

Partial dynamical symmetries in the $j = 9/2$ shell: Progress and puzzles

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We present new results concerning the properties of solvable states of four particles in a $j = 9/2$ shell which have seniority $v = 4$ and angular momentum $I = 4$ or 6 . In particular, based on properties of certain coefficients of fractional parentage, we show that the number of pairs coupled to angular momentum I is equal to one for these states. Similar results can be derived entirely analytically for four particles in a $j = 7/2$ shell.

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I. STATEMENT OF THE PROBLEM

It is known since long that a rotationally-invariant Hamiltonian describing identical fermions in a single- j shell has eigenstates with good seniority for any interaction between the particles as long as $j \leq 7/2$. Proofs of this statement can be found in the books of de-Shalit and Talmi [1] and Talmi [2]. This property of seniority conservation is no longer valid for all eigenstates in a $j = 9/2$ shell but it turns out that some selected eigenstates have good seniority for an arbitrary interaction. More specifically, it was noted in Refs. [3,4] that with the techniques used to calculate coefficients of fractional parentage (CFPs) in Ref. [5] (i.e., the diagonalization of the pairing interaction) the two $v = 4$ states with total angular momentum I (denoted, e.g., as $|j^4, 1, v = 4, I\rangle$ and $|j^4, 2, v = 4, I\rangle$) have the same energy, that is, they are degenerate. Hence, any linear combination of the two which we may define as

$$|j^4, a, v = 4, I\rangle = \alpha|j^4, 1, v = 4, I\rangle + \beta|j^4, 2, v = 4, I\rangle, \quad (1)$$

$$|j^4, b, v = 4, I\rangle = -\beta|j^4, 1, v = 4, I\rangle + \alpha|j^4, 2, v = 4, I\rangle,$$

would be equally valid as eigenstates. If, instead of a pairing interaction, one diagonalizes an arbitrary but seniority-conserving interaction (e.g., a δ interaction) then the two states become well defined. Both states are eigenstates of any interaction which conserves seniority. In addition, one of the linear combinations, say $|j^4, a, v = 4, I\rangle$, has the interesting property that it cannot mix with the $v = 2$ state even if an interaction is used that does not conserve seniority. Hence this state satisfies the property

$$M_{2a} \equiv \langle j^4, v = 2, I | \hat{V} | j^4, a, v = 4, I \rangle = 0, \quad (2)$$

for an arbitrary interaction \hat{V} in the $j = 9/2$ shell. For notational convenience the states $|j^4, i, v = 4, I\rangle$ henceforth shall be denoted in short as $|j^4 v_i I\rangle$, $i = a, b$.

The property (2) was proven numerically in Ref. [3] for four particles in a $j = 9/2$ shell coupled to total angular momentum $I = 4$ or $I = 6$. Subsequently, the states in question were shown to be solvable (i.e., to have a simple closed energy expression) in Ref. [8] by means of a symbolic computation in Mathematica. The purpose of this paper is to provide analytic proofs of some of these results based on generic properties of CFPs discussed in the next section. Parts of these proofs

were already given by one of us [4] but are repeated here for completeness.

II. RELATIONS BETWEEN COEFFICIENTS OF FRACTIONAL PARENTAGE

Our analytic proofs are based on known special properties of CFPs which are repeated here for completeness. First, we note the following relation between v -to- $(v + 1)$ -particle and $(v + 1)$ -to- $(v + 2)$ -particle CFPs:

$$\begin{aligned} & [j^{v+1}(v+1, \alpha_1 J_1) j J] j^{v+2} v \alpha J \\ &= (-1)^{J+j-J_1} \sqrt{\frac{2(2J_1+1)(v+1)}{(2J+1)(v+2)(2j+1-2v)}} \\ & \times [j^v(v\alpha J) j J_1] j^{v+1} v + 1, \alpha_1 J_1]. \end{aligned} \quad (3)$$

This relation has been derived in the books of de-Shalit and Talmi [1] and Talmi [2]; for example, see Eq. (19.31) of Ref. [2].

Second, we will use the equivalent of the Redmond recursion relation [6] but for CFPs classified by the seniority quantum number v and for which there are no redundancies. This modified relation reads [7]

$$\begin{aligned} & (n+1) \sum_{v_s} [j^n(v_0 J_0) j I_s] j^{n+1} v_s I_s [j^n(v_1 J_1) j I_s] j^{n+1} v_s I_s \\ &= \delta_{v_0, v_1} \delta_{J_0, J_1} + n(-1)^{J_0+J_1} \sqrt{(2J_0+1)(2J_1+1)} \\ & \times \sum_{v_2 J_2} \left\{ \begin{matrix} J_2 & j & J_1 \\ I_s & j & J_0 \end{matrix} \right\} [j^{n-1}(v_2 J_2) j J_0] j^n v_0 J_0 \\ & \times [j^{n-1}(v_2 J_2) j J_1] j^n v_1 J_1. \end{aligned} \quad (4)$$

Note that the sum on the left-hand side of this identity runs over all seniorities v_s but that the total angular momentum I_s is fixed.

III. THE $j = 9/2$ SHELL

A. Analytic proofs

As noted above, for four identical particles in a $j = 9/2$ shell there are two $v = 4$ states with $I = 4$ or $I = 6$. They can be written in terms of three-particle states in the usual way

with three-to-four-particle CFPs,

$$|j^4 v_i I\rangle = \sum_{v_3 J_3} [j^3(v_3 J_3) j I] |j^4 v_i I\rangle |j^3 v_3 J_3, j; I\rangle, \quad (5)$$

where the state on the right-hand side results from the coupling of the angular momentum J_3 of the first three particles with the last particle's angular momentum j to total angular momentum I . For $v_i = 4$ the intermediate seniority of the first three particles necessarily must be $v_3 = 3$. We now focus on the intermediate state with $J_3 = j$. Given the expansion (1), the following relation holds:

$$\begin{aligned} [j^3(v=3, J=j) j I] |j^4 v_a I\rangle \\ = \alpha [j^3(v=3, J=j) j I] |j^4 v_1 I\rangle \\ + \beta [j^3(v=3, J=j) j I] |j^4 v_2 I\rangle. \end{aligned} \quad (6)$$

We can always choose the coefficients α and β such that the CFP on the left-hand side vanishes. In other words, we *define* the special states $|j^4 v_a I\rangle$ ($I = 4, 6$) such that

$$[j^3(v=3, J=j) j I] |j^4 v_a I\rangle = 0. \quad (7)$$

Furthermore, using the proportionality relationship (3), we can deduce the following property:

$$[j^4(v_a I) j J] |j^5, v=3, J=j\rangle = 0. \quad (8)$$

The result (8) will be crucial in the proof of the property (2) to which we now turn.

In order to prove that the matrix element M_{2a} of Eq. (2) vanishes for any interaction, we must show that

$$M_{2a}(\lambda) = 0, \quad \text{for } \lambda = 0, 2, 4, 6, 8, \quad (9)$$

where $M_{2a}(\lambda)$ is the matrix element for a single component \hat{V}_λ of the interaction defined as

$$\hat{V} = \sum_{\lambda} v_{\lambda} \hat{V}_{\lambda}, \quad v_{\lambda} \equiv \langle j^2; \lambda | \hat{V} | j^2; \lambda \rangle. \quad (10)$$

We obtain an expression for $M_{2a}(\lambda)$ in two steps. First, we eliminate one of the four particles and get an expression in terms of three-particle matrix elements:

$$\begin{aligned} M_{2a}(\lambda) = 2 \sum_{v_3 v'_3 J_3} [j^3(v_3 J_3) j I] |j^4, v=2, I\rangle \\ \times [j^3(v'_3 J_3) j I] |j^4 v_a I\rangle \langle j^3 v_3 J_3 | \hat{V}_{\lambda} | j^3 v'_3 J_3 \rangle. \end{aligned} \quad (11)$$

The second CFP in the sum vanishes for $J_3 = j$ by construction and only the terms with $v_3 = v'_3 = 3$, $J_3 \neq j$ survive. Therefore, the summation may henceforth be considered as unrestricted in $v_3 = v'_3$ and J_3 . The three-particle matrix element in turn can be expressed in terms of a two-to-three-particle CFP,

$$\langle j^3 v_3 J_3 | \hat{V}_{\lambda} | j^3 v_3 J_3 \rangle = 3 [j^2(\lambda) j J_3] |j^3 v_3 J_3|^2, \quad (12)$$

which is obtained in closed form from the relation (4) for $n = 2$,

$$[j^2(\lambda) j J_3] |j^3 v_3 J_3|^2 = \frac{1}{3} + \frac{2}{3} (2\lambda + 1) \left\{ \begin{matrix} J_3 & j & \lambda \\ & j & j & \lambda \end{matrix} \right\}. \quad (13)$$

Putting everything together we obtain for the λ component of the interaction

$$\begin{aligned} M_{2a}(\lambda) = 6 \sum_{v_3 J_3} [j^3(v_3 J_3) j I] |j^4, v=2, I\rangle \\ \times [j^3(v_3 J_3) j I] |j^4 v_a I\rangle \\ \times \left[\frac{1}{3} + \frac{2}{3} (2\lambda + 1) \left\{ \begin{matrix} J_3 & j & \lambda \\ & j & j & \lambda \end{matrix} \right\} \right]. \end{aligned} \quad (14)$$

The first " $\frac{1}{3}$ " term in the square brackets vanishes because of orthogonality of the CFPs [1,2]. The second " $\frac{2}{3}$ " term in the square brackets can be evaluated for $\lambda = I$ by use of the Redmond relation (4) for $n = 4$ which gives

$$\begin{aligned} M_{2a}(\lambda = I) \equiv \langle j^4, v=2, I | \hat{V}_{\lambda=I} | j^4 v_a I \rangle \\ = 5 \sum_{v_s} [j^4(v=2, I) j I_s] |j^5 v_s, I_s = j\rangle \\ \times [j^4(v_a I) j I_s] |j^5 v_s, I_s = j\rangle = 0. \end{aligned} \quad (15)$$

The sum vanishes because the only term that contributes has $v_s = 3$ for which the second CFP is zero according to the property (8).

We can use this fact to prove the result (2), that is, that the mixing matrix M_{2a} vanishes for a general interaction in the $j = 9/2$ shell. Recall that for identical particles there are five two-body interaction matrix elements in this shell corresponding to $\lambda = 0, 2, 4, 6$, and 8. Furthermore, there are four independent seniority-conserving two-body operators and any linear combination of them with arbitrary coefficients conserves seniority. (The number of seniority-violating interactions is one less than the number of $J = j$ states for three identical particles [1,2] and there are two three-particle states with $J = 9/2$ for $j = 9/2$.) Four independent seniority-conserving interactions are, for example, a constant interaction, a pairing interaction, the two-body part of the \hat{J}^2 operator, and the δ interaction. But we have just found a seniority-violating interaction which does not admix $|j^4, v=2, I\rangle$ and $|j^4 v_a I\rangle$, namely the interaction $\hat{V}_{\lambda=I}$. Hence, we can express the five two-body interaction matrix elements in terms of these five interactions which will not admix the $|j^4, v=2, I\rangle$ and $|j^4 v_a I\rangle$ states. Indeed all $M_{2a}(\lambda)$ vanish.

B. Further proofs

Let us consider the matrix element between the seniority-four state $|j^4 v_a I\rangle$ and the state $|j^4 v_b I\rangle$ that is orthogonal to it. For this matrix element we obtain the expression

$$\begin{aligned} M_{ab}(\lambda = I) \equiv \langle j^4 v_a I | \hat{V}_{\lambda=I} | j^4 v_b I \rangle \\ = 5 \sum_{v_s} [j^4(v_a I) j I_s] |j^5 v_s, I_s = j\rangle \\ \times [j^4(v_b I) j I_s] |j^5 v_s, I_s = j\rangle. \end{aligned} \quad (16)$$

Although we do not have an analytic proof that this matrix element is zero, we can examine the consequences of its vanishing. The seniority quantum number v_s in the sum assumes the values 1, 3, and 5. For $v_s = 1$ the CFP in Eq. (16) is zero because one cannot obtain $v = 1$ from four particles

with $v = 4$ coupled to a single particle. That the first CFP in the sum (16) vanishes for $v_s = 3$ was shown in Eq. (8). Hence, in order for $M_{ab}(\lambda = I) = 0$ to be true, we must have that either the CFP of the state a or that of the state b vanishes. This can be verified by consulting tables of CFPs [1,5], putting in the appropriate coefficients α and β to make the CFP (8) vanish, and subsequently checking that with the same α and β the CFP with $v = 5$ vanishes as well. In other words, we note from the different tables for $j = 9/2$ that the following property holds for the states $|j^4, 1, v = 4, I\rangle$ and $|j^4, 2, v = 4, I\rangle$ (where 1 and 2 to refer to an arbitrary labeling):

$$\frac{[j^4(1, v = 4, I)jJ|j^5, v = 3, J = j]}{[j^4(2, v = 4, I)jJ|j^5, v = 3, J = j]} = \frac{[j^4(1, v = 4, I)jJ|j^5, v = 5, J = j]}{[j^4(2, v = 4, I)jJ|j^5, v = 5, J = j]}, \quad (17)$$

but we have no analytical proof of it. We note in passing that, because of the relation (3), this is equivalent to the property

$$\frac{[j^4(1, v = 4, I)jJ|j^5, v = 5, J = j]}{[j^4(2, v = 4, I)jJ|j^5, v = 5, J = j]} = \frac{[j^3(v = 3, J = j)jI|j^4, 1, v = 4, I]}{[j^3(v = 3, J = j)jI|j^4, 2, v = 4, I]}. \quad (18)$$

We conclude that $\langle j^4 v_a I | \hat{V}_\lambda | j^4 v_b I \rangle = 0$ for $\lambda = I$, but we note that this has yet to be shown for a general interaction $\lambda \neq I$. It has been shown empirically in Ref. [3] that the special state $|j^4 v_a I\rangle$ is an eigenstate for any interaction—seniority conserving or not. This means that there is no coupling of this state with the other $v = 4$ state via any interaction but as yet we have no analytic proof of this property.

Next we consider the energy of the $|j^4 v_a I\rangle$ states. In Ref. [8] closed energy expressions were obtained with use of MATHEMATICA,

$$\begin{aligned} E[(9/2)^4 v_a, I = 4] &= \frac{68}{33} v_2 + v_4 + \frac{13}{15} v_6 + \frac{114}{55} v_8, \\ E[(9/2)^4 v_a, I = 6] &= \frac{19}{11} v_2 + \frac{12}{13} v_4 + v_6 + \frac{336}{143} v_8. \end{aligned} \quad (19)$$

With minor modifications of what has been done up to now, we can explain why the coefficient of $v_{\lambda=I}$ is one. This in fact means that the number of pairs with angular momentum I is equal to one for these states. At this point we keep the discussion for general j and I and assume that an analytic expression is available for $E[j^4 v_a I]$ which is linear in the two-body matrix elements v_λ ,

$$E[j^4 v_a I] = \sum_\lambda x_\lambda v_\lambda, \quad (20)$$

where the coefficients x_λ depend implicitly on j and I . The application of this relation for each component \hat{V}_λ of the interaction leads to the identity $x_\lambda = \langle j^4 v_a I | \hat{V}_\lambda | j^4 v_a I \rangle$. Via an argument analogous to the preceding discussion this matrix

element can be written as

$$\begin{aligned} x_\lambda &= 6 \sum_{v_3 J_3} [j^3(v_3 J_3)jI|j^4 v_a I]^2 \\ &\times \left[\frac{1}{3} + \frac{2}{3}(2\lambda + 1) \left\{ \begin{matrix} J_3 & j & \lambda \\ j & j & \lambda \end{matrix} \right\} \right]. \end{aligned} \quad (21)$$

For $\lambda = I$ the sum can be carried out:

$$\begin{aligned} x_{\lambda=I} &= 2 + 4(2\lambda + 1) \sum_{v_3 J_3} [j^3(v_3 J_3)jI|j^4 v_a I]^2 \left\{ \begin{matrix} J_3 & j & I \\ j & j & I \end{matrix} \right\} \\ &= 2 - 1 + 5 \sum_{v_s} [j^4(v_a I)jI_s|j^5 v_s, I_s = j]^2, \end{aligned} \quad (22)$$

where use has been made of the normalization property of the CFPs and the Redmond relation (4) for $n = 4$, in the first and second step, respectively. With similar arguments as before one can show that the sum in Eq. (22) vanishes, leading to the final result

$$x_{\lambda=I} = \langle j^4 v_a I | \hat{V}_{\lambda=I} | j^4 v_a I \rangle = 1. \quad (23)$$

In summary of this subsection, we have three results: $\langle j^4(v_a I)jJ|j^5, v = 5, J = j\rangle = 0$, $\langle j^4 v_a I | \hat{V}_{\lambda=I} | j^4 v_b I \rangle = 0$, and $\langle j^4 v_a I | \hat{V}_{\lambda=I} | j^4 v_a I \rangle = 1$, and any two of them follow from the third.

IV. THE $j = 7/2$ SHELL

These properties can be used to find the energy expressions of the solvable states. To illustrate the procedure, we consider four identical particles in the $j = 7/2$ shell in which case there is at most a single $v = 4$ state for a given total angular momentum I which again we denote as $|j^4 v_a I\rangle$. The results derived above for the $j = 9/2$ shell are equally valid for $j = 7/2$. The property (15) is trivial since any interaction is diagonal in seniority in this shell. The property (23) follows from the fact that in the sum in Eq. (22) we necessarily have $v_s = 1$ since five particles in the $j = 7/2$ shell are equivalent to three holes which must have seniority $v = 1$ for $J = j$. The CFP therefore must vanish since a $v = 1$ five-particle state cannot have four of the particles coupled to seniority four.

The result (23) can be put to good use as follows. We assume that the energy of the $|j^4 v_a I\rangle$ state can be written as a linear expression (20) in the two-body matrix elements v_λ . The unknown coefficients x_λ can be determined by choosing different interactions (defined by the two-body matrix elements v_λ) for which the energy $E[j^4 v_a I]$ is known. Four such interactions are available:

- (i) The pairing interaction which is obtained for $v_0 = 1$ and $v_2 = v_4 = v_6 = 0$ and yields the energy $E[j^4 v_a I] = 0$.
- (ii) The constant interaction which is obtained for $v_0 = v_2 = v_4 = v_6 = 1$ and yields the energy $E[j^4 v_a I] = 6$.
- (iii) The two-body part of \hat{J}^2 which is obtained for $v_\lambda = \lambda(\lambda + 1) - 2j(j + 1)$ and yields the energy $E[j^4 v_a I] = I(I + 1) - 4j(j + 1)$.
- (iv) A single $\hat{V}_{\lambda=I}$ component which is obtained for $v_{\lambda=I} = 1$. According to the preceding discussion the energy is $E[j^4 v_a I] = 1$.

For $j = 7/2$ and $I = 2$ or 4 we have thus a system of four linear equations

$$\begin{aligned} x_0 &= 0, \\ x_0 + x_2 + x_4 + x_6 &= 6, \\ -\frac{63}{2}x_0 - \frac{51}{2}x_2 - \frac{23}{2}x_4 + \frac{21}{2}x_6 &= I(I+1) - 63, \\ x_I &= 1, \end{aligned} \quad (24)$$

which can be solved for the unknown coefficients x_λ to give the expressions

$$\begin{aligned} E[(7/2)^4, v=4, I=2] &= v_2 + \frac{42}{11}v_4 + \frac{13}{11}v_6, \\ E[(7/2)^4, v=4, I=4] &= \frac{7}{3}v_2 + v_4 + \frac{8}{3}v_6, \end{aligned} \quad (25)$$

which is also what is obtained via conventional techniques based on CFPs.

V. CONCLUDING REMARK

We have reported on some progress in the understanding of the peculiar occurrence of a partial dynamical symmetry in the $j = 9/2$ shell and have shown it to be the consequence of general properties of CFPs. The matter is not fully settled yet since we still lack an analytic proof of the relation (17). Also, although we have a simple derivation of energy expressions in the $j = 7/2$ shell, this is not yet the case for the solvable states in the $j = 9/2$ shell.

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- [1] A. de-Shalit and I. Talmi, *Nuclear Shell Theory* (Academic Press, New York, 1963).
 [2] I. Talmi, *Simple Models of Complex Nuclei. The Shell Model and Interacting Boson Model* (Harwood Academic, Chur, Switzerland, 1993).
 [3] A. Escuderos and L. Zamick, Phys. Rev. C **73**, 044302 (2006).

- [4] L. Zamick, Phys. Rev. C **75**, 064305 (2007).
 [5] B. F. Bayman and A. Lande, Nucl. Phys. **77**, 1 (1966).
 [6] P. Redmond, Proc. R. Soc. London A **222**, 84 (1954).
 [7] L. Zamick and A. Escuderos, Ann. Phys. (NY) **321**, 987 (2006).
 [8] P. Van Isacker and S. Heinze, Phys. Rev. Lett. **100**, 052501 (2008).