Bohr Hamiltonian for collective low-lying vibrational states of well deformed nuclei

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(Received 8 January 2008; revised manuscript received 8 April 2008; published 20 June 2008)

It is shown that the Bohr Hamiltonian in the intrinsic frame used for the description of the low-lying vibrational states in the well-deformed nuclei can be presented in two different forms. In the first form the Hamiltonian has three different mass coefficients but the quadrupole transition operator has a standard form with one parameter only. The second form of the Hamiltonian can be derived by a transformation from β and γ to the new variables. In this form the Hamiltonian contains only one mass coefficient but the quadrupole operator takes a different form with three parameters. It is shown also that this Hamiltonian can describe a situation when a collectivity of the vibrational states is rather low, but their excitation energies are relatively small, while the $E2$ transitions inside the bands are very strong.

DOI: 10.1103/PhysRevC.77.064317 PACS number(s): 21.10.Re, 21.60.Ev, 23.20.Lv, 27.70.+q

I. INTRODUCTION

The Bohr Hamiltonian [1] being applied to the description of collective nuclear properties usually has a very simple form of the kinetic energy term which is determined by one constant mass coefficient for all three modes of excitation: rotation, β- and γ-vibrations. A more general form of the kinetic energy term was introduced in [2,3] and applied in [4–7]. In [2,3] this kinetic energy term was also transformed into the intrinsic frame where it was shown that the mass coefficients could be different for different modes of motion. It is necessary to note also that in [8] the Grodzins products for the ground, β-, and γ-bands are presented with different mass coefficients for different bands, however, without an indication of the Hamiltonian which gave this result.

For the case when the collective Hamiltonian is written in terms of the Bohr's β- and γ-variables earlier we found that the assumption of a common mass coefficient for three bands contradicts, in the case of the well-deformed nuclei, the experimental data on energies and $E2$ transition probabilities. Namely, in [9,10] it was shown that significantly different mass coefficients for rotational and γ-vibrational motion are needed to explain experimental data on Grodzins products for the ground and the γ-bands. It was also shown in [10] that in order to resolve this contradiction it is necessary to consider the Bohr Hamiltonian not with a constant mass coefficient but with a mass tensor having also nonzero components with angular momentum $L = 2$ and 4. In this case in the limit of the well-deformed axially symmetric nucleus we obtain different mass coefficients for rotation ($B_{\rm rot}$), γ-vibrations (B_γ), and β-vibrations (B_β).

At the same time it is well known that dealing with a Hamiltonian having a coordinate dependent mass tensor is very difficult. It is much more practical to perform a transformation of coordinates in such a way as to obtain the Hamiltonian with one constant mass coefficient [11]. This is done in the present paper for the case of the well-deformed axially symmetric nuclei. However, as a consequence of this coordinate transformation the potential energy and the quadrupole transition operator are changed.

There is another question which we also want to discuss below. In the phenomenological collective quadrupole model all dynamical variables, namely the Euler angles and the β- and γ-shape variables, are considered by definition as the collective ones, i.e., describing a motion of many nucleons. At the same time, in RPA calculations performed for the well-deformed axially symmetric nuclei [12–14], a small number of components exhaust the structure of the γ-phonon. As a measure for the collectivity of the quadrupole state we consider the value of the corresponding $E2$ transition probability from the ground state to the vibrational state. In the case of the Bohr Hamiltonian with a constant mass coefficient the rotational and vibrational degrees of freedom are not completely decoupled because the Grodzins products for different rotational bands are expressed through the same mass coefficient.

We should mention here that the Grodzins products can be derived separately for the ground, β-, and γ-bands only in the case of the well-deformed axially symmetric nuclei in which rotation, β-, and γ-vibrations are decoupled. Indeed, in these nuclei the amplitudes of the β- and γ-oscillations around equilibrium values are relatively small. For instance, the ratio $\langle(\beta-\beta_0)^2\rangle/\beta_0^2$ which is equal to the ratio of $B(E2)$'s: $B(E2;0^+_{\rm g.s.}\rightarrow 2^+_\beta)/B(E2;0^+_{\rm g.s.}\rightarrow 2^+_{\rm g.s.})$ takes the values from 0.014 to less than 0.005 in the well-deformed nuclei, and for $\beta_0 = 0.3$ the amplitude of the β-oscillations as a rule is not larger than 0.03. As a consequence the quadrupole transition operator can be presented as a sum of three terms. The first term produces transitions inside the same rotational band only. The second term excites only the γ-band and the third term excites only the β-band acting on the ground state band.

If, by varying some parameters of the Hamiltonian, we decrease the $E2$ transition probability for the transition from the ground state to the β- or γ-vibrational state, i.e., decrease the collectivity of the vibrational state in agreement with the RPA results, we automatically increase the energy of the vibrational state because the product of the energy and

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the corresponding $B(E2)$ is inversely proportional to the mass coefficient B. But if we do not want to change the result for the ground state band we should keep B fixed. For instance, the moment of inertia is proportional to B. On the other hand in the RPA calculations performed for the well-deformed axially symmetric nuclei the description of the vibrational modes is completely decoupled from the description of the rotational motion.

Let us consider this in more detail. If the mass coefficient has the same value for all three bands a collectivity of the β- and γ-vibrations is fixed in the sense that the products

$$E(2^+_{\beta,\gamma})B(E2;0^+_{gr}\to 2^+_{\beta,\gamma}) = c\cdot\frac{\hbar^2 q^2}{B}, \tag{1}$$

where $c=1$ for γ-band, $c=0.5$ for β-band and $q=\frac{3}{4\pi}eZr_0^2A^{2/3}$ are fixed by the properties of the ground state rotational band through the mass coefficient B. It follows from Eq. (1) that a decrease of $B(E2;0^+_{gr}\to 2^+_{\beta,\gamma})$ can go only in parallel with unrestricted increase of $E(2^+_{\beta,\gamma})$. This is not the case in RPA calculations where the energies of the 2^+_γ and of the 0^+_β states cannot be higher than the energies of the lowest two-quasiparticle state with a corresponding value of K. Moreover, when the energy of the β- or γ-phonon is close to the two-quasiparticle energy a small change in the phonon energy produces a tremendous change in the $B(E2)$ value. This argument shows that we need more than one mass parameter.

We have mentioned above that the appearance in the Bohr Hamiltonian applied to the consideration of the well-deformed axially symmetric nuclei of three different mass coefficients means that the mass tensor contains not only the monopole component but also components with angular momentum $L=2$ and 4. But these last two components are unavoidably shape-dependent. This has important consequences for transitional nuclei. However, in the present paper we only consider the consequences of this fact for the well-deformed nuclei, which were partly discussed above.

II. TWO EQUIVALENT FORMS OF THE BOHR HAMILTONIAN AND THE CORRESPONDING QUADRUPOLE OPERATOR

In this section we transform the Bohr Hamiltonian and the quadrupole operator by changing the dynamical variables, thus going from a Hamiltonian H^I with three different mass coefficients and the usual Bohr quadrupole operator $Q^I_{2\mu}$ which we call the Hamiltonian and the quadrupole operator in the first form to a Hamiltonian with one common mass coefficient called the Hamiltonian in the second form H^{II} but with a transformed quadrupole operator $Q^{II}_{2\mu}$ expressed in terms of the new variables which will depend on B_β and B_γ. Based on the transformed Hamiltonian H^{II} we will show in a very simple way that the Bohr Hamiltonian with different $B_{\rm rot}$, B_β, and B_γ resolves the contradiction with the results obtained in the framework of RPA, i.e., describes a situation where the collectivity of the vibrational states is low. This means that the $E2$ reduced transition probabilities from the ground state to the 2^+_γ and the 2^+_β states are very small, but their excitation energies have values typical for the collective states, and the $E2$ transitions inside the bands continue to be strong.

TABLE I. The average values of the ratio $B_\gamma/B_{\rm rot}$ for the rare earth elements. The experimental data are taken from [27].

Element	$B_\gamma/B_{\rm rot}$	Element	$B_\gamma/B_{\rm rot}$
^{152,154}Sm	3.82	$^{162-168}$Er	4.18
$^{156-160}$Gd	3.66	$^{168-176}$Yb	6.22
$^{158-164}$Sm	4.00	$^{174-180}$Hf	2.97

As is shown in [10] in the case of the well-deformed axially symmetric nuclei, where K is a good quantum number, the Bohr Hamiltonian can be written in the following form which we call in this paper as the form I:

$$H^I = H_{\rm rot} + H^I_\gamma + H^I_\beta, \tag{2}$$

where

$$H_{\rm rot} = \frac{\hbar^2}{6B_{\rm rot}\beta_0^2}\left(\hat{L}^2 - \hat{L}_3^2\right), \tag{3}$$

$$H^I_\gamma = -\frac{\hbar^2}{2B_\gamma\beta_0^2}\frac{1}{\gamma}\frac{\partial}{\partial\gamma}\gamma\frac{\partial}{\partial\gamma} + \frac{\hbar^2}{2B_\gamma}\frac{\hat{L}_3^2}{4\beta_0^2\gamma^2} + \frac{1}{2}C_\gamma\beta_0^2\gamma^2, \tag{4}$$

$$H^I_\beta = -\frac{\hbar^2}{2}\left(\frac{1}{B_\beta}\frac{\partial^2}{\partial\beta^2} + \frac{2}{B_\gamma}\frac{1}{\beta}\frac{\partial}{\partial\beta} + \frac{2}{B_\beta}\frac{1}{\beta}\frac{\partial}{\partial\beta}\right) + \frac{1}{2}C_\beta(\beta-\beta'_0)^2. \tag{5}$$

Compared to our previous paper [10] we used a new definition above of $C_\gamma((C_\gamma)_{old} = C_\gamma\beta_0^4)$. The importance of using three different mass coefficients is related to the fact that the average value of $B_\gamma/B_{\rm rot}$ for rare earth nuclei is 4 (see Table I) and that the value of $B_\beta/B_{\rm rot}$ is even lager (Table II). The ratios of the mass coefficients shown in Tables I and II have been obtained from the ratios of the corresponding Grodzins products of the

TABLE II. The experimental values of the ratio $B_\beta/B_{\rm rot}$ for some of the rare earth and actinide nuclei. The experimental data are taken from [27].

Nucleus	$E(2^+_\beta$(in KeV)	$B_\beta/B_{\rm rot}$
^{154}Sm	1178	6.4 ± 1.6
^{156}Gd	1129	11.7 ± 1.1
^{158}Gd	1260	20.2 ± 2.6
^{158}Dy	1086	4.0 ± 1.0
^{160}Dy	1350	9.6 ± 1.2
^{170}Er	960	30.4 ± 3.3
^{168}Yb	1233	4.1 ± 0.5
^{170}Yb	1139	6.9 ± 1.3
^{172}Yb	1466	31.1 ± 1.3
^{174}Hf	900	3.7 ± 1.0
^{176}Hf	1227	6.7 ± 1.3
^{178}Hf	1277	16.2 ± 3.7
^{230}Th	678	6.8 ± 3.1
^{232}Th	774	2.3 ± 1.0
^{238}U	966	17.2 ± 3.0
^{238}Pu	983	17.2 ± 5.3

transition energies and the corresponding $B(E2)$'s:

$$E(2_i^+)B(E2;0^+_{\text{g.s.}} \to 2_i^+) \quad i = \text{g.s.}, \beta, \gamma. \tag{6}$$

It is seen from Table II that the experimental errors are large in some cases. We also mention that there are well known difficulties in the identification of the β-vibrational states [15–19]. We have considered the lowest excited 0^+ states as β-vibrational states. The exception is the case of ^{172}Yb where the second excited 0^+ state having a larger $B(E2)$ for the transition to the ground state has been considered as the β-vibrational state. It is also necessary to keep in mind that the lowest 0^+ excited state in the well-deformed nuclei can be a mixture of the β-vibrational and the pairing vibrational states and this decreases the collectivity of this state, i.e., increases the value of B_β. Of course in the case of transitional nuclei the collective potential cannot be taken as above in the form of the harmonic oscillator centered at the equilibrium values of deformation. It should be taken in a more complicated form as it follows from the Strutinsky type calculations or as it is approximated analytically, e.g., in [20–26].

The difference between β_0' and β_0 is due to the terms in H_β linear in $\partial/\partial\beta$. However this difference is of the order of the square of the ratio of the energy of the 2_1^+ state to the energy of the β-vibrational state, i.e., the correction is of the order of 1% and can be neglected.

In the case of the well-deformed nuclei the expression for the quadrupole transition operator appropriate for H^I can be presented as

$$Q^I_{2\mu} = Q_{\text{rot},2\mu} + Q^I_{\gamma,2\mu} + Q^I_{\beta,2\mu}, \tag{7}$$

where

$$\begin{aligned} Q_{\text{rot},2\mu} &= q\beta_0 D^2_{\mu 0}, \\ Q^I_{\gamma,2\mu} &= q\beta_0 \frac{1}{\sqrt{2}}\left(D^2_{\mu 2} + D^2_{\mu -2}\right)\gamma, \\ Q^I_{\beta,2\mu} &= q(\beta - \beta_0) D^2_{\mu 0}. \end{aligned} \tag{8}$$

Let us transform the Hamiltonian and the quadrupole operator (2)–(8) to the form II by a transformation of the coordinates. Since in the RPA calculations the frequencies ω_γ and ω_β have an upper limit it is convenient to express C_γ and C_β through ω_γ and ω_β:

$$C_\gamma = B_\gamma \omega_\gamma^2 \tag{9}$$

and

$$C_\beta = B_\beta \omega_\beta^2. \tag{10}$$

Let us introduces dimensionless parameters η_γ and η_β characterizing the ratios of the mass coefficients:

$$\begin{aligned} B_\gamma &= \eta_\gamma B_{\text{rot}}, \\ B_\beta &= \eta_\beta B_{\text{rot}}. \end{aligned} \tag{11}$$

Now it is easy to see from Eqs. (4) and (8) that introducing a new variable by scaling γ,

$$\tilde{\gamma} = \sqrt{\eta_\gamma}\gamma, \tag{12}$$

we obtain for H^I_γ and $Q^I_{\gamma,2\mu}$ the following expressions:

$$\begin{aligned} H^I_\gamma \to H^{II}_\gamma &= -\frac{\hbar^2}{2B_{\text{rot}}\beta_0^2}\frac{1}{\tilde{\gamma}}\frac{\partial}{\partial\tilde{\gamma}}\tilde{\gamma}\frac{\partial}{\partial\tilde{\gamma}} \\ &\quad + \frac{\hbar^2}{8B_{\text{rot}}\beta_0^2}\frac{\hat{L}_3^2}{\tilde{\gamma}^2} + \frac{1}{2}B_{\text{rot}}\omega_\gamma^2\beta_0^2\tilde{\gamma}^2, \end{aligned} \tag{13}$$

$$Q^I_\gamma \to Q^{II}_{\gamma,2\mu} = \frac{1}{\sqrt{\eta_\gamma}} q\beta_0 \frac{1}{\sqrt{2}}\left(D^2_{\mu 2} + D^2_{\mu -2}\right)\tilde{\gamma}. \tag{14}$$

We see that in terms of the new collective variable, H^{II}_γ does not contain the factor η_γ but $Q^{II}_{\gamma,2\mu}$ contains this factor. So, the normalized wave functions depend only on $\tilde{\gamma}$ but the value of $B(E2;0^+_{gr} \to 2^+_\gamma)$ is proportional to $1/\eta_\gamma$. In other words, we obtain the γ-vibrational state with the fixed energy equal to $\hbar\omega_\gamma$ but with a value of $B(E2)$ to the ground band scaled by factor $1/\eta_\gamma$. In the case when $B_\gamma \gg B_{\text{rot}}$ a transition from the γ to the ground band is very weak.

The β-vibrational part of the Hamiltonian and of the quadrupole operator can be considered in an analogous way. The details are given in Appendix A. The result is the following: Introducing a new variable

$$\tilde{\beta} = \sqrt{\eta_\beta}(\beta - \beta_0), \tag{15}$$

we obtain [see Eqs. (A10) and (A11)]

$$H^I_\beta \to H^{II}_\beta = -\frac{\hbar^2}{2B_{\text{rot}}}\frac{\partial^2}{\partial\tilde{\beta}^2} + \frac{1}{2}B_{\text{rot}}\omega_\beta^2\tilde{\beta}^2, \tag{16}$$

$$Q^I_{\beta,2\mu} \to Q^{II}_{\beta,2\mu} = \frac{1}{\sqrt{\eta_\beta}} q\tilde{\beta} D^2_{\mu 0}. \tag{17}$$

We see that, as in the case of γ-vibrations, the Hamiltonian describing β-vibrations expressed in terms of the new variable does not depend on B_β. However, $Q^{II}_{\beta,2\mu}$ is proportional to $\eta_\beta^{-1/2}$, and when B_β becomes very large $B(E2;0^+_{gr} \to 2^+_\beta)$ becomes very small, describing a situation of low collectivity of the β-band, although with a fixed energy of the β-vibrational 0^+ state.

Summarizing the results derived above we obtain the Bohr Hamiltonian in the form

$$H^{II} = H_{\text{rot}} + H^{II}_\gamma + H^{II}_\beta, \tag{18}$$

where H^{II}_γ and H^{II}_β are given by Eqs. (13) and (16). We mention, that H_{rot} does not change its form under our coordinate transformation. The Hamiltonian H^{II} contains only one mass coefficient B_{rot}. However the quadrupole operator expressed in the same variables becomes more complicated

$$\begin{aligned} Q^{II}_{2\mu} = \Bigg(& Q_{\text{rot},2\mu} + q\frac{1}{\sqrt{\eta_\gamma}}\beta_0\frac{1}{\sqrt{2}}\left(D^2_{\mu 2} + D^2_{\mu -2}\right)\tilde{\gamma} \\ & + q\frac{1}{\sqrt{\eta_\beta}}\tilde{\beta}D^2_{\mu 0}\Bigg). \end{aligned} \tag{19}$$

and consists of three terms with three independent parameters each for each term. The situation with two forms of the collective Hamiltonian and the quadrupole operator is illustrated by Table III.

TABLE III. The number of the parameters in the kinetic energy part of the Hamiltonian ($\mathcal{N}_H$) and in the quadrupole operator ($\mathcal{N}_Q$) for two different forms of the Hamiltonian and the quadrupole operator.

Operator	$\mathcal{N}_H$	Operator	$\mathcal{N}_Q$
H^{I}	3	$Q^{I}_{2\mu}$	1
H^{II}	1	$Q^{II}_{2\mu}$	3

III. SUMMARY

We have considered a Hamiltonian and a quadrupole operator expressed in terms of the Bohr β- and γ-variables and applied for a description of the collective quadrupole motion in even-even well-deformed axially symmetric nuclei. This Hamiltonian has three different mass coefficients for the three excitation modes. By a scaling of the collective variables we have transformed the Bohr Hamiltonian in such a way that instead of the Hamiltonian with the three different mass coefficients we obtain the Hamiltonian with one mass coefficient. However, as a consequence of this coordinate transformation, the potential energy and the quadrupole transition operator are changed. It indicates the possibility that also in a general case the collective Hamiltonian might be transformed by a suitable coordinate transformation to the form with the kinetic energy term having one mass coefficient, however, with modified potential energy and quadrupole operator. This would be very useful because it is much more convenient to perform calculations with this form of the Hamiltonian.

For the case when B_β and B_γ are much larger than $B_{\rm rot}$, which is the case for the rare earth nuclei, we have shown that the eigensolutions of the Schrödinger equation contain three independent bands with fixed excitation energies and very weak $E2$ transitions between the bands. This was achieved by transforming the Hamiltonian to the new collective variables which differ from the previous ones by the scaling factors. This property that enables us to vary $B(E2)$ via B_β and B_γ also gives an intuitive feeling for the mass parameters. At this point it becomes obvious that using only one mass coefficient is a significant approximation which is not realized in the experiment. It shows also that the Bohr Hamiltonian with the three different mass coefficients applied to the well-deformed axially symmetric nuclei can describe a situation where β- and γ-bands are connected to the ground band by very weak $E2$ transitions, in accordance with the RPA results, but where at the same time the excitation energies are smaller than the pairing gap while $E2$ transitions inside the bands are strong. As we have seen above this situation is described by the Hamiltonian and the quadrupole operator in the form II which is, in fact, more suitable for a description of the situation with strong intraband and weak interband transitions.

ACKNOWLEDGMENTS

The authors are grateful to Drs. A. Gelberg, J. Jolie, N. Pietralla, W. Scheid, and Dipl. Phys. C. Scholl for useful discussions. One of us (R.V.J.) would like to thank colleagues from the University of Cologne for their kind hospitality. This work was supported by the DFG (Germany) under Contract Br 799/13-1 and by a DFG East European-Germany Collaboration grant.

APPENDIX A

The Schrödinger equation for the β-dependent part of the total wave function is

$$H_\beta \psi(\beta) = E_\beta \psi(\beta). \tag{A1}$$

Let us introduce a new function $\tilde{\psi}(\beta)$ which is determined as

$$\psi(\beta) = \beta^a \tilde{\psi}(\beta), \tag{A2}$$

where the parameter a is determined so as to cancel in H_β terms proportional to $\frac{1}{\beta}\frac{\partial}{\partial\beta}$

$$a = -\left(1 + \eta_\gamma^{-1}\right)\eta_\beta. \tag{A3}$$

We obtain

$$\tilde{H}_\beta \tilde{\psi}(\beta) = E_\beta \tilde{\psi}(\beta), \tag{A4}$$

where

$$\tilde{H}_\beta = -\frac{\hbar^2}{2B_{\rm rot}}\frac{1}{\eta_\beta}\frac{\partial^2}{\partial\beta^2} + V(\beta) \tag{A5}$$

and

$$V(\beta) = \frac{\hbar^2}{2B_{\rm rot}}\left(\frac{1}{\eta_\gamma} + 1\right)\left(\frac{\eta_\beta}{\eta_\gamma} + \eta_\beta - 1\right)\frac{1}{\beta^2} + \frac{1}{2}B_{\rm rot}\omega_\beta^2\eta_\beta(\beta - \beta_0')^2. \tag{A6}$$

Let us approximate $V(\beta)$ by an oscillator potential. This is justified for the well-deformed nuclei whose potential energy has a sufficiently deep minimum at the equilibrium value of deformation. If we neglect a constant term which produces an equal shift of all energies we obtain

$$V(\beta) \approx \frac{1}{2}\left(1 + 3\frac{\beta_0 - \beta_0'}{\beta_0}\right)B_{\rm rot}\omega_\beta^2\eta_\beta(\beta - \beta_0)^2, \tag{A7}$$

where

$$\beta_0 = \beta_0' + \beta_0\left(\frac{\hbar^2/B_{\rm rot}\beta_0^2}{\hbar\omega_\beta}\right)^2\left(1 + \frac{1}{\eta_\gamma}\right) \times \left(1 + \frac{1}{\eta_\gamma} - \frac{1}{\eta_\beta}\right). \tag{A8}$$

The second term in the last expression is two orders of magnitude smaller than the first one and can be neglected. Introducing a new variable

$$\tilde{\beta} = \sqrt{\eta_\beta}(\beta - \beta_0), \tag{A9}$$

we obtain for $\tilde{H}_\beta$ and $Q^\beta_{2\mu}$

$$\tilde{H}_\beta = -\frac{\hbar^2}{2B_{\rm rot}}\frac{\partial^2}{\partial\tilde{\beta}^2} + \frac{1}{2}B_{\rm rot}\omega_\beta^2\tilde{\beta}^2, \tag{A10}$$

$$Q^\beta_{2\mu} = \frac{1}{\sqrt{\eta_\beta}}q\tilde{\beta}D^2_{\mu 0}. \tag{A11}$$

APPENDIX B

For completeness we present below the formulas for the energies and the $B(E2)$'s:

$$E(2^+_{gr}) = \frac{\hbar^2}{B_{\rm rot}\beta_0^2}, \tag{B1}$$

$$E(0^+_{\beta}) = \hbar\sqrt{\frac{C_\beta}{B_\beta}}, \tag{B2}$$

$$E(2^+_{\gamma}) = \hbar\sqrt{\frac{C_\gamma}{B_\gamma}}, \tag{B3}$$

$$B(E2; 0^+_{gr} \to 2^+_{gr}) = q^2\beta_0^2, \tag{B4}$$

where

$$q = \frac{3}{4\pi} eZr_0^2 A^{2/3}. \tag{B5}$$

$$B(E2; 0^+_{gr} \to 2^+_{\beta}) = \frac{1}{2}q^2\frac{\hbar}{\sqrt{B_\beta C_\beta}}, \tag{B6}$$

$$B(E2; 0^+_{gr} \to 2^+_{\gamma}) = q^2\frac{\hbar}{\sqrt{B_\gamma C_\gamma}}, \tag{B7}$$

$$B(E2; 2^+_{\gamma} \to 2^+_{gr}) = \frac{10}{7}B(E2; 2^+_{\gamma} \to 0^+_{gr}). \tag{B8}$$

An additional factor 1/2 in Eq. (B6) compared to Eqs. (B4) and (B7) takes into account the fact that from five quadrupole degrees of freedom one is related to β-vibrations but two are related to γ-vibrations and two to rotation [8].

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