Chiral condensate in a constant electromagnetic field at $\mathcal{O}(p^6)$

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We examine the shift in the chiral condensate due to a constant electromagnetic field at $\mathcal{O}(p^6)$ using SU(2) chiral perturbation theory and a realistic $M_\pi = 140$ MeV. We find that this value differs significantly from the value calculated using $M_\pi = 0$, while the magnitude of the two-loop correction is unclear due to the uncertainty in the experimentally determined value of the relevant $\mathcal{L}_6$ LEC.

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I. INTRODUCTION

In QCD, the chiral condensate is important as it is the order parameter of chiral symmetry breaking. As such, its behavior is key in understanding QCD in extreme conditions. Finite temperature and pressure effects have been studied extensively. On the other hand, comparatively few efforts have focused on the effects of finite electromagnetic fields. Prior studies have been done either using chiral perturbation theory (χPT) [1–3], the effective theory for low-energy QCD, or models compatible with the large-N_C expansion [4,5] (the NJL model, in particular [6–9]). The method used in this paper is χPT. This approach has the advantage of being model independent and systematic, but has the disadvantage of containing a number of undetermined parameters (low-energy constants or LECs). We extend prior work done at the $M_\pi = 0$ limit [2,3] and at one loop [10] to two loops for $M_\pi = 140$ MeV.

Other calculations using χPT have focused on $M_\pi = 0$, and while this may be of interest theoretically, it has at best a narrow window of validity [10]. In addition, it is unlikely to find in nature a real electric or magnetic field with $eE \gg M_\pi^2$ or $eH \gg M_\pi^2$, which is required for the approximation to be reasonable. Incidentally, the opposite limit, $M_\pi^2 \gg eH$, might be of more interest in the sense that it can be produced in the laboratory, but in such a regime the shift in the condensate is miniscule.

Continuing a calculation to higher orders in an expansion is always of interest at least in the trivial sense of finding a more precise result. In this case, large-N_C QCD [11,12] provides another possible motivation for why the $\mathcal{O}(p^6)$ result might be of interest. Low-energy constants (LECs) of the same chiral order will have different orders of N_C, depending upon the number of flavor traces in the term they multiply. This can be understood as follows: a flavor trace corresponds to a quark loop in the analogous QCD calculation, and large-N_C counting rules indicate quark loops are down by a power of N_C, and the LEC is the only parameter available to absorb this difference.

This large-N_C dependence of the LECs can provide hints to the convergence of the chiral expansion. There are several processes which have been calculated at two-loop order, most of which show a close match between the $\mathcal{O}(p^6)$ results and the experimental results [13]. The $\mathcal{O}(p^6)$ correction required to achieve this agreement varies, but in particular, the $\mathcal{O}(p^6)$ calculation of the process $\gamma\gamma \to \pi^0\pi^0$ is strikingly more accurate than the $\mathcal{O}(p^4)$ result [14]. This is also the process which provides a (very rough) estimate of the LEC we use here, so it is conceivable that the $\mathcal{O}(p^6)$ correction will be important in our case, as well.

We will first proceed with a brief overview of SU(2)χPT. We then follow with an analytical calculation of the shift in the chiral condensate due to a magnetic field at $\mathcal{O}(p^6)$, and finally, a numerical analysis of the shift for general electromagnetic fields.

II. CHIRAL PERTURBATION THEORY

A. Basics

This section is a brief summary of the chiral perturbation theory notation that will be used in this paper. For a detailed description of the theory, see the original papers by Gasser and Leutwyler [1] or a number of reviews (Refs. [15,16], for example). Treatment of the chiral Lagrangian to $\mathcal{O}(p^6)$ can be found in [13,17,18].

The building blocks that are used to construct the chiral lagrangian include $U = u(\phi)^2$, containing the dynamical pion fields, and external fields s, p, a_μ, and v_μ. We will consider here SU(2) flavor symmetry with $m_u \approx m_d$, with external fields corresponding to only a constant electromagnetic field and quark masses. We then have

$$\begin{aligned}\chi &= 2B(s + ip) = 2B\mathcal{M},\\ r_\mu &= l_\mu = -eQA_\mu = -e\left(\frac{\tau_3}{2}\right)A_\mu.\end{aligned} \tag{1}$$

We also need to define the covariant derivative as

$$D_\mu U = \partial_\mu U - ir_\mu U + iUl_\mu. \tag{2}$$

From these, we define the following operators that will contribute to the terms relevant in this paper (in the general SU(n) notation):

$$\begin{aligned}u_\mu &= i\{u^\dagger(\partial_\mu - ir_\mu)u - u(\partial_\mu - il_\mu)u^\dagger\},\\ \chi_\pm &= u^\dagger\chi u^\dagger \pm u\chi^\dagger u,\\ f_\pm^{\mu\nu} &= uF_L^{\mu\nu}u^\dagger \pm u^\dagger F_R^{\mu\nu}u,\\ \chi_-^\mu &= u^\dagger D^\mu\chi u^\dagger - uD^\mu\chi^\dagger u,\end{aligned} \tag{3}$$

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which are defined

$$F_R^{\mu\nu} = \partial^\mu r^\nu - \partial^\nu r^\mu - i[r^\mu, r^\nu],$$
$$F_L^{\mu\nu} = \partial^\mu l^\nu - \partial^\nu l^\mu - i[l^\mu, l^\nu]. \tag{4}$$

Note that in the case we are considering, $F_R^{\mu\nu} = F_L^{\mu\nu}$.

U can be parametrized in several ways, but we will use the Weinberg parametrization

$$U = \sigma + \frac{i\pi^a\tau^a}{F}, \quad \sigma^2 + \frac{\vec{\pi}^2}{F^2} = 1. \tag{5}$$

Here, π^a are still the dynamical fields, and σ is represented as an expansion in terms of π^a from the second equation.

Using these definitions, $\mathcal{L}_2$ and $\mathcal{L}_4$ can be written as follows (in the general SU(N) form), where $\langle A\rangle$ denotes the trace of A [14]:

$$\begin{aligned}\mathcal{L}_2 &= \frac{F^2}{2}\langle u_\mu u^\mu + \chi_+\rangle,\\ \mathcal{L}_4 &= \frac{l_1}{4}\langle u^\mu u_\mu\rangle^2 + \frac{l_2}{4}\langle u_\mu u_\nu\rangle\langle u^\mu u^\nu\rangle + \frac{l_3}{16}\langle\chi_+\rangle^2\\ &\quad + \frac{il_4}{4}\langle u_\mu \chi_-^\mu\rangle - \frac{l_5}{2}\langle f_-^{\mu\nu} f_{-\mu\nu}\rangle\\ &\quad + \frac{il_6}{4}\langle f_+^{\mu\nu}[u_\mu, u_\nu]\rangle - \frac{l_7}{16}\langle\chi_-\rangle^2\\ &\quad + \text{contact terms.}\end{aligned} \tag{6}$$

The calculation here is up to $\mathcal{O}(p^6)$, which means that we will be using the $\mathcal{L}_2$ Lagrangian up to two loops and the $\mathcal{L}_4$ Lagrangian up to one loop; the $\mathcal{L}_6$ Lagrangian will also contribute at tree level. This Lagrangian has been calculated in Ref. [18], and has more terms than we will list (112 for SU(n) and 53 for SU(2)). Fortunately, only one of these (in SU(2)) will be relevant for our calculation, as we will see later, and it can be expressed as

$$\mathcal{L}_6 = c_{34}\langle\chi_+ f_{+\mu\nu} f_+^{\mu\nu}\rangle + \sum_{i\neq 34} c_i P_i. \tag{7}$$

B. Renormalization

Renormalization of the theory to $\mathcal{O}(p^4)$ was calculated in Ref. [1]. It has also more recently been calculated for the $\mathcal{O}(p^6)$ Lagrangian in [17].

Using the SU(2) LECs, the renormalized couplings can be written

$$\begin{aligned} l_i &= (c\mu)^{d-4}\left(l_i^r + \gamma_i\Lambda\right)\\ \Lambda &= \frac{1}{16\pi^2(d-4)},\\ \gamma_1 &= \frac{1}{3}, \quad \gamma_2 = \frac{2}{3}, \quad \gamma_3 = -\frac{1}{2}, \quad \gamma_4 = 2,\\ \gamma_5 &= -\frac{1}{6}, \quad \gamma_6 = -\frac{1}{3}, \quad \gamma_7 = 0.\end{aligned} \tag{8}$$

When $M \neq 0$, these can be expressed in terms of scale-independent parameters as

$$l_i^r = \frac{\gamma_i}{32\pi^2}\left(\bar{l}_i + \log\frac{M^2}{\mu^2}\right). \tag{9}$$

The renormalization of the $\mathcal{L}_6$ term that we will be using later can be expressed similarly in terms of the renormalized LECs from $\mathcal{L}_4$ as

$$\begin{aligned} c_i &= \frac{(c\mu)^{2(d-4)}}{F^2}\left(c_i^r(\mu,d) - \gamma_i^{(2)}\Lambda^2 - (\gamma_i^{(1)} + \gamma_i^{(L)}(\mu,d))\Lambda\right),\\ \gamma_{34}^{(L)} &= -l_5^r + \frac{1}{2}l_6^r, \quad \gamma_{34}^{(1)} = \gamma_{34}^{(2)} = 0.\end{aligned} \tag{10}$$

III. CALCULATION OF Σ FROM VACUUM ENERGY

The term in the QCD Lagrangian which is relevant to Σ is $m_q\bar{q}q$. Since we know that $\Sigma \sim \langle\bar{q}q\rangle$, in the isospin limit of $m_u = m_d = \hat{m}$ we can calculate the condensate from the vacuum energy as

$$\Sigma = -\frac{\partial\epsilon_{\text{vac}}}{\partial\hat{m}}. \tag{11}$$

To first order in M_π^2/F_π^2, the Gell-Mann–Oakes–Renner relation $F_\pi^2 M_\pi^2 = \Sigma(m_u + m_d)$ [19] applies, and can be used to calculate the shift in the condensate. Unfortunately, we will need the next order result, which will introduce some ambiguity as follows. Neither the chiral condensate or the quark mass can be defined independently; only their product has a physical meaning. In the language of χPT [1],

$$2\hat{m}\Sigma = F^2M^2\left\{1 + \frac{M_\pi^2}{32\pi^2F_\pi^2}(4\bar{h}_1 - \bar{l}_3) + \mathcal{O}(M_\pi^4)\right\}. \tag{12}$$

The ambiguity here is codified in the unphysical LEC $\bar{h}_1$, which will vary according to the renormalization convention. This is an ambiguity in the definition of Σ. In order to avoid this difficulty, we choose to normalize our results according to the quantity Σ_0, which we define by

$$2\hat{m}\Sigma_0 = F_\pi^2 M_\pi^2, \tag{13}$$

with F_π and M_π at their physical values. We will thus express our results in terms of $\Delta\Sigma/\Sigma_0$, where $\Delta\Sigma \equiv \Sigma(H) - \Sigma(H = 0)$. As we will see, this ratio is unambiguous at the order to which we work.

We will also need the relationships between M and M_π as well as F and F_π ($M^2 \equiv 2B\hat{m}$). The difference between the lowest-order $\mathcal{O}(p^4)$ result and the $\mathcal{O}(p^6)$ result is $\mathcal{O}(p^2)$; we therefore only need one order of corrections [1]:

$$\begin{aligned} M_\pi^2 &= M^2\left[1 - \frac{M^2}{32\pi^2F^2}\bar{l}_3 + \mathcal{O}(M^4)\right],\\ F_\pi &= F\left[1 + \frac{M^2}{16\pi^2F^2}\bar{l}_4 + \mathcal{O}(M^4)\right].\end{aligned} \tag{14}$$

This will only be applicable in the first term of the expansion, which is two powers of momentum less than the maximum order for the calculation.

Our object is to calculate the condensate for the case of a constant electromagnetic field at $\mathcal{O}(p^6)$ in χPT. As described above, we can accomplish this by calculating the vacuum energy to the same order. As noted by Ref. [3], this calculation will involve two-loop diagrams with $\mathcal{L}_2$ vertices, one-loop diagrams with an $\mathcal{L}_4$ vertex, and tree-level diagrams with an $\mathcal{L}_6$

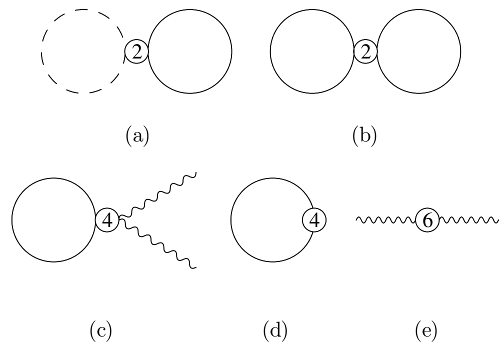


FIG. 1. Diagrams contributing to the vacuum energy shift due to an electromagnetic field. Dashed lines denote π^0 and solid lines denote $\pi^\pm$.

vertex. Diagrams which contribute to the vacuum energy will contain only external photon lines coming from the constant EM field (which is the "vacuum" in this case). Insertions of the electromagnetic field at the $\mathcal{L}_2$ level are calculated as part of the propagator of the $\pi^\pm$. Thus, the $\mathcal{O}(p^4)$ calculation corresponds roughly to a closed single propagator and was calculated in Ref. [10]. Contributing diagrams must be dependent on the electromagnetic field, either through a direct insertion or through the propagator of the $\pi^\pm$.

With these criteria, we find that the diagrams contributing to our calculation are as pictured in Fig. 1. Returning, then, to the χPT Lagrangian, we find that only the terms proportional to l_3, l_5, and l_6 can contribute from the $\mathcal{L}_4$ Lagrangian, and only the term proportional to c_{34} (as we anticipated above) can contribute to the vacuum energy from $\mathcal{L}_6$.

We simplify the chiral Lagrangians for SU(2) and $m_u = m_d$ up to the relevant terms in $\mathcal{L}_6$ [3], including only terms which will contribute to the diagrams in Fig. 1:

$$\begin{aligned}
\mathcal{L}_2 &= \frac{1}{2}(\partial_\mu\pi^0)^2 - \frac{M^2(\pi^0)^2}{2} - M^2\pi^+\pi^- \\
&\quad + (\partial_\mu\pi^+ + ieA_\mu\pi^+)(\partial^\mu\pi^- - ieA^\mu\pi^-) \\
&\quad + \frac{1}{2F^2}[\pi^0\partial_\mu\pi^0 + \partial_\mu(\pi^+\pi^-)]^2 \\
&\quad - \frac{M\pi^2}{8F^2}[2\pi^+\pi^- + (\pi^0)^2]^2, \\
\mathcal{L}_4 &= -\frac{2l_5}{F^2}(eF_{\mu\nu})^2\pi^+\pi^- \\
&\quad - \frac{2il_6}{F^2}eF_{\mu\nu}[\partial^\mu\pi^-\partial^\nu\pi^+ + ieA^\mu\partial^\nu(\pi^+\pi^-)] \\
&\quad - 2l_3\frac{M^4}{F^2}\pi^+\pi^-, \\
\mathcal{L}_6 &= 4c_{34}M^2(eF_{\mu\nu})^2.
\end{aligned} \tag{15}$$

Here, a term proportional to l_3 have been added to the Lagrangian from Ref. [3], which are down by an order of M^2 but have the same overall chiral order.

As a first case, we will work with the case of pure magnetic fields, where $(eF_{\mu\nu})^2 = 2(eH)^2$. This simplifies the calculations and allows us to obtain an analytic result. We will later generalize to an arbitrary combination of E and H fields for numerical analysis.

The propagator for a scalar particle in a constant H field was first calculated in Ref. [23], and here we use the convenient form also used in Ref. [3]:

$$\begin{aligned}
D^H(x,y) &= \Phi(x,y)\int\frac{d^4k}{(2\pi)^4}e^{ik(x-y)}D^H(k), \\
D^H(k) &= \int_0^\infty \frac{ds}{\cosh(eHs)}e^{-s(k_\parallel^2 + k_\perp^2\frac{\tanh eHs}{eHs} + M^2)},
\end{aligned} \tag{16}$$

where $\Phi(x,y) = \exp\{ie\int_y^x A_\mu(z)dz_\mu\}$, $k_\parallel^2 = k_3^2 + k_4^2$ and $k_\perp^2 = k_1^2 + k_2^2$.

We will also need the scalar propagator

$$\begin{aligned}
D(0) \equiv D(x,x) &= \int\frac{d^dk}{k^2+M^2} \\
&= 2M^2(c\mu)^{d-4}\left[\Lambda + \frac{1}{32\pi^2}\log\frac{M^2}{\mu^2}\right], \\
\Lambda &= \frac{1}{16\pi^2(d-4)}.
\end{aligned} \tag{17}$$

$D(0)$ and $D^H(0) \equiv D^H(x,x)$ are both divergent quantities, whereas $D^H(0) - D(0)$ is finite:

$$\begin{aligned}
D^{\Delta H}(0) &\equiv D^H(0) - D(0) \\
&= -\frac{eH}{16\pi^2}\int_0^\infty\frac{dx}{x^2}e^{-\beta x}\left(1 - \frac{x}{\sinh x}\right)
\end{aligned} \tag{18}$$

with $\beta = M^2/eH$. This is the same integral as was calculated in Ref. [10], and can be expressed analytically as

$$\begin{aligned}
D^{\Delta H}(0) &= -\frac{eH}{16\pi^2}I_H(\beta) \\
I_H(\beta) &= \log(2\pi) + \beta\log\left(\frac{\beta}{2}\right) - \beta - 2\log\Gamma\left(\frac{1+\beta}{2}\right).
\end{aligned} \tag{19}$$

With these, the diagrams in Fig. 1 can be calculated fairly straightforwardly to be [3]

$$\begin{aligned}
\epsilon_{1(a)}^{(2)} &= \frac{M^2}{2F^2}D(0)D^H(0), \\
\epsilon_{1(b)}^{(2)} &= \frac{1}{F^2}D^H(0)\int\frac{d^dk}{(2\pi)^d}(k^2+M^2)D^H(k), \\
\epsilon_{1(c)}^{(2)} &= \frac{2(eH)^2}{F^2}(2l_5 - l_6)D^H(0), \\
\epsilon_{1(d)}^{(2)} &= 2l_3\frac{M^4}{F^2}D^H(0), \\
\epsilon_{1(e)}^{(2)} &= -8c_{34}M^2(eH)^2.
\end{aligned} \tag{20}$$

Here, we have used $\epsilon_{1(x)}^{(2)}$ to denote the refer to the (second-order) vacuum energy contribution from the diagram in Fig. 1(x), with $\epsilon^{(2)}$ denoting the total vacuum energy contribution from the second order calculation.

The only one of these diagrams which is not expressed solely in terms of $D^H(0)$ and $D(0)$ is $\epsilon_{1(b)}^{(2)}$, which vanishes generally as well as in the $M_\pi = 0$ case.

In order to make the divergences and scale-dependence explicit, we first make the substitution $D^H(0) = D^{\Delta H}(0) + D(0)$. Any term which is dependent on neither H nor $D^H(0)$ can then be reabsorbed into the vacuum energy; we are only

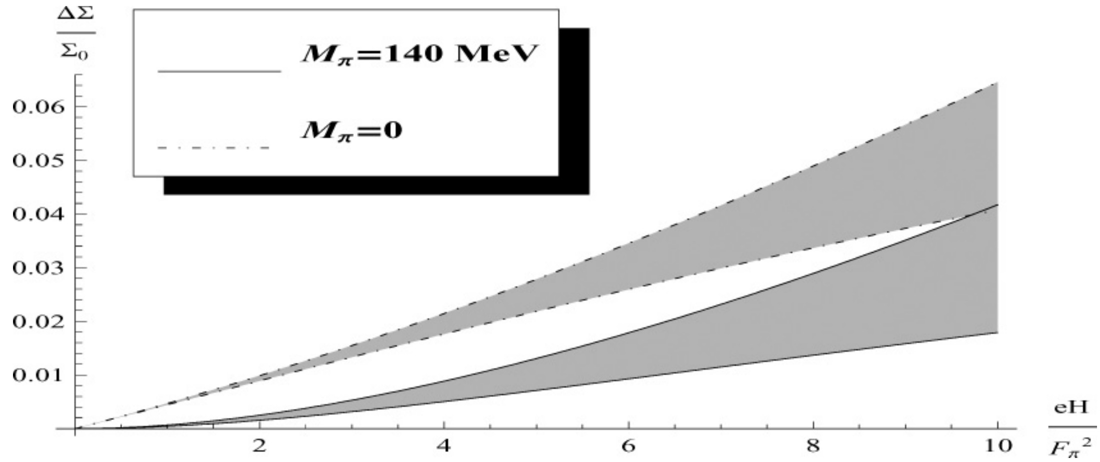


FIG. 2. A comparison of the shift due to a pure magnetic field in the $M_\pi = 0$ case to the $M_\pi = 140$ MeV case. Shaded regions indicate uncertainty due to the $\mathcal{L}_6$ constant d^r.

looking for the shift due to H. After making this substitution, we see that $\epsilon^{(2)}_{1(a)}$ is divergent and canceled by a counterterm generated by l_3. $\epsilon^{(2)}_{1(c)}$ has both a finite piece, which will contribute to the calculation, and a divergent piece, which is canceled by a counterterm in c_{34}. $\epsilon^{(2)}_{1(d)}$ and $\epsilon^{(2)}_{1(e)}$ are finite, aside from the aforementioned counterterms.

Combining, then, all of these terms, we find the vacuum energy to be

$$\epsilon^{(2)}(H) = -\frac{(eH)^3}{(16\pi^2)^2F^2} \times \left\{ I_H(\beta)\left[\frac{1}{3}(\bar{l}_6-\bar{l}_5) - \frac{\beta^2}{2}\bar{l}_3\right] + \beta\bar{d}(M^2)\right\}, \quad (21)$$

where we have defined the scale-independent quantity

$$\bar{d}(M^2) = 8(16\pi^2)^2c^r_{34} - \frac{1}{3}(\bar{l}_6-\bar{l}_5)\log\left(\frac{M^2}{\mu^2}\right). \quad (22)$$

We then substitute Eq. (14) to find M_π^2 from M^2 in the first-order term. We find that the $\bar{l}_3$ term cancels, and that the correction to F does not play a role (as it only appears at the second order). Then, taking a derivative, and applying the Gell-Mann–Oakes–Renner relation as above, with the first-order corrections for the $\mathcal{O}(p^4)$ term, we find ($\beta_\pi = M_\pi^2/eH$)

$$\frac{\Delta\Sigma(H)}{\Sigma_0} = \frac{eH}{16\pi^2F_\pi^2}I_H(\beta_\pi) + \left(\frac{eH}{16\pi^2F_\pi^2}\right)^2\left\{-\frac{1}{3}(\bar{l}_6-\bar{l}_5) \times\left[1+\log 2+\psi\left(\frac{1+\beta_\pi}{2}\right)\right] + \bar{d}(eH)\right\}, \quad (23)$$

with $\psi(x) \equiv \frac{d}{dx}\log\Gamma(x)$.

Taking $\beta \to 0$, $\psi(\frac{1}{2}) = -\gamma_e$, and the shift we find agrees with the expression found in Ref. [3] for the case $M_\pi = 0$.

For the case of an E field, we can make the substitution $H \to iE$ and get a similar analytic expression. The $E \cdot H \neq 0$ case is somewhat more complicated, but we will write an integral expression which we can later evaluate numerically.

For a convenient parametrization of the general case, we introduce the variables ϕ and f such that with $\mathcal{F} = \frac{H^2-E^2}{2} =$

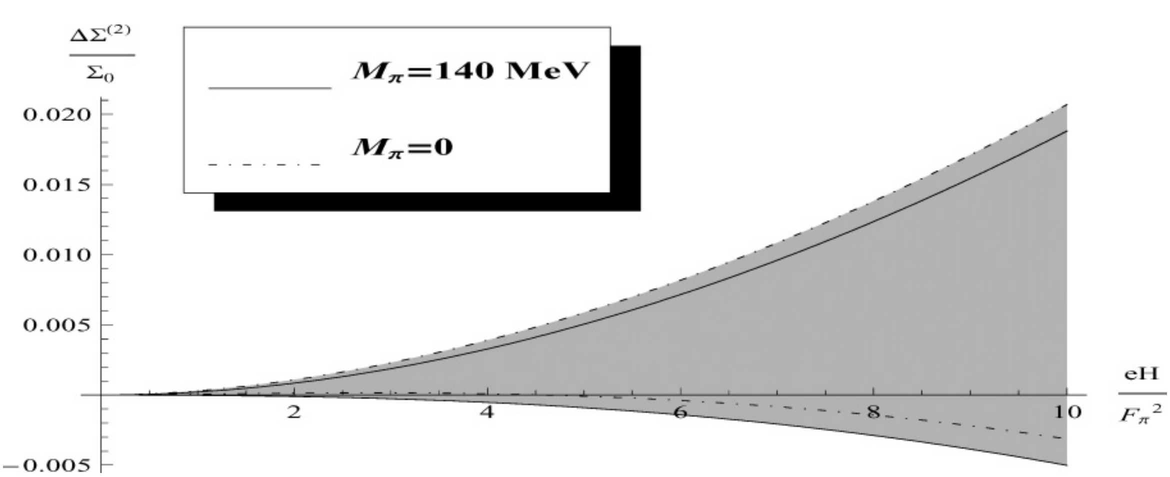
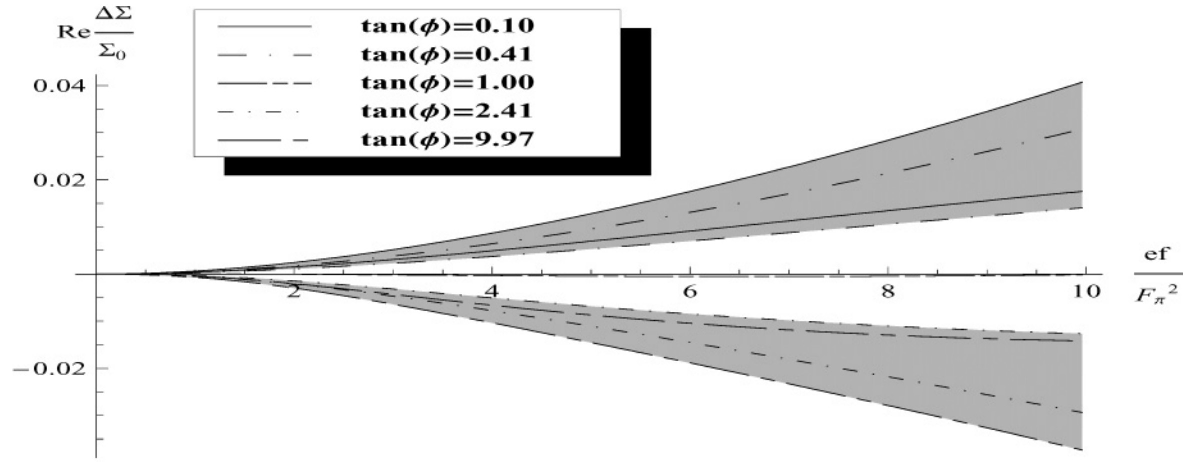
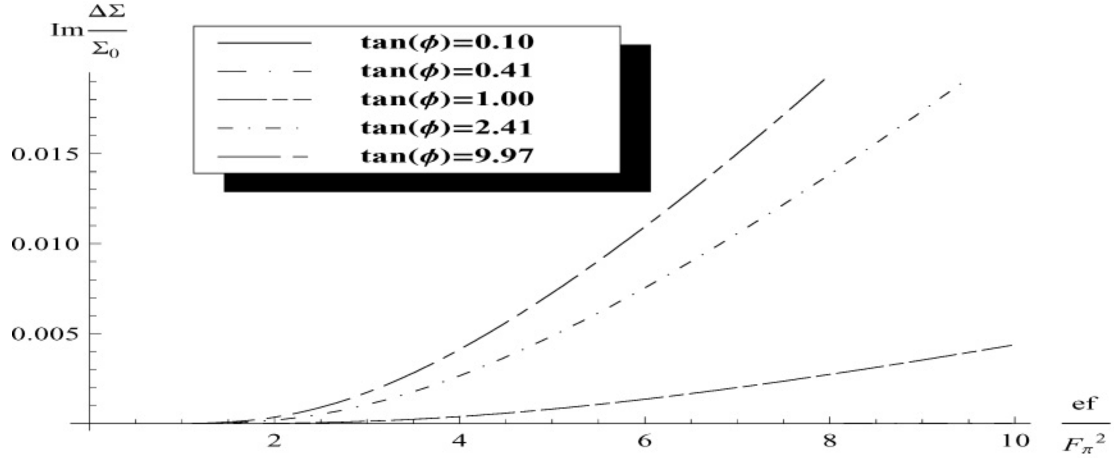


FIG. 3. A comparison of the shift due to a pure magnetic field in the $M_\pi = 0$ case to the $M_\pi = 140$ MeV case, using only the $\mathcal{O}(p^6)$ portion. Shaded regions indicate uncertainty due to the $\mathcal{L}_6$ constant d^r.



(a)



(b)

FIG. 4. The imaginary and real parts of the total value of the shift in the condensate due to general E and H fields, with f and ϕ as defined in the text. Shading depicts uncertainty due to c_{34}.

$\frac{1}{4}F_{\mu\nu}^2$ and $\mathcal{G} = \vec{E}\cdot\vec{H}$ [10],

$$\mathcal{F} = \frac{f^2\cos(2\phi)}{2} \quad \mathcal{G} = \frac{f^2\sin(2\phi)}{2}. \tag{24}$$

Expressed in terms of these variables, the shift in the condensate due to an arbitrary combination of fields will become ($\beta_f = M_\pi^2/ef$)

$$\frac{\Delta\Sigma(\mathcal{F},\mathcal{G})}{\Sigma_0} = \frac{ef}{16\pi^2F_\pi^2}I_{EH}(\beta_f,\phi) + \left(\frac{ef}{16\pi^2F_\pi^2}\right)^2\cos 2\phi \times\left\{\frac{1}{3}(\bar{l}_6-\bar{l}_5)(I_{EH}{}'(\beta_f,\phi)-1)+\bar{d}(M_\pi^2)\right\}, \tag{25}$$

$$I_{EH}(\beta_f,\phi) = \int_0^\infty \frac{dz}{z^2}e^{-\beta_f z} \times\left[1-\frac{z^2\sin 2\phi}{2\sin(z\sin\phi)\sinh(z\cos\phi)+i\epsilon}\right].$$

This is the same integral as in Ref. [10], and as before we have had to avoid some potential ambiguity. The poles in the integrand indicate an instability in the system, which is interpreted as due to pair creation in an electric field [23]. We have chosen to regulate the divergence in a manner which has an imaginary part corresponding to this pair creation. The magnitude of the imaginary part indicates the importance of this instability, though some caution is warranted in interpreting it quantitatively. This issue was discussed in more detail in Ref. [10].

Equations (23) and (25) are the principal results of this work.

IV. NUMERICAL RESULTS

Because the behavior of the theory is encoded in the LECs, a real-world interpretation of the low-energy behavior requires the use of measured LECs. For the $\mathcal{L}_4$ LECs, this is straightforward, as these are individually determined with

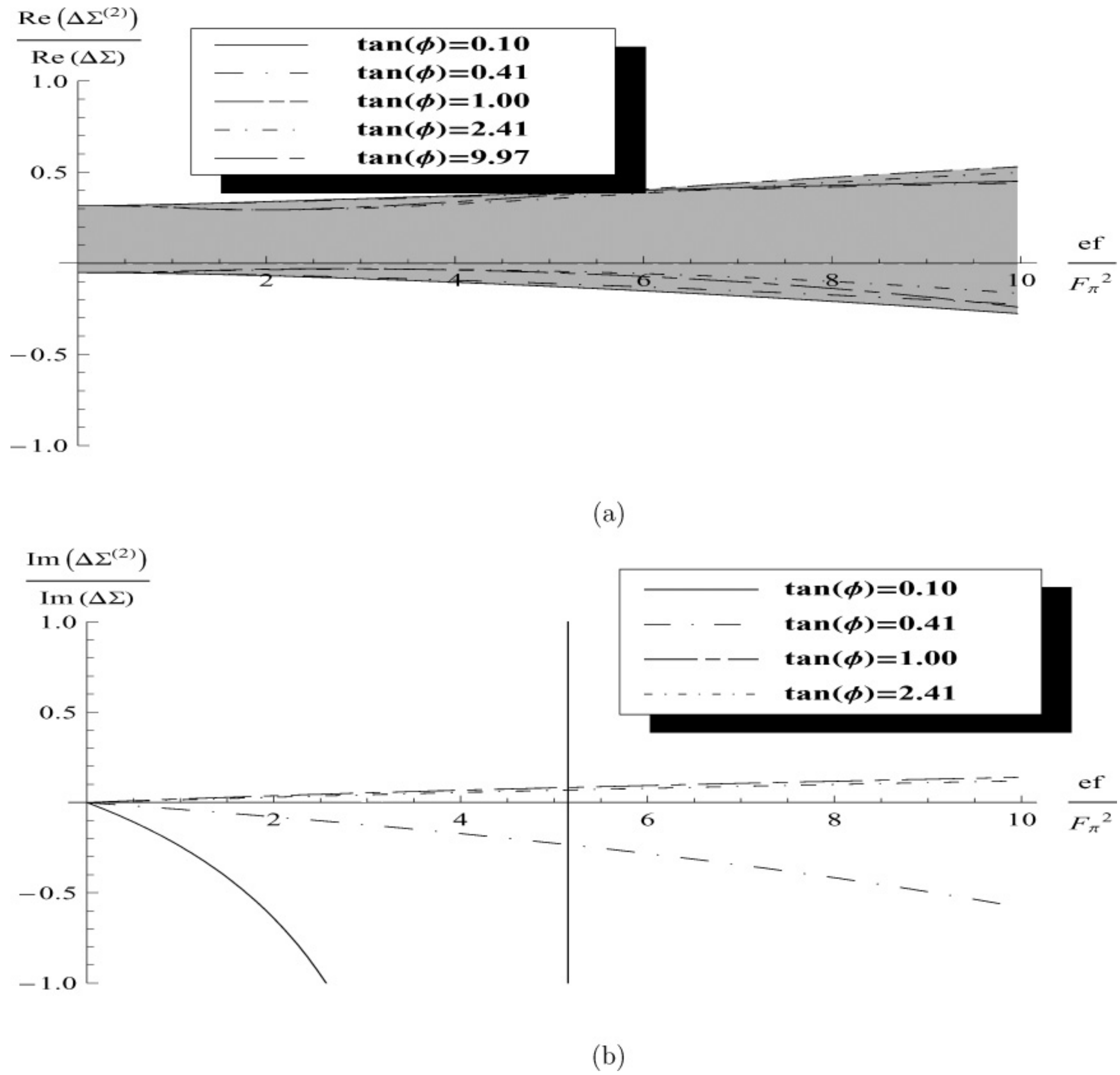


FIG. 5. The imaginary and real parts of the ratio of the shift at two loops to the total shift for the case of general E and H fields, with f and ϕ as defined in the text. Shading depicts uncertainty due to c_{34}.

relatively small error. On the other hand, $\mathcal{L}_6$ LECs, such as c_{34}, are more problematic, as there are in general many more LECs than easily measurable processes to determine them. These LECs are often estimated (at a particular scale) by resonance exchange. Unfortunately, c_{34} in particular is difficult to extract, as the resonance processes to which it contributes involve only scalar exchange, and because it appears squared in these processes, its sign is undetermined. This resonance exchange occurs at a scale $M_\rho = 768$ [14], and it is the (scale-dependent) value determined by experiment that has an undetermined sign. The scale-independent $\bar{d}$ is positive in both cases.

The values we use for these constants are [14,24]

$$\begin{aligned} \bar{l}_6 - \bar{l}_5 &= 3.0 \pm 0.3, \\ d^r &\equiv 8(16\pi^2)^2 c_{34}^r = \pm 1.5 \pm 1.5. \end{aligned} \tag{26}$$

With these experimental values, we can plot realistic values of the shift in the condensate. In Fig. 2, for the case of a pure magnetic field, we compare the value calculated for a finite M_π to that for $M_\pi = 0$. It is clear that these values are significantly different, as in the $\mathcal{O}(p^4)$ case [10]. In Fig. 3, we have plotted the magnitude of the $\mathcal{O}(p^6)$ portion alone. The magnitude of the $\mathcal{O}(p^6)$ shift is not so different in the massive vs the massless case, however, because the magnitude of the total shift is less in the massive case, this correction is potentially more significant. In this and the figures following, we have chosen to extend our results up to $ef = 290$ MeV (the expansion parameter is $\Lambda_H = 4\pi F_\pi = 1.2$ GeV).

Here, we use the same numerical trick as in Ref. [10] to extract the principal value of the integral numerically. We remove the singularities due to the poles located at z_i with residue R_i by subtracting the expression

$$i \sum_n R_n(z_n) \left(\frac{1}{z - z_n} - \frac{1}{z + z_n} \right). \tag{27}$$

The principal value of the integral of this expression is zero, but it has a singularity at z_i which exactly cancels the singular behavior of the integrand in I_{EH}.

Using this method, we plot the total shift in the condensate up to $\mathcal{O}(p^6)$ from a general E and H field in Fig. 4, and in Fig. 5, we plot the ratio of the added correction at $\mathcal{O}(p^6)$ to the total shift. In these plots, we have included a shaded region to indicate the possible values for the shift based on a range for d^r of $(-3, 3)$.

The asymptotic expression for the shift as $\beta_\pi \to \infty$ (for an H field, which also provides some qualitative insight to other cases) is

$$\frac{\Delta\Sigma(H)}{\Sigma_0} = \frac{eH}{16\pi^2 F_\pi^2}\left(\frac{F_\pi^2}{6M_\pi^2} - \frac{\bar{l}_6 - \bar{l}_5}{48\pi^2} + \frac{\bar{d}}{16\pi^2}\right), \quad (28)$$

which is, of course, zero for $\beta_\pi \to \infty$ $(H \to 0)$. This expression encodes low-energy behavior for a more realistic regime, namely, that of the actual pion mass and a very small magnetic field. We see from Fig. 5 and in Eq. (28) that the (unknown) sign of d^r has a profound impact on the importance of the $\mathcal{O}(p^6)$ calculation. The shift in the condensate for a positive d^r is significant enough even as $f \to 0$, whereas the shift for a negative one is negligible up to large values of ef/F_π^2.

Another notable feature is that the contribution to the imaginary part is larger at $\mathcal{O}(p^6)$ order as a pure H field is approached (while the total imaginary part is going to zero). Except in this case where the total imaginary part is negligible, the fraction of the imaginary part at two loops will be much less significant than its real counterpart in regimes where the chiral expansion would be expected to converge ($ef \sim M_\pi^2$ or below).

The calculation as a whole will only be valid when the real part is significantly larger than the imaginary part. When the imaginary part dominates, the system will break down due to the instability from pair creation.

V. DISCUSSION AND CONCLUSIONS

We have studied the shift in the chiral condensate due to an electromagnetic field using chiral perturbation theory, which is a powerful tool for analyzing the low-energy behavior of QCD. Our analysis was done at $\mathcal{O}(p^6)$ with $M_\pi = 140$ MeV. It is obvious that the inclusion of a nonzero pion mass greatly affects the result.

The importance of the $\mathcal{O}(p^6)$ correction is less clear. Large-N_C reasoning coupled with the results of model-based calculations give circumstantial evidence that it could play an important role in the final result. However, because the sign of the relevant LEC at $\mathcal{L}_6$ is undetermined by experiment, its effect at $\mathcal{O}(p^6)$ could be significant or virtually irrelevant.

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