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Nuclear α-particle condensates: Definitions, occurrence conditions, and consequences

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There has been a recent flurry of interest in the possibility of condensates of α particles in nuclei. In this Rapid Communication we discuss occurrence conditions for such states. Using the quantality condition of Mottelson we show that condensates are only marginally expected in α-particle states. We proceed to demonstrate that few-body nuclear condensates are ill defined and emphasize the conflict between α-localization and α-condensate formation. We also explore the connection between Ikeda diagrams, linear chains, and Tonks-Girardeau gases. Our findings show that no new information is contained in the approximations of nuclear states as α-cluster condensates. Furthermore, condensates of more than three α particles are very unlikely to exist due to couplings to other degrees of freedom.

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Introduction. The idea of α particles as essential constituents in the structure of nuclei arises from the small radius, the relatively large binding, and the spin saturation of both neutrons and protons. Attempts were made in the early days of nuclear physics to construct nuclear structure from α particles and valence nucleons [1–3]. In general these attempts were largely unsuccessful because the nucleon distances within and between different α particles are comparable. Thus there are no compelling reasons for clusterization of nucleons into α particles.

However, α-cluster models were able to explain many properties of specific light nuclei [4]. This was highlighted by the prediction of a 3α-structure near threshold [5] which is crucial for the nuclear synthesis of the heavy elements in stars. This Hoyle state was soon found experimentally [6] and its properties established in a microscopic cluster model [7] where it was first characterized as a "gaslike" structure. This structure is confirmed in details in numerous theoretical works [8–10]. A radius about 30% larger than that of the ground state of ^{12}C is also reproduced by use of an approximate wave function consisting of an antisymmetrized product of three Gaussians each containing an α particle [11]. Moreover, the ground state of ^{8}Be was in the same approximation described as a gaslike structure of two α particles [12], and the established structure [13,14] was again essentially recovered. The fundamental continuum resonance properties were suppressed by use of boxlike boundary conditions effectively supplying a confining external field.

These approximations were presented as novel discoveries of condensates consisting of two and three α particles. The last 5 years have witnessed surprisingly large efforts invested in both investigations of the accuracy of the approximation in Ref. [11] and extensions to similar simple models for other nuclei [15,16]. The aim seems to be a search for α-cluster condensates in nuclei. The inspiration is from atomic physics where Bose-Einstein condensates (BEC) of cold atoms are routinely made and manipulated by external fields [17]. Related theoretical investigations are also abundant; see, e.g., the review in Ref. [18].

The concept of BEC is well-defined for macroscopic systems of cold atoms and molecules. Extensions to self-bound quantum systems with a small number of particles are not straightforward. The purpose of this Rapid Communication is to discuss the concept of few-body nuclear condensates, give definitions, compare to cold atomic gases, show the conflict between localized α-cluster models and α condensation, derive occurrence conditions, and investigate consequences.

Concepts. To find α-cluster condensates in nuclei two conditions must be met, i.e., (i) the nucleons must be confined in α clusters and (ii) these α particles must form a condensate. Although α-cluster models in general are unsuccessful we shall assume that (i) holds. In addition, to ensure nucleon antisymmetry, the α-α distance must, on average, be larger than the diameter of the α particle.

The classic definition of an ideal Bose-Einstein condensate is a collection of identical particles in the same quantum state. This implies that the independent particle model gives an accurate description that obviously is true for noninteracting particles in an external field. For interacting particles the mean-field description is valid when Mottelson's quantality condition is met [19], i.e.,

$$\Lambda_{\rm Mot} = \frac{\hbar^2}{m c_{\rm min}^2 |V_{\rm min}|} > 0.1 - 0.2, \qquad (1)$$

where m is the mass of the particles and $c_{\rm min}$ is the distance between two particles when the total two-body potential has its minimum value $V_{\rm min}$. When $\Lambda_{\rm Mot}$ is small the attractive potential dominates over the kinetic energy and the particles are confined to the attractive pockets, i.e., localization or solid structure. When $\Lambda_{\rm Mot}$ is large the particles cannot be confined by the attraction and the mean-field model is appropriate.

The condition for Bose-Einstein condensates is that the deBroglie wavelength λ_{dB} of the motion must be larger than the distance $c_{\rm min}$ to the nearest neighbor, i.e.,

$$1 < \Lambda_{\rm bec} \equiv \frac{\lambda_{dB}}{c_{\rm min}} = \frac{2\pi\hbar}{c_{\rm min}\sqrt{2m|V_{\rm min}|}} = \pi\sqrt{2\Lambda_{\rm Mot}}. \qquad (2)$$

Thus $\Lambda_{\rm bec}^2 = 2\pi^2\Lambda_{\rm Mot}$ implying that the quantality inequality in Eq. (1) separating solid and mean-field structures is equivalent to the condition for breakdown of the classical gas regime in statistical mechanics.

We can evaluate these conditions for the α-α potential V without any bound states parametrized in Ref. [20] as an attractive and a repulsive Gaussian of different ranges (r_a, r_r) and strengths (V_a, V_r), i.e.,

$$V(r) = V_r \exp\left(-r^2/r_r^2\right) - V_a \exp\left(-r^2/r_a^2\right), \qquad (3)$$

where the minimum $V_{\min} \approx 5$–8 MeV (including the Coulomb energy of ≈ 2 MeV) for $c_{\min} \approx 2.5$–3.0 fm and $\Lambda_{\mathrm{Mot}} \approx 0.1$–$0.2$. Thus α particles would be in the mean-field range but with a strong tendency to localize.

Symmetry requirements. Wave functions describing self-bound few-body structures must be invariant under translations and rotations. The connection to conditions for condensate formation is most easily illustrated by use of an N-body wave function Ψ expressed as products of identical Gaussian single-particle wave functions, i.e.,

$$\Psi(\{\boldsymbol{r}_i\}) = (b\sqrt{\pi})^{-3N/2} \exp\left[-\sum_{i=1}^{N} r_i^2/(2b^2)\right], \qquad (4)$$

where $\boldsymbol{r}_i$ is the ith coordinate. This as well as all other mean-field wave functions violate translation invariance or, equivalently, momentum conservation, which is restored by integrating $\Psi(\{\boldsymbol{r}_i - \boldsymbol{R}'\}) \exp(i\boldsymbol{P}\cdot\boldsymbol{R}')$ over all $\boldsymbol{R}'$. The solution Ψ_{int} of lowest energy has $\boldsymbol{P} = 0$ which for Eq. (4) results in

$$\Psi_{\mathrm{int}}(\{\boldsymbol{r}_i\}) = (b\sqrt{\pi})^{-3(N-1)/2} \exp[-\rho^2/(2b^2)], \qquad (5)$$

$$\rho^2 \equiv \sum_{i=1}^{N} q_i^2 = \frac{1}{N}\sum_{i<j} r_{ij}^2 = \sum_{i=1}^{N} r_i^2 - N\boldsymbol{R}^2, \qquad (6)$$

$$\boldsymbol{q}_i \equiv \boldsymbol{r}_i - \boldsymbol{R}, \quad \boldsymbol{r}_{ij} \equiv \boldsymbol{r}_i - \boldsymbol{r}_j, \quad \boldsymbol{R} \equiv \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{r}_i, \qquad (7)$$

where the coordinates now are measured from the common center-of-mass (c.m.) $\boldsymbol{R}$. This wave function is invariant under rotations around the center-of-mass.

In contrast to the mean-field solution the particles can be correlated and the wave function Ψ_{loc} in a body-fixed coordinate system localized in distributions around preferred points $\boldsymbol{R}_k$, i.e.

$$\Psi_{\mathrm{loc}}(\{\boldsymbol{r}_i\}) \propto \sum_p \exp\left\{-\sum_{i=1}^{N} [\boldsymbol{r}_i - \boldsymbol{R}_{p(i)}]^2/(2B^2)\right\}, \qquad (8)$$

where the normalization is omitted and full symmetry is achieved by the summation over all permutations p of the set of numbers $\{1, 2, \ldots, N\}$. The translational invariance is restored precisely as for Eq. (4), i.e., the wave function is obtained from Eq. (8) by the substitution $\boldsymbol{r}_i \to \boldsymbol{q}_i$ in Eq. (6) and a corresponding change of normalization constant. The rotational invariance is broken for Ψ_{loc} in Eq. (8) but recovered for states of zero angular momentum by linear combinations of all spatial rotations of Ψ_{loc}.

Condensate assessment. To decide if a given wave function describes a condensate we apply different available definitions. We illustrate again with Gaussian wave functions which overestimates the degree of factorization of the N-body wave function. One necessary criterion for a condensate is that the one-body density matrix must have an eigenvalue λ comparable in size to N [21]. For a mean-field product solution the condensate fraction is $c_f = \lambda/N = 1$. However, the one-body density matrix is ill defined for self-bound systems of a finite number of particles. This is due to the center-of-mass motion that decouples completely for correct translationally invariant solutions. An appropriate center-of-mass wave function could be chosen to allow the usual definition of the one-body density matrix. The choice could be such that the condensate fraction c_f is optimized that would be equivalent to adding an external field as in atomic physics. For Eq. (5) this recovers the product wave function of all coordinates in Eq. (4) where the center-of-mass motion is completely ignored.

Instead of using all particle coordinates relative coordinates could be used and an internal one-body density matrix, $n(\boldsymbol{q}, \boldsymbol{q}')$, defined in Refs. [15,22–24]. Following Ref. [24], i.e., inserting $\boldsymbol{q}_N = -\sum_{i=1}^{N-1} \boldsymbol{q}_i$ in Eq. (5), we get

$$n(\boldsymbol{q}, \boldsymbol{q}') \propto \int d^3\boldsymbol{q}_2 d^3\boldsymbol{q}_3 \cdots d^3\boldsymbol{q}_{N-1} |\Psi_{\mathrm{int}}|^2$$
$$\propto \exp\left[-\frac{\boldsymbol{q}^2 + \boldsymbol{q}'^2}{b^2} + \frac{(N-2)(\boldsymbol{q}' + \boldsymbol{q})^2}{(N-1)4b^2}\right], \qquad (9)$$

where $\boldsymbol{q}$ and $\boldsymbol{q}'$ refer to $\boldsymbol{q}_1$. The condensate fraction obtained through the largest eigenvalue is then [25] $c_f = 8/(1 + \sqrt{2 - 2/N})^3$, which decreases with N from 1 for $N = 2$ toward about 0.57 for large N. However, the choice of internal coordinates is arbitrary [25] and we could as well choose $\boldsymbol{q}_1$ supplemented by a set of $N - 1$ *independent* Jacobi coordinates. Then the density matrix corresponding to Eq. (5) would factorize and give $c_f = 1$.

For α clusters these options can by appropriate choices lead to large condensate fractions for rather accurate cluster wave functions. This is because approximate factorization easily arises at smaller distances where the potential minimum resembles a harmonic oscillator and the related s-wave solutions resemble Gaussians.

Instead of using the eigenvalues of the density matrix a condensate criterion could be that the one-body (internal) density matrix should factorize at large distances [21]. This criterion is extremely difficult to fulfill because the correct nuclear wave functions never factorize *at large distances* as shown in Ref. [26]. Thus, at best only properties at intermediate distances could possess condensate properties with this criterion.

Yet another condensate criterion is that all particles occupy the same quantum state [25]. This implies that removal of one particle should leave the single-particle wave functions completely unchanged. However, for a finite number of particles the remaining interacting particles would reorganize into a different structure. This criterion would be extremely difficult to test directly.

Localization. The α-cluster models and the quantality parameter in Eqs. (1) and (2) suggest that localization, or crystal features of the wave function, may be important. The resulting condensate fraction depends strongly on the degree of localization as we can see explicitly by computing the one-body density matrix for Eq. (8). We assume very narrow nonoverlapping Gaussians and an appropriate center-of-mass

motion and obtain

$$n(\boldsymbol{r},\boldsymbol{r}') = (B\sqrt{\pi})^{-3/2}\sum_{k=1}^{N}\exp\{-[(\boldsymbol{r}-\boldsymbol{R}_k)^2 + (\boldsymbol{r}'-\boldsymbol{R}_k)^2]/(2B^2)\}, \quad (10)$$

which has N equally large eigenvalues while all others are zero. This is a condensate fraction of $1/N$ corresponding to one single-particle state for each of the N particles. However, after restoration of rotational symmetry only eigenvalues zero remain. If the widths, B, of the Gaussians increase and they begin to overlap with each other one eigenvalue separates out and becomes finite. Increasing the width leads to increasing overlap with a product wave function like Eq. (4). We can quantify by computing the overlap between the factorized and localized wave functions in Eqs. (4) and (8), i.e.,

$$\langle\Psi|\Psi_{\rm loc}\rangle = \left(\frac{2bB}{b^2+B^2}\right)^{3N/2}\exp\left(-\frac{\sum_{k=1}^{N}R_k^2}{2b^2+2B^2}\right), \quad (11)$$

which only is close to unity when $b\sim B$ and either $R_i/B \ll 1$ or $R_i/b \ll 1$. Equation (11) is also obtained by replacing $\Psi_{\rm loc}$ with the rotationally invariant wave function. Thus a substantial condensate fraction requires that the overlap with Eq. (4) is large. However, the spatial extension must be large to ensure definition (i) of nonoverlapping α particles.

Condensates from cluster models. The well-known structure of the Hoyle state in ^{12}C has about 90% overlap with α particles in relative s waves around the center-of-mass [15,22,23,27]. This corresponds to an eigenvalue of about 0.7 [22,23] in agreement with our upper bound of $c_f = 0.80$ derived from Eq. (5). At the same time α-cluster models show α-particle density distributions localized around specific points in space [27]. Reconciling these results, where apparently both the localization and large condensate fraction are present in the same wave function, is possible only with large widths of the localized wave in Eq. (8). This effectively recovers the independent particle wave function in Eq. (4) where the α particles are sufficiently separated to remove the need for nucleon antisymmetrization. These arguments show that the α-condensate states proposed should be regarded as merely an approximation to existing nuclear α-cluster states.

A crucial question is whether a condensate structure can be experimentally distinguished from other structures. To address this question computed and measured electron scattering on ^{12}C was compared in Ref. [27]. The conclusions are that α-cluster models of the Brink-type [4] and the α-condensate states of Ref. [11] predict virtually indistinguishable cross sections and charge distributions. In addition, these models and results from more elaborate microscopic calculations [27] give precisely the same charge density at large distances. Thus the classical cluster parametrization supplemented by nucleon antisymmetry [11] is apparently accurate to about 90% for the Hoyle state. However, it is important to realize that fulfilling an ambiguous definition has very little to do with true condensates that can be diagnosed only through properties of the wave functions and not by density distributions. In particular, observable coherence properties of the many-body states are necessary to separate cluster states from condensates. Both this and the localization discussion above strongly indicate that no new consequences arise from approximating cluster states with "condensates."

Condensate identity. The approximation as a condensate wave function of a quantum state rapidly gets invalid with increasing nuclear mass. This is seen from a sequence of four arguments. First the approximation as a condensate wave function is related to a restricted part of the full Hilbert space. Variational computations of condensates assume a class of wave functions with parameters determined by minimizing the energy. When the Hilbert space is extended to include other degrees of freedom the solution must remain essentially unchanged. An analogy is found in the s-wave neutron strength function that is broad and distributed over a large number of many-body states. This is reflected in the lack of neutron halo states at excitations around the neutron binding energy B_n [28,29]. To maintain the condensate character, the residual coupling $V_{c,n}$ of an α-condensate state $|c\rangle$ to the true many-body continuum nuclear states $|n\rangle$ must all be smaller than their energy difference [29].

The second argument in the sequence is that this approximation gets increasingly worse with increasing excitation energy because the density of states increases. A "clean" condensate wave function must be more and more "smeared out" over the true many-body states and at some excitation energy the condensate wave function no longer describes a state of the nucleus. Third, α condensates are postulated at the threshold for disintegration into α particles. Fourth, this is at an excitation energy E^* of about 7 MeV for ^{12}C and increasing by about 7 MeV for each additional α particle, i.e., $E^* \approx 7(A/4-2)$ MeV.

We first estimate an average $V_{\rm av}$ of $V_{c,n}$ by using a nucleon-nucleon potential of range b and strength V_0 between the two states with similar radii $R = bA^{1/3}$. Both residual kinetic and potential energy contributions are then proportional to the number of nucleons A, i.e., $|V_{\rm av}| \approx AS_0$. The radii in $|c\rangle$ and $|n\rangle$ must be comparable if the attraction of short range has to keep the condensate spatially confined in competition with the repulsive Coulomb interaction. The condition for maintaining the condensate character is then $|V_{\rm av}| < D$, where D is the average level spacing.

We also estimate $V_{\rm av}$ by replacing $|c\rangle$ by the state $|\alpha,(A-4)\rangle$, consisting of an α particle and the ground state, $|(A-4)\rangle$, of the $(A-4)$ system. These wave functions are similar, because the condensate consists of α particles, but they are clearly not identical, because the ground state of A-4 cannot be accurately described by an α-cluster model. However, they are similar in the approximations of harmonic oscillator potentials or Gaussian wave functions. The differences in the coupling matrix elements from using $|c\rangle$ and $|\alpha,(A-4)\rangle$ can then on average be expected to deviate much less than an order of magnitude. With many states $|n\rangle$ in the average this leads as in Refs. [29,30] to the estimate of $V_{\rm av} \approx W_\alpha$, where W_α is the strength of the imaginary part of the α-nucleus optical potential. As in Ref. [29] we conclude that the condensate resembles one of the many-body states when $W_\alpha < D$.

From the imaginary α-nucleus potential we can estimate the spreading width Γ_α of an α-particle state on the true many-nucleon states. For nucleons the spreading width of a

single-particle state of energy ϵ, $\Gamma_{\rm sp}$, is in Fermi-liquid transport theory [31–33] found to be $\Gamma_{\rm sp} = [(\epsilon-\mu)^2+\pi^2T^2]/\Gamma_0$ where μ is the Fermi level and T the temperature. The constant Γ_0 is related to the imaginary potential and the estimate ≈33-MeV results in $\Gamma_{\rm sp} \approx 1.5$ MeV for $T = 0$ and an energy equal to the nucleon separation energy [32,33]. Analogously we estimate Γ_α for α particles moving in the medium of nucleons. The α-nucleus separation energy is about 7 MeV, which with $\Gamma_0 \approx 33$ MeV again results in $\Gamma_\alpha = 1.5$ MeV for $T = 0$. The finite temperature is obtained from the average excitation energy, $E^* = aT_c^2$, at the threshold for fragmentation into free α particles. With the level-density parameter $a = A/10$ MeV we get $T_c \approx 4$ MeV $\sqrt{1-8/A}$, where A is the nucleon number. In total we get the estimate $\Gamma_\alpha \approx 6$ MeV almost independent of nucleon number. This corresponds to a strength $W_\alpha \approx 3$ MeV for the appropriate increasing excitation energy.

We estimate D in the Fermi gas model adjusted phenomenologically to excitation energies $E^* \approx B_n$, i.e., $D \approx D_0 \exp[-2\sqrt{a(E^*-2\Delta)}]$, where $\Delta \approx 12\ \text{MeV}/\sqrt{A}$ is the pairing gap Δ. The level spacing, $D_0 \approx 20\ \text{MeV}/A$, for $E^* = 2\Delta$ is essentially equal to the single-particle level spacing which is the correct limit. The extremely simple expression for D can only be an average over many nuclei at energies where many excited states are present.

The conditions $|V_{\rm av}| < D$ and $W_\alpha < D$ are then

$$S_0 A \text{ or } W_\alpha < D_0 \exp[-2\sqrt{a(E^*-2\Delta)}]. \qquad (12)$$

A very low limit of both S_0 and W_α is 1 MeV [34]. With a very small value of $E^* = 2\Delta$ we get the conservative estimate of preservation of condensate identity $A < \sqrt{20} < 5$ or $A < 20$, respectively. The spreading width estimate, even reduced by a factor of 2, is also larger than the level distance for $A < 14$. These estimates are valid when a sufficient number of excited states contributes in the average around the threshold energy. This is fulfilled for all nuclei heavier than ^{12}C, including ^{16}O. These conditions for survival of the condensate structures are almost always violated, and the violation increases exponentially with excitation energy.

Tonks-Giradeau structures. The linear chain structures of α particles at the breakup threshold, Ikeda diagrams [35], are conceptually similar to the one-dimensional atomic condensates called Tonks-Giradeau structures [36,37]. The latter have been realized with bosonic Rubidium atoms in optical traps that have strong repulsive interactions in the 1D geometry. This near impenetrability makes the system behave like a 1D Fermi gas in many aspects. This is analoguous to impenetrable α particles in 1D cluster structures. The corresponding states have been searched for and for many years the Hoyle state in ^{12}C was the favorite candidate. This state is now claimed as a condensate with a completely different structure. No observable has been found to distinguish between these structures that in any case both are approximated as three-body cluster states.

To assess whether such linear structures could exist in nuclei we turn to the two conditions in Eqs. (1) and (2). The α-α Coulomb energy is unimportant compared to V in Eq. (3). It does not change the condition but it also cannot provide the confining external field allowing a mean-field condensate-like solution. Hence a linear chain structure is possible only with localized α particles. The linear chain structure may also be destroyed by couplings to other degrees of freedom. As for three-dimensional condensates we estimate the survival probability to be very small for any excitation energy above 2Δ.

Conclusions. The existence of Bose-Einstein condensates of α particles assumes first that nucleons clusterize into α particles and, second, that a condensate is formed. The quantality condition of Mottelson indicates that α particles marginally prefer independent particle motion over correlation. We show that definitions of condensates of very few particles are ambiguous and lead to disparate condensate fractions. The origin is conflicts among mean-field solutions, correlations, and translational or rotational symmetries and between definitions related to short- and long-range behavior. The differences between nuclear and atomic condensates are few versus macroscopic number of particles, dilute versus high density, self-bound system versus external confining field, and ambiguous versus rigorous definitions.

In conclusion, we have found that the concept of a nuclear condensate is of little use. The recent theoretical claims of nuclear α condensates refer to well-known cluster states and can be regarded as merely an approximation to such states. No observable differences can be constructed to distinguish these alleged novel structures from ordinary cluster states. Occurrence of one- and three-dimensional nuclear condensates of more than three particles at higher excitation energies are very unlikely. They would either be completely unstable and vanished into the continuum, or the α-condensate structure would cease to exist due to spreading over many-nucleon states. In any case if traces remain they are nothing else than parts of ordinary α-cluster states.

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