Abundance of ground states with positive parity

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We investigate analytically and numerically a random-matrix model for m fermions occupying ℓ_1 single-particle states with positive parity and ℓ_2 single-particle states with negative parity and interacting through random two-body forces that conserve parity. The single-particle states are completely degenerate and carry no further quantum numbers. We compare spectra of many-body states with positive and with negative parity. We show that in the dilute limit defined by $m, \ell_{1,2} \to \infty$ and $m/\ell_{1,2} \to 0$, ground states with positive and negative parity occur with equal probability. Differences in the ground-state probabilities are, thus, a finite-size effect and are mainly due to different dimensions of the Hilbert spaces of either parity.

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I. MOTIVATION AND AIM

Johnson *et al.* [1] observed that in the two-body random ensemble (TBRE) of the nuclear shell model, ground states with spin zero occur much more frequently than expected from their statistical weight. That observation caused considerable theoretical activity (see the reviews in Refs. [2,3]). A similar preponderance for states with positive parity was found in Ref. [4]. We wish to explore the reason for that preponderance. We focus attention on parity (rather than spin) because that quantum number is analytically more easily accessible. We use a model with spinless fermions that interact via random two-body forces. The degenerate single-particle states carry no orbital angular momentum quantum number but have either positive or negative parity. The model is a modified version of EGOE(2), the embedded two-body ensemble of Gaussian random matrices [5]. We investigate the model by using both an analytical approach and numerical simulations. The analytical approach evaluates traces of powers of the Hamiltonian up to very high order and uses results of Refs. [6,7] to estimate the position of the ground state. The numerical simulations involve diagonalization of matrices drawn at random from the ensemble and can be done only for Hamiltonian matrices of sufficiently small dimension, whereas the analytical approach is suited also for large-dimensional matrices.

To motivate our focus on traces of the Hamiltonian, we recall in Sec. II how the ground-state energy was estimated in Refs. [6,7]. That method is used and compared with numerical simulations in Sec. V. Prior to that, we define our model in Sec. III. The first and second moments of the Hamiltonian are calculated for both parities in Sec. IV. After presenting our numerical results, we investigate our model in the limit of large matrix dimension N in Sec. VI. We show that for $N \to \infty$, both the first and second moments of the Hamiltonian have the same values for either parity. Combining that fact with the well-known result [5] that the shape of the average spectrum is asymptotically ($N \to \infty$) Gaussian, we conclude that ground states of either parity are equally likely. In Sec. VII we show that the strong correlations found asymptotically for the first and second moments extend to higher (but not to all) moments. We discuss the implications of that result for correlations between the spectral fluctuation properties of positive- and negative-parity states and show that the result reinforces our conclusions. We conclude with a summary and discussion.

II. SIMPLE ESTIMATE FOR THE GROUND-STATE ENERGY

To estimate the ground-state energy, we use with proper modifications the method introduced for states with spin in Ref. [6] and improved in Ref. [7]. Let H denote the Hamiltonian of the system, $\mathcal{P}_\pm$ the projectors onto states with positive and negative parity, and $E_{\rm ground}(\pm)$ the energies of the lowest state with positive or negative parity. We estimate $E_{\rm ground}(\pm)$ by writing

$$E_{\rm ground}(\pm) = {\rm nTr}(H\mathcal{P}_\pm) - r_\pm \sigma_\pm. \tag{1}$$

The symbol nTr stands for the normalized trace (the actual trace divided by the dimension $N_\pm$ of Hilbert space), and the width σ is defined as

$$\sigma_\pm^2 = {\rm nTr}(H^2\mathcal{P}_\pm). \tag{2}$$

In Ref. [6], the analog of Eq. (1) was used without the first term on the right-hand side. That term was added in Ref. [7]. It represents the fluctuations of the centroid of the spectrum. Inclusion of that term improves the agreement with numerical simulations: the fluctuations of the parameter r are reduced. Equation (1) has a simple interpretation: shell-model spectra have nearly Gaussian shape [5] and thus are essentially characterized by the centroid and the width. The distance of the lowest state from the centroid of the spectrum is given by a multiple $r_\pm$ of the width. In the case of spin, the stochastic fluctuations of r were found to be small, so that r can be considered a constant. In Ref. [7], an explicit expression for r was obtained by fitting the results of numerical calculations. It reads

$$r = \sqrt{0.99 \ln N + 0.36}. \tag{3}$$

We actually prefer to determine $r_\pm$ by a fit to numerical data. In Sec. V we compare the result with Eq. (3). We also compare the numerically determined probability of finding a ground state of given parity with predictions derived from Eqs. (1) and (2).

III. MODEL

We consider a system of m spinless fermions distributed over a set of degenerate single-particle states. There are ℓ_1 states of positive parity and ℓ_2 states of negative parity, with associated creation and destruction operators $a^\dagger_{1\mu}, a_{1\mu}$ ($\mu = 1, 2, \ldots, \ell_1$) and $a^\dagger_{2\rho}, a_{2\rho}$ ($\rho = 1, 2, \ldots, \ell_2$), respectively. The single-particle states carry no further quantum numbers. The many-body states of the system have positive (negative) parity if the number m_2 of fermions in negative-parity states is even (odd). The total numbers N_+ and N_- of positive- and negative-parity states are

$$\begin{aligned} N_+ &= \sum_{m_1,m_2} \delta_{m_1+m_2,m}\delta_{m_2,\mathrm{even}} \binom{\ell_1}{m_1}\binom{\ell_2}{m_2}, \\ N_- &= \sum_{m_1,m_2} \delta_{m_1+m_2,m}\delta_{m_2,\mathrm{odd}} \binom{\ell_1}{m_1}\binom{\ell_2}{m_2}. \end{aligned} \tag{4}$$

The Hamiltonian H is a sum of two-body interactions that conserve parity,

$$\begin{aligned} H = {} & \frac{1}{4}\sum_{\mu\nu\rho\sigma} V^{(1)}_{\mu\nu;\rho\sigma} a^\dagger_{1\mu} a^\dagger_{1\nu} a_{1\sigma} a_{1\rho} + \frac{1}{4}\sum_{\mu\nu\rho\sigma} V^{(2)}_{\mu\nu;\rho\sigma} a^\dagger_{2\mu} a^\dagger_{2\nu} a_{2\sigma} a_{2\rho} \\ & + \frac{1}{4}\sum_{\mu\nu\rho\sigma} X^{(1)}_{\mu\nu;\rho\sigma} (a^\dagger_{1\mu} a^\dagger_{1\nu} a_{2\sigma} a_{2\rho} + a^\dagger_{2\rho} a^\dagger_{2\sigma} a_{1\nu} a_{1\mu}) \\ & + \sum_{\mu\nu\rho\sigma} X^{(2)}_{\mu\nu;\rho\sigma} a^\dagger_{1\mu} a^\dagger_{2\rho} a_{2\sigma} a_{1\nu}. \end{aligned} \tag{5}$$

The ranges of the summation indices depend in an obvious way on the creation operators and matrix elements on which they appear. The two-body matrix elements obey the symmetry relations

$$\begin{aligned} V^{(1)}_{\mu\nu;\rho\sigma} &= V^{(1)}_{\rho\sigma;\mu\nu} = -V^{(1)}_{\nu\mu;\rho\sigma} = \left(V^{(1)}_{\mu\nu;\rho\sigma}\right)^*, \\ V^{(2)}_{\mu\nu;\rho\sigma} &= V^{(2)}_{\rho\sigma;\mu\nu} = -V^{(2)}_{\nu\mu;\rho\sigma} = \left(V^{(2)}_{\mu\nu;\rho\sigma}\right)^*, \\ X^{(1)}_{\mu\nu;\rho\sigma} &= -X^{(1)}_{\nu\mu;\rho\sigma} = -X^{(1)}_{\mu\nu;\sigma\rho} = \left(X^{(1)}_{\mu\nu;\rho\sigma}\right)^*, \\ X^{(2)}_{\mu\nu;\rho\sigma} &= \left(X^{(2)}_{\mu\nu;\rho\sigma}\right)^*. \end{aligned} \tag{6}$$

An ensemble of Hamiltonians is obtained when we consider the matrix elements in Eq. (5) as Gaussian-distributed random variables. We assume that the $V^{(1)}_{\mu\nu;\rho\sigma}$ are not correlated with the $V^{(2)}_{\mu'\nu';\rho'\sigma'}$ and likewise for the pairs $V^{(i)}_{\mu\nu;\rho\sigma}, X^{(k)}_{\mu'\nu';\rho'\sigma'}$ for $i = 1, 2$ and $k = 1, 2$, and for the pair $X^{(1)}_{\mu\nu;\rho\sigma}, X^{(2)}_{\mu'\nu';\rho'\sigma'}$. All matrix elements have zero mean values. For the variances, we define pairs of indices α, β by writing $\alpha = \{\mu\nu\}$ and likewise for β, and have for $i = 1, 2$,

$$\begin{aligned} \overline{V^{(i)}_{\alpha;\beta} V^{(i)}_{\alpha';\beta'}} &= v^2(\delta_{\alpha\alpha'}\delta_{\beta\beta'} + \delta_{\alpha\beta'}\delta_{\beta\alpha'}), \\ \overline{X^{(1)}_{\alpha;\beta} X^{(1)}_{\alpha';\beta'}} &= v^2\delta_{\alpha\alpha'}\delta_{\beta\beta'}. \end{aligned} \tag{7}$$

The bar denotes the average over the ensemble, and $\delta_{\alpha\beta}$ stands for $(\delta_{\mu\mu'}\delta_{\nu\nu'} - \delta_{\mu\nu'}\delta_{\nu\mu'})$, etc. The matrix elements $X^{(2)}$ do not possess any symmetry properties and obey

$$\overline{X^{(2)}_{\mu\nu;\rho\sigma} X^{(2)}_{\mu'\nu';\rho'\sigma'}} = v^2\delta_{\mu\mu'}\delta_{\nu\nu'}\delta_{\rho\rho'}\delta_{\sigma\sigma'}. \tag{8}$$

Without loss of generality, we put $v^2 = 1$ in the sequel.

IV. CALCULATION OF nTr(H) AND nTr(H^2)

These two traces are needed for the evaluation of Eqs. (1) and (2). The only nonvanishing contributions to the two traces arise from terms in H and in H^2 which leave the number of fermions in every single-particle state unchanged. These terms are found by using Wick contractions of the creation and annihilation operators in the expressions for H and H^2. We indicate the omission of all other terms by an arrow. For H we obtain

$$\begin{aligned} H \to {} & \frac{1}{2}\sum_{\mu\nu} V^{(1)}_{\mu\nu;\mu\nu} n_{1\mu} n_{1\nu} + \frac{1}{2}\sum_{\mu\nu} V^{(2)}_{\mu\nu;\mu\nu} n_{2\mu} n_{2\nu} \\ & + \sum_{\mu\rho} X^{(2)}_{\mu\mu;\rho\rho} n_{1\mu} n_{2\rho}. \end{aligned} \tag{9}$$

Here $n_{i\mu}$ is the number operator for state $(i\mu)$ with $i = 1, 2$.

The diagonal element of $n_{1\mu}n_{1\nu}$ taken between one of the states with m_1 fermions in positive-parity single-particle states and m_2 fermions in negative-parity single-particle states vanishes unless both states (1μ) and (1ν) are occupied, in which case the matrix element equals unity. There are altogether $\binom{\ell_1-2}{m_1-2}\binom{\ell_2}{m_2}$ such states. We consider separately the normalized traces over the positive- and negative-parity many-body states. We recall that $\mathcal{P}_\pm$ are the projection operators onto the many-body states with positive and negative parity. We obtain

$$\begin{aligned} &\mathrm{nTr}(H\mathcal{P}_+) \\ &= \frac{1}{2N_+}\sum_{\mu\nu} V^{(1)}_{\mu\nu;\mu\nu} \sum_{m_1 m_2} \delta_{m_1+m_2,m}\,\delta_{m_2,\mathrm{even}} \binom{\ell_1-2}{m_1-2}\binom{\ell_2}{m_2} \\ &\quad + \frac{1}{2N_+}\sum_{\mu\nu} V^{(2)}_{\mu\nu;\mu\nu} \sum_{m_1 m_2} \delta_{m_1+m_2,m}\delta_{m_2,\mathrm{even}} \binom{\ell_1}{m_1}\binom{\ell_2-2}{m_2-2} \\ &\quad + \frac{1}{N_+}\sum_{\mu\rho} X^{(2)}_{\mu\mu;\rho\rho} \sum_{m_1 m_2} \delta_{m_1+m_2,m}\delta_{m_2,\mathrm{even}} \binom{\ell_1-1}{m_1-1} \\ &\quad \times \binom{\ell_2-1}{m_2-1}, \\ &\mathrm{nTr}(H\mathcal{P}_-) \\ &= \frac{1}{2N_-}\sum_{\mu\nu} V^{(1)}_{\mu\nu;\mu\nu} \sum_{m_1 m_2} \delta_{m_1+m_2,m}\,\delta_{m_2,\mathrm{odd}} \binom{\ell_1-2}{m_1-2}\binom{\ell_2}{m_2} \\ &\quad + \frac{1}{2N_-}\sum_{\mu\nu} V^{(2)}_{\mu\nu;\mu\nu} \sum_{m_1 m_2} \delta_{m_1+m_2,m}\delta_{m_2,\mathrm{odd}} \binom{\ell_1}{m_1}\binom{\ell_2-2}{m_2-2} \end{aligned}$$

$$+\frac{1}{N_-}\sum_{\mu\rho}X^{(2)}_{\mu\mu;\rho\rho}\sum_{m_1m_2}\delta_{m_1+m_2,m}\delta_{m_2,\mathrm{odd}}\binom{\ell_1-1}{m_1-1}\times\binom{\ell_2-1}{m_2-1}. \tag{10}$$

Both traces are seen to depend on the same three uncorrelated random variables,

$$z_1=\sum_{\mu\nu}V^{(1)}_{\mu\nu;\mu\nu}\,,\quad z_2=\sum_{\mu\nu}V^{(2)}_{\mu\nu;\mu\nu}\,,\quad z_3=\sum_{\mu\rho}X^{(2)}_{\mu\mu;\rho\rho}. \tag{11}$$

As sums of uncorrelated random variables with equal Gaussian distributions, z_1, z_2, and z_3 have Gaussian distributions with mean values zero and second moments $[\ell_1(\ell_1-1)/4]$, $[\ell_2(\ell_2-1)/4]$, and $\ell_1\ell_2$, respectively. Thus, the distribution of the traces in Eq. (10) is completely known.

The pattern that emerges in Eq. (10) will be seen to apply quite generally to traces of arbitrary powers of H: the traces are sums of products. The first factor in each product depends only on the random variables and *is the same for both parities*. The second factor differs for states of positive and states of negative parity but *is independent of the random variables*. That general pattern will be decisive for our understanding of the preponderance of ground states with positive parity.

We turn to $\mathrm{Tr}(H^2)$. The following terms yield nonzero contributions: the square of the first term on the right-hand side of Eq. (5), the square of the second term, the product of the first and second terms, the square of the third term, and the square of the fourth term. We consider these terms in turn.

In the square of the first term, there appear the two matrix elements $V^{(1)}$ with their associated creation and annihilation operators. Wick contraction is possible in three different ways: (i) We contract the two creation and the two annihilation operators associated with the same matrix element. That is the same procedure as used in formula (9) and yields a total of four contraction patterns. (ii) We contract one of the two creation operators associated with the first matrix element with an annihilation operator associated with the same matrix element, and the other with an annihilation operator associated with the second matrix element. That yields a total of 16 contraction patterns. (iii) We contract the two creation operators associated with the first matrix element with the two annihilation operators associated with the second matrix element. That yields a total of four contraction patterns. It is straightforward to check that because of the fermionic anticommutation rules and the symmetry properties of Eq. (6), the different contraction patterns in each of the three groups yield identical results. For the square of the second term on the right-hand side of Eq. (5), these considerations apply likewise. For the product of the first and second terms, only the contraction patterns used in formula (9) are possible. In the square of the third term on the right-hand side of Eq. (5), only the product of the two terms in round brackets gives a nonvanishing contribution, with obvious contraction patterns. In the square of the fourth term, the same three possibilities occur as in the square of the first term. Altogether this yields

$$\begin{aligned}
H^2\to&\left(\sum_{i=1}^{2}\sum_{\alpha\beta}\frac{1}{2}V^{(i)}_{\alpha\beta;\alpha\beta}n_{i\alpha}n_{i\beta}\right)^2\\
&+\sum_{i=1}^{2}\sum_{\alpha\beta\beta'}V^{(i)}_{\alpha\beta;\alpha\beta}V^{(i)}_{\alpha\beta';\alpha\beta'}n_{i\alpha}n_{i\beta}n_{i\beta'}\\
&+\sum_{i=1}^{2}\sum_{\alpha\beta\alpha'\beta'}V^{(i)}_{\alpha\beta;\alpha'\beta}V^{(i)}_{\alpha\beta';\alpha'\beta'}n_{i\alpha}n_{i\beta}(1-n_{i\alpha'})n_{i\beta'}\\
&+\sum_{i=1}^{2}\left(\frac{1}{2}\sum_{\alpha\beta}\left(V^{(i)}_{\alpha\beta;\alpha\beta}\right)^2n_{i\alpha}n_{i\beta}\right.\\
&+\sum_{\alpha\beta\alpha'}\left(V^{(i)}_{\alpha\beta;\alpha'\beta}\right)^2n_{i\alpha}n_{i\beta}(1-n_{i\alpha'})\\
&\left.+\frac{1}{4}\sum_{\alpha\beta\alpha'\beta'}\left(V^{(i)}_{\alpha\beta;\alpha'\beta'}\right)^2n_{i\alpha}n_{i\beta}(1-n_{i\alpha'})(1-n_{i\beta'})\right)\\
&+\frac{1}{4}\sum_{\alpha\beta\alpha'\beta'}\left(X^{(1)}_{\alpha\beta;\alpha'\beta'}\right)^2(n_{1\alpha}n_{1\beta}(1-n_{2\alpha'})(1-n_{2\beta'})\\
&\qquad+(1-n_{1\alpha})(1-n_{1\beta})n_{2\alpha'}n_{2\beta'})\\
&+\left(\sum_{\mu\rho}X^{(2)}_{\mu\mu;\rho\rho}n_{1\mu}n_{2\rho}\right)^2\\
&+\sum_{\mu\nu\rho\sigma}X^{(2)}_{\mu\nu;\rho\rho}X^{(2)}_{\nu\mu;\sigma\sigma}n_{1\mu}(1-n_{1\nu})n_{2\rho}n_{2\sigma}\\
&+\sum_{\mu\nu\rho\sigma}X^{(2)}_{\mu\mu;\rho\sigma}X^{(2)}_{\nu\nu;\sigma\rho}n_{1\mu}n_{1\nu}n_{2\rho}(1-n_{2\sigma})\\
&+\sum_{\mu\nu\rho\sigma}X^{(2)}_{\mu\nu;\rho\sigma}X^{(2)}_{\nu\mu;\sigma\rho}n_{1\mu}(1-n_{1\nu})n_{2\rho}(1-n_{2\sigma}).
\end{aligned}\tag{12}$$

Before working out the trace of this expression, it is useful to rearrange it in such a way that in all summations, no two summation indices take the same values. This yields

$$\begin{aligned}
H^2\to&\sum_{i=1}^{2}\left(2\sum_{\alpha\beta}\left(V^{(i)}_{\alpha\beta;\alpha\beta}\right)^2n_{i\alpha}n_{i\beta}\right.\\
&+2\sum_{\alpha\beta\beta'}\left(V^{(i)}_{\alpha\beta;\beta'\beta}\right)^2n_{i\alpha}n_{i\beta}(1-n_{i\beta'})\\
&+2\sum_{\alpha\beta\beta'}(1-\delta_{\beta\beta'})V^{(i)}_{\alpha\beta;\alpha\beta}V^{(i)}_{\alpha\beta';\alpha\beta'}n_{i\alpha}n_{i\beta}n_{i\beta'}\\
&+\frac{1}{4}\sum_{\alpha\beta\alpha'\beta'}\left(V^{(i)}_{\alpha\beta;\alpha'\beta'}\right)^2n_{i\alpha}n_{i\beta}(1-n_{i\alpha'})(1-n_{i\beta'})\\
&+\frac{1}{4}\sum_{\alpha\beta\alpha'\beta'}(1-\delta_{\alpha\alpha'})(1-\delta_{\alpha\beta'})(1-\delta_{\beta\alpha'})(1-\delta_{\beta\beta'})\\
&\times V^{(i)}_{\alpha\beta;\alpha\beta}V^{(i)}_{\alpha'\beta';\alpha'\beta'}n_{i\alpha}n_{i\beta}n_{i\alpha'}n_{i\beta'}\\
&+\sum_{\alpha\beta\alpha'\beta'}(1-\delta_{\alpha\alpha'})(1-\delta_{\beta\beta'})V^{(i)}_{\alpha\beta;\alpha'\beta}V^{(i)}_{\alpha\beta';\alpha'\beta'}
\end{aligned}$$

$$
\begin{aligned}
&\times n_{i\alpha} n_{i\beta} (1 - n_{i\alpha'}) n_{i\beta'} \Big) \\
&+ \frac{1}{2} \sum_{\alpha\beta\alpha'\beta'} V^{(1)}_{\alpha\beta;\alpha\beta} V^{(2)}_{\alpha'\beta';\alpha'\beta'} n_{1\alpha} n_{1\beta} n_{2\alpha'} n_{2\beta'} \\
&+ \frac{1}{4} \sum_{\alpha\beta\alpha'\beta'} \left(X^{(1)}_{\alpha\beta;\alpha'\beta'} \right)^2 \Big(n_{1\alpha} n_{1\beta} (1 - n_{2\alpha'}) (1 - n_{2\beta'}) \\
&+ (1 - n_{1\alpha})(1 - n_{1\beta}) n_{2\alpha'} n_{2\beta'} \Big) \\
&+ \sum_{\mu\rho} \left(X^{(2)}_{\mu\mu;\rho\rho} \right)^2 n_{1\mu} n_{2\rho} \\
&+ \sum_{\mu\mu'\rho} (1 - \delta_{\mu\mu'}) X^{(2)}_{\mu\mu;\rho\rho} X^{(2)}_{\mu'\mu';\rho\rho} n_{1\mu} n_{1\mu'} n_{2\rho} \\
&+ \sum_{\mu\rho\rho'} (1 - \delta_{\rho\rho'}) X^{(2)}_{\mu\mu;\rho\rho} X^{(2)}_{\mu\mu;\rho'\rho'} n_{1\mu} n_{2\rho} n_{2\rho'} \\
&+ \sum_{\mu\mu'\rho\rho'} (1 - \delta_{\mu\mu'})(1 - \delta_{\rho\rho'}) X^{(2)}_{\mu\mu;\rho\rho} \\
&\times X^{(2)}_{\mu'\mu';\rho'\rho'} n_{1\mu} n_{1\mu'} n_{2\rho} n_{2\rho'} \\
&+ \sum_{\mu\nu\rho} X^{(2)}_{\mu\nu;\rho\rho} X^{(2)}_{\nu\mu;\rho\rho} n_{1\mu} (1 - n_{1\nu}) n_{2\rho} \\
&+ \sum_{\mu\nu\rho\sigma} (1 - \delta_{\rho\sigma}) X^{(2)}_{\mu\nu;\rho\rho} X^{(2)}_{\nu\mu;\sigma\sigma} n_{1\mu} (1 - n_{1\nu}) n_{2\rho} n_{2\sigma} \\
&+ \sum_{\mu\rho\sigma} X^{(2)}_{\mu\mu;\rho\sigma} X^{(2)}_{\mu\mu;\sigma\rho} n_{1\mu} n_{2\rho} (1 - n_{2\sigma}) \\
&+ \sum_{\mu\nu\rho\sigma} (1 - \delta_{\mu\nu}) X^{(2)}_{\mu\mu;\rho\sigma} X^{(2)}_{\nu\nu;\sigma\rho} n_{1\mu} n_{1\nu} n_{2\rho} (1 - n_{2\sigma}) \\
&+ \sum_{\mu\nu\rho\sigma} X^{(2)}_{\mu\nu;\rho\sigma} X^{(2)}_{\nu\mu;\sigma\rho} n_{1\mu} (1 - n_{1\nu}) n_{2\rho} (1 - n_{2\sigma}).
\end{aligned}
\tag{13}
$$

In calculating the trace, we observe that the number of nonequal summation indices in the terms on the right-hand side of Eq. (13) determines the weight factors. The result is

$$
\begin{aligned}
&\mathrm{nTr}(H^2 \mathcal{P}_+) \\
&= \sum_{m_1 m_2} \delta_{m_1+m_2,m} \delta_{m_2,\mathrm{even}} \\
&\times \Bigg\{ \frac{2}{N_+} \sum_{\alpha\beta} \left(V^{(1)}_{\alpha\beta;\alpha\beta} \right)^2 \binom{\ell_1 - 2}{m_1 - 2} \binom{\ell_2}{m_2} \\
&+ \frac{2}{N_+} \sum_{\alpha\beta} \left(V^{(2)}_{\alpha\beta;\alpha\beta} \right)^2 \binom{\ell_1}{m_1} \binom{\ell_2 - 2}{m_2 - 2} \\
&+ \frac{2}{N_+} \sum_{\alpha\beta\beta'} (1 - \delta_{\alpha\beta'}) \left(V^{(1)}_{\alpha\beta;\beta'\beta} \right)^2 \binom{\ell_1 - 3}{m_1 - 2} \binom{\ell_2}{m_2} \\
&+ \frac{2}{N_+} \sum_{\alpha\beta\beta'} (1 - \delta_{\alpha\beta'}) \left(V^{(2)}_{\alpha\beta;\beta'\beta} \right)^2 \binom{\ell_1}{m_1} \binom{\ell_2 - 3}{m_2 - 2} \\
&+ \frac{2}{N_+} \sum_{\alpha\beta\beta'} (1 - \delta_{\beta\beta'}) V^{(1)}_{\alpha\beta;\alpha\beta} V^{(1)}_{\alpha\beta';\alpha\beta'} \binom{\ell_1 - 3}{m_1 - 3} \binom{\ell_2}{m_2} \\
&+ \frac{2}{N_+} \sum_{\alpha\beta\beta'} (1 - \delta_{\beta\beta'}) V^{(2)}_{\alpha\beta;\alpha\beta} V^{(2)}_{\alpha\beta';\alpha\beta'} \binom{\ell_1}{m_1} \binom{\ell_2 - 3}{m_2 - 3} \\
&+ \frac{1}{4N_+} \sum_{\alpha\beta\alpha'\beta'} (1 - \delta_{\alpha\alpha'})(1 - \delta_{\alpha\beta'})(1 - \delta_{\beta\alpha'})(1 - \delta_{\beta\beta'}) \\
&\times \left(V^{(1)}_{\alpha\beta;\alpha'\beta'} \right)^2 \binom{\ell_1 - 4}{m_1 - 2} \binom{\ell_2}{m_2} \\
&+ \frac{1}{4N_+} \sum_{\alpha\beta\alpha'\beta'} (1 - \delta_{\alpha\alpha'})(1 - \delta_{\alpha\beta'})(1 - \delta_{\beta\alpha'})(1 - \delta_{\beta\beta'}) \\
&\times \left(V^{(2)}_{\alpha\beta;\alpha'\beta'} \right)^2 \binom{\ell_1}{m_1} \binom{\ell_2 - 4}{m_2 - 2} \\
&+ \frac{1}{4N_+} \sum_{\alpha\beta\alpha'\beta'} (1 - \delta_{\alpha\alpha'})(1 - \delta_{\alpha\beta'})(1 - \delta_{\beta\alpha'})(1 - \delta_{\beta\beta'}) \\
&\times V^{(1)}_{\alpha\beta;\alpha\beta} V^{(1)}_{\alpha'\beta';\alpha'\beta'} \binom{\ell_1 - 4}{m_1 - 4} \binom{\ell_2}{m_2} \\
&+ \frac{1}{4N_+} \sum_{\alpha\beta\alpha'\beta'} (1 - \delta_{\alpha\alpha'})(1 - \delta_{\alpha\beta'})(1 - \delta_{\beta\alpha'})(1 - \delta_{\beta\beta'}) \\
&\times V^{(2)}_{\alpha\beta;\alpha\beta} V^{(2)}_{\alpha'\beta';\alpha'\beta'} \binom{\ell_1}{m_1} \binom{\ell_2 - 4}{m_2 - 4} \\
&+ \frac{1}{N_+} \sum_{\alpha\beta\alpha'\beta'} (1 - \delta_{\alpha\alpha'})(1 - \delta_{\beta\beta'}) V^{(1)}_{\alpha\beta;\alpha'\beta} V^{(1)}_{\alpha\beta';\alpha'\beta'} \\
&\times \binom{\ell_1 - 4}{m_1 - 3} \binom{\ell_2}{m_2} \\
&+ \frac{1}{N_+} \sum_{\alpha\beta\alpha'\beta'} (1 - \delta_{\alpha\alpha'})(1 - \delta_{\beta\beta'}) V^{(2)}_{\alpha\beta;\alpha'\beta} V^{(2)}_{\alpha\beta';\alpha'\beta'} \\
&\times \binom{\ell_1}{m_1} \binom{\ell_2 - 4}{m_2 - 3} \\
&+ \frac{1}{2N_+} \sum_{\alpha\beta\alpha'\beta'} V^{(1)}_{\alpha\beta;\alpha\beta} V^{(2)}_{\alpha'\beta';\alpha'\beta'} \binom{\ell_1 - 2}{m_1 - 2} \binom{\ell_2 - 2}{m_2 - 2} \\
&+ \frac{1}{4N_+} \sum_{\alpha\beta\alpha'\beta'} \left(X^{(1)}_{\alpha\beta;\alpha'\beta'} \right)^2 \Bigg[\binom{\ell_1 - 2}{m_1 - 2} \binom{\ell_2 - 2}{m_2} \\
&+ \binom{\ell_1 - 2}{m_1} \binom{\ell_2 - 2}{m_2 - 2} \Bigg] \\
&+ \frac{1}{N_+} \sum_{\mu\rho} \left(X^{(2)}_{\mu\mu;\rho\rho} \right)^2 \binom{\ell_1 - 1}{m_1 - 1} \binom{\ell_2 - 1}{m_2 - 1} \\
&+ \frac{1}{N_+} \sum_{\mu\mu'\rho} (1 - \delta_{\mu\mu'}) X^{(2)}_{\mu\mu;\rho\rho} X^{(2)}_{\mu'\mu';\rho\rho} \\
&\times \binom{\ell_1 - 2}{m_1 - 2} \binom{\ell_2 - 1}{m_2 - 1}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{N_+}\sum_{\mu\rho\rho'}(1-\delta_{\rho\rho'})X^{(2)}_{\mu\mu;\rho\rho}X^{(2)}_{\mu\mu;\rho'\rho'}\binom{\ell_1-1}{m_1-1}\\
&\times\binom{\ell_2-2}{m_2-2}\\
&+\frac{1}{N_+}\sum_{\mu\mu'\rho\rho'}(1-\delta_{\mu\mu'})(1-\delta_{\rho\rho'})X^{(2)}_{\mu\mu;\rho\rho}X^{(2)}_{\mu'\mu';\rho'\rho'}\\
&\times\binom{\ell_1-2}{m_1-2}\binom{\ell_2-2}{m_2-2}\\
&+\frac{1}{N_+}\sum_{\mu\nu\rho}(1-\delta_{\mu\nu})X^{(2)}_{\mu\nu;\rho\rho}X^{(2)}_{\nu\mu;\rho\rho}\binom{\ell_1-2}{m_1-1}\binom{\ell_2-1}{m_2-1}\\
&+\frac{1}{N_+}\sum_{\mu\nu\rho\sigma}(1-\delta_{\mu\nu})(1-\delta_{\rho\sigma})X^{(2)}_{\mu\nu;\rho\rho}X^{(2)}_{\nu\mu;\sigma\sigma}\\
&\times\binom{\ell_1-2}{m_1-1}\binom{\ell_2-2}{m_2-2}\\
&+\frac{1}{N_+}\sum_{\mu\rho\sigma}(1-\delta_{\rho\sigma})X^{(2)}_{\mu\mu;\rho\sigma}X^{(2)}_{\mu\mu;\sigma\rho}\binom{\ell_1-1}{m_1-1}\\
&\times\binom{\ell_2-2}{m_2-1}\\
&+\frac{1}{N_+}\sum_{\mu\nu\rho\sigma}(1-\delta_{\mu\nu})(1-\delta_{\rho\sigma})X^{(2)}_{\mu\mu;\rho\sigma}X^{(2)}_{\nu\nu;\sigma\rho}\\
&\times\binom{\ell_1-2}{m_1-2}\binom{\ell_2-2}{m_2-1}\\
&+\frac{1}{N_+}\sum_{\mu\nu\rho\sigma}(1-\delta_{\mu\nu})(1-\delta_{\rho\sigma})X^{(2)}_{\mu\nu;\rho\sigma}X^{(2)}_{\nu\mu;\sigma\rho}\\
&\times\binom{\ell_1-2}{m_1-1}\binom{\ell_2-2}{m_2-1}\Bigg\}. \qquad (14)
\end{aligned}
$$

For $\mathrm{nTr}(H^2\mathcal{P}_-)$ we find exactly the same expression except that the second Kronecker δ in the first line on the right-hand side of Eq. (14) is replaced by $\delta_{m_2,\mathrm{odd}}$, and that N_+ is replaced everywhere by N_-.

As in the case of $\mathrm{Tr}(H)$, the trace of H^2 is a sum of terms each of which is the product of two factors. One factor depends only on the random variables and is the same for both parities. The distribution of these factors can be worked out and is not done here. Some of the factors are correlated with each other. The other factor is a weight factor, which is a sum over products of binomial factors. It does not depend on the random variables and is not obviously the same for the two parities. Our result would not apply in the case of states with spin where the linear or bilinear forms containing the random variables will depend on the total spin. Assuming that Eqs. (1)–(3) hold, we conclude that a preponderance of ground states with even parity—if it exists—can have only one of two causes: it may be due to differences between the nonstatistical weight factors or to differences in the scale factors r_+ and r_-. [We recall that according to Eq. (3), the latter depend on the matrix dimensions $N_\pm$.]

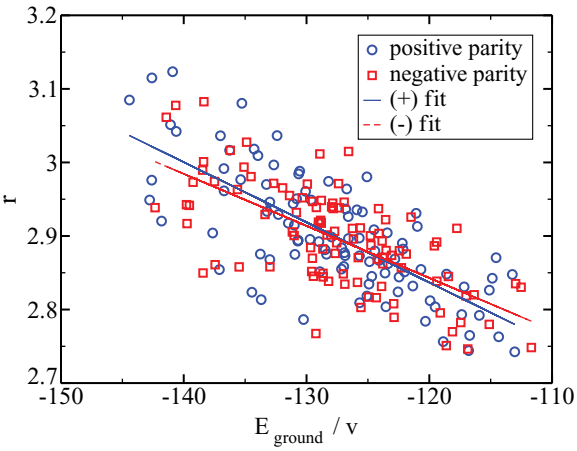

FIG. 1. (Color online) Scale factors $r_\pm$ of Eq. (1) for a system of $m = 9$ fermions on $\ell_1 = 9$ single-particle orbitals with positive parity and $\ell_2 = 9$ orbitals with negative parity as a function of the ground-state energies $E_{\mathrm{ground}}(\pm)$.

V. NUMERICAL RESULTS

For a test of Eq. (1), we perform numerical simulations. To this end, we consider several systems that differ in the parameters ℓ_1, ℓ_2, and m. For each set of parameters, we set up the matrix corresponding to the Hamiltonian (5) in a space of Slater determinants. The Gaussian-distributed two-body matrix elements are computed by a pseudorandom number generator, and the ground-state energies $E_{\mathrm{ground}}(\pm)$ are obtained from a numerical diagonalization of the Hamiltonian matrix. For the largest dimensional matrices, we employ the ARPACK package [8] in the diagonalization. In addition to the ground-state energy, we also compute the normalized traces $\mathrm{nTr}(H^k\mathcal{P}_\pm)$ for $k = 1, 2$. Our ensemble consists of 100 random Hamiltonians for each set of parameters ℓ_1, ℓ_2, and m, and we record the ground-state energies $E_{\mathrm{ground}}(\pm)$ and the first two moments $\mathrm{nTr}(H^k\mathcal{P}_\pm)$ (with $k = 1, 2$) of the parity-projected Hamiltonian for each member of the ensemble. We employ Eq. (1) and determine the scale factors $r_\pm$ that relate the ground-state energy to the first and second moment of the Hamiltonian by fit. An example is shown for the set of parameters $m = \ell_1 = \ell_2 = 9$ in Fig. 1. The results obtained for the scale factors (with their rms variances) are shown in Table I. The table also shows the probability p_+ that the ground state has positive parity. Inspection of Table I shows that the parity of the ground state is very sensitive to $r_\pm$. A small difference in the scale factors $r_\pm$ is more strongly correlated with the parity of the ground state than is a small difference in the numbers $N_\pm$ of many-body basis states.

Once the scale factors are determined, we can test how well the right-hand side of Eq. (1) can be used to determine the parity of the ground state. Our results show that the application of Eq. (1) with an energy-independent scale factor does not yield reliable predictions. Indeed, Fig. 1 suggests that a linear relation $r_\pm E_{\mathrm{ground}}(\pm) = a_\pm + b_\pm E_{\mathrm{ground}}(\pm)$ should describe the data more accurately. Again, we determine the coefficients $a_\pm$ and $b_\pm$ by fit, and then employ the right-hand side of Eq. (1) with the energy-dependent scale factor to determine the parity

TABLE I. Results of numerical simulations. Here, m, ℓ_1, and ℓ_2 denote the number of fermions and the number of single-particle levels with positive and negative parity, respectively. $N_\pm$ is the number of many-body states with the indicated parity, and $r_\pm$ denote the scale factors. p_+ denotes the probability that the ground state has positive parity, while p_+(est) is the probability that the estimated ground state has positive parity [based on Eq. (1) with a scale factor that is a polynomial of degree one in the energy].

ℓ_1	ℓ_2	m	N_+	N_-	r_+	r_-	p_+	p_+(est)
6	6	6	452	472	2.39 ± 0.12	2.43 ± 0.12	0.18	0.00
7	7	5	1001	1001	2.42 ± 0.08	2.42 ± 0.08	0.47	0.49
7	7	6	1484	1519	2.51 ± 0.08	2.55 ± 0.08	0.20	0.03
7	7	7	1716	1716	2.61 ± 0.08	2.60 ± 0.08	0.48	0.56
9	9	5	4284	4284	2.47 ± 0.06	2.47 ± 0.06	0.55	0.54
10	8	5	4312	4256	2.48 ± 0.06	2.46 ± 0.05	0.84	1.00
9	7	8	6435	6435	2.77 ± 0.07	2.76 ± 0.08	0.54	0.58
8	8	8	6470	6400	2.78 ± 0.07	2.74 ± 0.08	0.83	1.00
10	6	8	6390	6480	2.73 ± 0.08	2.77 ± 0.09	0.18	0.00
9	9	9	24310	24310	2.90 ± 0.08	2.90 ± 0.07	0.52	0.57
8	10	9	24240	24380	2.87 ± 0.07	2.91 ± 0.07	0.20	0.00
7	11	9	24310	24310	2.88 ± 0.07	2.89 ± 0.07	0.50	0.27

of the ground state as

$$\text{sign}\left(\frac{\text{nTr}(H\mathcal{P}_-) - a_-\sigma_-}{1 + b_-\sigma_-} - \frac{\text{nTr}(H\mathcal{P}_+) - a_+\sigma_+}{1 + b_+\sigma_+}\right). \quad (15)$$

Though this estimate is not correct for each individual member of the ensemble, it yields reasonably reliable predictions for the estimated probability p_+(est) of finding a ground-state with positive parity. Our results for this probability are shown in the last column of Table I.

VI. DILUTE LIMIT

In canonical random-matrix theory, attention is usually focused on the limit of large matrix dimension. We follow suit by considering our model in the "dilute limit" [5] defined by $\ell_{1,2}, m \to \infty$ and $m/\ell_{1,2} \to 0$. In practice, we compute the leading order of expressions of interest under the strong conditions $1 \ll m \ll \ell_{1,2}$. We show that the weight factors appearing in the traces of H^k with $k = 1, 2$ for positive and negative parity become asymptotically equal. That statement holds not only for $\ell_1 = \ell_2$ but also for $\ell_1 \neq \ell_2$.

Equations (10) and (14) show that for the positive-parity states, all weight factors have the general form

$$\sum_{m_1,m_2} \delta_{m_1+m_2,m}\delta_{m_2,\text{even}} \frac{1}{N_+}\binom{\ell_1 - \alpha_1}{m_1 - \beta_1}\binom{\ell_2 - \alpha_2}{m_2 - \beta_2}, \quad (16)$$

with $\alpha_1, \alpha_2, \beta_1, \beta_2$ small positive integers. We evaluate the sums in Eq. (16) and the corresponding sums defining N_+ with the help of Stirling's formula, $n! \approx \exp\{n \ln n - n\}$. With μ integer, we write $m_2 = 2\mu$, $m_1 = m - 2\mu$ and have for the numerator of Eq. (16) [all terms except for $(N_+)^{-1}$]

$$\begin{aligned}\sum_\mu &\exp\{(\ell_1 - \alpha_1)\ln(\ell_1 - \alpha_1) - (m - 2\mu - \beta_1)\\ &\times \ln(m - 2\mu - \beta_1) - (\ell_1 - \alpha_1 - m + 2\mu + \beta_1)\\ &\times \ln(\ell_1 - \alpha_1 - m + 2\mu + \beta_1) + (\ell_2 - \alpha_2)\\ &\times \ln(\ell_2 - \alpha_2) - (2\mu - \beta_2)\ln(2\mu - \beta_2)\\ &- (\ell_2 - \alpha_2 - 2\mu + \beta_2)\ln(\ell_2 - \alpha_2 - 2\mu + \beta_2)\}. \quad (17)\end{aligned}$$

We write the sum as an integral over μ. The integrand assumes its maximum value at

$$\mu_+^{(0)} = \frac{1}{2}\frac{(m - \beta_1)(\ell_2 - \alpha_2) + \beta_2(\ell_1 - \alpha_1)}{\ell_1 - \alpha_1 + \ell_2 - \alpha_2}. \quad (18)$$

With $\delta\mu = \mu - \mu_0$, expansion around the maximum yields the negative-definite quadratic form

$$\begin{aligned}&-\frac{2(\delta\mu)^2}{m - 2\mu_+^{(0)} - \beta_1} - \frac{2(\delta\mu)^2}{\ell_1 - \alpha_1 - m + 2\mu_+^{(0)} + \beta_1}\\ &\quad - \frac{2(\delta\mu)^2}{2\mu_+^{(0)} - \beta_2} - \frac{2(\delta\mu)^2}{\ell_2 - \alpha_2 - 2\mu_+^{(0)} + \beta_2} = \frac{1}{2}\frac{(\delta\mu)^2}{\tau^2}. \quad (19)\end{aligned}$$

Here the last equation defines the width τ. For $\ell_1 \gg 1$, $\ell_2 \gg 1$, and $m \gg 1$, we have $\mu_0 \gg 1$. For the dilute limit, we neglect terms of higher order, and the resulting integral is Gaussian. We extend the integration from $-\infty$ to $+\infty$. The numerator of expression (16) becomes

$$\begin{aligned}&\sqrt{2\pi}\tau \exp\{(\ell_1 - \alpha_1)\ln(\ell_1 - \alpha_1) + (\ell_2 - \alpha_2)\ln(\ell_2 - \alpha_2)\}\\ &\times \exp\{-(m - 2\mu_+^{(0)} - \beta_1)\ln(m - 2\mu_+^{(0)} - \beta_1)\}\\ &\times \exp\{-(2\mu_+^{(0)} - \beta_2)\ln(2\mu_+^{(0)} - \beta_2)\}\\ &\times \exp\{-(\ell_1 - \alpha_1 - m + 2\mu_+^{(0)} + \beta_1)\\ &\times \ln(\ell_1 - \alpha_1 - m + 2\mu_+^{(0)} + \beta_1)\}\\ &\times \exp\{-(\ell_2 - \alpha_2 - 2\mu_+^{(0)} + \beta_2)\\ &\times \ln(\ell_2 - \alpha_2 - 2\mu_+^{(0)} + \beta_2)\}. \quad (20)\end{aligned}$$

Using the same approximations to calculate N_+, we obtain a result of the form (20) but with $\alpha_1, \alpha_2, \beta_1, \beta_2$ everywhere (including the definitions of τ and $\mu_+^{(0)}$) replaced by zero.

We turn to the negative-parity states. For these states, the word "even" in Eq. (16) is replaced by "odd" and N_+

by N_-. The calculation is completely analogous except for the replacements $\beta_1 \to \beta_1 + 1$ and $\beta_2 \to \beta_2 - 1$. For the maximum of the integrand, that implies that $2\mu_-^{(0)} = 2\mu_+^{(0)} - 1$. As a consequence, the terms $2\mu^{(0)} + \beta_1$ and $2\mu^{(0)} - \beta_2$ have the same values for states with positive and negative parity. This in turn implies that the widths τ and the terms in the exponential in expression (20) have the same values for states with positive and negative parity. It follows that in our approximation, every weight factor for states with positive parity has the same value as the corresponding weight factor for states with negative parity. This result is valid beyond the Gaussian approximation used in obtaining Eq. (20). Indeed, the fundamental form (17) depends on μ only through the invariant combinations $\mu + \beta_1$ and $\mu - \beta_2$. Modifications can arise only in cases where the limits of integration (which depend on α_1, α_2, β_1, and β_2) play a role, i.e., for small values of ℓ_1, ℓ_2, or m.

We have shown that in the dilute limit and for every realization of our random-matrix model, both the first and the second moments of H coincide in leading order for states with positive and for states with negative parity. The same is true of the matrix dimensions N_+ and N_-. Thus for every realization, our Eqs. (1)–(3) predict equal values for the ground-state energies for both parities. How reliable is that prediction? We recall that in the dilute limit, the average spectrum of the embedded random two-body ensemble [EGOE(2)] is Gaussian [5]. The proof given in Ref. [5] applies likewise to our model. We expect, therefore, that in the dilute limit and to a very high degree of approximation, the spectrum of any given realization of the ensemble also has a Gaussian shape. (For a single realization, the shape of the spectrum is defined by taking local averages over a number $n \ll N_\pm$ of neighboring levels.) That expectation rests on the plausible assumption that our random-matrix model is ergodic, at least in the dilute limit, and implies that for every realization, our Eqs. (1)–(3) become even better approximations as the matrix dimension increases. We conclude that the probabilities for ground states of positive and negative parity are equal in the dilute limit. That conclusion holds with the following proviso. A preference for ground states of, say, positive parity might occur if the local spectral fluctuation properties of the two ensembles are locked in such a way that the positive-parity ground state fluctuates more often toward smaller energies than does its opposite number. In the next section, we exclude that possibility. We do so by investigating higher moments of H.

VII. SPECTRAL FLUCTUATIONS

Given the coincidence of both the first and second moments of H for states of either parity in the dilute limit, we ask: Does that coincidence extend to all higher moments so that the local spectral fluctuation properties of both ensembles are completely locked? We approach the answer by studying higher moments of H.

We consider $\mathrm{nTr}(H^k \mathcal{P}_\pm)$ for k integer and $k \geqslant 3$. These traces are now shown to have the same structure as the first and second moments of H: each trace is a sum of terms each of which is the product of a monomial (or polynomial) of order k in the two-body matrix elements (the same for the projectors $\mathcal{P}_+$ and $\mathcal{P}_-$) and a weight factor that does not depend on the random variables but may have a different value for positive and negative parity.

We proceed as in Sec. IV but are interested only in the general form of the result. The operator H^k is a monomial of order k in the matrix elements $V^{(1)}$, $V^{(2)}$, $X^{(1)}$, $X^{(2)}$. Each matrix element carries four indices. Thus, in H^k there occur $4k$ independent summations over single-particle level indices. Nonvanishing contributions to the trace of H^k arise only from Wick-contracted terms. Each pairwise Wick contraction of a creation and an annihilation operator in H^k produces a factor of the form $n_{1\alpha}$, $(1 - n_{1\alpha})$, $n_{2\alpha}$, or $(1 - n_{2\alpha})$, as the case may be. At the same time, two summation indices become equal. After all Wick contractions are done, H^k contains at most $2k$ independent summations over level indices. [That number may be smaller than $2k$, since two or more of the resulting factors $n_{1\alpha}$, $(1 - n_{1\alpha})$, $n_{2\alpha}$, or $(1 - n_{2\alpha})$ may carry the same index.] By using the identity $n^2 = n$ for the number operator, the Wick-contracted H^k can be written in such a way that the summation indices on all such factors are different. For $k = 2$, that was done in Eq. (13). We consider a single term resulting from this procedure and denote by k_1, k_2, k_3, k_4 the powers of the four types of factors (in the same sequence as listed above) in that term. The maximum power with which all factors jointly can appear is $2k$, so that $k_1 + k_2 + k_3 + k_4 \leqslant 2k$. Clearly we must also have $k_1 + k_2 \leqslant \ell_1$ and $k_3 + k_4 \leqslant \ell_2$. We conclude that a general term in the Wick-contracted form of H^k, characterized by the four integers k_1, k_2, k_3, k_4 as constrained above, has the form

$$\sum_{\alpha_1,\alpha_2,\ldots,\alpha_{k_1}} \sum_{\beta_1,\beta_2,\ldots,\beta_{k_2}} \sum_{\gamma_1,\gamma_2,\ldots,\gamma_{k_3}} \sum_{\delta_1,\delta_2,\ldots,\delta_{k_4}} \left\{ \prod_{r=1}^{k_1} n_{1\alpha_r} \prod_{s=1}^{k_2} (1 - n_{1\beta_s}) \prod_{t=1}^{k_3} n_{2\gamma_t} \prod_{u=1}^{k_4} (1 - n_{2\delta_u}) \times f_{\alpha_1,\ldots,\alpha_{k_1};\beta_1,\ldots,\beta_{k_2};\gamma_1,\ldots,\gamma_{k_3};\delta_1,\ldots,\delta_{k_4}} \right\}. \tag{21}$$

The sums in this expression are jointly constrained by the condition that no two summation indices are equal. The form of the function f depends upon the value of k. f is a monomial of order k in the matrix elements V^1, V^2, X^1, X^2. These carry the summation indices. The Wick contraction of H^k yields a sum of terms of the form (21). For the calculation of $\mathrm{nTr}(H^k \mathcal{P}_\pm)$, we observe that the expression

$$\Pi_\pm(k_1, k_2, k_3, k_4) = \mathrm{nTr} \left\{ \prod_{r=1}^{k_1} n_{1\alpha_r} \prod_{s=1}^{k_2} (1 - n_{1\beta_s}) \prod_{t=1}^{k_3} n_{2\gamma_t} \prod_{u=1}^{k_4} (1 - n_{2\delta_u}) \mathcal{P}_\pm \right\} \tag{22}$$

does not depend on the values of the indices $\alpha_1, \ldots, \delta_{k_4}$. Therefore, the normalized traces of the projections of the

expression (21) are given by

$$\sum_{\alpha_1,\ldots,\alpha_{k_1}} \sum_{\beta_1,\ldots,\beta_{k_2}} \sum_{\gamma_1,\ldots,\gamma_{k_3}} \sum_{\delta_1,\ldots,\delta_{k_4}} f_{\alpha_1,\ldots,\alpha_{k_1};\beta_1,\ldots,\beta_{k_2};\gamma_1,\ldots,\gamma_{k_3};\delta_1,\ldots,\ldots,k_4} \times \Pi_{\pm}(k_1,k_2,k_3,k_4). \quad (23)$$

Expression (23) shows that the results derived in Sec. IV for $\mathrm{nTr}(H\mathcal{P}_{\pm})$ and for $\mathrm{nTr}(H^2\mathcal{P}_{\pm})$ hold for arbitrary powers k of H: each trace $\mathrm{nTr}(H^k\mathcal{P}_{\pm})$ is a sum of terms; every term in the sum is the product of two factors. The first factor contains the random variables and is the same for the states with positive and with negative parity. The second factor, a weight factor, may depend on parity. Thus, we have shown that the Hamiltonians for positive- and negative-parity states are very highly correlated.

This is a remarkable result in its own right. Indeed, with increasing values of ℓ_1 and ℓ_2, the matrix dimensions N_+ and N_- grow approximately like $[(\ell_1+\ell_2)/m]^m$, while the number of two-body matrix elements only grows like $(\ell_1+\ell_2)^4$. Thus, for $(\ell_1+\ell_2) > m^2$, the matrix dimensions become asymptotically very much larger than the number of independent matrix elements. Still, in the sense of Eq. (23), the two Hamiltonians remain totally correlated.

We turn to the weight factors appearing in Eq. (23) and show that these are also asymptotically equal. Our statement applies up to a maximum value of k, which we determine approximately. The weight factors $\Pi_{\pm}(k_1,k_2,k_3,k_4)$ are explicitly given by

$$\Pi_+(k_1,k_2,k_3,k_4) = \frac{1}{N_+} \sum_{m_1,m_2} \delta_{m_1+m_2,m}\delta_{m_2,\mathrm{even}} \binom{\ell_1-k_2}{m_1-k_1}\binom{\ell_2-k_4}{m_2-k_3}, \quad (24)$$

and

$$\Pi_-(k_1,k_2,k_3,k_4) = \frac{1}{N_-} \sum_{m_1,m_2} \delta_{m_1+m_2,m}\,\delta_{m_2,\mathrm{odd}} \binom{\ell_1-k_2}{m_1-k_1}\binom{\ell_2-k_4}{m_2-k_3}. \quad (25)$$

In the summations over m_1, m_2, we obviously must have $m_1 \geqslant k_1$ and $m_2 \geqslant k_3$. Since m_1 and m_2 are both bounded by m, that condition in fact limits k_1 and k_3. It is obvious that for large values of k, the two weight factors cannot always be equal. Consider, for instance, the case $k_1 = 0$, $k_3 = m$. Then we have $m_1 = 0$ and $m_2 = m$. That implies that $\Pi_+(k_1,k_2,k_3,k_4) = 0$, $\Pi_-(k_1,k_2,k_3,k_4) \neq 0$ if m is odd, and $\Pi_-(k_1,k_2,k_3,k_4) = 0$, $\Pi_+(k_1,k_2,k_3,k_4) \neq 0$ if m is even. To avoid such cases, we must have $k < m$. Even then Π_+ and Π_- may differ. This happens when the bounds on the summation indices in Eqs. (24) and (25) are relevant. We avoid these cases by choosing $k \ll m$. We recall that the asymptotic regime is characterized by the relations $1 \ll m \ll \ell_1, \ell_2$. We thus require that m is sufficiently large to accommodate the relation $k \ll m$ and yet allows k to assume values large compared to unity. With these assumptions, the arguments used above for $k = 1, 2$ show that $\Pi_+ = \Pi_-$.

We have shown that in the asymptotic regime and for all k with $k \leqslant k_0$, the moments $\mathrm{Tr}(H^k\mathcal{P}_{\pm})$ pairwise have the same values for states with positive and with negative parity. Here k_0 obeys $1 \ll k_0 \ll m$. That conclusion does not depend on assuming any symmetry such as $\ell_1 = \ell_2$. We have also shown that for $k \gg k_0$, the moments differ. As k increases, the bounds on the summations over products of binomial factors become ever more important. As a consequence, the differences between moments for states with positive and negative parity increase with k. That statement is relevant for the local spectral fluctuation properties of both ensembles. Indeed, it is known [9] that such fluctuation properties depend on the very highest moments of H. In the limit of infinite matrix dimension, there exists a clear separation between the overall shape of the spectrum (defined by averaging over an energy interval that is large compared to the average level spacing d) and the local spectral fluctuations (defined on a scale of order d). Since the moments of H for states of positive and negative parity differ for $k \gg k_0$, we conclude that the local fluctuation properties of both ensembles are uncorrelated in the dilute limit, even though the moments of H for both parities coincide up to $k \approx k_0$. This excludes the possibility mentioned in Sec. VI that the local spectral fluctuation properties of the two ensembles are locked in such a way that the positive-parity ground state fluctuates more often toward smaller energies than does its opposite number or vice versa and completes the proof that in the dilute limit, ground states of either parity carry equal probabilities.

VIII. SUMMARY AND DISCUSSION

We have shown that in the dilute limit, ground states of either parity carry equal probabilities. That conclusion is based on the following facts. (i) The spectra are asymptotically Gaussian, and Eqs. (1)–(3) become asymptotically strictly valid. (ii) The first and second moments of H and the dimensions of the Hamiltonian matrices become asymptotically equal for either parity so that Eqs. (1)–(3) predict equal probabilities for either parity. (iii) The local spectral fluctuation properties of the two spectra are asymptotically uncorrelated because very high moments have different values. Fact (iii) excludes a locking of these fluctuations.

Deviations from equal ground-state probabilities thus are finite-size effects. For values of the parameters m, ℓ_1, and ℓ_2 that are sufficiently small for numerical simulations, we have indeed found such deviations. They occur whenever the dimensions N_+ and N_- differ. Conversely, for $N_+ = N_-$, we have not found significant deviations from equal probabilities. The small fluctuations found for $r_{\pm}$ in the fits to the data show that Eqs. (1) and (2) are approximately valid: they do predict correctly which parity has the higher probability to furnish the ground state. The values of the predicted probabilities are semiquantitatively correct.

Calculations using the two-body random ensemble (TBRE) reported in Ref. [10] displayed correlations between spectra carrying different quantum numbers. (The TBRE uses the shell model, confines itself to a major shell, assumes that the two-body matrix elements are random variables, and conserves angular momentum and parity. For a review of the TBRE, see Ref. [11]). One may argue that the results of Ref. [10] contradict our present findings. This is not the case: numerical calculations using the TBRE are necessarily restricted to small

matrix dimensions, while our argument for independence of spectral fluctuation properties of states of positive and negative parity applies only in the dilute limit, i.e., for infinite matrix dimension. Still, the question persists as to whether in the limit of a large matrix dimension, correlations between spectra carrying different quantum numbers would likewise disappear in the TBRE. We cannot answer that question analytically at the present time. In our model, moments of H are calculated using Wick contraction. That fact allows us to go to the dilute limit. In the TBRE, a similar simplification does not seem to exist. Similarities between the two models make us expect, however, that our result also applies to the TBRE. The calculations in Sec. VI show that in our model, the difference between high moments taken for states of positive and negative parity is due to the finite size of the single-particle basis. A similar (but stronger) restriction exists also in the TBRE, because each major shell is made up of a number of different subshells, each with a finite number of single-particle states. That fact and angular-momentum coupling effects cause even the low moments of H in the TBRE to differ for different spin states (while these moments are identically equal in our model). We expect that difference to increase as the power of H increases. Our argument clearly fails for bosons where the occupation numbers of the single-particle states are not restricted, and the sp-boson models reviewed in Ref. [4] do indeed display different patterns.

Our results may have interesting implications for the statistical theory of nuclear reactions, wherein lies the following open question: Are S-matrix elements carrying different quantum numbers, such as total spin, uncorrelated? The assumption that they are uncorrelated is always used in the theory and is consistent with the observed symmetry of compound-nucleus cross sections about 90° in the center-of-mass system. Still, the assumption is not obviously valid for a realistic random-matrix model of nuclear reactions. Normally the statistical theory of nuclear reactions uses the Gaussian orthogonal ensemble (GOE). It would be more realistic to use instead the TBRE. But then it is the same set of random variables that govern scattering matrix elements carrying different quantum numbers, just as in the model considered above, the same random two-body matrix elements govern the Hamiltonians for states of different parity. To approach the question, we observe that for orthogonally invariant ensembles, universality holds also for elements of the scattering matrix carrying identical quantum numbers [12]. That statement implies that correlations between such elements depend only on local spectral fluctuation properties. This conclusion is supported by the explicit calculation in Ref. [13] of the correlation function of a pair of S-matrix elements: aside from the strength of the coupling to the open channels, the correlation depends solely on the value of the local mean level density. If we assume that these statements carry over to the TBRE, and if we further assume that in the TBRE, just as in the model studied above, the local spectral fluctuation properties of spectra carrying different quantum numbers are uncorrelated in the limit of a large matrix dimension, we are led to the conclusion that S-matrix elements carrying different quantum numbers are, likewise, uncorrelated. The limit of a large matrix dimension is appropriate, because the resonances relevant in the statistical theory correspond to states above the ground state.

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[1] C. W. Johnson, G. F. Bertsch, and D. J. Dean, Phys. Rev. Lett. **80**, 2749 (1998).
[2] V. Zelevinsky and A. Volya, Phys. Rep. **391**, 311 (2004).
[3] Y. M. Zhao, A. Arima, and N. Yoshinaga, Phys. Rep. **400**, 1 (2004).
[4] Y. M. Zhao, A. Arima, N. Shimizu, K. Ogawa, N. Yoshinaga, and O. Scholten, Phys. Rev. C **70**, 054322 (2004).
[5] K. F. Mon and J. B. French, Ann. Phys. (NY) **95**, 90 (1975).
[6] T. Papenbrock and H. A. Weidenmüller, Phys. Rev. Lett. **93**, 132503 (2004).
[7] N. Yoshinaga, A. Arima, and Y. M. Zhao, Phys. Rev. C **73**, 017303 (2006); J. J. Shen, Y. M. Zhao, A. Arima, and N. Yoshinaga, *ibid.* **77**, 054312 (2008).
[8] R. B. Lehoucq, D. C. Sorensen, and C. Yang, ARPACK *Users' Guide: Solution of Large-Scale Eigenvalue Problems with Implicitly Restarted Arnoldi Methods* (SIAM Publications, Philadelphia, 1998).
[9] T. A. Brody, J. Flores, J. B. French, P. A. Mello, A. Pandey, and S. S. M. Wong, Rev. Mod. Phys. **53**, 385 (1981).
[10] T. Papenbrock and H. A. Weidenmüller, Phys. Rev. C **73**, 014311 (2006).
[11] T. Papenbrock and H. A. Weidenmüller, Rev. Mod. Phys. **79**, 997 (2007).
[12] G. Hackenbroich and H. A. Weidenmüller, Phys. Rev. Lett. **74**, 4118 (1995).
[13] J. J. M. Verbaarschot, M. R. Zirnbauer, and H. A. Weidenmüller, Phys. Rep. **129**, 367 (1985).