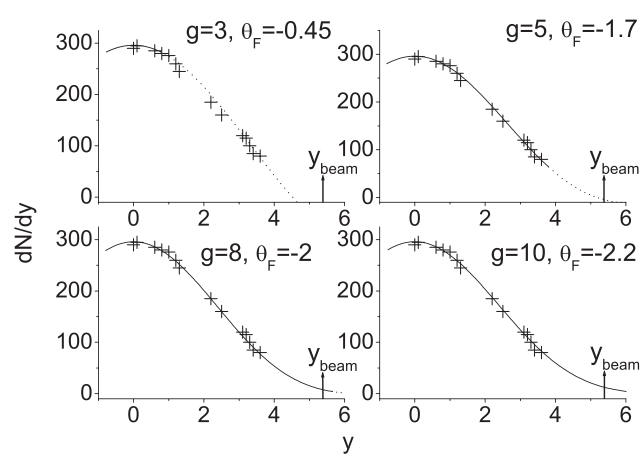

FIG. 1. BRAHMS data fitted with the hydrodynamic formula. The data are taken from Ref. [16], and they correspond to charged pions in cental Au + Au collisions at $\sqrt{s_{NN}} = 200$ GeV. The solid line corresponds to the physical region $\frac{y^2}{g} \leqslant \theta_F^2$, while the dotted portion corresponds to its analytic continuation $\frac{y^2}{g} > \theta_F^2$. The small vertical line marks the experimental beam rapidity $Y_{\rm beam} \sim Y/2$.

we present the BRAHMS data fitted with Eq. (82), for four pairs of g and θ_F values, reported on the figure.

In these plots, the solid line corresponds to the physically meaningful region $\frac{y^2}{g} \leqslant \theta_F^2$, while the dotted line corresponds to the analytic continuation of formula (82) in the region $\frac{y^2}{g} > \theta_F^2$, where the applicability of solution (82) is theoretically questionable.

The phenomenological application appears to be correct for quite different values of the speed of sound $c_s \equiv g^{-1/2}$. The overall form of the curves is satisfactory. For the first curve at $c_s = 1/\sqrt{3}$ (i.e., the conformal case), however, the analytic continuation beyond $\frac{y^2}{g} \leqslant \theta_F^2$ is soon reached.[11] We will comment on this remark later on. Indeed, when decreasing the speed of sound, e.g., for $g = 5$, the physical domain $\frac{y^2}{g} \leqslant \theta_F^2$ extends in rapidity.

Some comments on these results are in order.

(i) It has long been well-known [6], and confirmed more recently, that a Gaussian fit to the data

$$\frac{dS}{dy}(y) \sim e^{-y^2/Y} \tag{83}$$

with a variance $\sqrt{Y}$, as predicted by Landau [3], was reasonably verified. We noticed that, indeed, expression (82) has an approximate Gaussian form, but it does not correspond, except for very large θ_F, to the expansion of the exact entropy distribution near $y = 0$, as in the original argument [3] which was based on an asymptotic approximation.[12] Hence, the subasymptotic features of the full solution plays an important phenomenological role.

(ii) There is apparently no track of the transition between the physical regime $\frac{y^2}{g} \leqslant \theta_F^2$ and its analytic continuation, described by the dotted lines in Fig. 1. This is related to the mathematical property of the general solution (72) expressed using a Laplace transform. In short, the $I_{0,1}$ Bessel functions are transformed into $J_{0,1}$ with the same argument up to a factor i, without discontinuity.

(iii) This transition is, however, meaningful. In fact, one knows that the lines $\frac{y}{\sqrt{g}} = \mp\theta_F$ delineate different regions of the hydrodynamic regime. Discontinuities and thus shock or Riemann waves may occur at these boundaries, called characteristics of the equation [18]. Hence, some other solutions may branch at this point (see, e.g., Refs. [9,10]). However, our results do not depend on the specific form of these other solutions such as the shock waves considered in Refs. [9,10].

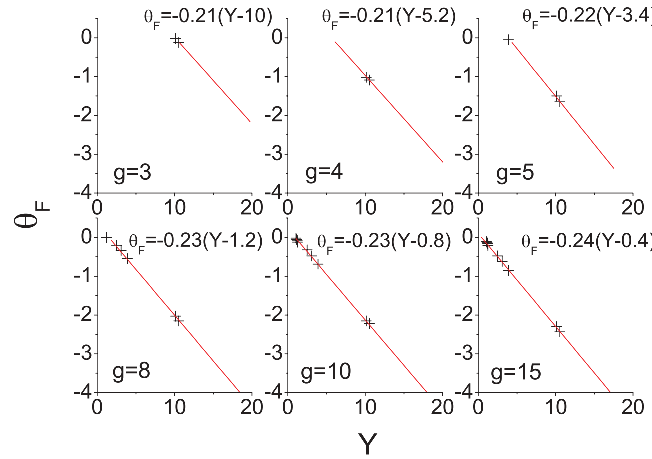

FIG. 2. (Color online) Hydrodynamic parameter θ_F as a function of Y. We describe the dependence for six values of g, in a large range starting from the canonical value 3 which would correspond to a conformal fluid in (3+1) dimensions.

B. Energy dependence of the multiplicity distributions

Going a step further, we would like to interpret the energy dependence (i.e., the Y dependence) of the (1+1) solution for the entropy flow compared with multiplicity data. For this purpose, we make use of the Gaussian fits reported[13] in Ref. [16] for different sets of data ranging from those of the BNL Alternating Gradient Synchrotron (AGS) to those of RHIC.

In Fig. 2, we give the determination of θ_F as a function of Y which gives a good description of the Gaussian fits with the variance taken from Ref. [16]. As in the previous study of BRAHMS data, we performed this fit for six different values of g. As shown in Fig. 2, the corresponding relation is clearly linear. We can write

$$\theta_F = -\kappa(Y - Y_0). \tag{84}$$

As shown on the figure, the constant term Y_0 depends appreciably on g, whereas the slope $\kappa \approx 0.2$ remains only slightly dependent on it.

On a physical ground, the linear relation (84) has a reasonable interpretation. The initial temperature of the medium is expected to grow as a power $\kappa \leqslant 1$ of the incident energy. One finds approximately $T_0/T_F = e^{-\theta_F} \sim e^{0.22(Y-Y_0)}$. Hence, the more energy available, the longer the hydrodynamic evolution lasts. At smaller speeds of sound, the hydrodynamic evolution has to occur on a larger temperature interval to describe the same entropy distribution, as would be expected.

Moreover, there is a physical argument, analogous to the one proposed by Landau [3], for the existence of a linear relation (84) between the temperature ratio and the total c.m. energy. Assuming the approximate validity of the Bjorken relation[14] $\tau_0/\tau_F \approx (T_F/T_0)^g$, where τ_0 (τ_F) are the initial

[11] One may also note that the curve indicates a violation of positivity before the kinematic limit.

[12] It is indeed easy to verify that for phenomenological values of θ_F, this approximation does not work in the data range.

[13] In fact, the prediction (83) fits reasonably well, but we used instead the actual best-fit determination of the variances provided in Ref. [16].

[14] This relation, properly stating, is exact only for the Bjorken boost-invariant flow. However, one expects that it remains approximately valid in the central region of more general flows (see, e.g., Ref. [11].).

(final) proper times of the (1+1) hydrodynamic evolution and reporting in Eq. (84), one finds

$$\ln(\tau_0/\tau_F) \approx \kappa g(Y - Y_0). \tag{85}$$

Indeed, following Ref. [3], the separation proper time from (1+1) hydrodynamics to the (1+3) regime is of order $\ln \Delta\tau_s \sim 12Y$, where Δ is the typical transverse size of the initial particles. Assuming that we can approximate $\Delta\tau_s$ by τ_0/τ_F, and taking into account the Bjorken flow approximation, formula (85) is suggestive of the proportionality property. We leave the precise values for κ, τ_0/τ_F, and Y_0 to a further determination of g, since the data we discussed do not prefer a precise value of κg (to be compared with 1/2 obtained for $\ln \Delta\tau_s$).

An interesting consequence of the linear relation (84) between θ_F and Y at fixed g is the possibility of relating the general hydrodynamic entropy distribution (72) to the one-particle inclusive cross section and thus to the appropriate scattering amplitudes. These are not easy to formulate in the hydrodynamic formalism. Being more specific, let us transform Eq. (72) in terms of the energy dependence using κ as the coefficient of proportionality in Eq. (84). We then obtain

$$\frac{dN}{dy}(y,Y) \propto \int_{\gamma_0 - i\infty}^{\gamma_0 + i\infty} \frac{d\gamma}{2\pi i}\, e^{-\kappa(Y-Y_0)(\gamma + \frac{g-1}{2})} \times [(\gamma + g/2)^2 - 1/4]\tilde{K}(\gamma)\, \frac{e^{-\frac{|y|}{\sqrt{g}}\sqrt{\gamma^2 - \frac{(g-1)^2}{4}}}}{\sqrt{\gamma^2 - \frac{(g-1)^2}{4}}}. \tag{86}$$

Formula (86) shows that the characteristic hydrodynamic kernel $\mathcal{Q}$ [see Eq. (73)] appears also as the kernel of the Laplace transform in Y of the one-particle inclusive cross section, up to a redefinition of the conjugate moment $\omega = \kappa\gamma$ of the total rapidity Y. This relation may be useful in comparing various theories and models for scattering amplitudes of high-energy collisions with the predictions of hydrodynamic evolution.

VI. CONCLUSIONS AND OUTLOOK

Let us summarize the results of our study. From the theoretical point of view, we have the following results:

(i) We have recalled and reformulated the derivation of the Khalatnikov potential and equation in terms of light-cone variables. This allows us to formulate the initial nonlinear problem of (1+1) hydrodynamics of a perfect fluid in terms of solutions of a linear equation. As an application, using the Green's function formalism, we derive the general form of the solution for constant speed of sound.

(ii) Expressing the flow of entropy through fixed-temperature hypersurfaces, we provide general and simple expressions of the entropy flow dS/dy in terms of the Khalatnikov potential.[15]

(iii) We check and illustrate the simplicity of the formulas obtained for dS/dy by applying the formalism to some exact hydrodynamic solutions which were not using the Khalatnikov formulation, such as the Bjorken flow and the less straightforward example of the harmonic flows of Ref. [11].

(iv) We use our formalism to find the entropy flow for the subclass of solutions for which the hydrodynamic evolution dominates over the influence of initial conditions. A characteristic example of such flows is the one studied long ago by Landau and Belenkij [9], corresponding to full stopping initial conditions. We provide an exact expression for the related entropy flow.

As a phenomenological application, we discuss the relevance of the full stopping entropy flow for modern heavy-ion experiments which was advocated, e.g., in Refs. [6,7].

(i) The exact expression of dS/dy for the Belenkij-Landau solution, depending only on the ratio T_F/T_0 and on the speed of sound c_s, is in agreement with the shape of the multiplicity distribution of particles $dN/dy(y,Y)$ observed in heavy-ion reactions, with a linear relation between the temperature ratio and the total rapidity $\ln(T_0/T_F) = \kappa[Y - Y_0(c_s)]$.

(ii) However, comparing our exact results with the asymptotic Gaussian predictions [3,6] for the multiplicity distributions, we find that nonasymptotic contributions play an important role in the phenomenological description.

(iii) The speed of sound, which is the remaining parameter in our study, is not determined by the multiplicity distribution, since the phenomenological description seems satisfactory for a rather large range of the parameter $g \equiv 1/c_s^2$. However, even if one does not see any sizable effect on the curve for $dS/dy(y,T)$, one notices that the physical domain of the hydrodynamic expansion is restricted by the condition $y^2 \leqslant g\theta_F^2$, especially for a speed of sound as large as the conformal one $c_s = 1/\sqrt{3}$.

This summary of conclusions leads to a few comments on possible further developments of our approach. Some of them are technical but could provide further insight into the features of (1+1) hydrodynamics. First, it should be useful to study in detail a larger set of solutions. Second, implementing the Cooper-Frye formalism [17] directly in terms of the Khalatnikov potential could refine the hypothesis of a fixed final temperature T_F. Also, the investigation of the entropy flow through other hypersurfaces, in particular the proper-time ones (cf. Ref. [11]) would be welcome, in particular to allow for a straightforward implementation of fixed proper-time initial conditions.

From the phenomenological point-of-view, it is important to develop the comparison of the (1+1) approach with the data and include more corrections to the idealized dominance of the longitudinal motion. One question could be settled at least phenomenologically, which is the determination of the best fit for the speed-of-sound parameter, which is presently rather free. Also, including a viscosity contribution is another important issue, together with the investigation of the entropy flows with varying speeds of sound.

[15] After completing this paper, we noticed a related study in Ref. [19].

One may ask about the meaning of the transitions on the lines $y^2 = g\theta_F^2$, which appear even in the physical rapidity region. Mathematically, they are the Riemann characteristics of the Khalatnikov equations, and as such, they are regions where discontinuities may appear [18]. Indeed, these characteristics were used in the old studies [9,10] to connect boundary Riemann waves to the domain of dynamical hydrodynamic evolution. What their meaning is, if any, in today's understanding of high-energy collisions is an interesting open question.

From a more conceptual point of view, our study of the entropy flow and its dependence on rapidity may have some impact on recent studies [13,14] of the anti–de Sitter-space/conformal-field-theory (AdS/CFT) correspondence. It relates the hydrodynamics of a fluid, whose microscopic description is the one of a gauge field theory, with the string theory in a higher dimensional space where the Einstein equations govern the gravitational properties of its low-energy regime. The actual realizations of the duality correspondence for a collective flow require boost invariance and thus are limited to the Bjorken flow. This flow contains an infinite energy and is thus of limited relevance. Knowing the analytic form of more physical solutions should be helpful in deriving their dual gravitational backgrounds, which is *a priori* a formidable task.

ACKNOWLEDGMENTS

We thank Andrzej Bialas and Jean-Yves Ollitrault for useful suggestions. One of us (E.N.S.) thanks Institut de Physique Théorique, CEA, for its hospitality during the preparation of the present work.

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