Bose-Einstein condensation in the relativistic pion gas: Thermodynamic limit and finite size effects

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We consider the Bose-Einstein condensation (BEC) in a relativistic pion gas. The thermodynamic limit when the system volume V goes to infinity as well as the role of finite size effects are studied. At $V \to \infty$ the scaled variance for particle-number fluctuations, $\omega = \langle \Delta N^2 \rangle / \langle N \rangle$, converges to finite values in the normal phase above the BEC temperature, $T > T_C$. It diverges as $\omega \propto V^{1/3}$ at the BEC line $T = T_C$, and $\omega \propto V$ at $T < T_C$ in a phase with the BE condensate. Possible experimental signals of the pion BEC in finite systems created in high-energy proton-proton collisions are discussed.

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I. INTRODUCTION

Pions are spin-zero bosons. They are the lightest hadrons copiously produced in high-energy collisions. There were several suggestions to search for the Bose-Einstein condensation (BEC) of π mesons (see, e.g., Ref. [1]). However, no clear experimental signals were found until now. Most of previous proposals of the pion BEC signals were based on an increase of the pion momentum spectra in the low-(transverse) momentum region. These signals appear to be rather weak and they are contaminated by resonance decays to pions. In our recent article [2] it was suggested that the pion number fluctuations strongly increase and may give a prominent signal of approaching the BEC. This can be achieved by selecting special samples of collision events with high pion multiplicities.

In the present article we study the dependence of different physical quantities on the system volume V and, in particular, their behavior at $V \to \infty$. As any other phase transition, the BEC phase transition has a mathematical meaning in the thermodynamic limit (TL) $V \to \infty$. To define rigorously this limit one needs to start with a finite volume system. Moreover, the finite size effects are important for an experimental search of the pion BEC fluctuation effects proposed in Ref. [2]. The size of the pion number fluctuations in the region of the BEC is restricted by a finite system volume V. To be definite and taking in mind the physical applications, we consider the ideal pion gas. However, the obtained results are more general and can be also applied to other Bose gases.

The article is organized as follows. In Sec. II we consider the BEC in the TL. Here we emphasize some specific effects of the BEC in relativistic gases. Section III presents a systematic study of the finite size effects for average quantities and for particle-number fluctuations. In Sec. IV we discuss the finite size restrictions on the proposed fluctuation signals of the BEC in high-energy collisions with large pion multiplicities. A summary, presented in Sec. V, closes the article.

II. BEC IN THERMODYNAMIC LIMIT

A. Phase diagram

We consider the relativistic ideal gas of pions. The occupation numbers, $n_{\mathbf{p},j}$, of single quantum states, labeled by three-momenta $\mathbf{p}$, are equal to $n_{\mathbf{p},j} = 0, 1, \ldots, \infty$, where index j enumerates three isospin pion states, π^+, π^-, and π^0. The grand canonical ensemble (GCE) average values, fluctuations, and correlations are as follows [3]:

$$\langle n_{\mathbf{p},\,j} \rangle = \frac{1}{\exp[(\sqrt{\mathbf{p^2} + m^2}\mu_j)/T] - 1}, \quad (1)$$

$$\langle (\Delta n_{\mathbf{p},j})^2 \rangle \equiv \langle (n_{\mathbf{p},j} - \langle n_{\mathbf{p},j} \rangle)^2 \rangle = \langle n_{\mathbf{p},j} \rangle (1 + \langle n_{\mathbf{p},j} \rangle) \equiv \upsilon^2_{\mathbf{p},j}, \quad (2)$$

$$\langle \Delta n_{\mathbf{p},j} \Delta n_{\mathbf{k},i} \rangle = \upsilon^2_{\mathbf{p},j} \delta_{\mathbf{pk}} \delta_{ji},$$

where the relativistic energy of one-particle states is taken as $\epsilon_{\mathbf{p}} = (\mathbf{p^2} + m^2)^{1/2}$ with $m \cong 140$ MeV being the pion mass (we neglect a small difference between the masses of charged and neutral pions), T is the system temperature, and chemical potentials are $\mu_+ = \mu + \mu_Q$, $\mu_- = \mu - \mu_Q$, and $\mu_0 = \mu$, for π^+, π^-, and π^0, respectively. In Eq. (1) there are two chemical potentials: μ_Q regulates an electric charge and μ a total number of pions. In this article we follow our proposal of Ref. [2] and discuss a pion system with $\mu_Q = 0$.[1] This corresponds to zero electric charge Q, which is defined by a difference between the number of π^+ and π^- mesons, $Q = N_+ - N_- = 0$. The total pion number density is equal to:

$$\rho(T, \mu) \equiv \rho_+ + \rho_- + \rho_0 = \frac{\sum_{\mathbf{p},\,j} \langle n_{\mathbf{p},\,j} \rangle}{V} \cong \frac{3}{2\pi^2} \int_0^\infty \frac{p^2 dp}{\exp[(\sqrt{p^2 + m^2} - \mu)/T]^{-1}} \equiv \rho^*(T, \mu) = \frac{3T\,m^2}{2\,\pi^2} \sum_{n=1}^{\infty} \frac{1}{n}\, K_2(n\,m/T) \exp(n\,\mu/T), \quad (3)$$

where $\rho_j = \langle N_j \rangle / V$, with $j = +, -, 0$, are the pion number densities, V is the system volume, and K_2 is the modified Hankel function. In the TL, i.e., for $V \to \infty$, the sum over momentum states is transformed into the momentum integral, $\sum_{\mathbf{p}} \cdots = (V/2\pi^2) \int_0^\infty \cdots p^2 dp$. The particle-number density

[1]The BEC in the relativistic gas of “positive” and “negative” particles at $\mu_Q \to m$ was discussed in Refs. [4–8].

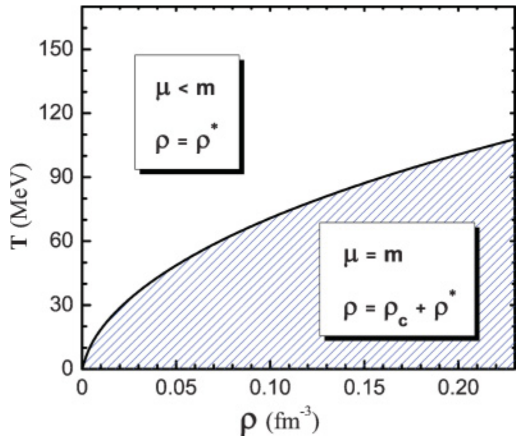

FIG. 1. (Color online) The phase diagram of the relativistic ideal pion gas with zero electric charge density $\rho_Q \equiv \rho_+ - \rho_- = 0$. The solid line shows the relation $T = T_C$ (4) of the BEC. Above this line, $T > T_C$ and there is a normal phase described by Eq. (3). Under this line, $T < T_C$ and there is a phase with the BE condensate described by Eq. (8).

ρ depends on T and μ, and volume V does not enter in Eq. (3). This is valid only at $\mu < m$. The number of particles at the zero-momentum level is then finite, and its contribution to particle-number density goes to zero in the TL. The inequality $\mu \leqslant m$ is a general restriction in the relativistic Bose gas, and $\mu = m$ corresponds to the BEC. Equation (3) gives the following relation between the BEC temperature T_C and total pion number density ρ [2]:

$$\rho = \frac{3\,T_C\,m^2}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n}K_2\left(n\,m/T_C\right)\exp(n\,m/T_C). \tag{4}$$

A phase diagram of the ideal pion gas in the ρ-T plane is presented in Fig. 1.

The line of the BEC phase transition is defined by Eq. (4), and it is shown by the solid line in Fig. 1. In the nonrelativistic limit, $T_C/m \ll 1$, using $K_2(x) \cong \sqrt{\pi/(2x)}\exp(-x)$ at $x \gg 1$, one finds from Eq. (4),

$$T_C \cong 2\pi[3\zeta(3/2)]^{-2/3}m^{-1}\rho^{2/3} \cong 1.592\,m^{-1}\rho^{2/3}, \tag{5}$$

whereas in the ultrarelativistic limit,[2] $T_C/m \gg 1$, one uses $K_2(x) \cong 2/x^2$ at $x \ll 1$, and Eq. (4) gives [2],

$$T_C \cong [3\zeta(3)/\pi^2]^{-1/3}\rho^{1/3} \cong 1.399\,\rho^{1/3}. \tag{6}$$

In Eqs. (5)–(6), $\zeta(k) = \sum_{k=1}^{\infty} n^{-k}$ is the Riemann ζ function [11], $\zeta(3/2) \cong 2.612$ and $\zeta(3) \cong 1.202$. Equation (5) corresponds to the well-known nonrelativistic result (see, e.g., Ref. [3]) with pion mass m and "degeneracy factor" 3.

The particle-number density is inversely proportional to the proper particle volume, $\rho \propto r^{-3}$. Then it follows from Eq. (5) for a ratio of the BEC temperature in the atomic gases, $T_C(A)$, to that in the pion gas, $T_C(\pi)$, in nonrelativistic approximation,

$$\frac{T_C(\pi)}{T_C(A)} \cong \frac{m_A}{m}\left(\frac{r_A}{r_\pi}\right)^2 \cong \frac{m_A}{m}10^{10}, \tag{7}$$

where $r_A \cong 10^{-8}$ cm and $r_\pi \cong 10^{-13}$ cm are typical radiuses of atom and pion, respectively, and m_A is the mass of an atom. The Eq. (7) shows that $T_C(\pi) \gg T_C(A)$, and this happens due to $r_\pi \ll r_A$.

Equation (3) gives the total pion number density at $V \to \infty$ in the normal phase $T > T_C$ without BE condensate. At $T < T_C$ the total pion number density becomes a sum of two terms,

$$\rho = \rho_C + \rho^*(T, \mu = m). \tag{8}$$

The second term in the right-hand side of Eq. (8) is given by Eq. (3). The BE condensate ρ_C defined by Eq. (8) corresponds to a macroscopic (proportional to V) number of particles at the lowest quantum level $p = 0$.

To obtain the asymptotic expansion of $\rho^*(T,\mu)$ given by Eq. (3) at $\mu \to m - 0$ we use the identity $[\exp(y) - 1]^{-1} = [\mathrm{cth}(y/2) - 1]/2$ and variable substitution, $p = \sqrt{2}m^{1/2}(m - \mu)^{1/2}x$, similar to those in a nonrelativistic gas [12]. Then one finds,

$$\begin{aligned}
&\rho^*(T,m) - \rho^*(T,\mu)\\
&= \frac{3m^{3/2}}{\sqrt{2}\pi^2}(m-\mu)^{3/2}\int_0^\infty x^2dx\\
&\quad\times\left[\mathrm{cth}\frac{\sqrt{2m(m-\mu)x^2+m^2}-m}{2T}\right.\\
&\quad\left.-\,\mathrm{cth}\frac{\sqrt{2m(m-\mu)x^2+m^2}-\mu}{2T}\right]\\
&\cong \frac{3m^{3/2}}{\sqrt{2}\pi^2}(m-\mu)^{3/2}\int_0^\infty x^2dx\left[\mathrm{cth}\frac{(m-\mu)x^2}{2T}\right.\\
&\quad\left.-\,\mathrm{cth}\frac{(m-\mu)(x^2+1)}{2T}\right]\\
&\cong \frac{6Tm^{3/2}}{\sqrt{2}\pi^2}(m-\mu)^{1/2}\int_0^\infty x^2dx\left[\frac{1}{x^2}-\frac{1}{x^2+1}\right]\\
&= \frac{3Tm^{3/2}}{\sqrt{2}\pi}(m-\mu)^{1/2}.
\end{aligned} \tag{9}$$

At constant density, $\rho^*(T,\mu) = \rho^*(T = T_C, \mu = m)$, one finds in the TL at $T \to T_C + 0$,

$$\begin{aligned}
&\rho^*(T = T_C, \mu = m)\\
&= \rho^*(T,\mu) \cong \rho^*(T,\mu = m) - \frac{3Tm^{3/2}}{\sqrt{2}\pi}(m-\mu)^{1/2}\\
&\cong \rho^*(T = T_C, \mu = m) + \left.\frac{d\rho(T,\mu = m)}{dT}\right|_{T=T_C}\\
&\quad\times(T - T_C) - \frac{3T_Cm^{3/2}}{\sqrt{2}\pi}(m-\mu)^{1/2}.
\end{aligned} \tag{10}$$

[2]The BE condensate formed in the ultrarelativistic regime has been considered in Refs. [9,10] as a *dark matter* candidate in cosmological models.

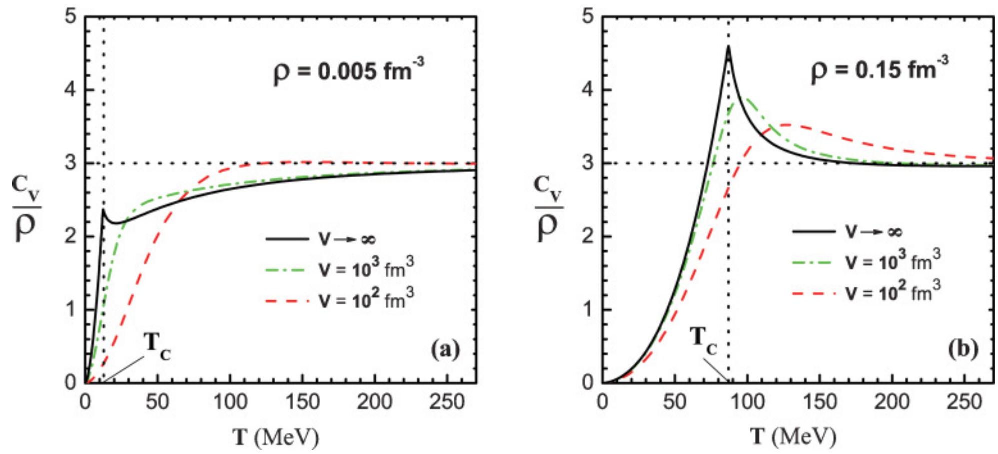

FIG. 2. (Color online) The solid lines demonstrate the temperature dependence of c_V/ρ [Eq. (14)] in the TL at fixed values of $\rho = 0.005$ and 0.15 fm^{-3}. The c_V/ρ in the finite system is described by Eq. (34) (see next section). The dashed lines correspond to $V = 10^2$ fm^3 and the dashed-dotted lines to $V = 10^3$ fm^3, respectively.

Using Eq. (10) one finds the function $\mu(T)$ at $T \to T_C + 0$,

$$m - \mu(T) \cong \frac{2\pi^2}{9T_C^2 m^3}\left[\frac{d\rho(T, \mu = m)}{dT}\Big|_{T=T_C}\right]^2 \times (T - T_C)^2, \quad (11)$$

In the nonrelativistic limit $m/T \gg 1$ one finds, $\rho(T, \mu = m) \cong 3\zeta(3/2)[mT/(2\pi)]^{3/2}$, similar to Eq. (5). Using Eq. (11) one then obtains [12],

$$\frac{m - \mu(T)}{T_C} \cong \frac{9\zeta^2(3/2)}{16\pi} \times \left(\frac{T - T_C}{T_C}\right)^2. \quad (12)$$

In the ultrarelativistic limit, $T/m \gg 1$, one finds, $\rho(T, \mu = m) \cong 3\zeta(3)\, T^3/\pi^2$, similar to Eq. (6). This gives,

$$\frac{m - \mu(T)}{T_C} \cong \frac{18\zeta^2(3)}{\pi^2}\left(\frac{T_C}{m}\right)^3 \times \left(\frac{T - T_C}{T_C}\right)^2. \quad (13)$$

Thus, $\mu(T) \to m$ and $d\mu/dT \to 0$ at $T \to T_C + 0$, both $\mu(T)$ and $d\mu/dT$ are continuous functions at $T = T_C$.

B. Specific heat at fixed particle-number density

The standard description of the BEC phase transition in a nonrelativistic Bose gas is discussed in terms of the specific heat per particle at finite volume, C_V/N (see, e.g., Refs. [3, 12–14]). The relativistic analog of this quantity is:

$$\frac{c_V}{\rho} \equiv \frac{1}{\rho}\left(\frac{\partial \varepsilon}{\partial T}\right)_\rho. \quad (14)$$

The energy density in the TL equals to:

$$\varepsilon(T, \mu) = \rho_C\, m + \varepsilon^*(T, \mu) = \rho_C\, m + \frac{3}{2\pi^2} \times \int_0^\infty p^2 dp \frac{\sqrt{m^2 + p^2}}{\exp[(\sqrt{m^2 + p^2} - \mu)/T] - 1}$$
$$= \rho_C\, m + \frac{3T^2 m^2}{2\pi^2}\sum_{n=1}^\infty \left\{\frac{1}{n^2}K_2\left(\frac{nm}{T}\right) + \frac{m}{2nT}\left[K_1\left(\frac{nm}{T}\right) + K_3\left(\frac{nm}{T}\right)\right]\right\}\exp\left(\frac{n\mu}{T}\right), \quad (15)$$

where $\rho_C = 0$ and $\mu \leqslant m$ at $T \geqslant T_C$, whereas $\rho_C > 0$ and $\mu = m$ at $T < T_C$. The high- and low-temperature behavior of c_V/ρ can be easily found. At $T \to \infty$ and fixed ρ, both the Bose effects and particle mass become inessential. The energy density (15) behaves as $\varepsilon \cong 3T\rho$, thus, $c_V/\rho \cong 3$. Note that in a nonrelativistic gas at $T_C \ll T \ll m$, one finds $\varepsilon \cong (m + 3T/2)\rho$. Thus, the "high-temperature nonrelativistic limit" would give $c_V/\rho \cong 3/2$. At $T_C \gg T \to 0$ the behavior of ε at fixed total particle-number density, $\rho = \rho_C + \rho^*(T, \mu = m)$, is given by,

$$\varepsilon = \rho m + \frac{9\zeta(5/2)}{2}\left(\frac{m}{2\pi}\right)^{3/2} T^{5/2}. \quad (16)$$

This leads to:

$$\left(\frac{c_V}{\rho}\right)_{T\to 0} \cong \frac{45\zeta(5/2)}{4\rho}\left(\frac{m}{2\pi}\right)^{3/2} T^{3/2} \propto T^{3/2} \to 0. \quad (17)$$

Figure 2 shows the temperature dependence of c_V/ρ at fixed ρ. As seen from Fig. 2, c_V/ρ (14) has a maximum at $T = T_C$. The c_V/ρ is a continuous function of T, whereas its temperature derivative has a discontinuity at $T = T_C$. This discontinuity emerges in the TL $V \to \infty$ and can be classified as a third-order phase transition. To estimate the value of c_V/ρ (14) at $T = T_C$ we start from $T > T_C$ when the contribution from $p = 0$ level to c_V/ρ (14) equals zero in the TL and then consider the limit $T \to T_C + 0$. We discuss separately the nonrelativistic and ultrarelativistic approximations.

Using the asymptotic, $K_\nu(x) \cong \sqrt{\pi/(2x)}\exp(-x)[1 + (4\nu^2 - 1)/8x]$ at $x \gg 1$ [11], one finds a nonrelativistic limit of Eqs. (3) and (15), respectively,

$$\rho(T, \mu) \cong 3\left(\frac{mT}{2\pi}\right)^{3/2}\left[\mathrm{Li}_{3/2}(z) + \frac{15T}{8\,m}\mathrm{Li}_{5/2}(z)\right], \quad (18)$$

$$\varepsilon(T,\mu) = 3\left(\frac{mT}{2\pi}\right)^{3/2} m\left[\mathrm{Li}_{3/2}(z) + \frac{27T}{8m}\mathrm{Li}_{5/2}(z)\right] = \rho\, m + 3\left(\frac{m}{2\pi}\right)^{3/2} T^{5/2}\frac{3}{2}\mathrm{Li}_{5/2}(z). \qquad (19)$$

where $z \equiv \exp[-(m-\mu)/T]$ and $\mathrm{Li}_k(z) = \sum_{k=1}^{\infty} z^n/n^k$ is the polylogarithm function [15]. At $T = T_C$ it follows, $z = 1$ and $\mathrm{Li}_k(1) = \zeta(k)$. Using Eqs. (18)–(19), and (12) one finds at $T = T_C \ll m$,

$$\left(\frac{c_V}{\rho}\right)_{T=T_C} \cong \frac{15}{4}\frac{\zeta(5/2)}{\zeta(3/2)} \cong 1.926. \qquad (20)$$

Using the asymptotic expansion, $K_\nu(x) \cong \frac{1}{2}\Gamma(\nu)(x/2)^{-\nu}$ at $x \ll 1$ [11], one finds from Eq. (15) in the ultrarelativistic limit $T \geqslant T_C \gg m$,

$$\varepsilon(T,\mu) \cong \frac{3T^4}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{3}{n^4}\left(1+\frac{n\mu}{T}\right)\right] = \frac{9\zeta(4)}{\pi^2}T^4 + \frac{9\zeta(3)}{\pi^2}T^3\mu. \qquad (21)$$

From Eqs. (21) and (6) it then follows at $m \ll T = T_C$,

$$\left(\frac{c_V}{\rho}\right)_{T=T_C} \cong 12\frac{\zeta(4)}{\zeta(3)} \cong 10.805. \qquad (22)$$

Different pion number densities correspond to different values of the BEC temperature T_C. Equations (20) and (22) show that c_V/ρ goes at $T \to \infty$ to its limiting value 3 from below, if T_C is "small," and from above, if T_C is "large" (see Fig. 2).

Note that at the BEC in atomic gases the number of atoms is conserved. Thus, the temperature dependence of c_V/ρ for the system of atoms at fixed ρ can be straightforwardly measured. This is much more difficult for the pion gas. There is no conservation law of the number of pions, and the special experimental procedure is needed to form the statistical ensemble with fixed number of pions.

III. FINITE SIZE EFFECTS

A. Chemical potential at finite volume

The standard introduction of ρ_C with Eq. (8) is rather formal. To have a more realistic picture, one needs to start with finite volume system and consider the limit $V \to \infty$ explicitly. The main problem is that the substitution, $\sum_{\mathbf{p}} \cdots \cong (V/2\pi^2)\int_0^\infty \cdots p^2 dp$, becomes invalid below the BEC line. We consider separately the contribution to the total pion density from the two lower quantum states,

$$\rho \cong \frac{1}{V}\sum_{\mathbf{p},j}^{\infty}\langle n_{\mathbf{p},j}\rangle = \frac{3}{V}\frac{1}{\exp\left[(m-\mu)/T\right]-1} + \frac{3}{V}\frac{6}{\exp[(\sqrt{m^2+p_1^2}-\mu)/T]-1} + \frac{3}{2\pi^2}\int_{p_1}^{\infty} p^2 dp \frac{1}{\exp[(\sqrt{m^2+p^2}-\mu)/T]-1}. \qquad (23)$$

The first term in the right-hand side of Eq. (23) corresponds to the lowest momentum level $p = 0$, the second one corresponds to the first excited level $p_1 = 2\pi V^{-1/3}$ with the degeneracy factor 6, and the third term approximates the contribution from levels with $p > p_1 = 2\pi V^{-1/3}$. Note that this corresponds to free particles in a box with periodic boundary conditions (see, e.g., Ref. [16]). At any finite V the equality $\mu = m$ is forbidden as it would lead to the infinite value of particle-number density at $p = 0$ level.

At $T < T_C$ in the TL one expects a finite nonzero particle density ρ_C at the $p = 0$ level. This requires $(m-\mu)/T \equiv \delta \propto V^{-1}$ at $V \to \infty$. The particle-number density at the $p = p_1$ level can be then estimated as,

$$\rho_1 = \frac{18V^{-1}}{\exp[(\sqrt{m^2+p_1^2}-\mu)/T]-1} \cong \frac{18V^{-1}}{\delta + p_1^2/(2mT)} \propto \frac{V^{-1}}{V^{-2/3}} = V^{-1/3}, \qquad (24)$$

and it goes to zero at $V \to \infty$. Thus, the second term in the right-hand side of Eq. (23) can be neglected in the TL. One can also extend the lower limit of integration in the third term in the right-hand side of Eq. (23) to $p = 0$, as the region $[0, p_1]$ contributes as $V^{-1/3} \to 0$ in the TL and can be safely neglected. Therefore, we consider the pion number density and energy density at large but finite V in the following form:

$$\rho \cong \frac{3}{V}\frac{1}{\exp\left[(m-\mu)/T\right]-1} + \rho^*(T,\mu), \qquad (25)$$

$$\varepsilon \cong \frac{3}{V}\frac{m}{\exp\left[(m-\mu)/T\right]-1} + \varepsilon^*(T,\mu). \qquad (26)$$

Thus, at large V, the zero-momentum level defines completely the finite size effects of the pion system.

The behavior of $\rho^*(T,\mu)$ at $\mu \to m$ can be found from Eq. (9). At large V, Eq. (25) takes then the following form,

$$\rho \cong \frac{3}{V\delta} + \rho^*(T,\mu) \cong \frac{3}{V\delta} + \rho^*(T,\mu=m) - \frac{3}{\sqrt{2}\pi}(mT)^{3/2}\sqrt{\delta}. \qquad (27)$$

The Eq. (27) can be written as,

$$A\,\delta^{3/2} + B\,\delta - 1 = 0, \qquad (28)$$

where

$$A = \frac{V}{\sqrt{2}\pi}(mT)^{3/2} \equiv a(T)V, \qquad (29)$$

$$B = \frac{V}{3}[\rho - \rho^*(T,\mu=m)] \equiv b(T)V. \qquad (30)$$

The Eq. (28) for δ has two complex roots and one real root. An asymptotic behavior at $V \to \infty$ of the physical (real) root can be easily found. At $T < T_C$ it follows from Eq. (30) that $b(T) > 0$, and one finds from Eq. (28) at large V,

$$\delta \cong \frac{1}{b}V^{-1}. \qquad (31)$$

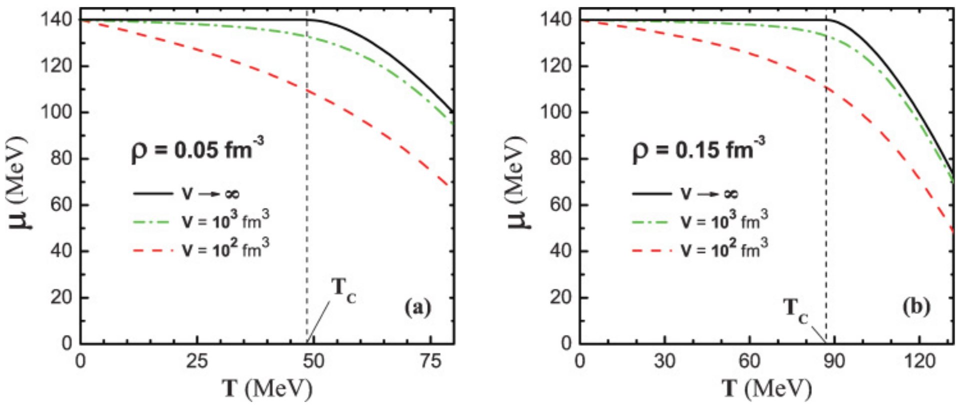

FIG. 3. (Color online) The chemical potential μ as a function of temperature T at fixed particle-number density ρ. The solid line presents the behavior in the TL $V \to \infty$. The dashed line corresponds to $V = 10^2$ fm^3 and the dashed-dotted line to $V = 10^3$ fm^3. The vertical dotted line indicates the BEC temperature T_C. The left panel corresponds to $\rho = 0.05$ fm^{-3} and the right one to $\rho = 0.15$ fm^{-3}.

From Eq. (30) one finds that $b = 0$ at $T = T_C$. In this case, Eq. (28) gives,

$$\delta \cong \frac{1}{a^{2/3}} V^{-2/3}. \qquad (32)$$

The Eq. (28) can be also used at $T > T_C$, if T is close to T_C, thus, $\delta \ll 1$. In this case it follows from Eq. (30) that $b(T) < 0$, and one finds from Eq. (28),

$$\delta \cong \frac{b^2}{a^2}. \qquad (33)$$

Thus, δ is small but finite at $V \to \infty$, and μ remains smaller than m in the TL. The temperature dependence of chemical potential $\mu = \mu(T)$ for $V = 10^2$ fm^3 and 10^3 fm^3 at fixed pion number density ρ is shown in Fig. 3.

The value of c_V/ρ (14) at finite volume V is calculated as,

$$\begin{aligned}\frac{c_V}{\rho} &\equiv \frac{1}{\rho}\left(\frac{\partial \varepsilon}{\partial T}\right)_{\rho,V} = \frac{3}{\rho\, V} \frac{\exp[(m-\mu)/T]}{(\exp[(m-\mu)/T]-1)^2} \\ &\times \frac{m(m-\mu+\mu' T)}{T^2} + \frac{3}{2\pi^2 \rho} \\ &\times \int_0^\infty p^2 dp \frac{\exp[(\sqrt{m^2+p^2}-\mu)/T]}{(\exp[(\sqrt{m^2+p^2}-\mu)/T]-1)^2} \\ &\times \frac{\sqrt{m^2+p^2}(\sqrt{m^2+p^2}-\mu+\mu' T)}{T^2}, \end{aligned} \qquad (34)$$

where $\mu' = (\partial\mu/\partial T)_{\rho,V}$. The temperature dependence of c_V/ρ (34) at several fixed values of V is shown in Fig. 2 by the dashed and dashed-dotted lines.

B. Particle-number fluctuations

The variance of particle-number fluctuations in the GCE at finite V is:

$$\begin{aligned}\langle \Delta N^2 \rangle &\equiv \langle (N - \langle N \rangle)^2 \rangle = \sum_{\mathbf{p},j} \langle n_{\mathbf{p},j}\rangle (1 + \langle n_{\mathbf{p},j}\rangle) \\ &\cong \frac{3}{\exp[(m-\mu)/T]-1} + \frac{3}{\{\exp[(m-\mu)/T]-1\}^2} \\ &+ \frac{3V}{2\pi^2}\int_0^\infty p^2 dp\, \frac{\exp[(\sqrt{m^2+p^2}-\mu)/T]}{\{\exp[(\sqrt{m^2+p^2}-\mu)/T]-1\}^2}, \end{aligned} \qquad (35)$$

where the first two terms in the right-hand side of Eq. (35) correspond to particles of the lowest level $p = 0$, and the third term to particles with $p > 0$. We will use the scaled variance,

$$\omega = \frac{\langle \Delta N^2\rangle}{\langle N \rangle}, \qquad (36)$$

as the measure of particle-number fluctuations. The numerical results for the scaled variance are shown in Fig. 4. At $T > T_C$ the parameter δ goes to the finite limit (33) at $V \to \infty$. This leads to the finite value of ω (36) in the TL. At $T \leqslant T_C$ one finds from Eq. (35) in the TL,

$$\langle \Delta N^2 \rangle \cong 3\delta^{-2} + \tfrac{3}{2} V a \delta^{-1/2}\,. \qquad (37)$$

This gives

$$\omega \equiv \frac{\langle \Delta N^2\rangle}{\langle N\rangle} \cong 3\rho^{-1}V^{-1}\delta^{-2} + \frac{3}{2} a \rho^{-1}\delta^{-1/2}\,, \qquad (38)$$

where $a = a(T)$ is defined in Eq. (29). The substitution of δ in Eq. (38) from (31), gives for $T < T_C$ and $V \to \infty$,

$$\omega \cong 3b^2\rho^{-1}V + \tfrac{3}{2} a b^{1/2} \rho^{-1} V^{1/2} \equiv \omega_C + \omega^*. \qquad (39)$$

The ω_C in the right-hand side of Eq. (39) is proportional to V and corresponds to the particle-number fluctuations in the BE condensate, i.e., at the $p = 0$ level, $\omega_C \cong \sum_j \langle (\Delta n_{p=0,j})^2\rangle / \langle N \rangle$. The second term, ω^*, is proportional to $V^{1/2}$. It comes from the fluctuation of particle numbers at $p > 0$ levels, $\omega^* = \sum_{\mathbf{p},j;p>0} \langle (\Delta n_{\mathbf{p},j})^2 \rangle / \langle N \rangle$. At $T \to 0$, one finds $a \to 0$ and $b \to \rho/3$. This gives the maximal value of the scaled variance, $\omega = \rho V/3 = \langle N \rangle/3$, for given ρ and V values.

The substitution of δ in Eq. (38) from Eq. (32), gives for $T = T_C$ and $V \to \infty$,

$$\omega \cong 3a^{4/3}\rho^{-1}V^{1/3} + \tfrac{3}{2}a^{4/3}\rho^{-1}V^{1/3} \equiv \omega_C + \omega^*. \qquad (40)$$

Figure 5 demonstrates the ratios ρ_C/ρ and ω_C/ω as the functions of T for $V = 10^2, 10^3, 10^4$ fm^3, and at $V \to \infty$. In the TL $V \to \infty$, one finds $\rho_C \to 0$ at $T \geqslant T_C$. The value of ρ_C starts to increase from zero at $T = T_C$ to ρ at $T \to 0$. Thus, ρ_C remains a continuous function of T in the TL. In contrast to this, both ω_C and ω^* have discontinuities at $T = T_C$. They both go to infinity in the TL $V \to \infty$. The ω_C/ω ratio equals to zero at $T > T_C$, "jumps" from 0 to 2/3 at $T = T_C$, and further continuously approaches 1 at $T \to 0$. At $T = T_C$ the contribution of $p = 0$ level to particle density, ρ_C, is negligible

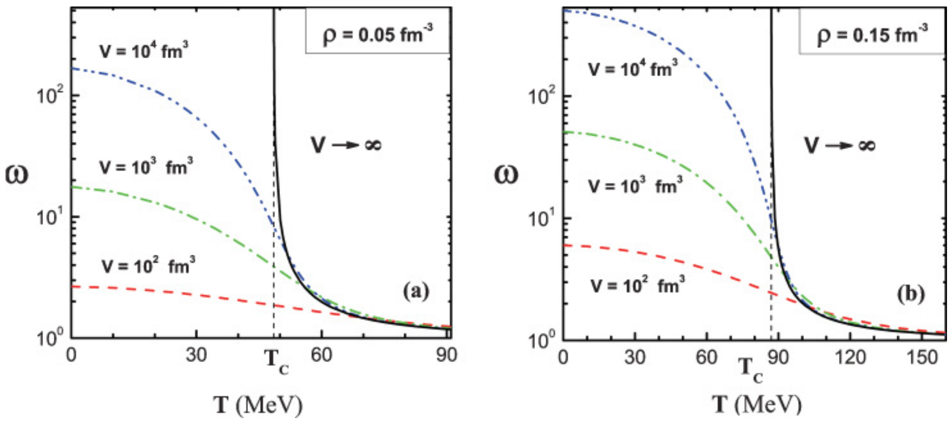

FIG. 4. (Color online) The dashed lines show the GCE scaled variance (36) for the pion gas as a function of temperature T for $V = 10^4$ fm^3, 10^3 fm^3, 10^2 fm^3 (from top to bottom). The vertical dotted line indicates the BEC temperature T_C. The solid line shows the ω (36) in the TL $V \to \infty$. The left panel corresponds to $\rho = 0.05$ fm^{-3} and the right one to $\rho = 0.15$ fm^{-3}.

at $V \to \infty$, but the scaled variance ω_C from this level equals 2/3 of the total scaled variance ω and diverges as $V^{1/3}$. We conclude this section by stressing that the particle-number fluctuations expressed by the scaled variance ω looks as a very promising quantity to search for the BEC in the pion gas.

IV. BEC FLUCTUATION SIGNALS IN HIGH MULTIPLICITY EVENTS

In the GCE, the scaled variances for different charge pion states, $j = +, -, 0$, are equal to each other and equal to the

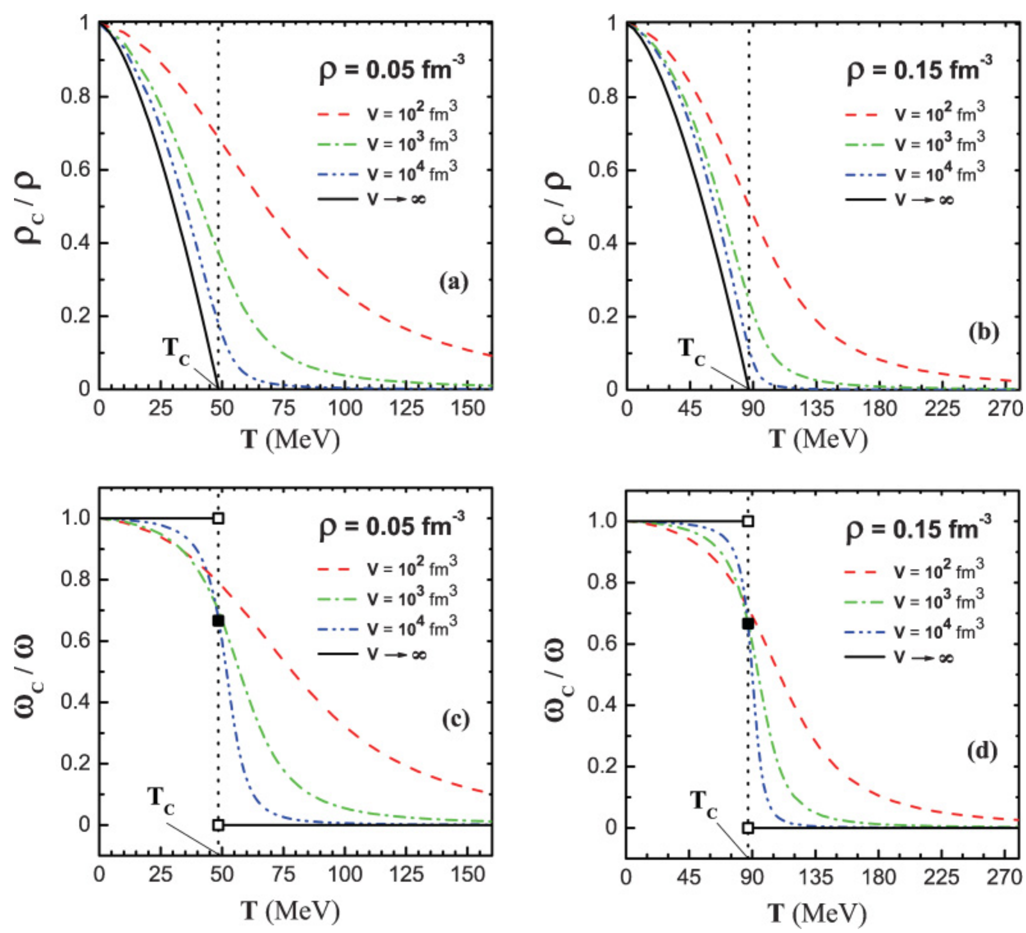

FIG. 5. (Color online) The upper panel shows the ratio of condensate particle-number density to the total particle-number density, ρ_C/ρ, as functions of T for $V = 10^2$, 10^4, 10^4 fm^3, and in the TL $V \to \infty$. The lower panel shows the ratio of particle-number fluctuations in condensate to the total particle-number fluctuations, ω_C/ω, as functions of T for the same volumes. The vertical dotted line indicates the BEC temperature T_C. The left panel corresponds to $\rho = 0.05$ fm^{-3} and the right one to $\rho = 0.15$ fm^{-3}.

scaled variance ω for total number of pions,

$$\omega^j = 1 + \frac{\sum_{\mathbf{p},j} \langle n_{\mathbf{p},j}\rangle^2}{\sum_{\mathbf{p},j} \langle n_{\mathbf{p},j}\rangle} = \omega. \qquad (41)$$

There is a qualitative difference in the properties of the mean multiplicity and the scaled variance of multiplicity distribution in statistical models. In the case of the mean multiplicity results obtained with the GCE, canonical ensemble, and microcanonical ensemble (MCE) approach each other in the TL. One refers here to the thermodynamical equivalence of the statistical ensembles. It was recently found [17–20] that corresponding results for the scaled variance are different in different ensembles, and thus the scaled variance is sensitive to conservation laws obeyed by a statistical system. The differences are preserved in the thermodynamic limit. Therefore, the pion number densities are the same in different statistical ensembles, but this is not the case for the scaled variances of pion fluctuations. The pion number fluctuations in the system with fixed electric charge, $Q = 0$, total pion number, N, and total energy, E, should be treated in the MCE. The volume V is one more MCE parameter.

The MCE microscopic correlators equal to (see also Refs. [2,18]):

$$\begin{aligned}\langle \Delta n_{\mathbf{p},j} \Delta n_{\mathbf{k},i}\rangle_{mce} &= \upsilon^2_{\mathbf{p},j}\delta_{\mathbf{pk}}\delta_{ji} - \upsilon^2_{\mathbf{p},j}\,\upsilon^2_{\mathbf{k},i}\left[\frac{q_j q_i}{\Delta(q^2)}\right.\\ &\left.+\frac{\Delta(\epsilon^2) + \epsilon_{\mathbf{p}}\epsilon_{\mathbf{k}}\,\Delta(\pi^2) - (\epsilon_{\mathbf{p}}+\epsilon_{\mathbf{k}})\Delta(\pi\epsilon)}{\Delta(\pi^2)\Delta(\epsilon^2) - (\Delta(\pi\epsilon))^2}\right], \qquad (42)\end{aligned}$$

where $q_+ = 1, q_- = -1, q_0 = 0$, $\Delta(q^2) = \sum_{\mathbf{p},j} q_j^2 \upsilon^2_{\mathbf{p},j}$, $\Delta(\pi^2) = \sum_{\mathbf{p},j} \upsilon^2_{\mathbf{p},j}$, $\Delta(\epsilon^2) = \sum_{\mathbf{p},j} \epsilon^2_{\mathbf{p}}\upsilon^2_{\mathbf{p},j}$, $\Delta(\pi\epsilon) = \sum_{\mathbf{p},j} \epsilon_{\mathbf{p}}\upsilon^2_{\mathbf{p},j}$. Note that the first term in the right-hand side of Eq. (42) corresponds to the GCE (2). From Eq. (42) one notices that the MCE fluctuations of each mode $\mathbf{p}$ are reduced, and the (anti-)correlations between different modes $\mathbf{p} \neq \mathbf{k}$ and between different charge states appear. This results in a suppression of scaled variance $\omega_{\rm mce}$ in a comparison with the corresponding one ω in the GCE. Note that the MCE microscopic correlators (42), although being different from that in the GCE, are expressed with the quantities calculated in the GCE. The straightforward calculations lead to the following MCE scaled variance for π^0 mesons [2]:

$$\omega^0_{\rm mce} = \frac{\sum_{\mathbf{p},\mathbf{k}} \langle \Delta n_{\mathbf{p},0}\Delta n_{\mathbf{k},0}\rangle_{\rm mce}}{\sum_{\mathbf{p}} \langle n_{\mathbf{p},0}\rangle} \cong \frac{2}{3}\omega. \qquad (43)$$

Due to conditions, $N_+ \equiv N_-$ and $N_+ + N_- + N_0 \equiv N$, it follows, $\omega^{\pm}_{\rm mce} = \omega^0_{\rm mce}/4 = \omega/6$ and $\omega^{\rm ch}_{\rm mce} = \omega^0_{\rm mce}/2 = \omega/3$, where $N_{\rm ch} \equiv N_+ + N_-$.

The pion number fluctuations can be studied in high-energy particle and/or nuclei collisions. To search for the BEC fluctuation signals one needs the event-by-event identifications of both charge and neutral pions. Unfortunately, in most event-by-event studies, only charge pions are detected. In this case the global conservation laws would lead to the strong suppression of the particle-number fluctuations, see also Ref. [2], and no anomalous BEC fluctuations would be seen.

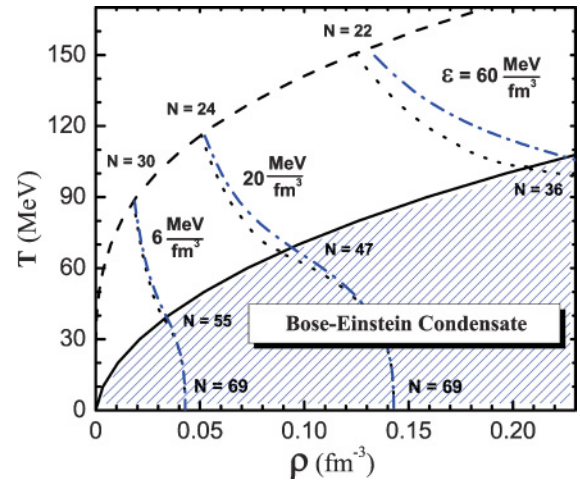

FIG. 6. (Color online) The phase diagram of the ideal pion gas with zero net electric charge. The dashed line corresponds to $\rho = \rho^*(T, \mu = 0)$ and the solid line to the BEC $T = T_C$ (4), both calculated in the TL $V \to \infty$. The dashed-dotted lines present the trajectories in the ρ-T plane with fixed energy densities, $\varepsilon = 6, 20, 60$ MeV/fm^3, calculated for the finite pion system with total energy $E = 9.7$ GeV according to Eq. (26). The dotted lines show the same trajectories calculated in the TL $V \to \infty$. The total numbers of pions N marked along the dashed-dotted lines correspond to three points: $\mu = 0, T = T_C$, and $T = 0$ for $E = 9.7$ GeV.

As an example we consider the high π-multiplicity events in $p + p$ collisions at the beam energy of 70 GeV (see Ref. [21]). In the reaction $p + p \to p + p + N$ with small final proton momenta in the center-of-mass system, the total center-of-mass (c.m.) energy of created pions is $E \cong \sqrt{s} - 2m_p \cong 9.7$ GeV. The estimates [22] reveal a possibility to accumulate the samples of events with fixed $N = 30 \div 50$ and have the full pion identification. Note that for this reaction the kinematic limit is $N^{\rm max} = E/m_\pi \cong 69$. To define the MCE pion system one needs to assume the value of V, in addition to given fixed values of $Q = 0$, $E \cong 9.7$ GeV, and N. The T and μ parameters of the GCE can be then estimated from the following equations,

$$E = V\varepsilon(T, \mu; V), \quad N = V\rho(T, \mu; V). \qquad (44)$$

In calculating the ε and ρ in Eq. (44) we take into account the finite volume effects according to Eqs. (25) and (26) as discussed in Sec. II. Several "trajectories" with fixed energy density are shown in Fig. 6 starting from the line $\mu = 0$ in the pion gas in the ρ-T phase diagram. The MCE scaled variance of π^0 number fluctuations, $\omega^0_{\rm mce}$, increases with increasing of N. The maximal value it reaches at $T \to 0$,

$$\omega^{0\,\max}_{\rm mce} \cong \frac{2}{3}(1 + \langle N_0\rangle^{\max}) = \frac{2}{3}\left(1 + \frac{N^{\max}}{3}\right) \cong 16. \qquad (45)$$

In Fig. 7, $\omega^0_{\rm mce}$ is shown as the function of N. Different possibilities of fixed energy densities and fixed particle-number densities are considered. One way or another, an increase of N leads to a strong increase of the fluctuations of N_0 and $N_{\rm ch}$ numbers due to the BEC effects.

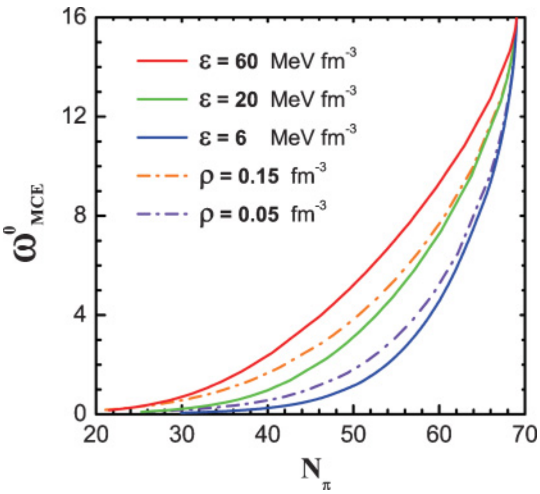

FIG. 7. (Color online) The scaled variance of neutral pions in the MCE is presented as the function of the total number of pions N. Three solid lines correspond to different energy densities, $\varepsilon = 6, 20, 60$ MeV/fm^3 (from bottom to top), calculated according to Eq. (26). Two dashed-dotted lines correspond to different particle-number densities, $\rho = 0.05, 0.15$ fm^{-3} (from bottom to top), calculated according to Eq. (25). The scaled variance $\omega^0_{\rm mce}$ is given by Eq. (43), with ω (36) and $\langle \Delta N^2 \rangle$ (35). The total energy of the pion system is assumed to be fixed, $E = 9.7$ GeV.

The large fluctuations of $N_0/N_{\rm ch} = f$ ratio were also suggested (see, e.g., Ref. [23]) as a possible signal for the disoriented chiral condensate (DCC). The DCC leads to the distribution of f in the form, $dW(f)/df = 1/(2\sqrt{f})$. The thermal Bose gas corresponds to the f distribution centered at $f = 1/2$. Therefore, f distributions from BEC and DCC are very different, and this gives a possibility to distinguish between these two phenomena.

V. SUMMARY

The idea for searching the pion BEC as an anomalous increase of the pion number fluctuations was suggested in our previous article [2]. The fluctuation signals of the BEC have been discussed in Ref. [2] in the thermodynamic limit. At $V \to \infty$, it follows, $\omega = \infty$ at $T \leqslant T_C$. This is evidently not the case for the finite systems. At finite V the scaled variance ω of the pion number fluctuation is finite for all possible combinations of the statistical system parameters. The ω demonstrates different dependence on the system volume V in different parts of the ρ-T phase diagram. In the TL $V \to \infty$, it follows that ω converges to a finite value at $T > T_C$. It increases as $\omega \propto V^{1/3}$ at the BEC line $T = T_C$, and it is proportional to the system volume, $\omega \propto V$, at $T < T_C$. The statistical model description gives no answer on the value of V for given E and N. The system volume remains a free model parameter. Thus, the statistical model does not suggest an exact quantitative predictions for the N dependence of $\omega^0_{\rm mce}$ and $\omega^{\pm}_{\rm mce}$ in the sample of high-energy collision events. However, the qualitative prediction looks rather clear: with increasing of N the pion system approaches the conditions of the BEC. One observes an anomalous increase of the scaled variances of neutral and charged pion number fluctuations. The size of this increase is restricted by the finite size of the pion system. In turn, a size of the created pion system (maximal possible values of N and V) should increase with the collision energy.

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[1] J. Zimanyi, G. Fai, and B. Jakobsson, Phys. Rev. Lett. **43**, 1705 (1979); I. N. Mishustin *et al.*, Phys. Lett. **B276**, 403 (1992); C. Greiner, C. Gong, and B. Müller, Phys. Lett. **B316**, 226 (1993); S. Pratt, Phys. Lett. **B301**, 159 (1993); T. Csörgö and J. Zimanyi, Phys. Rev. Lett. **80**, 916 (1998); A. Bialas and K. Zalewski, Phys. Rev. D **59**, 097502 (1999); Yu. M. Sinyukov, S. V. Akkelin, and R. Lednicky, *Proceedings*, edited by T. Csorgo *et al.* (World Scientific, 1998), p. 66, [arXiv:nucl-th/9909015]; R. Lednicky, V. Lyuboshitz, K. Mikhailov, Y. Sinyukov, A. Stavinsky, and B. Erazmus, Phys. Rev. C **61**, 034901 (2000).

[2] V. V. Begun and M. I. Gorenstein, Phys. Lett. **B653**, 190 (2007); V. Begun and M. I. Gorenstein, *Proceedings of 4th International Workshop Critical Point and Onset of Deconfinement*, 9–13 July 2007, Darmstadt, Germany (arXiv:0709.1434[hep-ph]).

[3] L. D. Landau and E. M. Lifshitz, *Course of Theoretical Physics: Statistical Physics, 3rd edition, Part 1* (Pergamon Press, Oxford, 1980), Vol. 5.

[4] H. E. Haber and H. A. Weldon, Phys. Rev. Lett. **46**, 1497 (1981); Phys. Rev. D **25**, 502 (1982).

[5] J. I. Kapusta, *Finite-Temperature Field Theory* (Cambridge University Press, Cambridge, 1989).

[6] L. Salasnich, Nuovo Cimento B **117**, 637 (2002).

[7] D. T. Son and M. A. Stephanov, Phys. Rev. Lett. **86**, 592 (2001); Phys. At. Nucl. **64**, 834 (2001); [Yad. Fiz. **64**, 899 (2001)]; K. Splittorff, D. T. Son, and M. A. Stephanov, Phys. Rev. D **64**, 016003 (2001).

[8] V. V. Begun and M. I. Gorenstein, Phys. Rev. C **73**, 054904 (2006).

[9] J. Madsen, Phys. Rev. Lett. **69**, 571 (1992); J. Madsen, Phys. Rev. D **64**, 027301 (2001).

[10] D. Boyanovsky, H. J. de Vega, and N. G. Sanchez, Phys. Rev. D **77**, 043518 (2008).

[11] M. Abramowitz and I. A. Stegun, eds., *Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables*

(National Bureau of Standards, Applied Mathematics Series 55, Washington, DC, 1964).
[12] V. V. Tolmachev, *Theory of Bose Gas* (Moscow University Press, Moscow, 1969) (in Russian).
[13] Yu. B. Rumer and M. Sh. Rivkin, *Thermodynamics, Statistical Physics, and Kinetics* (Nauka, Moscow, 1972) (in Russian).
[14] W. Greiner, L. Neise, and H. Stöcker, *Thermodynamics and Statistical Mechanics* (Springer-Verlag, New York, 1995) (Corrected second printing, 1997).
[15] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and Series* (Nauka, Moscow, 1986).
[16] K. B. Tolpygo, *Thermodynamics and Statistical Physics* (Kiev University Press, Kiev, 1966) (in Russian).
[17] V. V. Begun, M. Gazdzicki, M. I. Gorenstein, and O. S. Zozulya, Phys. Rev. C **70**, 034901 (2004); V. V. Begun, M. Gorenstein, A. P. Kostyuk, and O. S. Zozulya, Phys. Rev. C **71**, 054904 (2005); V. V. Begun, M. Gorenstein, and O. S. Zozulya, Phys. Rev. C **72**, 014902 (2005); V. V. Begun, M. I. Gorenstein, A. P. Kostyuk, and O. S. Zozulya, J. Phys. G **32**, 935 (2006); A. Keränen, F. Becattini, V. V. Begun, M. I. Gorenstein, and O. S. Zozulya, J. Phys. G **31**, S1095 (2005); F. Becattini, A. Keränen, L. Ferroni, and T. Gabbriellini, Phys. Rev. C **72**, 064904 (2005); J. Cleymans, K. Redlich, and L. Turko, Phys. Rev. C **71**, 047902 (2005); J. Cleymans, K. Redlich, and L. Turko, J. Phys. G **31**, 1421 (2005).
[18] V. V. Begun, M. I. Gorenstein, M. Hauer, V. P. Konchakovski, and O. S. Zozulya, Phys. Rev. C **74**, 044903 (2006); V. V. Begun, M. Gazdzicki, M. I. Gorenstein, M. Hauer, V. P. Konchakovski, B. Lungwitz, Phys. Rev. C **76**, 024902 (2007).
[19] M. Hauer, V. V. Begun, and M. I. Gorenstein, arXiv: 0706.3290 (Submitted to Euro. Phys. J.).
[20] M. I. Gorenstein, *Proceedings of 4th International Workshop on Critical Point and Onset Deconfinement*, 9–13 July 2007, Darmstadt, Germany, arXiv:0709.1428[nucl-th]; V. V. Begun, *Proceedings of International Workshop on Relativistic Nuclear Physics: From Nuclotron to LHC Energies*, 18–22 June 2007, Kiev, Ukraine, arXiv:0711.2912[nucl-th].
[21] P. F. Ermolov *et al.*, Phys. At. Nucl. **67**, 108 (2004); V. V. Avdeichikov *et al.*, JINR-P1-2004-190 (2005), p. 45.
[22] V. A. Nikitin (private communication).
[23] J. P. Blaizot and A. Krzywicki, Phys. Rev. D **46**, 246 (1992); Acta Phys. Pol. B **27**, 1687 (1996); K. Rajagopal and F. Wilczek, Nucl. Phys. **B399**, 395 (1993); J. D. Bjorken, Acta Phys. Pol. B **28**, 2773 (1997).