Bohr Hamiltonian, mass coefficients, and the structure of well deformed axially symmetric nuclei

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It is shown that in the Bohr Hamiltonian the use of three different mass coefficients for the two vibrational and the rotational modes is very important for the correct description of the properties of the well deformed axially symmetric nuclei. Four parameters per nucleus are needed to describe the relative values of energies and $B(E2)$'s.

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I. INTRODUCTION

During the last few years the Bohr Hamiltonian [1] was frequently used for the description of the properties of the collective quadrupole states in even-even nuclei [2–4]. Different phenomenological potentials were used to simplify the solution of the Bohr Hamiltonian or even to obtain a solution in an analytical form [3,5–10]. We mention further that the Bohr Hamiltonian has been intensively used for the nuclear structure investigation in previous works, e.g., by Davydov [11,12], Greiner [13], Faessler [14], and Kumar and Baranger [15,16]. The last authors have used, however, a microscopical approach to calculate such ingredients of the Bohr Hamiltonian as the potential energy and the components of the mass tensor.

Among the important properties of the collective quadrupole states of the even-even well deformed nuclei we concentrate on in this paper are the reduced $E2$ transition probabilities between the β- and γ-vibrational and the ground state bands. The mass coefficients play an important role for the correct description of these data. As a rule, especially during the last few years, the Bohr Hamiltonian uses one common mass coefficient for rotational and β- and γ-vibrational modes. However, in our previous publications [17–19] it was shown, based on the data for the Grodzins-type relations for the $2^+_{\text{g.s.}}$, 2^+_β, and 2^+_γ states, that the mass coefficients for the vibrational modes are a factor 4, or even more, larger than the mass coefficient for the rotational motion. It means that the use in the Bohr Hamiltonian of the same mass coefficient for all collective modes will have as a consequence an overestimation by a factor 4 or more of the reduced $E2$ transition probabilities between the states of the β- and γ-vibrational and ground state bands (see, however, Ref. [19]).

In order to clarify why in the case of the well deformed nuclei the mass coefficients for β-vibrational (B_β), γ-vibrational (B_γ), and rotational (B_{rot}) modes are different, let us analyze the expressions for the mass coefficients obtained in the framework of the cranking model approach which is based on the mean field concept. The expressions for the mass coefficients for β- and γ-vibrational and rotational motion derived in the framework of the cranking approach differ in two respects. At first, the cranking expressions for the different mass coefficients are given by the sums over the particle-hole states (or two-quasiparticle states if pairing is taken into account) of the Nilsson potential having $K = 1$ for the rotational mass coefficient, $K = 2$ for the γ-vibrational mass coefficient, and $K = 0$ for the β-vibrational mass coefficient. Correspondingly, different two-quasiparticle energies and the matrix elements of the different single particle operators disturbing the nuclear mean field are presented in the expressions for the mass coefficients. In the expression for B_{rot} the matrix elements of the component of the single particle angular momentum operator corresponding to the rotational axis $\hat{j}_x$ are presented. In the expression for B_γ the component of the single particle quadrupole momentum operator with $K = 2$, i.e., $\frac{1}{\sqrt{2}}r^2(Y_{22} + Y_{2-2})$, and in the expression for B_β the component of the single particle quadrupole momentum operator with $K = 0$, i.e., r^2Y_{20} are presented.

The second difference is the presence in the energy denominators of the cranking formulas for B_β and B_γ of the nonzero vibrational frequencies which are of the order of 1 MeV in the case of the well deformed nuclei, i.e., quite close to the two-quasiparticle energies ($2\Delta \approx 1.5$ MeV). However, this frequency is absent in the expression for B_{rot} because the potential energy does not depend on the rotational angle.

First of all, applying the Bohr Hamiltonian with three different mass coefficients to a description of the well deformed nuclei, we have in mind a correct description of the interband $E2$ transitions between the states of the β- and γ-bands on one side and the states of the ground state band on the other side. As was shown in our previous publications, this aim can be achieved only if three different mass coefficients are used. However, it is necessary to consider how an introduction of the nonequal B_β, B_γ, and B_{rot} will influence a description of the other properties of the well deformed nuclei, such as the angular momentum dependence of the energies of the states belonging to the same rotational band and the intraband $E2$ transitions. These properties have been well described using a traditional Bohr Hamiltonian with one common mass coefficient for all excitation modes. For the correct description of these properties it is important to take into account the centrifugal stretching effect which increases the value of the equilibrium deformation with angular momentum. This is done in the Davydov-Chaban model [12], in the rotation-vibration

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coupling approach of Greiner and Faessler [14], and in the variable moment of inertia model [20]. Of course the important role played by the Coriolis antipairing effect is not excluded. To take this effect into account, it is necessary to include into the Hamiltonian the pairing degrees of freedom [21]. However, it can be done effectively by introducing an angular momentum dependence of the rotational mass coefficient. We have calculated these characteristics using the well known Davydov-Chaban procedure to solve approximately the Bohr Hamiltonian with three different mass coefficients.

Thus, the main aim of the present paper is to demonstrate that the correct description of the interband $E2$ transitions in the well deformed axially symmetric nuclei in the framework of the Bohr Hamiltonian is possible only in the case when three different mass coefficients for β-, γ-, and rotational motion are used. In addition we have considered the angular momentum dependence of the energies of the states belonging to the ground, β-, and γ-bands and the intraband $E2$ transitions and have shown that the quality of our description of these characteristics is quite satisfactory. Reference [9] is similar in intent to the present paper. It uses, however, a different potential, namely the Davidson potential, which allows an analytical treatment of the centrifugal stretching effect and one common mass parameter only.

II. THE HAMILTONIAN AND THE PARAMETERS USED IN THE CALCULATIONS

In the case of the well deformed axially symmetric nuclei where K is a good quantum number, the Bohr Hamiltonian with three different mass coefficients [18] can be written in the following form:

$$H = H_{\rm rot} + H_\gamma + H_\beta, \tag{1}$$

where

$$H_{\rm rot} = \frac{\hbar^2}{6B_{\rm rot}\beta^2}\left(\hat{\vec{L}}^2 - \hat{L}_3^2\right), \tag{2}$$

$$H_\gamma = -\frac{\hbar^2}{2B_\gamma\beta^2}\frac{1}{\gamma}\frac{\partial}{\partial\gamma}\gamma\frac{\partial}{\partial\gamma} + \frac{\hbar^2}{2B_\gamma}\frac{\hat{L}_3^2}{4\beta^2\gamma^2} + \frac{1}{2}C_\gamma\langle L, K \parallel \beta^2 \parallel L, K\rangle\gamma^2, \tag{3}$$

$$H_\beta = -\frac{\hbar^2}{2}\left(\frac{1}{B_\beta}\frac{\partial^2}{\partial\beta^2} + \frac{2}{B_\gamma}\frac{1}{\beta}\frac{\partial}{\partial\beta} + \frac{2}{B_\beta}\frac{1}{\beta}\frac{\partial}{\partial\beta}\right) + \frac{1}{2}C_\beta(\beta-\beta_0)^2, \tag{4}$$

where β_0 is the value of the equilibrium deformation for the states with $L = 0$. We have assumed above that the stiffness coefficient of the γ-vibrations is proportional to the average value of β^2 in the states under consideration having angular momentum L and its projection on the symmetry axis K. It is, of course, an approximation. In the general case the potential energy depends on the two invariants: β^2 and $\beta^3\cos 3\gamma$. In the case of small γ, the last invariant can be approximated by $\beta^3(1 - 9/2\gamma^2)$, i.e., γ^2 should be multiplied in the potential energy by β^3 or a more complicated function of β. However, in the case of the well deformed axially symmetric nuclei, which are the subject of our investigation, the wave functions of the collective states are located around the equilibrium value of β with a relatively small amplitude of the oscillations around this value. This equilibrium value can depend weakly on the angular momentum L because of the centrifugal stretching. For this reason we use a constant multiplier in the expression for the potential.

The terms of H_β linear, $\partial/\partial\beta$ can be removed from the Hamiltonian by a standard transformation of the wave function Ψ:

$$\Psi = \beta^{-(1+B_\beta/B_\gamma)}\Psi'. \tag{5}$$

From the Hamiltonian presented above we see that the effective angular momentum dependent potential for β motion is given by the expression where we put instead of the operator L_3^2 its eigenvalue K^2

$$U_{\rm eff}(\beta) = \frac{\hbar^2}{6B_{\rm rot}\beta^2}(\hat{\vec{L}}^2 - K^2) + \frac{1}{2}C_\beta(\beta-\beta_0)^2. \tag{6}$$

Acting as in the Davydov-Chaban and VMI models, we find the angular momentum dependent value of β, namely, $\beta_0(L, K)$, which minimizes the value of $U_{\rm eff}(\beta)$ and then expands $U_{\rm eff}(\beta)$ in degrees of $(\beta - \beta_0(L, K))$ taking into account only quadratic terms in $(\beta - \beta_0(L, K))$. Estimating the values of the different terms which appear due to the centrifugal stretching and also as the result of the iterations of the equation for $\beta_0(L, K)$, we keep only the most important terms. For instance, the correction to the stiffness coefficient for β vibrations C_β which is of the order of 10^{-3} for $L = 2$ is neglected below. As a result the angular momentum dependent value of the equilibrium deformation is given by the expression

$$\beta_0(L, K) = \beta_0\left(1 + \frac{\hbar^2(L(L+1) - K^2)}{3B_{\rm rot}\beta_0^2\cdot\hbar\sqrt{\frac{C_\beta}{B_\beta}}}\frac{\hbar}{\sqrt{B_\beta C_\beta}\beta_0^2}\right), \tag{7}$$

where we did not reduce the parameter B_β and did not combine C_β's and β_0's in one multiplier in order to demonstrate clearly that the centrifugal correction of the value of the equilibrium deformation is proportional to the ratio of the rotational energy to the energy of the β-vibrational state and to the square of the amplitude of the zero point β-fluctuations. The effect of the angular momentum dependence of the equilibrium value of β is not large. For instance in the case of ^{156}Gd for which $E(4^+_{\rm g.s.})/E(2^+_{\rm g.s.}) = 3.24$ $\beta_0(L = 2, K = 0) = 1.004\beta_0$, $\beta_0(L = 8, K = 0) = 1.05\beta_0$ and $\beta_0(L = 12, K = 0) = 1.11\beta_0$. However, taking this effect into account, we significantly improve the agreement of the calculated energies of the rotational states and the intraband reduced $E2$ transition probabilities with the experimental data.

Finally the Hamiltonian used for the calculations of the excitation energies and the reduced $E2$ transition probabilities

TABLE I. The values of four parameters used in the calculations.

Nucleus	$\hbar\sqrt{\frac{C_\beta}{B_\beta}}/\frac{\hbar^2}{B_{\rm rot}\beta_0^2}$	$\hbar\sqrt{\frac{C_\gamma}{B_\gamma}}/\frac{\hbar^2}{B_{\rm rot}\beta_0^2}$	$a\equiv\frac{\hbar}{\sqrt{B_\beta C_\beta}\beta_0^2}$	$g\equiv\frac{\hbar}{\sqrt{B_\gamma C_\gamma}\beta_0^2}$	$a(B_\beta=B_\gamma=B_{\rm rot})$	$g(B_\beta=B_\gamma=B_{\rm rot})$
^{154}Sm	13.40	17.23	0.0187	0.0184	0.0746	0.0580
^{156}Gd	11.79	12.63	0.0183	0.0250	0.0848	0.0792
^{172}Yb	13.20	18.23	0.0068	0.0063	0.0757	0.0549
^{182}W	11.36	11.88	0.0133	0.0248	0.0880	0.0849

takes the form

$$H=\frac{\hbar^2(L(L+1)-K^2)}{6B_{\rm rot}\beta_0^2(L,K)}-\frac{\hbar^2}{2B_\beta}\frac{\partial^2}{\partial(\beta-\beta_0(L,K))^2}+\frac{1}{2}C_\beta(\beta-\beta_0(L,K))^2-\frac{\hbar^2}{2B_\gamma\beta_0^2(L,K)}\left(\frac{1}{\gamma}\frac{\partial}{\partial\gamma}\gamma\frac{\partial}{\partial\gamma}\right)+\frac{1}{2}C_\gamma\beta_0(L,K)^2\gamma^2, \quad (8)$$

where we have substituted $\beta_0^2(L,K)$ instead of $\langle L,K\parallel\beta^2\parallel L,K\rangle$. Below, all $B(E2)$'s are calculated in units of $B(E2;2^+_{\rm g.s.}\to0^+_{\rm g.s.})$ and the energies are calculated in units of $E(2^+_{\rm g.s.})$.

As the input data necessary to fix the four unknown nondimensional parameters $\hbar\sqrt{\frac{C_\beta}{B_\beta}}/\frac{\hbar^2}{B_{\rm rot}\beta_0^2}$, $\hbar\sqrt{\frac{C_\gamma}{B_\gamma}}/\frac{\hbar^2}{B_{\rm rot}\beta_0^2}$, $a\equiv\frac{\hbar}{\sqrt{B_\beta C_\beta}\beta_0^2}$, and $g\equiv\frac{\hbar}{\sqrt{B_\gamma C_\gamma}\beta_0^2}$, we use the experimentally known values of $E(0^+_\beta)/E(2^+_{\rm g.s.})$, $E(2^+_\gamma)/E(2^+_{\rm g.s.})$, $B(E2;2^+_\beta\to0^+_{\rm g.s.})/B(E2;2^+_{\rm g.s.}\to0^+_{\rm g.s.})$, and $B(E2;2^+_\gamma\to0^+_{\rm g.s.})/B(E2;2^+_{\rm g.s.}\to0^+_{\rm g.s.})$ taking into account the known experimental errors. The nondimensional parameters mentioned above are determined solving a system of four nonlinear algebraic equations. The values of these parameters which we have used in the calculations are given in Table I.

If we assume that $B_\beta=B_\gamma=B_{\rm rot}$, then, since the number of the unknown parameters will be decreased by two, there will be no data needed on the reduced $E2$ transition probabilities to determine a and g. In this case these parameters are fixed by the energies of the $2^+_{\rm g.s.}$, 2^+_γ, and 0^+_β states. The values of a and g obtained under the assumption that all three mass coefficients coincide are also given in Table I where they are denoted as $a(B_\beta=B_\gamma=B_{\rm rot})$ and $g(B_\beta=B_\gamma=B_{\rm rot})$.

III. THE RESULTS OF CALCULATIONS AND COMPARISON TO EXPERIMENT

In this section we present the results of calculations of the reduced $E2$ transition probabilities and the energies of the collective states. The wave functions of the states of the ground state, β-, and γ-bands which are used for the calculations of the $E2$ reduced transition probabilities are given by the expressions:

$$\Psi_{\rm LM}({\rm g.s.})=\sqrt{\frac{2L+1}{8\pi^2}}D^L_{M0}\left(\frac{\hbar}{2\sqrt{B_\gamma C_\gamma}\beta_0^2(L,0)}\right)^{-1/2}\times\exp\left(-\frac{\sqrt{B_\gamma C_\gamma}\beta_0^2(L,0)}{2\hbar}\gamma^2\right)\left(\frac{\pi\hbar}{\sqrt{B_\beta C_\beta}}\right)^{-1/4}\times\exp\left(-\frac{\sqrt{B_\beta C_\beta}}{2\hbar}(\beta-\beta_0(L,0))^2\right), \quad (9)$$

$$\Psi_{\rm LM}(\beta)=\sqrt{\frac{2L+1}{8\pi^2}}D^L_{M0}\left(\frac{\hbar}{2\sqrt{B_\gamma C_\gamma}\beta_0^2(L,0)}\right)^{-1/2}\times\exp\left(-\frac{\sqrt{B_\gamma C_\gamma}\beta_0^2(L,0)}{2\hbar}\gamma^2\right)\left(\frac{\pi\hbar}{\sqrt{B_\beta C_\beta}}\right)^{-1/4}\times\left(\frac{2\sqrt{B_\beta C_\beta}}{\hbar}\right)^{1/2}(\beta-\beta_0(L,0))\times\exp\left(-\frac{\sqrt{B_\beta C_\beta}}{2\hbar}(\beta-\beta_0(L,0))^2\right), \quad (10)$$

$$\Psi_{\rm LM}(\gamma)=\sqrt{\frac{2L+1}{16\pi^2}}\left(D^L_{M2}+D^L_{M-2}\right)\left(\frac{\hbar^2}{2B_\gamma C_\gamma\beta_0^4(L,2)}\right)^{-1/2}\cdot\gamma\times\exp\left(-\frac{\sqrt{B_\gamma C_\gamma}\beta_0^2(L,2)}{2\hbar}\gamma^2\right)\left(\frac{\pi\hbar}{\sqrt{B_\beta C_\beta}}\right)^{-1/4}\times\exp\left(-\frac{\sqrt{B_\beta C_\beta}}{2\hbar}(\beta-\beta_0(L,2))^2\right). \quad (11)$$

TABLE II. The experimental values of the Grodzins type products $P_{\rm g.s.}=E(2^+_{\rm g.s.})B(E2;0^+_{\rm g.s.}\to2^+_{\rm g.s.})\times Z^{-2}A^{2/3}$, $P_\gamma=E(2^+_\gamma)B(E2;0^+_{\rm g.s.}\to2^+_\gamma)\times Z^{-2}A^{2/3}$, and $P_\beta=E(2^+_\beta)B(E2;0^+_{\rm g.s.}\to2^+_\beta)\times Z^{-2}A^{2/3}$ are given in the units KeV $\cdot\, e^2{\rm b}^2$ and the values of the ratios $B_\gamma/B_{\rm rot}$ and $B_\beta/B_{\rm rot}$ calculated using the relations [18] $B_\gamma/B_{\rm rot}=P_{\rm g.s.}/P_\gamma$ and $B_\beta/B_{\rm rot}=P_{\rm g.s.}/P_\beta$. The experimental data are taken from [22].

Nucleus	$P_{\rm g.s.}$	P_γ	P_β	$B_\gamma/B_{\rm rot}$	$B_\beta/B_{\rm rot}$
^{154}Sm	2.63	0.83	0.20	3.17	6.4
^{156}Gd	2.94	0.96	0.13	3.06	11.7
^{172}Yb	2.99	0.34	0.05	8.79	31.1
^{182}W	2.47	0.75	0.20	3.29	6.05

TABLE III. The calculated and experimental values of the $B(E2)$ for the transitions between the β- and the ground bands in ^{154}Sm and ^{156}Gd. The $B(E2)$'s are given in units of $B(E2; 2^+_{\text{g.s.}} \to 0^+_{\text{g.s.}})$. The results obtained using different values of B_β, B_γ, and B_{rot} are marked as Calc/$B_\beta \neq B_\gamma \neq B_{\text{rot}}$. The results obtained assuming that $B_\beta = B_\gamma = B_{\text{rot}}$ are marked as Calc/$B_\beta = B_\gamma = B_{\text{rot}}$. The experimental data are taken from [22].

$\frac{B(E2;I^+_\beta \to I^+_{\text{g.s.}})}{B(E2;2^+_{\text{g.s.}} \to 0^+_{\text{g.s.}})} \times 10^3$	^{154}Sm			^{156}Gd		
	Calc $B_\beta \neq B_\gamma \neq B_{\text{rot}}$	Calc $B_\beta = B_\gamma = B_{\text{rot}}$	Exp	Calc $B_\beta \neq B_\gamma \neq B_{\text{rot}}$	Calc $B_\beta = B_\gamma = B_{\text{rot}}$	Exp
$2^+_\beta \to 0^+_{\text{g.s.}}$	6.7	27.1	5.4(13)	6.3	29.2	3.4(3)
$4^+_\beta \to 2^+_{\text{g.s.}}$	5.6	21.6		4.7	20.5	
$6^+_\beta \to 4^+_{\text{g.s.}}$	2.9	9.7		1.9	6.3	
$2^+_\beta \to 2^+_{\text{g.s.}}$	13.3	54.0		13.0	61.7	18(2)
$4^+_\beta \to 4^+_{\text{g.s.}}$	12.1	49.1		11.8	56.0	
$6^+_\beta \to 6^+_{\text{g.s.}}$	11.9	48.1		11.6	54.9	
$0^+_\beta \to 2^+_{\text{g.s.}}$	61.7	251.2		62.5	298.0	
$2^+_\beta \to 4^+_{\text{g.s.}}$	43.8	180.1	25(6)	46.0	222.3	22(2)
$4^+_\beta \to 6^+_{\text{g.s.}}$	51.3	215.0		55.3	274.9	
$6^+_\beta \to 8^+_{\text{g.s.}}$	62.8	271.1		68.9	358.2	

The calculations are performed for four well deformed nuclei ^{154}Sm, ^{156}Gd, ^{172}Yb, and ^{182}W with the values of the $R_{4/2}$ ratio larger than 3.24.

First of all we present the results obtained for the interband reduced $E2$ transition probabilities since just these quantities are the most sensitive to the assumptions done concerning the relations between the mass coefficients. All formulas needed to calculate $B(E2)$'s are given in the Appendix. The results of calculations of the $B(E2; I'_\beta \to I_{\text{g.s.}})/B(E2; 2^+_{\text{g.s.}} \to 0^+_{\text{g.s.}})$ and $B(E2; I'_\gamma \to I_{\text{g.s.}})/B(E2; 2^+_{\text{g.s.}} \to 0^+_{\text{g.s.}})$ are presented in Tables III–VI.

As it is seen from the Tables III–VI the results obtained under the assumption that $B_\beta = B_\gamma = B_{\text{rot}}$ are, in average, ten times larger than the experimental values of $B(E2; I'_\beta \to I_{\text{g.s.}})/B(E2; 2^+_{\text{g.s.}} \to 0^+_{\text{g.s.}})$ and three times larger than the experimental values of $B(E2; I'_\gamma \to I_{\text{g.s.}})/B(E2; 2^+_{\text{g.s.}} \to 0^+_{\text{g.s.}})$. This is in correspondence with the results obtained in our previous publications [18,19] for the Grodzins type products

TABLE IV. The calculated and experimental values of the $B(E2)$ for the transitions between the β- and the ground bands in ^{172}Yb and ^{182}W. The $B(E2)$'s are given in units of $B(E2; 2^+_{\text{g.s.}} \to 0^+_{\text{g.s.}})$. The results obtained using different values of B_β, B_γ, and B_{rot} are marked as Calc/$B_\beta \neq B_\gamma \neq B_{\text{rot}}$. The results obtained assuming that $B_\beta = B_\gamma = B_{\text{rot}}$ are marked as Calc/$B_\beta = B_\gamma = B_{\text{rot}}$. The experimental data are taken from [22].

$\frac{B(E2;I^+_\beta \to I^+_{\text{g.s.}})}{B(E2;2^+_{\text{g.s.}} \to 0^+_{\text{g.s.}})} \times 10^3$	^{172}Yb			^{182}W		
	Calc $B_\beta \neq B_\gamma \neq B_{\text{rot}}$	Calc $B_\beta = B_\gamma = B_{\text{rot}}$	Exp	Calc $B_\beta \neq B_\gamma \neq B_{\text{rot}}$	Calc $B_\beta = B_\gamma = B_{\text{rot}}$	Exp
$2^+_\beta \to 0^+_{\text{g.s.}}$	2.4	27.3	1.1(1)	4.5	29.9	6.6(10)
$4^+_\beta \to 2^+_{\text{g.s.}}$	2.0	21.6		3.2	20.0	
$6^+_\beta \to 4^+_{\text{g.s.}}$	1.0	9.3		1.2	5.3	
$2^+_\beta \to 2^+_{\text{g.s.}}$	4.9	54.8		9.5	64.1	4.6(6)
$4^+_\beta \to 4^+_{\text{g.s.}}$	4.4	49.8		8.6	58.2	
$6^+_\beta \to 6^+_{\text{g.s.}}$	4.3	48.8		8.4	57.1	
$0^+_\beta \to 2^+_{\text{g.s.}}$	22.5	256.2		46.0	313.3	
$2^+_\beta \to 4^+_{\text{g.s.}}$	16.0	184.5	12(1)	34.1	236.5	13(1)
$4^+_\beta \to 6^+_{\text{g.s.}}$	18.8	221.2		41.3	295.6	
$6^+_\beta \to 8^+_{\text{g.s.}}$	22.8	279.7		51.6	389.4	

TABLE V. The calculated and experimental values of the $B(E2)$ for the transitions between the γ- and the ground bands in ^{154}Sm and ^{156}Gd. The $B(E2)$'s are given in units of $B(E2; 2^+_{\text{g.s.}} \rightarrow 0^+_{\text{g.s.}})$. The results obtained using different values of B_β, B_γ, and B_{rot} are marked as Calc/$B_\beta \neq B_\gamma \neq B_{\text{rot}}$. The results obtained assuming that $B_\beta = B_\gamma = B_{\text{rot}}$ are marked as Calc/$B_\beta = B_\gamma = B_{\text{rot}}$. The experimental data are taken from [22].

$\frac{B(E2;I^+_\gamma \rightarrow I^+_{\text{g.s.}})}{B(E2;2^+_{\text{g.s.}} \rightarrow 0^+_{\text{g.s.}})} \times 10^3$	^{154}Sm			^{156}Gd		
	Calc $B_\beta \neq B_\gamma \neq B_{\text{rot}}$	Calc $B_\beta = B_\gamma = B_{\text{rot}}$	Exp	Calc $B_\beta \neq B_\gamma \neq B_{\text{rot}}$	Calc $B_\beta = B_\gamma = B_{\text{rot}}$	Exp
$2^+_\gamma \rightarrow 0^+_{\text{g.s.}}$	18.4	59.0	18.4(29)	25.0	80.9	25.0(8)
$2^+_\gamma \rightarrow 2^+_{\text{g.s.}}$	26.2	83.3		35.5	113.7	38.7(13)
$2^+_\gamma \rightarrow 4^+_{\text{g.s.}}$	1.3	4.0	3.9(6)	1.8	5.4	4.1(2)
$3^+_\gamma \rightarrow 2^+_{\text{g.s.}}$	32.8	105.3		44.6	144.4	39.0(75)
$3^+_\gamma \rightarrow 4^+_{\text{g.s.}}$	13.0	40.9		17.7	55.4	27.2(53)
$4^+_\gamma \rightarrow 2^+_{\text{g.s.}}$	11.0	35.5		14.9	48.9	9.6(27)
$4^+_\gamma \rightarrow 4^+_{\text{g.s.}}$	32.1	102.2		43.6	139.6	53(16)
$4^+_\gamma \rightarrow 6^+_{\text{g.s.}}$	2.7	8.3		3.7	11.1	
$5^+_\gamma \rightarrow 4^+_{\text{g.s.}}$	29.3	94.4		39.8	129.7	43(43)
$5^+_\gamma \rightarrow 6^+_{\text{g.s.}}$	16.5	51.5		22.4	69.6	59(59)

TABLE VI. The calculated and experimental values of the $B(E2)$ for the transitions between the γ- and the ground bands in ^{172}Yb and ^{182}W. The $B(E2)$'s are given in units of $B(E2; 2^+_{\text{g.s.}} \rightarrow 0^+_{\text{g.s.}})$. The results obtained using different values of B_β, B_γ, and B_{rot} are marked as Calc/$B_\beta \neq B_\gamma \neq B_{\text{rot}}$. The results obtained assuming that $B_\beta = B_\gamma = B_{\text{rot}}$ are marked as Calc/$B_\beta = B_\gamma = B_{\text{rot}}$. The experimental data are taken from [22].

$\frac{B(E2;I^+_\gamma \rightarrow I^+_{\text{g.s.}})}{B(E2;2^+_{\text{g.s.}} \rightarrow 0^+_{\text{g.s.}})} \times 10^3$	^{172}Yb			^{182}W		
	Calc $B_\beta \neq B_\gamma \neq B_{\text{rot}}$	Calc $B_\beta = B_\gamma = B_{\text{rot}}$	Exp	Calc $B_\beta \neq B_\gamma \neq B_{\text{rot}}$	Calc $B_\beta = B_\gamma = B_{\text{rot}}$	Exp
$2^+_\gamma \rightarrow 0^+_{\text{g.s.}}$	6.3	55.8	6.3(5)	24.8	86.3	24.8(6)
$2^+_\gamma \rightarrow 2^+_{\text{g.s.}}$	9.0	78.8		35.3	121.2	49.2(13)
$2^+_\gamma \rightarrow 4^+_{\text{g.s.}}$	0.4	3.8	0.60(5)	1.8	5.7	0.2
$3^+_\gamma \rightarrow 2^+_{\text{g.s.}}$	11.2	99.7		44.2	154.1	
$3^+_\gamma \rightarrow 4^+_{\text{g.s.}}$	4.5	36.6		17.6	58.9	
$4^+_\gamma \rightarrow 2^+_{\text{g.s.}}$	3.8	33.6	33(24)	14.8	52.2	17.2(17)
$4^+_\gamma \rightarrow 4^+_{\text{g.s.}}$	11.0	96.7		43.3	148.8	75.9(73)
$4^+_\gamma \rightarrow 6^+_{\text{g.s.}}$	0.9	7.8		3.7	11.7	
$5^+_\gamma \rightarrow 4^+_{\text{g.s.}}$	10.0	89.4		39.5	138.4	
$5^+_\gamma \rightarrow 6^+_{\text{g.s.}}$	5.7	48.7		22.3	73.9	

TABLE VII. The calculated and experimental values of the $B(E2; (I+2)^+_{\text{g.s.}} \rightarrow I^+_{\text{g.s.}})$ given in units of $B(E2; 2^+_{\text{g.s.}} \rightarrow 0^+_{\text{g.s.}})$. The experimental data are taken from [22].

$\frac{B(E2;(I+2)^+_{\text{g.s.}} \rightarrow I^+_{\text{g.s.}})}{B(E2;2^+_{\text{g.s.}} \rightarrow 0^+_{\text{g.s.}})}$	^{154}Sm		^{156}Gd		^{172}Yb		^{182}W		Rigid rotor
	Calc	Exp	Calc	Exp	Calc	Exp	Calc	Exp	
$4^+_{\text{g.s.}} \rightarrow 2^+_{\text{g.s.}}$	1.44	1.40(5)	1.44	1.41(5)	1.43	1.42(10)	1.44	1.43(8)	1.43
$6^+_{\text{g.s.}} \rightarrow 4^+_{\text{g.s.}}$	1.61	1.67(7)	1.61	1.58(6)	1.59	1.51(14)	1.60	1.46(16)	1.57
$8^+_{\text{g.s.}} \rightarrow 6^+_{\text{g.s.}}$	1.72	1.83(11)	1.73	1.71(10)	1.67	1.89(19)	1.71	1.53(14)	1.65
$10^+_{\text{g.s.}} \rightarrow 8^+_{\text{g.s.}}$	1.82	1.81(11)	1.83	1.68(9)	1.74	1.77(11)	1.80	1.48(14)	1.69
$12^+_{\text{g.s.}} \rightarrow 10^+_{\text{g.s.}}$	1.91		1.93	1.60(16)	1.79		1.88		1.72

TABLE VIII. The calculated and experimental values of $R_{I/2} \equiv E(I^+)/E(2^+)$ for the states of the ground state band. The experimental data are taken from [22].

Nucleus	$E(2^+_{\text{g.s.}})$, KeV Exp	$R_{4/2}$		$R_{6/2}$		$R_{8/2}$		$R_{10/2}$		$R_{12/2}$	
		Calc	Exp	Calc	Exp	Calc	Exp	Calc	Exp	Calc	Exp
^{154}Sm	82	3.28	3.26	6.76	6.63	11.28	11.01	16.65	16.26	22.68	22.27
^{156}Gd	89	3.29	3.24	6.75	6.57	11.22	10.84	16.51	15.91	22.38	21.63
^{172}Yb	79	3.32	3.29	6.91	6.84	11.73	11.54	17.69	17.34	24.71	24.14
^{182}W	100	3.30	3.29	6.81	6.80	11.41	11.44	16.94	17.12	23.21	23.72
Rigid rotor		3.33		7		12		18.33		26	

for the β- and γ-bands which are inversely proportional to the corresponding mass coefficients. We see from these results that the correct description of the interband reduced $E2$ transition probabilities is impossible in the case of the well deformed nuclei if only one common mass coefficient for β-, γ-vibrational, and rotational modes is used in the Bohr Hamiltonian. This conclusion is supported by the results of calculations published in [9].

For completeness we have calculated the intraband $E2$ transition probabilities and the energies of the collective states although these quantities are not sensitive to the assumption concerning the relation between the mass coefficients. In Table VII the results of calculations of the intraband reduced transition probabilities for the ground state band are presented. Comparing the calculated results with the experimental data, the experimental errors should be taken into account. We see that an agreement with the experimental data is quite satisfactory.

The excitation energies of the states belonging to the ground, β-, and γ-bands are given in Tables VIII–X. They are calculated using the formulas

$$E(L_{\text{g.s.}}) = \frac{\hbar^2 L(L+1)}{6B_{\text{rot}}\beta_0^2(L,0)}, \tag{12}$$

$$E(L_\beta) = \hbar\sqrt{\frac{C_\beta}{B_\beta}} + \frac{\hbar^2 L(L+1)}{6B_{\text{rot}}\beta_0^2(L,0)}, \tag{13}$$

$$E(L_\gamma) = \hbar\sqrt{\frac{C_\gamma}{B_\gamma}} + \frac{\hbar^2 (L(L+1)-4)}{6B_{\text{rot}}\beta_0^2(L,2)}. \tag{14}$$

We see that the results obtained for the energies of the states belonging to the ground, β-, and γ-bands are in a quite satisfactory agreement with data. The deviations from the predictions of the rigid rotor model are due to the centrifugal stretching effect which is taken into account in this paper by using the Davydov-Chaban procedure which was suggested 50 years ago, and, as we see, works quite satisfactorily with the new data which appear since that time. The assumption that B_β, B_γ, and B_{rot} have different values is important only for the calculations of the interband $E2$ transition probabilities. The the intraband $E2$ reduced transition probabilities can be described under the assumption that $B_\beta = B_\gamma = B_{\text{rot}}$.

The values of the parameter "a" have been varied slightly around the values which follows from the experimental data on the interband transitions in order to get a better description of the centrifugal stretching effect.

IV. SUMMARY

We have considered a Bohr-type Hamiltonian and a quadrupole operator expressed in terms of Bohr's β- and γ-variables and applied them for the description of the collective quadrupole motion in four even-even well-deformed axially symmetric nuclei. This Hamiltonian has three different mass coefficients for the three excitation modes: β- and γ-vibrations and rotation. It is shown that without this assumption it is impossible to describe correctly the absolute values of the $E2$ reduced transition probabilities between the states of the vibrational bands and the states of the ground state band. It is very important to have a considerably extended set of high precision data for several $E2$ transitions of this type for the proper verification of the theoretical approach used.

TABLE IX. The calculated and experimental values of $R^\beta_{I/2} \equiv E(I^+_\beta)/E(2^+)$ for the states of the β-band. The experimental data are taken from [22].

Nucleus	$R^\beta_{0/2}$		$R^\beta_{2/2}$		$R^\beta_{4/2}$		$R^\beta_{6/2}$		$R^\beta_{8/2}$		$R^\beta_{10/2}$		$R^\beta_{12/2}$	
	Calc	Exp	Calc	Exp	Calc	Exp	Calc	Exp	Calc	Exp	Calc	Exp	Calc	Exp
^{154}Sm	13.40	13.40	14.40	14.37	16.68	16.32	20.16	19.23	24.68		30.05		36.08	
^{156}Gd	11.79	11.79	12.79	12.69	15.08	14.68	18.54	17.30	23.01	20.76	28.30	24.94	34.17	30.43
^{172}Yb	13.20	13.20	14.20	14.15	16.52	16.34	20.11	19.53	24.93	23.54	30.89	28.10	37.91	33.11
^{182}W	11.36	11.36	12.36	12.57	14.66	15.10	18.17		22.77		28.30		34.57	

TABLE X. The calculated and experimental values of $E(I^+_\gamma)/E(2^+)$. The experimental data are taken from [22].

	^{154}Sm		^{156}Gd		^{172}Yb		^{182}W	
I^γ	Calc	Exp	Calc	Exp	Calc	Exp	Calc	Exp
2	17.56	17.56	12.97	12.97	18.63	18.63	12.21	12.21
3	16.56	18.77	13.96	14.02	19.63	19.68	13.21	13.31
4	19.87	20.30	15.27	15.22	20.95	21.06	14.52	14.43
5	21.48	22.01	16.88	16.93	22.60	22.60	16.14	16.24
6	23.38	23.73	18.76	18.47	24.56		18.05	17.70
7	25.53	26.27	20.90	20.79	26.83		20.24	
8	27.93		23.27	22.60	29.39		22.68	21.80
9	30.55		25.85	25.28	32.24		25.35	
10	33.36		28.60	27.44	35.37		28.24	
11	36.33		31.50	30.19	38.76		31.31	
12	39.44		34.52	32.84	42.40		34.55	
13	42.65		37.63	35.67	46.28		37.92	

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APPENDIX

In this appendix we present the formulas used for the calculations of the $B(E2)$'s:

$$x(L,K) \equiv \frac{\beta_0(L,K)}{\beta_0}, \quad a \equiv \frac{\hbar}{\sqrt{B_\beta C_\beta}\beta_0^2}, \tag{A1}$$

$$\frac{B(E2; L'^+_{\text{g.s.}} \to L^+_{\text{g.s.}})}{B(E2; 2^+_{\text{g.s.}} \to 0^+_{\text{g.s.}})} = 5\left(C^{L0}_{L'020}\right)^2 \left(\frac{x(L',0)+x(L,0)}{x(2,0)+1}\right)^2 \cdot \left(\frac{x(L',0)x(L,0)(x(2,0)^2+1)}{x(2,0)(x(L',0)^2+x(L,0)^2)}\right)^2 \times \exp\left(-\frac{1}{a}((x(L',0)-x(L,0))^2-(x(2,0)-1)^2)\right), \tag{A2}$$

$$\frac{B(E2; L'^+_\beta \to L^+_{\text{g.s.}})}{B(E2; 2^+_{\text{g.s.}} \to 0^+_{\text{g.s.}})} = 5\left(C^{L0}_{L'020}\right)^2 \frac{a}{2}\left(\frac{2}{x(2,0)+1}\right)^2 \times \left(\frac{x(L',0)x(L,0)(x(2,0)^2+1)}{x(2,0)(x(L',0)^2+x(L,0)^2)}\right)^2 \times \left(1-0.5(x(L',0)^2-x(L,0)^2)/a\right)^2 \times \exp\left(-\frac{1}{a}((x(L',0)-x(L,0))^2-(x(2,0)-1)^2)\right), \tag{A3}$$

$$\frac{B(E2; L'^+_\gamma \to L^+_{\text{g.s.}})}{B(E2; 2^+_\gamma \to 0^+_{\text{g.s.}})} = 5\left(C^{L0}_{L'020}\right)^2 \times \left(\frac{x(L',2)^2x(L,0)(x(2,2)^2+1)(x(L',2)+x(L,0))}{x(2,2)^2(x(L',2)^2+x(L,0)^2)(x(2,2)+1)}\right)^2 \times \exp\left(-\frac{1}{a}((x(L',2)-x(L,0))^2-(x(2,2)-1)^2)\right). \tag{A4}$$

[1] A. Bohr, Mat. Fys. Medd. K. Dan. Vidensk. Selsk. **26**, no. 14, 1 (1952).
[2] P. Bizetti, in Proceedings of the Predeal International Summer School in Nuclear Physics, edited by A. A. Raduta, V. Baran, A. C. Georghe, and I. Ursu, 2006.
[3] F. Iachello, Phys. Rev. Lett. **87**, 052502 (2001).
[4] R. F. Casten and N. V. Zamfir, Phys. Rev. Lett. **87**, 052503 (2001).
[5] D. J. Rowe and C. Bahri, J. Phys. A **31**, 4947 (1998).
[6] N. Pietralla and O. M. Gorbachenko, Phys. Rev. C **70**, 011304(R) (2004).
[7] D. Bonatsos, D. Lenis, N. Minkov, P. P. Raychev, and P. A. Terziev, Phys. Rev. C **69**, 014302 (2004).
[8] L. Fortunato, Phys. Rev. C **70**, 011302(R) (2004).
[9] D. Bonatsos, E. M. McCutchan, N. Minkov *et al.*, Phys. Rev. C **76**, 064312 (2007).
[10] M. A. Caprio, Phys. Rev. C **72**, 054323 (2005).
[11] A. S. Davydov and G. F. Fillipov, Nucl. Phys. **8**, 237 (1958).

[12] A. S. Davydov and A. A. Chaban, Nucl. Phys. **20**, 499 (1960).
[13] J. M. Eisenberg and W. Greiner, *Nuclear Models*, Vol. 1 (North-Holland, Amsterdam, 1970).
[14] A. Faessler and W. Greiner, Z. Phys. **168**, 425 (1962).
[15] K. Kumar and M. Baranger, Nucl. Phys. **A110**, 529 (1968).
[16] K. Kumar, Nucl. Phys. **A231**, 189 (1974).
[17] R. V. Jolos and P. von Brentano, Phys. Rev. C **74**, 064307 (2006).
[18] R. V. Jolos and P. von Brentano, Phys. Rev. C **76**, 024309 (2007).
[19] R. V. Jolos and P. von Brentano, Phys. Rev. C **77**, 064317 (2008).
[20] M. A. J. Mariscotti, G. S. Goldhaber, and B. Buck, Phys. Rev. **178**, 1864 (1969).
[21] K. Zajac, L. Prochniak, K. Pomorski, S. G. Rohozinski, and J. Srebrny, Acta Phys. Pol. B **31**, 459 (2000).
[22] http://www.nndc.bnl.gov/nndc/ensdf/.