Wigner symmetry, large N_c, and renormalized one-boson exchange potentials

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Wigner symmetry in nuclear physics provides a unique example of a nonperturbative medium and long distance symmetry, a symmetry strongly broken at short distances. We analyze the consequences of such a concept within the framework of one-boson exchange potentials in NN scattering and keeping the leading N_c contributions. Phenomenologically successful relations between singlet $^1S_0$ and triplet $^3S_1$ scattering phase shifts are provided in the entire elastic region. We establish symmetry breaking relations among noncentral phase shifts which are successfully fulfilled by even-L partial waves and strongly violated by odd-L partial waves, in full agreement with large N_c requirements.

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I. INTRODUCTION

Symmetries have traditionally been very useful in nuclear physics partly because the force at the hadronic level is not well known at short distances [1–3]. In cases such as isospin, chiral, or heavy quark symmetry, the invariance can directly be traced from the fundamental QCD Lagrangian and formulated in terms of the underlying quark and gluonic degrees of freedom. In some other cases. the connection is less straightforward. Many years ago Wigner and Hund proposed [4,5] extending the spin and isospin $SU_S(2) \otimes SU_I(2)$ symmetry into the larger SU(4) group where the nucleon-spin states $p\uparrow$, $p\downarrow$, $n\uparrow$, $n\downarrow$ correspond to the fundamental representation, and hence providing a supermultiplet structure of nuclear energy levels. From a fundamental viewpoint, it is clear that SU(4) and more generally SU(6) (spin-flavor) symmetry cannot be realized exactly and dynamically due to the Coleman-Mandula theorem [6]. However, as a static symmetry, it yields interesting selection rules for nuclear transitions and response functions [7]. In addition, the corresponding SU(4) mass formula was found to be at least as good as the well-known Weizsäcker one [8,9]. Spin-orbit interaction of the shell model obviously violates the symmetry, and indeed a breakdown of SU(4) has been reported for heavier nuclei [10], while nuclear matter has been addressed in Ref. [11]. Double binding energy differences have been shown to be a useful test of the symmetry [12]. Recently, inequalities for light nuclei based on SU(4) and Euclidean path integrals have been derived by neglecting all but S-wave interactions [13].

Despite its relative success along the years, SU(4) symmetry has been treated as an accidental one within the traditional approach to nuclear physics and guessing its origin from QCD has been a subject of some interest in the last decade. Indeed, attempts to justify SU(4) spin-flavor symmetry from a more fundamental level have been carried out along several lines. Based on the limit of a large number of colors N_c of QCD [14,15], it was shown [16,17] that if the nucleon momentum scales as $p \sim N_c^0$, the nuclear potentials scale either as N_c or $1/N_c$, depending upon the particular spin-isospin channel, which shows that the NN force could be determined with $1/N_c^2$ relative accuracy. It was found that the leading potential would be SU(4) symmetric if the tensor force was neglected in addition, a plausible assumption for light nuclei where S waves dominate. Although these estimates are conducted directly in terms of quarks and gluons, quark-hadron duality allows one to reformulate these results in terms of purely hadronic degrees of freedom, providing a rationale for the one-boson-exchange (OBE) potential models [18], and the internal consistency of two- [19] and multiple-boson exchanges [20,21]. The analysis of sizes of volume integrals of phenomenologically successful potentials confirms the large-N_c expectations [22]. The large size of scattering lengths was regarded as a fingerprint of the SU(4) symmetry within an effective field theory (EFT) viewpoint [23] using the power divergent subtracted (PDS) scheme; singlet and triplet renormalized couplings coincide at the natural renormalization scale $\mu \sim m_\pi \gg 1/\alpha_s, 1/\alpha_t$, with α_s and α_t the scattering lengths, and a contact interaction makes sense in such a scaling regime. Resonance saturation based on the elimination of exchanged mesons in the OBE Bonn potential [18] at very low energies was also shown to reproduce the EFT approach and to agree numerically with the Wigner symmetry expectations [24]. According to Refs. [26–29], QCD might be close to a point where the effective theory had an SU(4) symmetry at zero energy as well as discrete scale invariance if the pion mass was larger than its physical value, around $m_\pi \sim 200$ MeV. This nice idea might be confirmed by recent fully dynamical lattice QCD determinations of the scattering lengths [30] and quenched lattice QCD evaluations of NN potentials [31,32] where indeed unphysical pion masses are probed.

While the proclaimed symmetry holds in a range where scale invariance sets in and EFT methods based on contact interactions can be applied [23,24], it is not obvious *what* the implications are for the lightest NN system itself for finite energies and for physical pion masses. In particular, the scale dependence of the contact interaction is modified when the finite range of the medium and long distance potential is taken into account. To be specific, low energy NN scattering is

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dominated by S waves in different (S, T) channels where spin and isospin are interchanged, $(1,0) = {}^1S_0 \leftrightarrow (0,1) = {}^3S_1$. Wigner SU(4) symmetry predicts *identical* interactions in both ${}^1S_0$ and ${}^3S_1$ channels. The above-mentioned identity of the ${}^1S_0$ and ${}^3S_1$ potentials holds also in the large-N_c expansion [16,17], so we take advantage of this fact by using the leading N_c-OBE potentials, which simplifies the discussion to a large extent as we discuss in Sec. II. In contrast, the corresponding phase shifts from partial wave analyses [33] are *very different* at all energies. We are thus confronted with an intriguing puzzle, since it is not at all obvious how the symmetry should be interpreted for the NN system; it would be difficult to understand otherwise the successes of SU(4) for light nuclei. A second puzzle arises from an embarrassing cohabitation of conflicts and agreements between large-N_c studies and Wigner symmetry. Despite the initial claim [16], a more complete analysis [17] could only justify the Wigner symmetry in even-L partial waves, while for odd-L a violation of the symmetry was expected. However, doing so requires neglecting the tensor force, which according to the Wigner symmetry should vanish, but it is a leading contribution to the potential in the large-N_c limit. Thus, while some pieces of the NN potential (such as spin orbit) are suppressed in both schemes, some others are not simultaneously small. These conflicts between the time-honored SU(4) Wigner symmetry and the QCD based large-N_c expansion for odd-L channels require an explanation and naturally pose the question on the validity of either framework.

In the present work, we analyze both puzzles by introducing the concept of a *long distance symmetry*[1] first to understand the meaning of Wigner symmetry in those cases where its validity complies with large-N_c expectations. This is a case where we expect the symmetry to be more robust. Once this is done, it is pertinent to elucidate the validity of the symmetry in those cases where a possible conflict with the large-N_c expansion arises. Our discussion is tightly linked to the coordinate space renormalization discussed in previous works [34,35]. This approach, while borrowing the physical motivation from EFT theories, provides a quantum mechanical framework in which the emphasis is placed on the nonperturbative aspects of the NN problem, a playground where the standard EFT viewpoint has encountered notorious difficulties. The method is reviewed in Sec. III for completeness. We find that for S waves, the Wigner symmetry holds in a much wider range than the applicability of a contact interaction suggests if the finite range of the interaction is incorporated. As a byproduct, we provide in Sec. IV *quantitative* predictions; the seemingly independent triplet and singlet S-wave phase shifts corresponding to isovector and isoscalar states, respectively, for the np system are shown to be neatly intertwined in the entire elastic region. A similar correlation can also be established between the ${}^1S_0$ virtual state and the ${}^3S_1$ deuteron bound state. Actually, we show how the symmetry may be visualized for large scattering lengths due to the onset of scale invariance. Symmetry breaking due to inclusion of further counterterms, tensor interaction, and spin-orbit interaction are discussed in Sec. V. We show how a sum rule for supermultiplet phase-shift splitting due to spin-orbit and tensor interactions is well fulfilled for noncentral L-even waves, and strongly violated in L-odd waves where a Serber-like symmetry holds instead. This pattern of SU(4)-symmetry breaking complies with the large-N_c expectations, a somewhat unexpected conclusion. Finally, in Sec. VI we provide our main conclusions and outlook for further work.

[1]Strictly speaking, we mean long and medium range effects, although we will be using long distance for brevity and to emphasize the renormalization aspect of the problem.

II. OBE POTENTIALS, LARGE N_c, AND WIGNER SYMMETRY

Our starting point is the field theoretical OBE model of the NN interaction [18], which includes all mesons with masses below the nucleon mass, i.e., π, $\sigma(600)$, η, $\rho(770)$, and $\omega(782)$. For the purpose of discussing SU(4) Wigner symmetry within the OBE framework (see Appendix A for a brief overview), we will deal here only with S waves, neglecting for the moment the S-D wave mixing stemming from the tensor force as required by Wigner symmetry. Our results of Sec. IV and estimates in Sec. V B will provide the *a posteriori* justification of this simplifying assumption. Noncentral waves and the role of spin-orbit as well as tensor force will be treated in Sec. V C as SU(4) breaking perturbations.

For S waves, the NN potential reads

$$V = V_C + \tau W_C + \sigma V_S + \tau\sigma W_S, \tag{1}$$

where $\tau = \tau_1 \cdot \tau_2 = 2T(T+1) - 3$ and $\sigma = \sigma_1 \cdot \sigma_2 = 2S(S+1) - 3$, and the Pauli principle requires $(-)^{S+T+L} = -1$. Thus, for the spin-singlet ${}^1S_0$ and spin-triplet ${}^3S_1$ states we get

$$V_s = V_C + W_C - 3V_S - 3W_S, \tag{2}$$

$$V_t = V_C - 3W_C + V_S - 3W_S, \tag{3}$$

To simplify the discussion, we will discard terms in the potential that are phenomenologically small. Actually, according to Refs. [16,17] in the leading large N_c, one has $V_C \sim W_S \sim N_c$, while $V_S \sim W_C \sim 1/N_c$. In terms of meson exchanges (see also Ref. [19]), one has the contributions

$$\begin{aligned} V_s(r) = V_t(r) = &-\frac{g_{\pi NN}^2 m_\pi^2}{16\pi M_N^2}\frac{e^{-m_\pi r}}{r} - \frac{g_{\sigma NN}^2}{4\pi}\frac{e^{-m_\sigma r}}{r} \\ &+ \frac{g_{\omega NN}^2}{4\pi}\frac{e^{-m_\omega r}}{r} - \frac{f_{\rho NN}^2 m_\rho^2}{8\pi M_N^2}\frac{e^{-m_\rho r}}{r} \\ &+ \mathcal{O}(N_c{}^{-1}), \end{aligned} \tag{4}$$

where $g_{\sigma NN}$ is a scalar-type coupling, $g_{\pi NN}$ a pseudoscalar derivative coupling, $g_{\omega NN}$ a vector coupling, and $f_{\rho NN}$ a tensor derivative coupling (see Ref. [18] for notation). Here, the scheme proposed in Ref. [36] of neglecting both energy and nonlocal corrections is realized explicitly. In principle, the large-N_c limit contains infinitely many multi-meson exchanges which decay exponentially with the sum of the exchanged meson masses. However, NN scattering in the elastic region below pion production threshold probes c.m.

momenta $p < p_{\max} = 400$ MeV. Given the fact that $1/m_\omega = 0.25$ fm $\ll 1/p_{\max} = 0.5$ fm, we expect heavier meson scales to be irrelevant, and in particular ω and ρ themselves are expected to be at most marginally important.[2] Note that, in any case, when $m_\omega = m_\rho$ the redundant combination $g^2_{\omega NN} - f^2_{\rho NN} m^2_\rho/(2M^2_N)$ appears, indicating a further source of cancellation between ρ and ω in this channel. Moreover, since the leading contributions to the potential are $\sim N_c$ and the subleading ones are $\sim 1/N_c$, the neglected terms are of relative $1/N_c^2$ order, so we might expect an *a priori* $\sim$10% accuracy.

The coincidence between ${}^1S_0$ and ${}^3S_1$ potentials complies with the Wigner SU(4) symmetry, which we review for completeness in Appendix A for the two-nucleon system. Modern high quality potentials [37] describing accurately NN scattering below pion production threshold show some traces of the symmetry for distances above 1.4–1.8 fm. Quenched lattice QCD evaluations of NN potentials for $m_\pi/m_\rho \sim 0.6$ [31,32] also yield similar ${}^1S_0$ and ${}^3S_1$ potentials for $r >$ 1.4 fm. Thus, at first sight, one might conclude that Wigner symmetry holds when OPE dominates and thus has a limited range of applicability. An important result of the present investigation, which will be elaborated in this paper, is that this is not necessarily so, provided the relevant scales of symmetry breaking are properly isolated with the help of renormalization ideas.

Let us analyze the consequences of the symmetry [Eq. (4)] within the standard approach to OBE potentials. The scattering phase shift $\delta_0(p)$ is computed by solving the (S-wave) Schrödinger equation in r-space, i.e.,

$$-u_p''(r) + M_N V(r) u_p(r) = p^2 u_p(r), \tag{5}$$

$$u_p(r) \to \frac{\sin(pr + \delta_0(p))}{\sin\delta_0(p)}, \tag{6}$$

with a regular boundary condition at the origin $u_p(0) = 0$. Moreover, for a short range potential such as the one in Eq. (4), one also has the effective range expansion (ERE)

$$p\cot\delta_0(p) = -\frac{1}{\alpha_0} + \frac{1}{2} r_0 p^2 + \cdots, \tag{7}$$

where the scattering length α_0 is defined by the asymptotic behavior of the zero-energy wave function as

$$u_0(r) \to 1 - \frac{r}{\alpha_0}, \tag{8}$$

and the effective range r_0 is given by

$$r_0 = 2\int_0^\infty dr \left[\left(1 - \frac{r}{\alpha_0}\right)^2 - u_0(r)^2\right]. \tag{9}$$

In the usual approach [18,38], everything is obtained from the potential assumed to be valid for $0 \leqslant r < \infty$. We note incidentally that the Wigner symmetry relation, Eq. (4), holds at *all* distances.[3] In addition, due to the *unnaturally large* $NN\,{}^1S_0$ scattering length ($\alpha_s \sim -23$ fm), any change in the potential $V \to V + \Delta V$ has a dramatic effect on α_0, since one obtains

$$\Delta\alpha_0 = \alpha_0^2 M_N \int_0^\infty \Delta V(r) u_0(r)^2\, dr, \tag{10}$$

and thus the potential parameters *must be fine-tuned*, and in particular the short distance physics. As discussed in Refs. [39,40], this short distance sensitivity is unnatural as long as the OBE potential does not truly represent a fundamental NN force at short distances. Indeed, the sensitivity manifests itself as tight constraints for the potential parameters when the ${}^1S_0$ phase shift is fitted, resulting in incompatible values of the coupling constants as obtained from other sources as NN scattering. Of course, there is nothing wrong with the need of a fine-tuning, as this is an unavoidable consequence of the large scattering length; the relevant point is whether this should be driven by a potential that will not be realistic at short distances.

In any case, in the traditional approach to NN potentials, we are confronted with a paradox: on the one hand, the symmetry seems to suggest that *somewhere* the phase shifts should coincide, while on the other hand, a fine-tuning is required because of the large scattering lengths. In the standard approach, if $V_s(r) = V_t(r)$, then $\delta_s(k) = \delta_t(k)$, and thus $\alpha_s = \alpha_t$, as one naturally expects. A straightforward explanation, of course, is to admit that the symmetry is strongly violated. This would make it difficult to understand how SU(4) can work at all for light nuclei if the simpler two nucleon system does not manifestly show the symmetry.

Before presenting our solution to this dilemma in the next section, let us note that a good condition for an approximate symmetry is that it be stable under symmetry breaking, otherwise a tiny perturbation $V_s(r) - V_t(r) = \Delta V(r) \neq 0$ would yield a large change, and this is precisely the bizarre situation we are bound to evolve because of the large scattering lengths. This suggests a clue to the problem; namely, we should provide a framework in which the highly potential-sensitive scattering length becomes a variable independent of the potential. More generally, we want to avoid the logical conclusion that a symmetry of the potential is a symmetry of the S matrix.[4] As we will explain below, the puzzle may be overcome by the concept of long distance symmetry: a symmetry that is only broken at short distances by a suitable boundary condition.

[2]This of course does not exclude explicit and leading N_c uncorrelated multiple pion exchanges, i.e., background nonresonant contributions in $\pi\pi$ or $\pi\rho$ scattering. We expect them not to be dominant once σ, ρ, and ω are included.

[3]In practice, strong form factors are included mimicking the finite nucleon size and reducing the short distance repulsion of the potential, but the regular boundary condition is always kept.

[4]This situation resembles the case of anomalies in quantum field theory, where the parallel statement would be that a symmetry of the Lagrangian becomes a symmetry of the S matrix, a conclusion that may be invalidated by the impossibility of preserving the symmetry by the necessary regularization of loop integrals. The present case is a bit more subtle, as it corresponds to the case of finite but ambiguous theories (see, e.g., Ref. [41]).

III. UNIVERSALITY RELATIONS AND RENORMALIZATION

We cut the Gordian knot by appealing to renormalization in coordinate space [34,35]. As we will show, this enables us to disentangle short and long distances in a way that the symmetry is kept at all nonvanishing distances. The main idea is to fix the scattering length independently of the potential by means of a suitable short distance boundary condition. As a result the undesirable dependence of observables on the potential is reduced at short distances, precisely the region where a determination of the NN force in terms of hadronic degrees of freedom becomes less reliable.

Let us review in the case of S waves the renormalization procedure in coordinate space pursued elsewhere [34,35] and which will prove particularly suitable in the sequel. This is fully equivalent to introducing one counterterm in the cutoff Lippmann-Schwinger equation in momentum space (see Ref. [42] for a detailed discussion on the connection). The superposition principle of boundary conditions implies

$$u_k(r) = u_{k,c}(r) + k\cot\delta_0 u_{k,s}(r), \tag{11}$$

with $u_{k,c}(r) \to \cos(kr)$ and $u_{k,s}(r) \to \sin(kr)/k$ for $r \to \infty$. At zero energy, $k \to 0$, and $\delta_0(k) \to -\alpha_0 k$ yields

$$u_0(r) = u_{0,c}(r) - \frac{1}{\alpha_0} u_{0,s}(r), \tag{12}$$

with $u_{0,c}(r) \to 1$ and $u_{0,s}(r) \to r$ for $r \to \infty$. Combining the zero- and finite-energy wave functions, we get

$$[u_k'(r)u_0(r) - u_0'(r)u_k(r)]|_{r_c}^{\infty} = k^2 \int_{r_c}^{\infty} u_k(r)u_0(r)\,dr, \tag{13}$$

where r_c is a short distance cutoff radius which will be removed at the end. To calculate the contribution from the term at infinity, we use the long distance behavior, Eq. (6). The integral and the boundary term at infinity yield two canceling δ functions. This corresponds to

$$\int_0^{\infty} u_k(r)u_p(r)dr = \frac{\pi\delta(k-p)}{2\sin^2\delta_0(k)}, \tag{14}$$

as can be readily seen. We are thus left with the boundary term at short distances; taking the limit $r_c \to 0$. we get

$$\lim_{r_c\to 0}[u_k'(r_c)u_0(r_c) - u_0'(r_c)u_k(r_c)] = 0. \tag{15}$$

Note that the regular solution $u_k(r_c) = u_0(r_c) = 0$ is a *particular* choice for $r_c = 0$. Writing out the orthogonality condition via the superposition principle at finite and zero energies, Eqs. (11) and Eq. (12), respectively, one gets

$$0 = \int_0^{\infty} dr \left[u_{0,c}(r) - \frac{1}{\alpha_0} u_{0,s}(r)\right] \times [u_{k,c}(r) + k\cot\delta_0(k)u_{k,s}(r)]. \tag{16}$$

Expanding the integrand and defining

$$\mathcal{A}(k) = \int_0^{\infty} dr u_{0,c}(r)u_{k,c}(r),$$
$$\mathcal{B}(k) = \int_0^{\infty} dr u_{0,s}(r)u_{k,c}(r),$$
$$\mathcal{C}(k) = \int_0^{\infty} dr u_{0,c}(r)u_{k,c}(r),$$
$$\mathcal{D}(k) = \int_0^{\infty} dr\, u_{0,s}(r)u_{k,s}(r), \tag{17}$$

we get the explicit formula

$$k\cot\delta_0(k) = \frac{\alpha_0\mathcal{A}(k) + \mathcal{B}(k)}{\alpha_0\mathcal{C}(k) + \mathcal{D}(k)}. \tag{18}$$

The functions $\mathcal{A}, \mathcal{B}, \mathcal{C}$, and $\mathcal{D}$ are even functions of k which depend *only on the potential*. Note that the dependence of the phase shift on the scattering length is wholly *explicit*; $\cot\delta_0$ is a bilinear rational mapping of α_0. Further, using Eq. (12), one gets the effective range

$$r_0 = A + \frac{B}{\alpha_0} + \frac{C}{\alpha_0^2}, \tag{19}$$

where

$$A = 2\int_0^{\infty} dr\left(1 - u_{0,c}^2\right), \tag{20}$$

$$B = -4\int_0^{\infty} dr(r - u_{0,c}u_{0,s}), \tag{21}$$

$$C = 2\int_0^{\infty} dr\left(r^2 - u_{0,s}^2\right), \tag{22}$$

depend only on the potential parameters. Again, the interesting thing is that all explicit dependence on the scattering length α_0 is displayed by Eq. (19).

We turn now to discuss the case of a bound state corresponding to the case of negative energy $E = -\gamma^2/M$, where γ is the wave number. The wave function behaves asymptotically as

$$u_\gamma(r) \to A_S e^{-\gamma r}, \tag{23}$$

and is chosen to fulfill the normalization condition

$$\int_0^{\infty} u_\gamma(r)^2 dr = 1. \tag{24}$$

In principle, such a state would be unrelated to the scattering solutions. An explicit relation may be determined from the orthogonality condition, which applied in particular to the zero-energy state yields

$$0 = \int_0^{\infty} dr\left[u_{0,c}(r) - \frac{1}{\alpha_0}u_{0,s}(r)\right]u_\gamma(r). \tag{25}$$

This generates a correlation between the scattering length α_0 and the bound state wave number γ,

$$\alpha_0(\gamma) = \frac{\int_0^{\infty} dr u_\gamma(r)u_{0,s}(r)}{\int_0^{\infty} dr u_\gamma(r)u_{0,c}(r)}. \tag{26}$$

We recall that the two independent zero-energy solutions, $u_{0,c}(r)$ and $u_{0,s}(r)$, depend only on the potential.

A trivial realization of the conditions discussed above is given by the case with no potential, $U(r) = 0$. Hence, the general solution for a positive energy state $E = k^2/M$ is given by

$$u_k(r) = \cot\delta_0(k)\sin(kr) + \cos(kr), \tag{27}$$

and using the low energy limit condition $\delta_0(k) \to -\alpha_0 k$, we obtain

$$u_0(r) = 1 - \frac{r}{\alpha_0}. \tag{28}$$

Orthogonality between zero- and finite-energy states yields, after evaluating the integrals,

$$k \cot \delta_0(k) = -\frac{1}{\alpha_0}, \tag{29}$$

and as a consequence the effective range vanishes $r_0 = 0$, in accordance with the fact that the range of the potential is zero. For a negative energy state $E = -\gamma^2/M$, the normalized bound state is

$$u_\gamma(r) = A_S e^{-\gamma r}, \quad A_S = \sqrt{2\gamma}. \tag{30}$$

Orthogonality between the zero-energy and the bound state, again, yields the correlation

$$\alpha_0 = 1/\gamma. \tag{31}$$

In the Appendix B we illustrate further the procedure in the case of weak potentials for which a form of perturbation theory may be applied for the case of *weak potentials* but *arbitrary* scattering lengths.

Before going further, we should ponder the need to take the limit $r_c \to 0$, which corresponds to eliminating the cutoff. We note that the potential $V(r)$ is used at *all* distances in both the standard approach, which involves the regular solution only, and the renormalization approach, which requires the regular as well as the irregular solution. However, the sensitivity to the short distance behavior of the potential is quite different: the standard approach displays much stronger dependence, while the renormalization approach is fairly independent of the hardly accessible short distance region, a feature that becomes evident perturbatively [see, e.g., Eq. (B6)]. This is in fact the key property that allows us to eliminate the cutoff in the renormalization approach. Thus, removing the cutoff does not mean that the OBE potential is believed to hold all the way down to the origin.

The procedure carried out before is described in purely quantum mechanical terms, but it can be mapped onto field theoretical terminology; it is equivalent to the method of introducing one counterterm in the cutoff Lippmann-Schwinger equation in momentum space [42,43]. Moreover, Eq. (12) represents the corresponding renormalization condition, which is chosen to be on-shell at zero energy. In the case of the bound state, the corresponding renormalization condition is given by Eq. (23) at negative energy. Imposing more than one renormalization condition, i.e., introducing more than one counterterm and removing the cutoff, presents some subtleties, which have been discussed in Refs. [35,42]. We will analyze below this issue in the present context (see Sec. V A).

IV. CENTRAL PHASES AND THE DEUTERON

A. Potential parameters

To proceed further, we fix the potential parameters, keeping in mind that the leading N_c nature of the potential embodies some systematic $1/N_c^2$ uncertainties. Of course, while we will use relations compatible with large-N_c scaling, the numerical values can only be fixed phenomenologically. The main point is that besides the σ-meson mass (see below), we may choose quite natural values for the masses and couplings unlike the usual OBE potentials [18]. As discussed at the end of Sec. II, the standard approach suffers from tight constraints reflecting the unnatural short distance sensitivity. In this regard, let us note that, as emphasized in Refs. [39,40], it is a virtue of the renormalization viewpoint, which we are applying here to the OBE potential, that the unwanted short distance sensitivity is largely removed, allowing a determination of the potential parameters using independent sources.

For definiteness, we take $g_{\pi NN} = 13.1$ and $g_{\sigma NN} = 10.1$, which are quite close to the Goldberger-Treiman values for σ and π, $g_{\sigma NN} = M_N/f_\pi$ and $g_{\pi NN} = g_A M_N/f_\pi$, respectively. We also take the SU(3) value $g_{\omega NN} = 3g_{\rho NN} - g_{\phi NN}$, which on the basis of the OZI rule $g_{\phi NN} = 0$, Sakurai's universality $g_{\rho NN} = g_{\rho\pi\pi}/2$, and the KSFR relation $2g^2_{\rho\pi\pi} f^2_\pi = m^2_\rho$ yields $g_{\omega NN} = N_c m_\rho/(2\sqrt{2} f_\pi) = 8.8$. The ρ tensor coupling is taken to be $f_{\rho NN} = \sqrt{2} M_N g_{\omega NN}/m_\rho = 15.5$, which cancels the vector meson contributions in the potential and yields $\kappa_\rho = f_{\rho NN}/g_{\rho NN} = 5.5$, a quite reasonable result [18].[5] Note that $1/N_c$ effects include not only other mesons but also finite width effects of σ and ρ, since for large N_c one has stable mesons, $\Gamma_\sigma, \Gamma_\rho \sim 1/N_c$.

For the masses, we take $m_\pi = 140$ MeV and $m_\omega = 783$ MeV. This fixes all parameters except m_σ (actually the real part), which we identify with the lightest $J^{PC} = 0^{++}$ meson $f_0(600)$. According to the recent analysis based on Roy equations, $m_\sigma - i\Gamma_\sigma/2 = 441^{+16}_{-8} - i272^{+9}_{-12}$ MeV [44]. A fit to the pn data of Ref. [37] in the $^1S_0$ channel yields $m_\sigma = 510(1)$ MeV, where the error is statistical. The fitted mass value differs by about 10% from the location of the real part of the resonance, in harmony with the expected $1/N_c^2$ corrections.[6] Although a more quantitative estimate of the large-N_c corrections to the potentials parameters would be very useful, for the present purposes of discussing Wigner symmetry in light of large N_c, it is more than sufficient. Thus, we make no attempt here to make any systematic expansion.

B. Low energy parameters and phase shifts

Clearly, in the traditional approach, if we have $V_s(r) = V_t(r)$ and impose the regular boundary condition, $u_s(0) = u_t(0) = 0$, the only possible solution is $\alpha_s = \alpha_t, r_s = r_t$, and $\delta_s(p) = \delta_t(p)$. However, in the renormalization approach, we allow *different* short distance boundary conditions

[5] As shown in previous work [39,40], the net vector meson exchange contribution corresponding to the combined repulsive coupling $g^2_{\omega NN} - f^2_{\rho NN} m^2_\rho/2M^2_N$ (referred there simply as $g^2_{\omega NN}$) cannot be pinned down accurately from a fit to the $^1S_0$ phase shift being compatible with zero within errors. This is due to the short distance insensitivity embodied by the renormalization approach.

[6] Actually, our estimate of the σ mass as a pole in the second Riemann sheet for $\pi\pi$ scattering for large N_c [40] yields the value $m_\sigma \sim 507$ MeV.

$u_s'(0^+)/u_s(0^+) \neq u_t'(0^+)/u_t(0^+)$,[7] and hence we may have $\alpha_s \neq \alpha_t$. Note that this corresponds to a breaking of the symmetry at *short* distances and hence postulating its validity at *long* distances. The previous equations imply straight away the following expressions for the effective ranges in the singlet and triplet channels:

$$
\begin{aligned}
r_s &= A + \frac{B}{\alpha_s} + \frac{C}{\alpha_s^2}, \\
r_t &= A + \frac{B}{\alpha_t} + \frac{C}{\alpha_t^2}.
\end{aligned} \tag{32}
$$

As already mentioned, the remarkable aspect of these two equations is that the coefficients A, B, C are *identical* in the triplet and singlet channels as long as $V_s(r) = V_t(r)$, thus the only difference resides in the numerical values of the scattering lengths α_s and α_t. Numerically, we obtain (everything in fm)

$$
\begin{aligned}
r_0 &= 1.3081 - \frac{4.5477}{\alpha_0} + \frac{5.1926}{\alpha_0^2} \qquad (\pi), \\
&= 1.5089 \text{ fm} \quad (\alpha_0 = \alpha_s) \quad (\text{exp. } 2.770 \text{ fm}), \\
&= 0.6458 \text{ fm} \quad (\alpha_0 = \alpha_t) \quad (\text{exp. } 1.753 \text{ fm}), \\
r_0 &= 2.4567 - \frac{5.5284}{\alpha_0} + \frac{5.7398}{\alpha_0^2} \qquad (\pi + \sigma), \\
&= 2.6989 \text{ fm} \quad (\alpha_0 = \alpha_s) \quad (\text{exp. } 2.770 \text{ fm}), \\
&= 1.5221 \text{ fm} \quad (\alpha_0 = \alpha_t) \quad (\text{exp. } 1.753 \text{ fm}),
\end{aligned} \tag{33}
$$

where the corresponding numerical values when the experimental $\alpha_s = -23.74$ and $\alpha_t = 5.42$ fm as well as the experimental values for the effective ranges have also been added. More generally, for any fixed potential, the correlation of r_0 on $1/\alpha_0$ is a parabola, which we plot in Fig. 1 for the OPE and OPE+σ. This dependence is universal to *all S* waves having the *same* potential, and from this viewpoint there is nothing in this curve making unnaturally large scattering lengths particularly different from smaller ones. The present analysis, however, does sheds no light on the origin of the large size of the α nor howα_s and α_t are interrelated.[8] In any case, as we see from Fig. 1, the experimental values fall strikingly almost on top of the curve, pointing toward a correct interpretation of the underlying symmetry.

We turn next to the phase shifts. According to Eq. (18), they are given in terms of the universal functions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ defined in Eq. (17) and presented in Fig. 2 in appropriate length units as a function of the c.m. momentum p in MeV for completeness. As we see, these functions are smooth. From them, the corresponding singlet and triplet phase shifts are obtained by

$$
\begin{aligned}
k \cot \delta_s &= \frac{\alpha_s \mathcal{A}(k) + \mathcal{B}(k)}{\alpha_s \mathcal{C}(k) + \mathcal{D}(k)}, \\
k \cot \delta_t &= \frac{\alpha_t \mathcal{A}(k) + \mathcal{B}(k)}{\alpha_t \mathcal{C}(k) + \mathcal{D}(k)},
\end{aligned} \tag{34}
$$

respectively. When the experimental scattering lengths $\alpha_s = -23.74$ and $\alpha_t = 5.42$ fm are taken, we can *fit* the singlet $^1S_0$ channel and *predict* the triplet $^3S_1$ channel. The result is shown in Fig. 3, and we see that the agreement is remarkably good, considering that we neglected the tensor force and the *a priori* $1/N_c^2$ systematic corrections to the potential. Note that the identity of the singlet and triplet potentials is not sufficient; the simple OPE fulfills this property but does not explain either phase shift. Actually, it shows that *both* failures are correlated.[9]

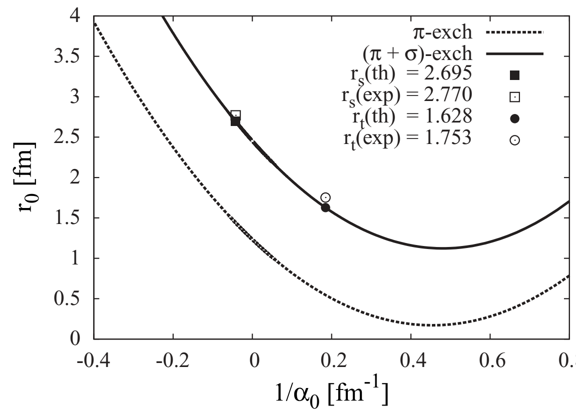


FIG. 1. Wigner correlation for the effective range r_0 (in fm) of a *np S* wave as a function of an arbitrary inverse scattering length α_0 in the case of the OPE and OPE+σ potentials. The parabolic shape is determined by a unique long distance potential. The points in the solid curve correspond to the two different values of the effective range r_s in the singlet $^1S_0$ and r_t in the triplet $^3S_1$ channels when the scattering length is taken to be $\alpha_s = -23.74$ fm and $\alpha_t = 5.42$ fm, respectively. Experimental points are also shown for comparison.

C. Renormalization and scale invariance

It is interesting to analyze our results in light of Refs. [16,23,24], where a square well potential, PDS, and sharp momentum cutoff were used to model the short distance contact interactions arising when all exchanged particles are integrated out. Here we are interested in the dependence on the arbitrary renormalization scale separating the contact and the extended particle exchange interaction, since they are not independent of each other; by keeping this scale dependence, we may enter the interaction region where, as we will show now, the symmetry can be visualized. We appeal

[7]The limit from above, $u(0^+) = \lim_{r_c \to 0^+} u(r_c)$ is really necessary to pick both the regular and irregular solutions. If one starts *exactly* from the origin, the only possible solution is the regular one.

[8]This is in fact a price we pay for the built-in short distance insensitivity. We note, however, that after Refs. [25–29], both scattering lengths might coincide for a pion mass around $m_\pi \sim 200$ MeV. As a consequence, QCD might be close to a point where the effective theory had a standard SU(4) symmetry at *zero* energy. Actually, in Ref. [26] the similarity between $^1S_0$ and $^3S_1$ phase shifts can be seen. This scenario would turn the long distance symmetry we propose for the physical pion mass into a standard symmetry for such an unphysical value of the pion mass.

[9]The reason why OPE fails at much lower energies in the $^1S_0$ channel than in the $^3S_1$ channel is because of a stronger short distance sensitivity of the channel with larger scattering length.

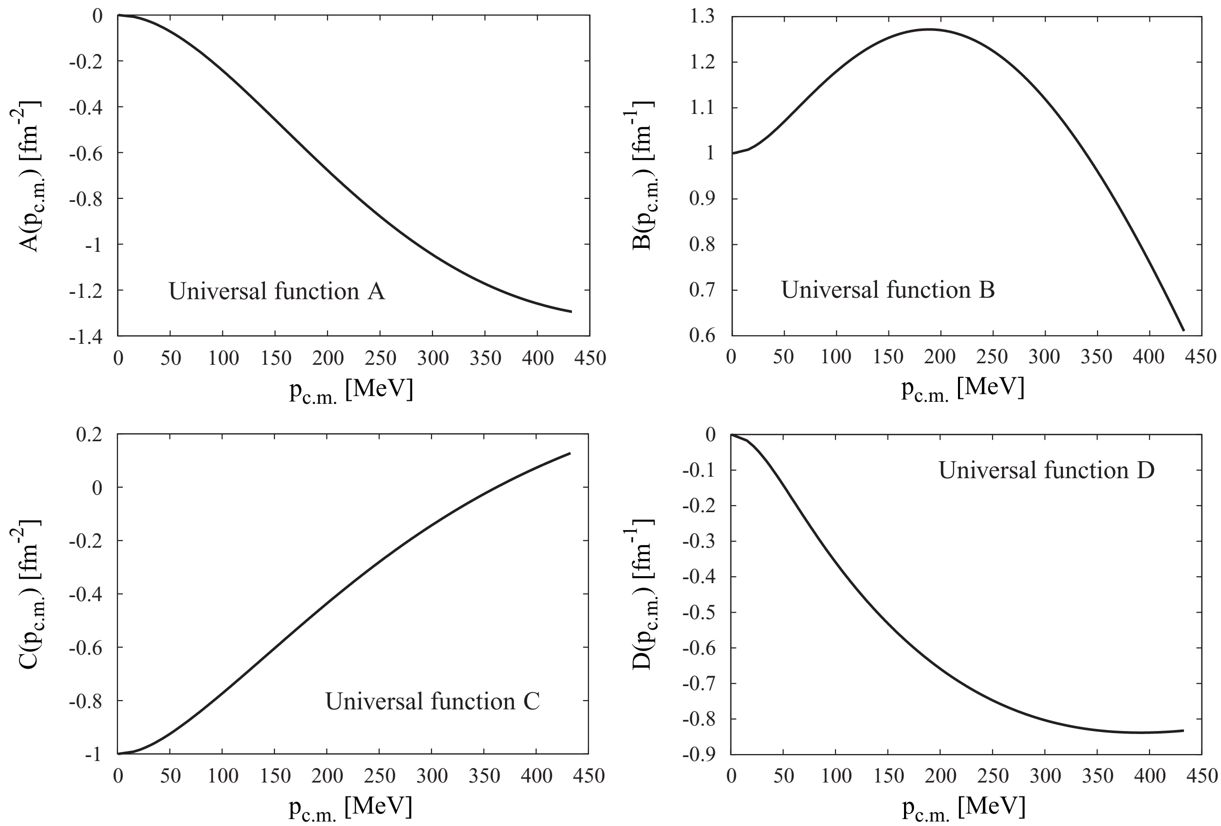


FIG. 2. Universal functions $\mathcal{A}, \mathcal{B}, \mathcal{C}$, and $\mathcal{D}$ defined by Eqs. (17) in appropriate length units as a function of the c.m. momentum p (in MeV). These functions depend only on the potential $V_s(r) = V_t(r)$, but are independent of the scattering length.

to the coordinate space version of the renormalization group [35,45] (for a momentum space version, see Ref. [46]), where the version of the Callan-Zymanzik equation for potential scattering reads

$$Rc_0'(R) = c_0(R)(1 - c_0(R)) + MR^2V(R), \qquad (35)$$

where $c_0(R) = Ru_0'(R)/u_0(R)$ is a suitable combination of the short distance boundary condition, and we have chosen for simplicity to work at zero energy.[10] The above equation provides the evolution of the boundary condition as a function of the distance R (the renormalization scale) in order to have a fixed scattering amplitude (see Ref. [35] for a thorough discussion). Clearly, at long distances $r \gg 1/m_\pi$, the potential becomes negligible and the equation is scale invariant, only broken by the renormalization condition which fixes the value of c_0 at some scale.[11] In fact, the solution of the above equation is given in terms of the scattering length α_0 in the infrared, $R \to \infty$ and $c_0(R) \to \alpha_0/R$. On the other hand, if the scattering length is large, we also have an intermediate regime with clear scale separation and

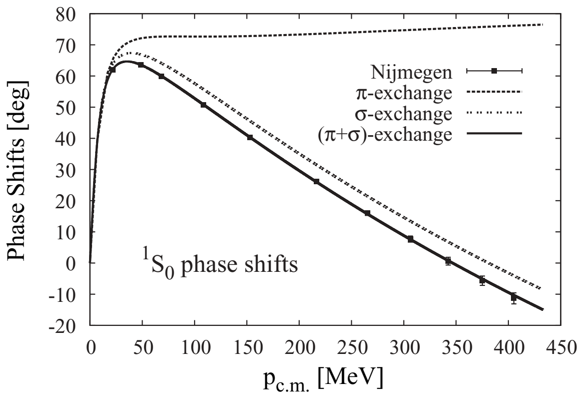
$$c_0(R) = \frac{\alpha_0}{R - \alpha_0} \sim -1, \quad 1/m_\pi \ll R \ll \alpha_0, \qquad (36)$$

indicating the onset of scale invariance [35]. This is in agreement with the PDS argument of Refs. [23] if the identification $\mu \sim 1/R$ is done. Eventually the infrared stable fixed point $c_0 \to 0$ will be achieved. Note, however, that $c_s(R) \sim c_t(R)$ in a much wider range, particularly in the scaling violating region, where the potential acts.

In the more conventional language of wave functions, the situation corresponds to a case where both wave functions $u_s(r) \sim u_t(r)$ for $r \ll \alpha_s, \alpha_t$. The situation is illustrated in Fig. 4 where the similarity in the range below 1 fm can clearly be seen and does not differ much from the solution $u_{0,c}(r)$ entering the superposition principle [Eq. (12)] and corresponding to the limit $\alpha_0 \to \pm\infty$. Note that the symmetry can be visualized within the range of the potential only when the scattering length is large because there exists the scaling regime $1/m_\pi \ll r \ll \alpha_0$, but the long distance correlations between the two S-wave channels due to the identity of potentials hold regardless of the unnatural size of the scattering lengths.

[10]The orthogonality conditions discussed above correspond to taking $c_p(R) \to c_0(R)$ for $R \to 0$.

[11]In Appendix C, we analyze a case where the dilatation symmetry of a $1/r^2$ potential *must* necessarily be broken by a renormalization condition.



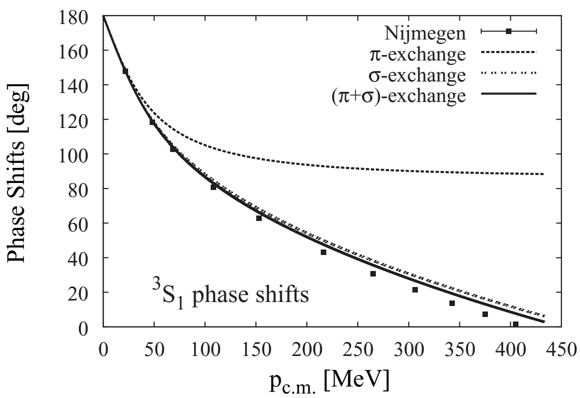


FIG. 3. Phase shifts (in degrees) for the *fitted* $^1S_0$ and *predicted* $^3S_1$ channels as a function of the c.m. momentum (in MeV). In both cases, the potential is the *same*, $V_s(r) = V_t(r)$, while the only difference is in the scattering lengths: in the singlet channel, $\alpha_s = -23.74$ fm; and in the triplet channel, $\alpha_t = 5.42$ fm, corresponding to a *different* short distance boundary condition. We also plot the cases with only 1σ exchange and 1π exchange for comparison. Data from Ref. [37].

D. Virtual and bound states

It is of course tempting to analyze the kind of features for the deuteron that may be obtained from this simplified picture where the tensor force is neglected from the start. The deuteron is determined by integrating in the Schrödinger equation with negative energy $E = -\gamma_d^2/M$ with $\gamma_d = 0.2316\ \mathrm{fm}^{-1}$ the wave number and imposing the long distance boundary condition, Eq. (23). We also compute the matter radius

$$r_m^2 = \frac{1}{4}\int_0^\infty r^2 u_d(r)^2 \tag{37}$$

and the $\mathcal{M}_{M1}$ matrix element

$$A_S \mathcal{M}_{M1} = \int_0^\infty dr u_d(r) u_{0,{}^1S_0}(r), \tag{38}$$

which correspond to the dominant magnetic contribution to neutron capture process $np \to \gamma d$ in the range of thermal neutrons (∼keV) in stars.[12] For the experimental $\gamma_d = 0.2316\ \mathrm{fm}^{-1}$, we get $A_S = 0.8643\ \mathrm{fm}^{-1/2}$ [exp. $0.8846(9)\ \mathrm{fm}^{-1/2}$] and $r_m = 1.9138$ fm [exp. 1.9754(9) fm] and $\mathcal{M}_{M1} = 4.0464$ fm (exp. 3.979 fm). As mentioned above, orthogonality between the bound state and the zero-energy state yields an explicit correlation between the triplet scattering length α_t and the deuteron wave number γ, i.e.,

[12]In this normalization, the total cross section is given by $\sigma_M(np \to \gamma d) = \pi\alpha(\mu_p - \mu_n)^2\sqrt{B/2E}(B/M_N)\gamma\mathcal{M}_{M1}^2$, where E is the neutron energy, and μ_p and μ_n the proton and neutron magnetic moments in units of the nuclear magneton, $\mu_N = e/2M_p$. We neglect meson exchange currents in the calculation of $\mathcal{M}_{M1}$.

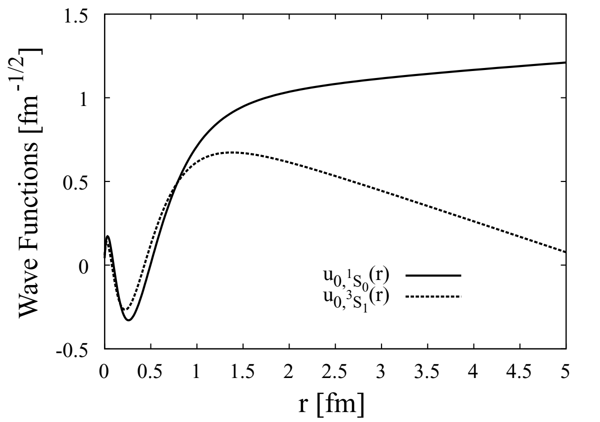


FIG. 4. Zero-energy S-wave radial functions for the singlet $^1S_0$ and triplet $^3S_1$ channels as a function of distance (in fm). The normalization is such that $u_{0,{}^1S_0} \to 1 - r/\alpha_s$ and $u_{0,{}^3S_1} \to 1 - r/\alpha_t$ with $\alpha_s = -23.74$ and $\alpha_t = 5.42$ fm the singlet and triplet scattering lengths, respectively. The potentials generating these wave functions are the same, $V_s(r) = V_t(r)$.

$$\alpha_t = \alpha_0(\gamma_d) = \left.\frac{\int_0^\infty dr\, u_\gamma(r) u_{0,s}(r)}{\int_0^\infty dr\, u_\gamma(r) u_{0,c}(r)}\right|_{\gamma=\gamma_d}. \tag{39}$$

Since the two independent zero-energy solutions, $u_{0,c}(r)$ and $u_{0,s}(r)$, depend only on the potential and hence are identical for the S-wave components of the singlet and triplet channels, this correlation is a consequence of the Wigner symmetry as well, as long as we take $V_s(r) = V_t(r)$. Note that taken as a function of the scattering length, the expression

$$\mathcal{M}(\gamma, \alpha_0) = \int_0^\infty dr\ u_\gamma(r) u_0(r) \tag{40}$$

yields *both* the orthogonality relation as well as $\mathcal{M}_{M1}$:

$$\begin{aligned} \mathcal{M}(\gamma_d, \alpha_t) &= 0, \\ \mathcal{M}(\gamma_d, \alpha_s) &= \mathcal{M}_{M1}. \end{aligned} \tag{41}$$

Actually, the dependence on the inverse scattering length is a straight line, which we show in Fig. 5. As we see, both conditions are very well fulfilled. Similarly to the previous case, the orthogonality between finite-energy states and the deuteron corresponds to the magnetic contribution to the photodisintegration of the deuteron. The result, however, does not differ much from the potential-less theory, and so we will not discuss it any further. For the experimental $\gamma_d = 0.2316\ \mathrm{fm}^{-1}$, we get $\alpha_t = 5.32$ fm. This value improves over the simple formula $\alpha_t = 1/\gamma = 4.31$ fm obtained from the case without potential, or the single OPE case where

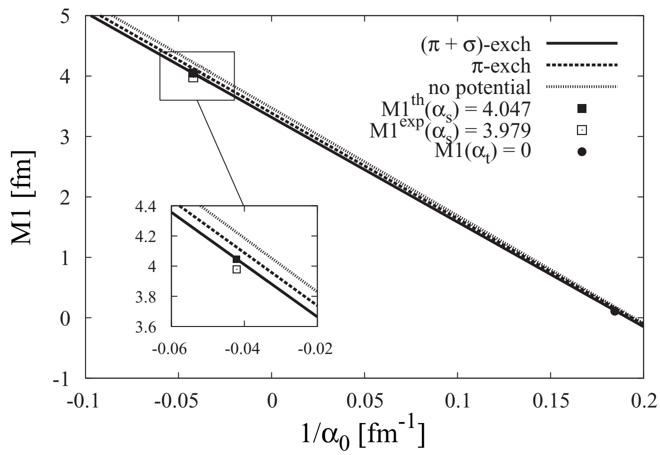


FIG. 5. Matrix element $\mathcal{M}(\gamma, \alpha)$ for $\gamma = \gamma_d$ as a function of the inverse scattering length α_0. This function should fulfill the properties $\mathcal{M}(\gamma_d, \alpha_s) = \mathcal{M}_{M1}$ (the neutron capture $M1$ matrix element) and $\mathcal{M}(\gamma_d, \alpha_t) = 0$ (the orthogonality relation between the deuteron and the zero-energy triplet state). The experimental values are highlighted as points.

$\alpha_t = 4.60$ fm. It is worth stressing that the *same* relation above yields the virtual state, a purely exponentially growing wave function, $u_v(r) \to e^{+\gamma_v r}$, in the singlet channel, yielding for $\alpha_s = -23.74$ fm, the value $\gamma_v = 0.042\ \mathrm{fm}^{-1}$. In other words, the function $\alpha_0(\gamma)$ fulfills $\alpha_0(\gamma_d) = \alpha_t$ and *simultaneously* $\alpha_0(\gamma_v) = \alpha_s$. Numerically, we get

$$\alpha_0(-0.042\ \mathrm{fm}^{-1}) = -23.74\,\mathrm{fm}, \tag{42}$$

$$\alpha_0(0.2265\,\mathrm{fm}^{-1}) = 5.42\ \mathrm{fm}. \tag{43}$$

In the region below 1 fm, the virtual state $u_v(r)$ and the deuteron bound state $u_d(r)$ look very much like the corresponding singlet and triplet zero-energy wave functions, respectively (see Fig. 4). Thus, $u_{0,{}^1S_0}(r) \sim u_v(r)$ and $u_{0,{}^3S_1}(r) \sim u_d(r)$ are consequences of the closeness of the poles to the real axis, either in the second or first Riemann sheets, respectively. However, $u_{0,{}^1S_0}(r) \sim u_{0,{}^3S_1}(r)$ and $u_v(r) \sim u_d(r)$ are further consequences of the identity of the potentials $V_s(r) = V_t(r)$.

V. SYMMETRY BREAKING

A. Symmetry breaking with two counterterms

An essential ingredient of the present analysis is the requirement of orthogonality between different energy states, which ultimately reflects the self-adjoint character of the Hamiltonian. This implies that for the Yukawa-like potentials we are dealing with, the only way to parametrize the unknown information at short distances is by allowing, besides the regular solution, the irregular one and fixing the appropriate combination by imposing a value of the scattering length as an independent renormalization condition. This may appear too restrictive, and in fact it is possible to renormalize using energy-dependent boundary conditions, a procedure essentially equivalent to imposing more renormalization conditions or counterterms. Although there are subtleties on how short distances should be parametrized in such way that the cutoff may be removed [35,42], the procedure in coordinate space turns out to be rather simple. In the case of two conditions, we would fix the scattering length α_0 and the effective range r_0 independently of the potential. The coordinate space procedure [35,42] consists of expanding the wave function in powers of the energy

$$u_p(r) = u_0(r) + p^2 u_2(r) + \cdots, \tag{44}$$

where $u_0(r)$ and $u_2(r)$ satisfy the equations

$$\begin{aligned} -u_0''(r) + MV(r)u_0(r) &= 0, \\ u_0(r) &\to 1 - r/\alpha_0, \end{aligned} \tag{45}$$

$$\begin{aligned} -u_2''(r) + MV(r)u_2(r) &= u_0(r), \\ u_2(r) &\to (r^3 - 3\alpha_0 r^2 + 3\alpha_0 r_0 r)/6\alpha_0. \end{aligned} \tag{46}$$

The asymptotic conditions correspond to fixing α_0 and r_0 as independent parameters (two counterterms). The matching condition at the boundary $r = r_c$ becomes energy dependent [35]

$$\frac{u_p'(r_c)}{u_p(r_c)} = \frac{u_0'(r_c) + p^2 u_2'(r_c) + \cdots}{u_0(r_c) + p^2 u_2(r_c) + \cdots}, \tag{47}$$

whence the corresponding phase shift may be deduced by integrating in Eqs. (45) and (46) and integrating out the finite-energy equation. It is worth mentioning that the energy-dependent matching condition, Eq. (47), is quite unique, since this is the only representation guaranteeing the existence of the limit $r_c \to 0$ for singular potentials [35]. In any case, if r_0 is fixed from the start to their experimental values in the singlet and triplet channels, the Wigner correlation given by Eq. (32) and generating the universal curve shown in Fig. 1 would not be predicted, and the symmetry between the ${}^1S_0$ and ${}^3S_1$ channels would be further hidden into the phase shifts. Note that the breaking of the symmetry with two counterterms is a short distance one when the cutoff is eliminated, $r_c \to 0$, since at any rate the potential is kept fixed and $V_s(r) = V_t(r)$ for any nonvanishing distance, $r \geq r_c > 0$. Thus, if we write

$$r_0 = A + \frac{B}{\alpha_0} + \frac{C}{\alpha_0^2} + r_0^{\mathrm{short}}, \tag{48}$$

with r_0^{short} the effect of the second counterterm, we would obtain

$$r_t - r_s \sim r_t^{\mathrm{short}} - r_s^{\mathrm{short}} + B\left[\frac{1}{\alpha_t} - \frac{1}{\alpha_s}\right] + \cdots, \tag{49}$$

where small $1/\alpha^2$ terms have been neglected. This yields $r_t^{\mathrm{short}} - r_s^{\mathrm{short}} \sim 0.1$ fm. Thus, while introducing no counterterm (trivial boundary condition) does not break the symmetry and yields identical phase shifts, $\delta_s(k) = \delta_t(k)$, introducing more than one counterterm (energy-dependent boundary condition) breaks the symmetry at the $\sim$10% level. As a consequence, we stick to the case of just one counterterm (energy-independent boundary condition).

B. Symmetry breaking due to tensor force

Of course, an interesting possibility that should be explored further is that of keeping the energy independence of the boundary condition and breaking the symmetry by introducing a long distance component of the potential, such as the tensor force, which would include the coupling of the $^3S_1$ wave treated here to the $^3D_1$ channel. Actually, this would correspond to take into account, as proposed in Ref. [17], the leading and complete large-N_c NN potential. In other words, while Wigner symmetry implies a vanishing tensor force, leading large-N_c does not necessarily require the tensor force to be small. To analyze this potential source of conflict, we consider the $^3S_1$ effective range parameter which incorporates a D-wave contribution stemming from S-D tensor force mixing and is given by

$$r_t = 2\int_0^\infty \left[\left(1-\frac{r}{\alpha_t}\right)^2 - u_{0,\alpha}(r)^2 - w_{0,\alpha}(r)^2\right] dr, \tag{50}$$

where the zero-energy S-wave function $u_{0,\alpha}(r) \to u_{0,{}^3S_1}(r)$ (discussed above) and the D-wave function $w_{0,\alpha}(r) \to 0$ when the tensor force is switched off, keeping α_t fixed. The corresponding tensor potential would include π and ρ exchange contributions characterized by the $g_{\pi NN}$ and $f_{\rho NN}$ couplings and diverges as $1/r^3$ at short distances. This situation resembles a previous OPE study [47], and a detailed account will be presented elsewhere [48]. There, it will be shown how the extension of the superposition principle and renormalization to the coupled-channel case yields in fact an identical analytical result as shown in Eq. (32) for the triplet (uncoupled) channel in the absence of tensor force. We will just quote here the numerical modification of the correlation relation coefficients for the triplet channel (the singlet $^1S_0$ is not modified), Eq. (32). Numerically, we obtain for $f_{\rho NN} = 17$ and $g_{\omega NN} = 9.86$,

$$r_t = 2.6199 - \frac{5.7843}{\alpha_t} + \frac{5.7608}{\alpha_t^2}, \tag{51}$$

which corresponds to a $\sim$10% breaking due to the tensor force. As we see, the coefficients in Eq. (33) are not modified much despite the singularity of the tensor force and its dominance at short distances. Actually, the dependence of the coefficients on the couplings responsible for the tensor force is moderate in a wide range. Therefore, while from the large-N_c viewpoint a large tensor force is not forbidden, we find the effect in the S wave to be numerically small, as implied by Wigner symmetry.

In this regard, it should be noted that a virtue of the renormalization approach is that, since the scattering lengths are always fixed, such a long distance symmetry breaking term only influences the region where the potential is resolved, and from this viewpoint the perturbation will be stable, i.e., the change will be small. Actually, in Ref. [47] a suitable form of perturbation theory in the tensor force was suggested based on the known smallness of the mixing angle ϵ_1, which stays below 2°–3°, in a wide energy range and is indeed smaller than the δ_β phase. It would be interesting to work out the consequences of such an approach when also ρ exchange is incorporated.

C. Symmetry breaking in noncentral waves

With the previous appealing interpretation of the Wigner symmetry as a long distance one for the S waves, we analyze the consequences for the phase shifts corresponding to partial waves at angular momentum larger than zero, $L > 0$. Unlike the S waves, we expect the dependence on the short distance behavior to be suppressed due to the centrifugal barrier, and the symmetry should become more evident. Note also that while a dissimilarity between phase shifts connected by the symmetry does not necessarily imply long distance symmetry breaking, an identity between phase shifts is a clear hint of the symmetry.

In the two-nucleon system, the Wigner symmetry implies the following relations for spin-isospin components of the antisymmetric sextet, $\mathbf{6}_A$, and the symmetric decuplet, $\mathbf{10}_S$, respectively (see Appendix A); thus we should have

$$\delta^{01}_{\rm LJ} = \delta^{10}_{\rm LJ} = \delta_L, \quad \text{even } L, \tag{52}$$

$$\delta^{00}_{\rm LJ} = \delta^{11}_{\rm LJ} = \delta_L, \quad \text{odd } L. \tag{53}$$

For P waves, for instance, we have the spin-singlet state $^1P_1$ and the spin triplets $^3P_0, {}^3P_1$, and $^3P_2$ which according to the symmetry should be degenerate, as they belong to the $\mathbf{10}_S$ supermultiplet. Inspection of the Nijmegen analysis [33] reveals that $^1P_1$ is very similar to $^3P_1$ at all energies, $|\delta_{^1P_1} - \delta_{^3P_1}| \sim 1^0$, but very different from the $^3P_0$ and $^3P_2$ phases. For D waves, associated to a $\mathbf{6}_A$ supermultiplet, we have a similarity between $^1D_2$ and $^3D_3$ phases $|\delta_{^1D_2} - \delta_{^3D_3}| \sim 1^0$ but, again, clear differences between the $^3D_1$ and $^3D_2$ ones. Clearly, the symmetry is broken in higher partial waves. In what follows, we want to determine whether our interpretation of a long distance symmetry that worked so successfully for S waves (see Sec. IV) holds also for noncentral phases.

It is well known that the spin-orbit interaction lifts the independence on the total angular momentum, via the operator $\vec{L}\cdot\vec{S}$. Moreover, the tensor coupling operator S_{12} mixes states with different orbital angular momentum. We proceed in first-order perturbation theory by using the Wigner symmetric distorted waves as the unperturbed states. In Appendix D we show this procedure explicitly. To first order in spin-orbit and tensor force perturbation, the following sum rule for the center of the $S = 1$ multiplet, denoted by δ^{10}_L and δ^{11}_L, and the $S = 0$ states, denoted as δ^{01}_L and δ^{00}_L, holds:

$$\begin{aligned}
\delta^{10}_L &\equiv \frac{\sum_{J=L-1}^{L+1}(2J+1)\delta^{10}_{\rm LJ}}{(2L+1)3} = \delta^{01}_{\rm LL} \equiv \delta^{01}_L,\\
\delta^{11}_{\rm LL} &\equiv \frac{\sum_{J=L-1}^{L+1}(2J+1)\delta^{11}_{\rm LJ}}{(2L+1)3} = \delta^{00}_{\rm LL} \equiv \delta^{00}_L,
\end{aligned} \tag{54}$$

In terms of these mean phases, Wigner symmetry is formulated for noncentral waves as

$$\delta_{^1P_1} = \tfrac{1}{9}\left(\delta_{^3P_0} + 3\delta_{^3P_1} + 5\delta_{^3P_2}\right), \tag{55}$$

$$\delta_{^1D_2} = \tfrac{1}{15}\left(3\delta_{^3D_1} + 5\delta_{^3D_2} + 7\delta_{^3D_3}\right), \tag{56}$$

$$\delta_{^1F_3} = \tfrac{1}{21}\left(5\delta_{^3F_2} + 7\delta_{^3F_3} + 9\delta_{^3F_4}\right), \tag{57}$$

$$\delta_{^1G_4} = \tfrac{1}{27}\left(7\delta_{^3G_3} + 9\delta_{^3G_4} + 11\delta_{^3G_5}\right). \tag{58}$$

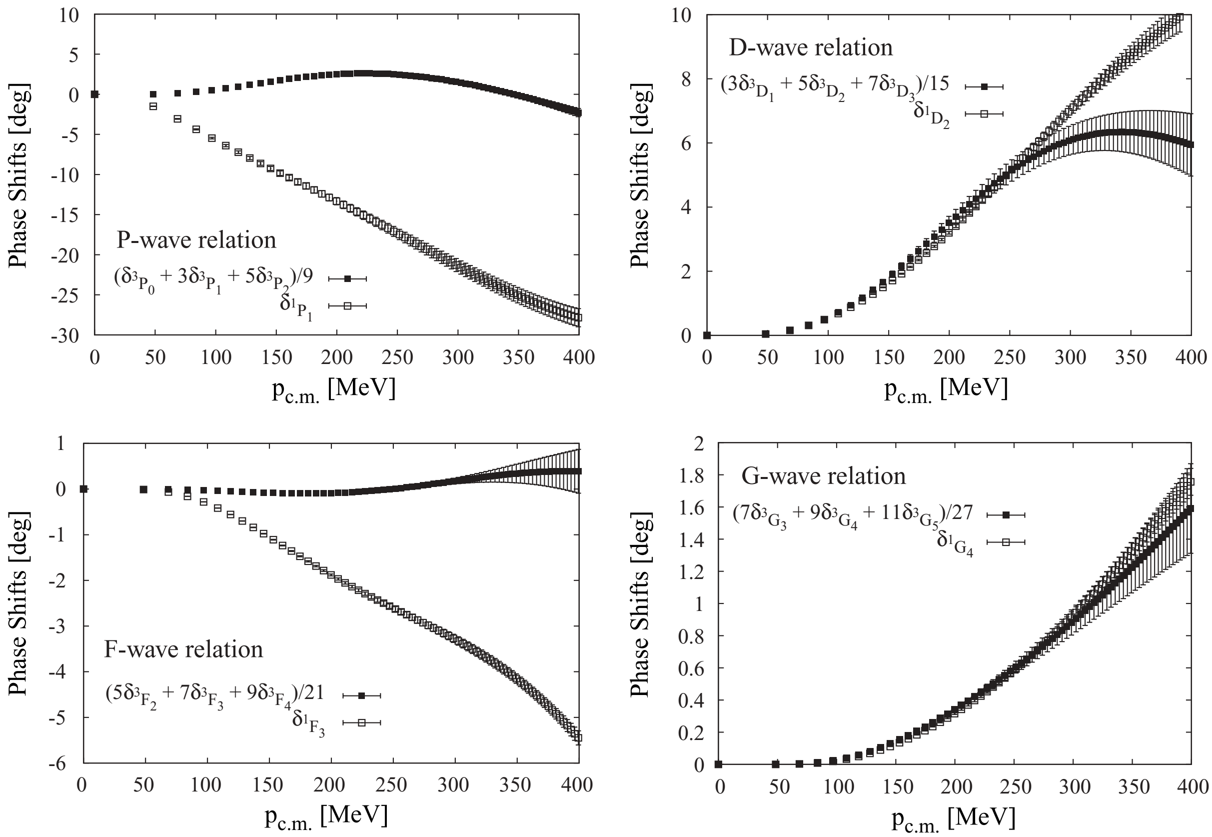


FIG. 6. Average values of the phase shifts [33] (in degrees) as a function of the c.m. momentum (in MeV) based on first-order spin-orbit coupling for P, D, F, and G waves. According to Wigner symmetry, $\delta_{^1L} = \delta_{^3L}$. Serber symmetry implies $\delta_{^3L} = 0$ for odd L. One sees that L-even waves satisfy Wigner symmetry while L-odd waves satisfy Serber symmetry.

These sum rules are true as long as the short distance breaking can be considered small, and for this reason we have not written down the sum rule for S waves. Furthermore, they hold also when the tensor force is added. In Fig. 6, we show the left- and right-hand sides of P, D, F, and G waves. As we see, the D waves fulfill this relation rather accurately up to $p \sim 250$ MeV and the G waves up to $p \sim 400$ MeV, while the P and F waves fail completely. Actually, at threshold, $\delta_L \to -\alpha_L p^{2L+1}$, and using the low energy parameters of the NijmII and Reid93 potentials [33] determined in Ref. [49], we get

$$
\begin{aligned}
\alpha_{^1P_1} &= \tfrac{1}{9}\left(\alpha_{^3P_0} + 3\alpha_{^3P_1} + 5\alpha_{^3P_2}\right),\\
(-2.46\ \text{fm}^3) &\quad (0.08\ \text{fm}^3),\\
\alpha_{^1D_2} &= \tfrac{1}{15}\left(3\alpha_{^3D_1} + 5\alpha_{^3D_2} + 7\alpha_{^3D_3}\right),\\
(-1.38\ \text{fm}^5) &\quad (-1.23\ \text{fm}^3),
\end{aligned} \tag{59}
$$

where the numerical values are displayed below the sum rules. In light of the previous discussions for the S waves, one reason for the discrepancy should be looked for in the short distance breaking of the symmetry for the D waves. Actually, the fact that D waves violate the sum rule at $p \sim 250$ MeV while the G waves show no violation up to $p \sim 400$ MeV agrees with our interpretation in the S waves that the Wigner symmetry is a long distance one, since higher partial waves are less sensitive to short distance effects. The case of P waves is different, since the ^1P potential and the ^3P potentials are very different. This pattern of symmetry breaking agrees with the findings of Ref. [17] based on the large-N_c expansion, where the central potential preserves the symmetry in L-even partial waves while it breaks the symmetry in the L-odd partial waves, since at leading order and neglecting the tensor force

$$V(r) = V_C(r) + \sigma\tau W_S(r) + \mathcal{O}(1/N_c), \tag{60}$$

so that for the lower L channels, we have

$$
\begin{aligned}
V_{^1S} &= V_{^3S} = V_C(r) - 3W_S(r) + \mathcal{O}(1/N_c),\\
V_{^1P} &= V_C(r) + 9W_S(r) + \mathcal{O}(1/N_c),\\
V_{^3P} &= V_C(r) + W_S(r) + \mathcal{O}(1/N_c),\\
V_{^1D} &= V_{^3D} = V_C(r) - 3W_S(r) + \mathcal{O}(1/N_c),
\end{aligned} \tag{61}
$$

so as we see, $V_{^3P} \neq V_{^1P}$, and thus it is obvious that $\delta_{^3P} \neq \delta_{^1P}$. One might check this further by proceeding as follows. In the case of odd waves such as the P waves, the proper comparison might be to take the ^3P potential and renormalize with the ^3P-mean scattering length, $\alpha_{^3P} = 0.08$ fm^3, and compare the result with the ^3P-mean phase shift.

We note that the initial claim of Ref. [16] on the validity of the Wigner symmetry based on the large-N_c expansion was restricted to purely center potentials, which do not faithfully distinguish the two irreducible representations, $\mathbf{10}_S$ and $\mathbf{6}_A$, of the SU(4) group for the NN system. Later on, the issue was qualified by a more complete study carried out in Ref. [17], which in fact could not justify the Wigner symmetry in odd-L partial waves, even when the tensor force was neglected. Although this appeared as a puzzling result, it is amazing to note that our calculations clearly show that the pattern of SU(4)-symmetry breaking supports a weak violation in even-L partial waves and a strong violation in the odd-L partial waves, *exactly* as the large-N_c expansion suggests.

D. Serber symmetry

On the other hand, from the odd waves, we see from Fig. 6 that the mean triplet phase is close to null, thus one might attribute this feature to an accidental symmetry in which the odd-wave potentials are likewise negligible. In the large-N_c limit, this means $V_C + W_S \gg V_C + 9W_S$, a fact which is well verified. For instance, at short distances, the Yukawa OBE potentials have Coulomb-like behavior $V \to C/(4\pi r)$ with the dimensionless combinations

$$\begin{aligned} C_{V_C+W_S} &= -g^2_{\sigma NN} + g^2_{\omega NN} + \frac{f^2_{\rho NN} m^2_\rho}{6M^2_N}, \\ C_{V_C+9W_S} &= -g^2_{\sigma NN} + g^2_{\omega NN} + \frac{3 f^2_{\rho NN} m^2_\rho}{2M^2_N}, \end{aligned} \tag{62}$$

where the small OPE contribution has been dropped. Numerically we get $C_{V_C+W_S} \sim 10$ and $C_{V_C+9W_S} \sim 300$ for reasonable choice of couplings. Although this approximate vanishing of triplet odd-wave potentials *is not* a consequence of large N_c, it is nevertheless reminiscent of the old and well-known Serber force [50], that is,

$$\begin{aligned} V_{\rm Serber}(r) = &\tfrac{1}{2}(1+P_M)\tfrac{1}{2}(1-P_\sigma)V_s(r) \\ &+ \tfrac{1}{2}(1-P_M)\tfrac{1}{2}(1+P_\sigma)V_t(r), \end{aligned} \tag{63}$$

with P_M the Majorana coordinate exchange operator. Because of the Pauli principle, $P_M P_\sigma P_\tau = -1$ with $P_\tau = (1+\tau)/2$ and $P_\sigma = (1+\sigma)/2$, the isospin and spin exchange yields vanishing potentials for spin-triplet and isospin-triplet channels and generates a scattering amplitude that is even in the c.m. scattering angle, a property which is approximately well fulfilled experimentally for proton-proton (pp) scattering. We call this property Serber symmetry for definiteness. After introducing spin-orbit coupling, we would get the sum rules to first order, i.e.,

$$\delta_{^3P} \equiv \tfrac{1}{9}\left(\delta_{^3P_0} + 3\delta_{^3P_1} + 5\delta_{^3P_2}\right) = 0, \tag{64}$$

$$\delta_{^3F} \equiv \tfrac{1}{21}\left(5\delta_{^3F_2} + 7\delta_{^3F_3} + 9\delta_{^3F_4}\right) = 0, \tag{65}$$

which is well fulfilled by the phase shifts [33] as shown in Fig. 6, where $\delta_{^3P} \ll \delta_{^1P}$ and $\delta_{^3F} \ll \delta_{^1F}$. In the large-N_c limit, we may comply with both Wigner symmetry in L-even waves and Serber symmetry in L-odd waves when $W_S(r) = -V_C(r)$, whence generally $V(r) = V_C(r)(1-\sigma\tau)$. Even if we neglect the small OPE effects, this will clearly not be exactly fulfilled unless one would require $m_\rho = m_\omega = m_\sigma$. Although there are schemes that explicitly verify such an identity between scalar and vector meson masses [51–53], at present, it is unclear whether the Serber symmetry which we observe in the NN system for spin-triplet and odd-L phase shifts could be formulated as a symmetry from the underlying QCD Lagrangian.

Our findings suggest that a pure large N_c in the absence of tensor force not only is compatible with the standard Wigner symmetry in the case of the dominant S waves and higher L-even channels, but also might be a competitive alternative for the L-odd waves where the usual Wigner symmetry is broken and Serber symmetry holds instead. Of course, it would be interesting to pursue the more complete situation including the tensor force from the start, a case which will be presented elsewhere [48].

E. NN level density in the continuum

Our results have some impact on hot nuclear matter at low densities. In the continuum, we may think of putting the two-nucleon system in a box and evaluating the corresponding level density when the infinite volume limit is taken. This is a standard problem in statistical mechanics which appears, e.g., in the calculation of the second virial coefficient contribution to the equation of state of a dilute quantum gas [54] (see Refs. [55,56] for recent applications to hot nuclear matter). The result is expressed as

$$\rho(E) = \frac{1}{2\pi i}{\rm Tr}\left[S(E)^\dagger \frac{dS(E)}{dE}\right] = \frac{1}{\pi}\frac{d\Delta_{NN}(E)}{dE}, \tag{66}$$

where $S(E)$ is the S matrix in all coupled channels, and the total phase Δ is defined by

$$\Delta_{NN}(E) = \sum_{S,T,J}(2J+1)(2T+1)\delta^{\rm ST}_{\rm LJ}(E). \tag{67}$$

For coupled channels, one should consider the corresponding eigenphases.[13] Defining the mean phase as

$$\delta^{\rm ST}_L(E) \equiv \frac{\sum_{J=L-S}^{L+S}(2J+1)\delta^{\rm ST}_{\rm LJ}(E)}{(2S+1)(2L+1)}, \tag{68}$$

corresponding to the phase-shift analog of the center of gravity of the supermultiplet [see also Eq. (54)], we get

$$\Delta_{NN}(E) = \sum_{S,T,J}(2S+1)(2L+1)(2T+1)\delta^{\rm ST}_L(E). \tag{69}$$

Thus, using the above relations, Eq. (58) for L-even waves and Eq. (65) for L-odd waves, featuring Wigner and Serber symmetries, respectively, we would get that mixed triplet-channel contributions either may be eliminated in terms of singlet ones for even-L or do not contribute for odd-L,

$$\Delta_{NN}(E) = 3\left(\delta_{^1S_0} + \delta_{^3S_1}\right) + 3\delta_{^1P_1} + 30\delta_{^1D_2} + \cdots. \tag{70}$$

For the neutron case, we have

$$\Delta_{nn}(E) = \delta_{^1S_0} + 5\delta_{^1D_2} + 9\delta_{^1G_4} + \cdots, \tag{71}$$

i.e., odd-L waves do not contribute. The lack of a P-wave contribution scaling as $\sim -\alpha_P p^3$ is compatible with the

[13] In the special case of NN scattering, one can also use the nuclear bar phase shifts due to the identity $\bar\delta_{^3(J-1)_J} + \bar\delta_{^3(J+1)_J} = \delta_{^3(J-1)_J} + \delta_{^3(J+1)_J}$. The concern spelled out in Ref. [55] that neglecting the mixing was an approximation is unjustified.

minimum observed in Ref. [55] for Δ_{nn} in the subthreshold region $E_{\mathrm{lab}} < 50$ MeV.

VI. CONCLUSIONS

At low energies, NN interactions are dominated by two S waves in different channels where spin-isospin (S, T) are interchanged: $(1, 0) \leftrightarrow (0, 1)$. Wigner SU(4) symmetry implies that the potentials in the $^1S_0$ and $^3S_1$ channels coincide and the tensor force vanishes, while the corresponding phase shifts from partial wave analyses are actually very different at all energies and show no evident trace of the identity of the potential, besides the qualitative fact that a weakly bound deuteron $^3S_1$ state and an almost bound virtual $^1S_0$ take place. Given that the nuclear force at short distances is fairly unknown, the validity of the symmetry to all distances would be at least questionable and could hardly be tested quantitatively. On the other hand, our lack of knowledge of the short distance physics should not be crucial at low energies, where the phase shifts are indeed quite dissimilar. Therefore, we propose to regard SU(4) as a medium and long distance symmetry which might be strongly broken at short distances and weakly broken at large distances. Using renormalization ideas in which the desirable short distance insensitivity is manifestly fulfilled, we have shown how the standard Wigner correlation between potentials indeed predicts one phase shift from the other in a nontrivial and successful way. Remarkably, using a large-N_c motivated one-boson exchange potential, we have proven that if one channel is described successfully, the other channel is unavoidably well reproduced within uncertainties which might be compatible with the disregard of the tensor force and the $1/N_c^2$ corrections to the potential. This long distance correlation holds also for the virtual singlet state and the deuteron bound state. Actually, the effects of symmetry breaking at long and short distances have been analyzed and the extension to higher partial waves has also been discussed, where a relation for phase shifts has been deduced.

Our calculations provide a justification for the use of Wigner symmetry in light nuclei solely on the basis of the NN interaction and suggest that a specific interpretation of the Wigner symmetry as a long distance one in conjunction with renormalization theory extends beyond the scaling region to a much wider range than assumed hitherto. It would be interesting to see how these ideas could be further exploited beyond the simple two-nucleon system.

However, key questions still remain: What is the origin of the accidental Wigner symmetry from the underlying fundamental QCD Lagrangian? And, moreover, under what conditions is this expected to be a useful symmetry? We find that not only is the large-N_c expansion in the absence of tensor force compatible with the standard Wigner symmetry in the case of the low energy dominant S waves and subdominant higher L-even partial waves, but it also may become a competitive alternative for the other L-odd partial waves where the usual Wigner symmetry is manifestly broken. These conclusions are remarkable, for they suggest that an unforeseen handle on the nature, applicability, and interpretation of a widely used approximate nuclear symmetry may be based on a QCD distinct pattern such as the large-N_c limit. Obviously, it would be very interesting to pursue further the study of the complete large-N_c potential with inclusion of the tensor force to verify this issue in more detail [48]. In our view, this would definitely provide useful insight into QCD inspired approximation schemes in nuclear physics.

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APPENDIX A: WIGNER SYMMETRY FOR NN

Wigner SU(4) spin-isospin symmetry consists of the following 15-generators [1–3]

$$T^a = \frac{1}{2}\sum_A \tau_A^a, \tag{A1}$$

$$S^i = \frac{1}{2}\sum_A \sigma_A^i, \tag{A2}$$

$$G^{ia} = \frac{1}{2}\sum_A \sigma_A^i \tau_A^a, \tag{A3}$$

where τ_A^a and σ_A^i are isospin and spin Pauli matrices for nucleon A, respectively, and T^a is the total isospin, S^i the total spin, and G^{ia} the Gamow-Teller transition operator. The quadratic Casimir operator reads

$$C_{\mathrm{SU}(4)} = T^a T_a + S^i S_i + G^{ia} G_{ia}, \tag{A4}$$

and a complete set of commuting operators can be taken to be $C_{\mathrm{SU}(4)}$, T_3 and S_z, G_{z3}. The fundamental representation has $C_{\mathrm{SU}(4)} = 4$ and corresponds to a single-nucleon state with a quartet of states $p\uparrow, p\downarrow, n\uparrow, n\downarrow$, with total spin $S = 1/2$ and isospin $T = 1/2$ represented as $\mathbf{4} = (S, T) = (1/2, 1/2)$. For two-nucleon states with good spin S and good isospin T, the Pauli principle requires $(-)^{S+T+L} = -1$ with L the angular momentum, thus

$$C_{\mathrm{SU}(4)}^{\mathrm{ST}} = \tfrac{1}{2}(\sigma + \tau + \sigma\tau) + \tfrac{15}{2}, \tag{A5}$$

where $\tau = \tau_1 \cdot \tau_2 = 2T(T+1) - 3$ and $\sigma = \sigma_1 \cdot \sigma_2 = 2S(S+1) - 3$, and the corresponding wave function is of the form

$$\Psi(\vec{x}) = \frac{u_L^{\mathrm{LS}}(r)}{r} Y_{LM_L}(\hat{x}) \chi^{SM_S} \chi^{TM_T}. \tag{A6}$$

One has two supermultiplets, whose Casimir values are

$$C_{\mathrm{SU}(4)}^{00} = C_{\mathrm{SU}(4)}^{11} = 9, \tag{A7}$$

$$C_{\mathrm{SU}(4)}^{01} = C_{\mathrm{SU}(4)}^{10} = 5, \tag{A8}$$

corresponding to an antisymmetric sextet $\mathbf{6}_A = (0,1) \oplus (1,0)$ when $L =$ even, and a symmetric decuplet $\mathbf{10}_S = (0,0) \oplus (1,1)$ when $L =$odd. The radial wave functions fulfill $u_L^{01}(r) = u_L^{10}(r)$ and $u_L^{00}(r) = u_L^{11}(r)$, respectively. This means that we have the following supermultiplets:

$$({}^1S_0, {}^3S_1)\,, ({}^1P_1, {}^3P_{0,1,2})\,, ({}^1D_2, {}^3D_{1,2,3}), \ldots \tag{A9}$$

When applied to the NN potential, the requirement of Wigner symmetry for *all* states implies

$$\begin{aligned} V_T &= W_T = V_{\rm LS} = W_{\rm LS} = 0, \\ W_S &= V_S = W_C, \end{aligned} \tag{A10}$$

so the potential may be written as

$$V = V_C + \left(2C^{\rm ST}_{\rm SU(4)} - 15\right) W_S. \tag{A11}$$

Note that the particular choice $W_S = 0$ corresponds to a spin-isospin independent potential, but in this case no distinction between the $\mathbf{6}_A$ and $\mathbf{10}_S$ supermultiplets arises. It is well known that the spin-orbit interaction lifts the total angular momentum independence. The Wigner symmetry does not distinguish between different total angular momentum values, so admitting that the potentials are different, we may define a common potential

$$V_{\rm LST}(r) \equiv \frac{\sum_{J=L-S}^{L+S} (2J+1) V_{\rm JST}(r)}{(2S+1)(2L+1)}, \tag{A12}$$

where similarly to the perturbation theory for energy levels where the center of a multiplet of states is predicted, the appropriate statistical weights related to the angular momentum have been used. The previous expression makes sense if the symmetry is broken linearly by spin-orbit coupling. In terms of these mean potentials, the symmetry would be

$$V_{{}^1L}(r) = V_{{}^3L}(r), \tag{A13}$$

or equivalently

$$V_{{}^1J_J}(r) = \frac{\sum_{J=L-1}^{L+1} (2J+1) V_{{}^3L_J}(r)}{3(2L+1)}. \tag{A14}$$

As mentioned in the paper, if the symmetry is taken literally at *all* distances, we should have $\delta_{{}^1L} = \delta_{{}^3L}$.

APPENDIX B: LONG DISTANCE PERTURBATION THEORY

We illustrate here a situation in which the potential may be treated in long distance perturbation theory and renormalized (for a somewhat similar approach for finite cutoffs, see, e.g., Ref. [57]). Unlike the standard perturbative approach, which usually does not hold in the presence of bound states, this expansion can deal with weakly bound states, provided this is the only one. This is in fact the case for the OPE potential for the parameters we use, applied to the deuteron state, for which we show the procedure here to first order. To analyze this situation, we vary the potential $V \to V + \Delta V$, so that

$$-\Delta u_k(r)'' + M \Delta V(r) u_k(r) + M V(r) \Delta u_k(r) = k^2 \Delta u_k(r), \tag{B1}$$

we use the previous wave functions $u_k(r)$ as the zeroth-order approximation, corresponding to taking $V(r) = 0$, and we solve for the first-order correction $\Delta u_k(r)$, the equation in which the asymptotic wave function corresponds to taking the phase shift $\delta + \Delta\delta$. Multiplying Eq. (5) by $\Delta u_k(r)$ and Eq. (B1) by $u_k(r)$, subtracting both equations, and integrating from r_c to ∞, we get

$$[-u_k' \Delta u_k + u_k \Delta u_k]\big|_{r_c}^{\infty} = \int_{r_c}^{\infty} dr \, \Delta U(r) u_k(r)^2. \tag{B2}$$

The lower limit term may be related to the variation of the boundary condition, whereas the upper limit term is related to the change in the phase shift $\Delta\delta$. To eliminate the cutoff, we subtract the zero-energy limit, $k \to 0$, and using the energy independence of the boundary condition, we get some cancellation since

$$\Delta \left(\frac{u_k'(r_c)}{u_k(r_c)} - \frac{u_0'(r_c)}{u_0(r_c)} \right) = 0. \tag{B3}$$

Finally, the result may be rewritten as

$$\Delta\,(k \cot\delta) = -\Delta \left(\frac{1}{\alpha_0} \right) + \int_{r_c}^{\infty} \Delta U(r) [u_k(r)^2 - u_0(r)^2]\, dr. \tag{B4}$$

If we fix the scattering length independently of the potential, we have $\Delta\alpha_0 = 0$, thus eliminating the first term of the right-hand side; and after taking the limit $r_c \to 0$, the result for the total (and renormalized) phase shift to first order in the potential reads

$$\begin{aligned} k \cot\delta_0(k) = -\frac{1}{\alpha_0} + \int_0^{\infty} dr \; M V(r) \\ \times \left(\left[\cos(kr) - \frac{\sin(kr)}{\alpha_0 k} \right]^2 \right. \\ \left. - \left[1 - \frac{r}{\alpha_0} \right]^2 \right) + \cdots. \end{aligned} \tag{B5}$$

The renormalized effective range is *entirely predicted* from the potential at all distances

$$r_0 = 4 \int_0^{\infty} dr \; r^2 M V(r) \left(1 - \frac{r}{\alpha_0} \right)^2 + \cdots. \tag{B6}$$

Note the extra power suppression at the origin when α_0 is fixed independently of the potential, indicating short distances become *less* important. The bound state can be obtained in a similar manner by replacing $u_k(r) \to u_\gamma(r)$, assuming that the binding energy is independent of the potential, $\Delta\gamma = 0$, and using orthogonality Eq. (B3) to the zero-energy state:

$$\frac{1}{\alpha_0} = \gamma + \int_0^{\infty} M V(r) [u_\gamma(r)^2 - u_0(r)^2]\, dr. \tag{B7}$$

This equation is implicit in both α_0 and γ, but we can make it perturbative explicitly, using that to first order $\alpha_0 \sim 1/\gamma$ in the zero-energy wave function $u_0(r) \sim 1 - \gamma r$, yielding

$$\frac{1}{\alpha_0} = \gamma + \int_0^{\infty} M V(r) [e^{-2\gamma r} - (1 - \gamma r)^2]\, dr. \tag{B8}$$

APPENDIX C: SCALE INVARIANCE AND RENORMALIZATION

We have suggested that Wigner symmetry is a long distance one. From a renormalization group (RG) viewpoint, this has a simple interpretation (for a discussion in coordinate space, see, e.g., Refs. [35,45]). It means finding a solution to the RG equations which break the symmetry of the equations. A very simple case illustrating this issue is provided by the problem

$$-u''(r)+\frac{g}{r^2}u(r)=k^2u(r). \tag{C1}$$

At zero energy, $k=0$, the solution is invariant under the scaling transformation $r\to\lambda r$. This property holds also at short distances, where the energy term on the right-hand side can be neglected. If we use the RG equation (35) for this particular case at short distances,

$$Rc_0'(R)=c_0(R)(1-c_0(R))+g. \tag{C2}$$

The scale symmetry becomes now evident: if $c_0(R)$ is a solution, then $c_0(\lambda R)$ is also a solution for *any* value of $\lambda\neq 0$. The solution must necessarily specify the value at a given scale $c_0(R_0)$, hence breaking explicitly the dilatation symmetry. This symmetry breaking is unavoidable. In Refs. [35,45] it is shown how, for $g<-1/4$, the breaking is lowered to the discrete subgroup of dilatations, and the connection to Russian doll renormalization. In the case of the Wigner symmetry for the ${}^1S_0$ and ${}^3S_0$ potentials discussed in the paper, the breaking is not unavoidable, and there exists in fact a very special choice where the symmetry can be preserved by taking identical boundary conditions at a given scale. Besides this particular solution, the identity between solutions $c_{0,s}(R)$ and $c_{0,t}(R)$ will generally be violated, although the relation from one scale to a different one $c_{0,s}(R_0)\to c_{0,s}(R)$ and $c_{0,t}(R_0)\to c_{0,t}(R)$ is governed by the *same* relation, Eq. (35).

It is worth noting the resemblance of the previous quantum-mechanical discussion with similar and well-known field theoretical concepts. The unavoidable breaking of the dilatation symmetry corresponds to an anomaly of the dilatation current. The optional choice of boundary conditions corresponds to the case of finite but ambiguous theories (see, e.g., Ref. [41]).

APPENDIX D: SPLITTING FORMULA FOR PHASE SHIFTS

We want to derive the splitting formula for phase shifts, Eq. (54) by using distorted wave perturbation theory. The coupled-channel Schrödinger equation for the relative motion reads

$$-\mathbf{u}''(r)+\left[\mathbf{U}(r)+\frac{\mathbf{L}^2}{r^2}\right]\mathbf{u}(r)=k^2\mathbf{u}(r), \tag{D1}$$

where $\mathbf{U}^{SJ}_{L,L'}(r)=2\mu_{np}\mathbf{V}^{SJ}_{L,L'}(r)$ is the coupled-channel matrix potential which for the total angular momentum $J>0$ can be written as

$$\mathbf{U}^{0J}(r)=U^{0J}_{JJ},\qquad \mathbf{U}^{1J}(r)=\begin{pmatrix}U^{1J}_{J-1,J-1}(r) & 0 & U^{1J}_{J-1,J+1}(r)\\ 0 & U^{1J}_{JJ}(r) & 0\\ U^{1J}_{J-1,J+1}(r) & 0 & U^{1J}_{J+1,J+1}(r)\end{pmatrix}. \tag{D2}$$

In Eq. (D1), $\mathbf{L}^2=\mathrm{diag}[L_1(L_1+1),\ldots,L_N(L_N+1)]$ is the angular momentum, $\mathbf{u}(r)$ is the reduced matrix wave function, and k the c.m. momentum. In the case at hand, $N=1$ for the spin-singlet channel with $L=J$, and $N=3$ for the spin-triplet channel with $L_1=J-1, L_2=J$, and $L_3=J+1$. For ease of notation, we will keep the compact matrix notation of Eq. (D1). At long distances, we assume the asymptotic normalization condition

$$\mathbf{u}(r)\to\hat{\mathbf{h}}^{(-)}(r)-\hat{\mathbf{h}}^{(+)}(r)\mathbf{S}, \tag{D3}$$

with $\mathbf{S}$ the standard coupled-channel unitary S matrix. For the spin-singlet state $S=0$, one has $L=J$ and hence the state is uncoupled, i.e.,

$$S^{0J}_{JJ}=e^{2i\delta^{0J}_J}, \tag{D4}$$

whereas for the spin-triplet state $S=1$, one has the uncoupled $L=J$ state

$$S^{1J}_{JJ}=e^{2i\delta^{1J}_J}, \tag{D5}$$

and the two channel coupled states $L,L'=j\pm 1$, which written in terms of the eigenphases are

$$S^{1J}=\begin{pmatrix}\cos\epsilon_J & -\sin\epsilon_J\\ \sin\epsilon_J & \cos\epsilon_J\end{pmatrix}\begin{pmatrix}e^{2i\delta^{1J}_{J-1}} & 0\\ 0 & e^{2i\delta^{1J}_{J+1}}\end{pmatrix}\times\begin{pmatrix}\cos\epsilon_J & \sin\epsilon_J\\ -\sin\epsilon_J & \cos\epsilon_J\end{pmatrix}. \tag{D6}$$

The corresponding out-going and in-going free spherical waves are given by

$$\hat{\mathbf{h}}^{(\pm)}(r)=\mathrm{diag}\big(\hat{h}^{\pm}_{L_1}(kr),\ldots,\hat{h}^{\pm}_{L_N}(kr)\big), \tag{D7}$$

with $\hat{h}^{\pm}_L(x)$ the reduced Hankel functions of order l, $\hat{h}^{\pm}_L(x)=xH^{\pm}_{L+1/2}(x)$ ($\hat{h}^{\pm}_0=e^{\pm ix}$), and satisfy the free Schrödinger equation for a free particle.

To determine the infinitesimal change of the S matrix, $\mathbf{S}\to\mathbf{S}+\Delta\mathbf{S}$, under a general deformation of the potential $\mathbf{U}(r)\to\mathbf{U}(r)+\Delta\mathbf{U}(r)$, we use Schrödinger's equation (D1) and the standard Lagrange identity adapted to this particular case to obtain

$$\left[\mathbf{u}(r)^\dagger\Delta\mathbf{u}'(r)-\mathbf{u}'(r)^\dagger\Delta\mathbf{u}(r)\right]'=\mathbf{u}(r)^\dagger\Delta\mathbf{U}(r)\mathbf{u}(r). \tag{D8}$$

The unitarity of the S matrix, $\mathbf{S}^\dagger\mathbf{S}=\mathbf{1}$, yields the condition $\Delta\mathbf{S}^\dagger\mathbf{S}+\mathbf{S}^\dagger\Delta\mathbf{S}=0$. We assume a mixed boundary condition at short distances, $r=r_c$, for the unperturbed coupled-channel potential, $\mathbf{U}(r)$,

$$\mathbf{u}'(r_c)+\mathbf{L}\mathbf{u}(r_c)=0, \tag{D9}$$

with $\mathbf{L}$ a self-adjoint matrix. After integration from the cutoff radius r_c to infinity and using the asymptotic form of the matrix wave function [Eq. (D3)], as well as the condition at the origin, Eq. (D9) yields

$$2ik\mathbf{S}^\dagger\Delta\mathbf{S} = \int_{r_c}^{\infty} dr\ \mathbf{u}(r)^\dagger\Delta\mathbf{U}(r)\mathbf{u}(r). \tag{D10}$$

If we take the Wigner symmetric states as the unperturbed problem, then $\mathbf{S}$, $\mathbf{U}(r)$, and $\mathbf{u}(r)$ become diagonal matrices, so that

$$\Delta\delta_{\rm JL}^{\rm ST} = -\frac{1}{2p}\int_{r_c}^{\infty} dr\ u_L^{\rm ST}(r)^\dagger\Delta\mathbf{U}(r)u_L^{\rm ST}(r), \tag{D11}$$

and the perturbed eigenphases become

$$\delta_{\rm JL}^{\rm ST} = \delta_L^{\rm ST} + \Delta\delta_{\rm JL}^{\rm ST}. \tag{D12}$$

Note that to this order the mixing phases vanish, $\Delta\epsilon_J = 0$. Identifying further $\Delta\mathbf{U}$ with the spin-orbit and the tensor potential, in the LS coupling the result may be written as

$$\begin{aligned}\delta_{\rm LJ}^{\rm ST} = \delta_L^{\rm ST} + \delta_{S,1}C_L^{\rm ST}\left(S_{12}^J\right)_{\rm LL} \\ + A_L^{\rm ST}[J(J+1) - L(L+1) - S(S+1)],\end{aligned} \tag{D13}$$

where $(S_{12}^J)_{J-1,J-1} = -2(J-1)/(2J+1)$, $(S_{12}^J)_{J,J} = 2$, and $(S_{12}^J)_{J+1,J+1} = -2(J+2)/(2J+1)$. Defining the supermultiplet coefficients $A_L = A_L^{10} = A_L^{01}$ and $B_L = A_L^{00} = A_L^{11}$,

$$\begin{aligned}\delta_{\rm LJ}^{10} = \delta_{\rm LJ}^{01} + A_L\,[J(J+1) - L(L+1) - 2] \\ + C_L\left(S_{12}^J\right)_{\rm LL},\end{aligned} \tag{D14}$$

$$\begin{aligned}\delta_{\rm LJ}^{11} = \delta_{\rm LJ}^{00} + B_L\,[J(J+1) - L(L+1) - 2] \\ + D_L\left(S_{12}^J\right)_{\rm LL},\end{aligned} \tag{D15}$$

we readily get the sum rule for phase shifts, Eq. (54). The above equations would yield a Lande-like interval rule between spin-triplet energy levels for the spin-orbit or the tensor potentials separately. For instance,

$$\begin{aligned}\delta_{^1P_1} &= \delta_P, \\ \delta_{^3P_0} &= \delta_P - 4D_1 - 4B_1, \\ \delta_{^3P_1} &= \delta_P + 2D_1 - 2B_1, \\ \delta_{^3P_2} &= \delta_P - \tfrac{2}{5}D_1 + 2B_1.\end{aligned} \tag{D16}$$

A further remark is in order, since the spin-orbit or tensor potentials may be singular at the origin. In such a case of singular perturbations, one computes the sum rule first and then removes the cutoff, $r_c \to 0$.

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