Three-nucleon bound state in a spin-isospin dependent three dimensional approach

S. Bayegan,* M. R. Hadizadeh,† and M. Harzchi‡
Department of Physics, University of Tehran, P. O. Box 14395-547, Tehran, Iran
(Received 19 November 2007; revised manuscript received 7 April 2008; published 30 June 2008)

A spin-isospin-dependent three-dimensional approach based on momentum vectors for formulation of the three-nucleon bound state is presented in this article. The three-nucleon Faddeev equations with two-nucleon interactions are formulated as a function of vector Jacobi momenta, specifically the magnitudes of the momenta and the angle between them with the inclusion of the spin-isospin quantum numbers, without employing a partial wave decomposition. As an application the spin-isospin-dependent Faddeev integral equations are solved with Bonn-B potential. Our result for the Triton binding energy with the value of −8.152 MeV is in good agreement with the achievements of the other partial wave based methods.

DOI: 10.1103/PhysRevC.77.064005 PACS number(s): 21.45.−v, 21.10.Hw, 21.30.−x, 27.10.+h

I. INTRODUCTION

During the past several years, several methods have been developed to solve the nonrelativistic Schrödinger equation accurately for few-nucleon bound states by using realistic nuclear potentials. These methods are the coupled-rearrangement-channel Gaussian-basis variational (CRCGV) [1], the stochastic variational (SV) [2], the hyperspherical harmonic variational (HH) [3], the Green's function Monte Carlo (GFMC) [4], the no-core shell model (NCSM) [5], the effective interaction hyperspherical harmonic (EIHH) [6] and the Faddeev. These calculational approaches are mostly based on a partial-wave (PW) decomposition. Stochastic and Monte Carlo methods, however, are performed directly using the position vectors in the configuration space. One of the most viable approaches appears to be the Faddeev method.

The calculations based on the Faddeev approach are performed after a PW expansion with phenomenological potentials either in the momentum space [7–12] or in the configuration space [13–18]. Recent bound-state calculations with the Faddeev approach have been done with the chiral potentials in the momentum space [19–22]. Experience in three-nucleon calculations shows that the standard treatment based on a PW decomposition is quite successful but also rather complex, because each building block related to involved operators requires extended algebra. The Faddeev calculations based on a PW decomposition, which includes the spin-isospin degrees of freedom, after truncation leads to a set of a finite number of coupled equations in two variables for the amplitudes and one needs a large number of partial waves to get converged results. In view of this large number of interfering terms it appears natural to give up such an expansion and work directly with the vector variables. On this basis three- and four-body bound states have recently been studied in a three-dimensional (3D) approach where the spin-isospin degrees of freedom have been neglected in the first attempt [23–27]. In the case of three-body bound state the Faddeev equations have been formulated for three identical bosons as a function of vector Jacobi momenta, with the specific stress upon the magnitudes of the momenta and the angle between them. Adding the spin-isospin to the 3D formalism is a major additional task, which will increase more degrees of freedom into the states and therefore will lead to a strictly finite number of coupled equations [28]. In this article we have attempted to implement this task by including the spin-isospin degrees of freedom in the $3N$ bound-state formalism. To this end we have formulated the Faddeev equations for the $3N$ bound state with the advantage of using the realistic NN forces. The presented 3D formalism in this article in comparison with the traditional PW formalism avoids the highly involved angular-momentum algebra occurring for the permutation operators. According to the spin-isospin states that have been taken into account, we have obtained the 8, 12, 16, and 24 coupled equations for a description of the $3N$ bound state, i.e., ^{3}H and ^{3}He. In this way, we solve the Faddeev integral equations for calculation of the Triton binding energy with Bonn-B potential. The input to our calculations is the two-body t matrix that has been calculated in an approach based on a helicity representation and depends on the magnitudes of the initial and final momenta and the angle between them [29].

This manuscript is organized as follows. In Sec. II we present the formalism. Meaning that we have derived the Faddeev equations and the $3N$ wave function in a realistic 3D scheme both as a function of Jacobi momenta vectors and the spin-isospin quantum numbers. Also the novel 3D representation of the Faddeev equations is contrasted with the corresponding traditional PW representation. In Sec. III we present our results for the triton binding energy and compare them with the results obtained from the PW calculations. To test our calculations the calculated expectation values of the Hamiltonian operator are compared to the obtained eigenvalue energies. Finally in Sec. IV a summary and an outlook will be presented.

*bayegan@khayam.ut.ac.ir
†hadizade@khayam.ut.ac.ir
‡harzchi@ut.ac.ir

II. FORMULATION FOR 3N BOUND STATE IN A 3D FADDEEV SCHEME

A. The Faddeev equations

The bound state of three pairwise-interacting nucleons is described by the Faddeev equation [11]:

$$|\psi\rangle \equiv |\psi_{12,3}\rangle = G_0 t P |\psi\rangle, \tag{1}$$

where G_0 is the free $3N$ propagator, t denotes the NN transition matrix determined by a two-body Lippman-Schwinger equation, and $P = P_{12}P_{23} + P_{13}P_{23}$ is the sum of cyclic and anticyclic permutations of the three nucleons. The total $3N$ wave function $|\Psi\rangle$ is composed of the three Faddeev components as:

$$|\Psi\rangle = (1 + P)|\psi\rangle. \tag{2}$$

The antisymmetry property of $|\psi\rangle$ under exchange of the interacting particles 1 and 2 guarantees that $|\Psi\rangle$ is totally antisymmetric. To solve Eq. (1) in the momentum space we introduce the $3N$ basis states in a 3D formalism as (see Fig. 1):

$$|\mathbf{pq}\,\alpha\rangle \equiv |\,\mathbf{pq}\,\alpha_S\alpha_T\rangle, \tag{3}$$

the basis states involve two standard Jacobi momenta $\mathbf{p}$ and $\mathbf{q}$ [11], and $|\alpha\rangle$ is the spin-isospin parts of the basis states, where the spin part is defined as:

$$|\alpha_S\rangle \equiv |[(s_1\,s_2)s_{12}\,s_3]SM_S\rangle \equiv |\left(s_{12}\,\frac{1}{2}\right)SM_S\rangle, \tag{4}$$

and the isospin part $|\alpha_T\rangle$ is similar to the spin part. As indicated in Fig. 1 the angular dependence explicitly appears in the Jacobi vector variables, whereas in a standard PW approach the angular dependence leads to two orbital angular-momentum quantum numbers, i.e., l_{12} and l_3 [11]. It indicates that in the present 3D formalism there is no coupling between the orbital angular momenta and the corresponding spin quantum numbers. Therefore we couple the spin quantum numbers s_{12} and s_3 to the total spin S and its third component M_S as $|(s_{12}\,s_3)SM_S\rangle$. For the isospin quantum numbers similar coupling scheme leads to the total isospin T, M_T as $|(t_{12}\,t_3)TM_T\rangle$.

To evaluate the transition and the permutation operators we need the free $3N$ basis states $|\mathbf{p}\,\mathbf{q}\,\gamma\,\rangle$, where

$$|\gamma\rangle \equiv |\gamma_S\gamma_T\,\rangle, \quad |\gamma_S\rangle \equiv \left|m_{s_1}m_{s_2}\,m_{s_3}\right\rangle. \tag{5}$$

The quantities $m_{s_i}(i = 1, 2, 3)$ are the third components of the spins of the three nucleons. The isospin part of the basis states $|\gamma_T\rangle$ is similar to the spin part. To achieve this aim when changing the $3N$ basis states $|\alpha\,\rangle$ to the free $3N$ basis states $|\gamma\rangle$ we need to calculate the following Clebsch-Gordan coefficients (see Appendix A):

$$\langle\gamma|\alpha\rangle = g_{\gamma\alpha} \equiv g^S_{\gamma\alpha}g^T_{\gamma\alpha} = \left\langle m_{s_1}m_{s_2}m_{s_3}\middle|\left(s_{12}\,\frac{1}{2}\right)SM_S\right\rangle \times \left\langle m_{t_1}m_{t_2}\,m_{t_3}\middle|\left(t_{12}\,\frac{1}{2}\right)TM_T\right\rangle. \tag{6}$$

The introduced basis states are complete and normalized as:

$$\sum_\xi \int d^3p \int d^3q\,|\mathbf{pq}\,\xi\,\rangle\,\langle\mathbf{pq}\xi\,| = \mathbf{1},\ \langle\mathbf{pq}\,\xi|\,\mathbf{p'q'}\xi'\rangle = \delta^3(\mathbf{p}-\mathbf{p'})\,\delta^3(\mathbf{q}-\mathbf{q'})\,\delta_{\xi\,\xi'}, \tag{7}$$

where ξ indicates α and γ quantum number sets. Now we can represent the Eq. (1) with respect to the basis states that have been already introduced in Eq. (3):

$$\langle\mathbf{p}\,\mathbf{q}\alpha\,|\psi\rangle = \sum_{\alpha'}\int d^3p' \int d^3q'\langle\mathbf{p}\,\mathbf{q}\alpha\,|G_0 t P|\mathbf{p'}\,\mathbf{q'}\alpha'\,\rangle \times \langle\mathbf{p'}\,\mathbf{q'}\alpha'|\psi\rangle. \tag{8}$$

For evaluating the Eq. (8), we need to evaluate the matrix elements of $\langle\mathbf{p}\,\mathbf{q}\alpha|G_0 t P|\mathbf{p'}\,\mathbf{q'}\alpha'\rangle$, toward this aim, it is convenient to insert the free $3N$ completeness relations as:

$$\begin{aligned}\langle\mathbf{p}\,\mathbf{q}\alpha|G_0 t P|\mathbf{p'}\,\mathbf{q'}\alpha'\rangle &= \sum_\gamma\sum_{\gamma'}\langle\,\alpha|\gamma\rangle\langle\mathbf{p}\,\mathbf{q}\,\gamma|G_0 t P|\mathbf{p'}\,\mathbf{q'}\gamma'\rangle\langle\gamma'|\alpha'\rangle \\ &= \sum_{\gamma,\gamma'} g_{\alpha\gamma}g_{\gamma'\alpha'}\langle\mathbf{p}\,\mathbf{q}\gamma|G_0 t P|\mathbf{p'}\,\mathbf{q'}\gamma'\rangle.\end{aligned} \tag{9}$$

For evaluating the matrix elements $\langle\mathbf{p}\,\mathbf{q}\,\gamma|G_0 t P|\mathbf{p'}\,\mathbf{q'}\gamma'\rangle$ we should insert again a free $3N$ completeness relation between the between the two-nucleon t matrix and the permutation operators as:

$$\begin{aligned}\langle\mathbf{p}\,\mathbf{q}\gamma|G_0 t P|\,\mathbf{p'}\,\mathbf{q'}\gamma'\rangle &= \frac{1}{E - \frac{p^2}{m} - \frac{3q^2}{4m}}\sum_{\gamma''}\int d^3p''\int d^3q'' \\ &\quad\times\langle\mathbf{p}\,\mathbf{q}\gamma|t|\mathbf{p''}\,\mathbf{q''}\gamma''\rangle\langle\mathbf{p''}\,\mathbf{q''}\gamma''|P|\mathbf{p'}\,\mathbf{q'}\gamma'\rangle,\end{aligned} \tag{10}$$

where the matrix elements of the two-body t matrix and the permutation operator P are evaluated separately as:

$$\begin{aligned}\langle\mathbf{p}\,\mathbf{q}\gamma|t|\mathbf{p''}\,\mathbf{q''}\gamma''\rangle &= \delta^3(\mathbf{q}-\mathbf{q''})\,\delta_{m_{s_3}m''_{s_3}}\delta_{m_{t_3}m''_{t_3}}\langle\mathbf{p}m_{s_1}m_{s_2}m_{t_1}m_{t_2}|t(\epsilon) \\ &\quad\times\left|\mathbf{p''}m''_{s_1}m''_{s_2}m''_{t_1}m''_{t_2}\right\rangle,\end{aligned} \tag{11}$$

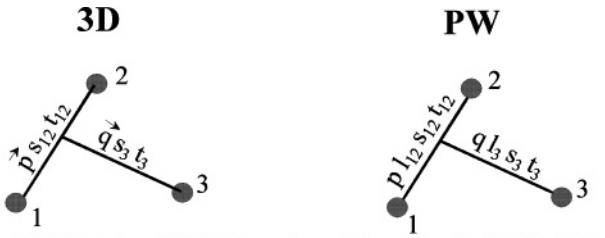

FIG. 1. Definition of the $3N$ basis states in the 3D approach in comparison with the corresponding basis states in the PW approach.

$$\begin{aligned}
&\langle \mathbf{p}''\,\mathbf{q}''\gamma''\,|P|\mathbf{p}'\,\mathbf{q}'\gamma'\rangle \\
&= \delta^3\left(\mathbf{p}'' + \frac{1}{2}\mathbf{p}' + \frac{3}{4}\mathbf{q}'\right)\delta^3\left(\mathbf{q}'' - \mathbf{p}' + \frac{1}{2}\mathbf{q}'\right) \\
&\quad \times \delta_{m''_{s_1}m'_{s_2}}\delta_{m''_{s_2}m'_{s_3}}\delta_{m''_{s_3}m'_{s_1}}\delta_{m''_{t_1}m'_{t_2}}\delta_{m''_{t_2}m'_{t_3}}\delta_{m''_{t_3}m'_{t_1}} \\
&\quad + \delta^3\left(\mathbf{p}'' + \frac{1}{2}\mathbf{p}' - \frac{3}{4}\mathbf{q}'\right)\delta^3\left(\mathbf{q}'' + \mathbf{p}' + \frac{1}{2}\mathbf{q}'\right) \\
&\quad \times \delta_{m''_{s_1}m'_{s_3}}\,\delta_{m''_{s_2}m'_{s_1}}\delta_{m''_{s_3}m'_{s_2}}\delta_{m''_{t_1}m'_{t_3}}\delta_{m''_{t_2}m'_{t_1}}\delta_{m''_{t_3}m'_{t_2}}, \qquad (12)
\end{aligned}$$

where the two-body subsystem energy in the NN t matrix is $\epsilon = E - \frac{3q^2}{4m}$.

In order to evaluate the matrix elements of the permutation operator P we have used the relation between the Jacobi momenta in the different $3N$ systems (312), (231), and (123). Inserting Eqs. (11) and (12) into Eq. (10) leads to:

$$\begin{aligned}
&\langle \mathbf{p}\,\mathbf{q}\gamma|G_0 t P|\mathbf{p}'\,\mathbf{q}'\gamma'\rangle \\
&= \frac{1}{E - \frac{p^2}{m} - \frac{3q^2}{4m}}\Bigg[\delta^3\left(\mathbf{q} - \mathbf{p}' + \frac{1}{2}\mathbf{q}'\right)\delta_{m_{s_3}m'_{s_1}}\delta_{m_{t_3}m'_{t_1}} \\
&\quad \times \left\langle \mathbf{p}m_{s_1}m_{s_2}m_{t_1}m_{t_2}\middle|t(\epsilon)\middle|\frac{-1}{2}\mathbf{q} - \mathbf{q}'m'_{s_2}m'_{s_3}m'_{t_2}m'_{t_3}\right\rangle \\
&\quad + \delta^3\left(\mathbf{q} + \mathbf{p}' + \frac{1}{2}\mathbf{q}'\right)\delta_{m_{s_3}m'_{s_2}}\,\delta_{m_{t_3}m'_{t_2}}\langle \mathbf{p}m_{s_1}m_{s_2}m_{t_1}m_{t_2}|t(\epsilon) \\
&\quad \times \left|\frac{1}{2}\mathbf{q} + \mathbf{q}'m'_{s_3}m'_{s_1}m'_{t_3}m'_{t_1}\right\rangle\Bigg]. \qquad (13)
\end{aligned}$$

Inserting Eq. (13) into Eq. (9) and consequently inserting into Eq. (8) and integrating over $\mathbf{p}'$ variable yields:

$$\begin{aligned}
\langle \mathbf{p}\,\mathbf{q}\alpha|\psi\rangle &= \frac{1}{E - \frac{p^2}{m} - \frac{3q^2}{4m}}\sum_{\gamma,\gamma',\alpha'} g_{\alpha\gamma}\,g_{\gamma'\alpha'}\int d^3q' \\
&\quad \times \Bigg[\left\langle \mathbf{p}m_{s_1}m_{s_2}m_{t_1}m_{t_2}\middle|t(\epsilon)\middle|\frac{-1}{2}\mathbf{q} - \mathbf{q}'m'_{s_2}m'_{s_3}m'_{t_2}m'_{t_3}\right\rangle \\
&\quad \times \delta_{m_{s_3}m'_{s_1}}\,\delta_{m_{t_3}m'_{t_1}}\left\langle \mathbf{q} + \frac{1}{2}\mathbf{q}'\,\mathbf{q}'\alpha'\middle|\psi\right\rangle \\
&\quad + \left\langle \mathbf{p}m_{s_1}m_{s_2}m_{t_1}m_{t_2}\middle|t(\epsilon)\middle|\frac{1}{2}\mathbf{q} + \mathbf{q}'m'_{s_3}m'_{s_1}m'_{t_3}m'_{t_1}\right\rangle \\
&\quad \times \delta_{m_{s_3}m'_{s_2}}\,\delta_{m_{t_3}m'_{t_2}}\left\langle -\mathbf{q} - \frac{1}{2}\mathbf{q}'\,\mathbf{q}'\alpha'\middle|\psi\right\rangle\Bigg]. \qquad (14)
\end{aligned}$$

Applying the permutation operator P_{12} action on the Faddeev component, the space, and the spin-isospin parts of the basis states results in:

$$\begin{aligned}
P_{12}|\psi\rangle &= -|\psi\rangle, \\
P_{12}|\mathbf{p}\,\mathbf{q}\rangle &= |-\mathbf{p}\,\mathbf{q}\rangle, \\
P_{12}|\alpha\rangle &= (-)^{s_1+s_2-s_{12}}(-)^{t_1+t_2-t_{12}}|\alpha\rangle = (-)^{s_{12}+t_{12}}|\alpha\rangle, \\
P_{12}|\gamma\rangle &= |m_{s_2}m_{s_1}m_{s_3}\,m_{t_2}m_{t_1}m_{t_3}\rangle, \qquad (15)
\end{aligned}$$

and consequently the following relations would be concluded:

$$\begin{aligned}
&\langle \mathbf{p}\,\mathbf{q}\alpha\,|\psi\rangle \\
&= -(-)^{s_{12}+t_{12}}\langle -\mathbf{p}\,\mathbf{q}\,\alpha\,|\psi\rangle, \left\langle \mathbf{p}m_{s_1}m_{s_2}m_{t_1}m_{t_2}\middle|t(\epsilon)\right. \\
&\quad \times \left.\middle|\mathbf{p}'m'_{s_1}m'_{s_2}m'_{t_1}m'_{t_2}\right\rangle \\
&= \left\langle \mathbf{p}m_{s_1}m_{s_2}m_{t_1}m_{t_2}\middle|t(\epsilon)P_{12}\middle|-\mathbf{p}'m'_{s_2}m'_{s_1}\,m'_{t_2}m'_{t_1}\right\rangle. \qquad (16)
\end{aligned}$$

Therefore, we can rewrite Eq. (14) as:

$$\begin{aligned}
\langle \mathbf{p}\,\mathbf{q}\alpha|\psi\rangle &= \frac{1}{E - \frac{p^2}{m} - \frac{3q^2}{4m}}\sum_{\gamma,\gamma',\alpha'} g_{\alpha\gamma}\,g_{\gamma'\alpha'}\int d^3q' \\
&\quad \times \Bigg\{\left\langle \mathbf{p}m_{s_1}m_{s_2}m_{t_1}m_{t_2}\middle|t(\epsilon)\middle|\frac{-1}{2}\mathbf{q} - \mathbf{q}'m'_{s_2}m'_{s_3}m'_{t_2}m'_{t_3}\right\rangle \\
&\quad \times \delta_{m_{s_3}m'_{s_1}}\,\delta_{m_{t_3}m'_{t_1}}\left\langle \mathbf{q} + \frac{1}{2}\mathbf{q}'\,\mathbf{q}'\alpha'\middle|\psi\right\rangle \\
&\quad + \langle \mathbf{p}m_{s_1}m_{s_2}\,m_{t_1}m_{t_2}|t(\epsilon)P_{12} \\
&\quad \times \left|\frac{-1}{2}\mathbf{q} - \mathbf{q}'m'_{s_1}m'_{s_3}m'_{t_1}m'_{t_3}\right\rangle\delta_{m_{s_3}m'_{s_2}}\,\delta_{m_{t_3}m'_{t_2}} \\
&\quad \times[-(-)^{s'_{12}+t'_{12}}]\left\langle \mathbf{q} + \frac{1}{2}\mathbf{q}'\,\mathbf{q}'\,\alpha'\middle|\psi\right\rangle\Bigg\} \\
&= \frac{1}{E - \frac{p^2}{m} - \frac{3q^2}{4m}}\sum_{\gamma,\gamma',\alpha'} g_{\alpha\gamma}\,g_{\gamma'\alpha'}\,\delta_{m_{s_3}m'_{s_1}}\delta_{m_{t_3}m'_{t_1}} \\
&\quad \times \int d^3q'\langle \mathbf{p}m_{s_1}m_{s_2}m_{t_1}m_{t_2}|t(\epsilon)(1 - P_{12}) \\
&\quad \times \left|\frac{-1}{2}\mathbf{q} - \mathbf{q}'m'_{s_2}m'_{s_3}m'_{t_2}m'_{t_3}\right\rangle\left\langle \mathbf{q} + \frac{1}{2}\mathbf{q}'\mathbf{q}'\alpha'\middle|\psi\right\rangle. \qquad (17)
\end{aligned}$$

The final derivation of Eq. (17) is made by the exchange of labels m'_{s_1}, m'_{t_1} to m'_{s_2}, m'_{t_2} and reverse of it in the second term as well as the following relation;

$$\begin{aligned}
g_{\gamma'\alpha'} &= (-)^{s'_{12}+t'_{12}}\left\langle m'_{s_2}m'_{s_1}\,m'_{s_3}\middle|\left(s'_{12}\,\frac{1}{2}\right)S'M'_S\right\rangle \\
&\quad \times \left\langle m'_{t_2}m'_{t_1}m'_{t_3}\middle|\left(t'_{12}\,\frac{1}{2}\right)T'M'_T\right\rangle. \qquad (18)
\end{aligned}$$

By introducing the physical representation of the two-body t matrix follows (see Appendix B);

$$\begin{aligned}
&{}_a\langle \mathbf{p}m_{s_1}m_{s_2}m_{t_1}m_{t_2}|t(\varepsilon)|\mathbf{p}'m'_{s_1}m'_{s_2}\,m'_{t_1}m'_{t_2}\rangle_a \\
&= \langle \mathbf{p}m_{s_1}m_{s_2}m_{t_1}m_{t_2}|t(\varepsilon)(1 - P_{12})|\mathbf{p}'\,m'_{s_1}m'_{s_2}m'_{t_1}m'_{t_2}\rangle, \qquad (19)
\end{aligned}$$

the three-dimensional Faddeev integral equations can be obtained as:

$$\begin{aligned}
\langle \mathbf{p}\,\mathbf{q}\alpha|\psi\rangle &= \frac{1}{E - \frac{p^2}{m} - \frac{3q^2}{4m}}\sum_{\gamma,\gamma',\alpha'} g_{\alpha\gamma}\,g_{\gamma'\alpha'}\delta_{m_{s_3}m'_{s_1}}\delta_{m_{t_3}m'_{t_1}} \\
&\quad \times \int d^3q'\,{}_a\langle \mathbf{p}m_{s_1}m_{s_2}m_{t_1}m_{t_2}|t(\epsilon) \\
&\quad \times \left|\frac{-1}{2}\mathbf{q} - \mathbf{q}'m'_{s_2}m'_{s_3}m'_{t_2}m'_{t_3}\right\rangle_a\left\langle \mathbf{q} + \frac{1}{2}\mathbf{q}'\,\mathbf{q}'\alpha'\middle|\psi\right\rangle. \qquad (20)
\end{aligned}$$

The Faddeev component $\langle \mathbf{p}\,\mathbf{q}\alpha|\psi\rangle$ is given as a function of Jacobi momenta vectors, $\mathbf{p}$ and $\mathbf{q}$, and also quantum number sets, α, as a solution of the spatial three-dimensional integral equations, Eq. (20). To solve this equation directly and without employing the PW projections, we have to define a coordinate system. It is convenient to choose the spin polarization direction parallel to the z axis and express the momentum vectors in this coordinate system. By these considerations we can rewrite Eq. (20) as:

$$\psi^{\alpha}(p\,q\,x_{pq}) = \frac{1}{E-\frac{p^2}{m}-\frac{3q^2}{4m}} \int_0^{\infty} dq' q'^2 \int_{-1}^{+1} dx_{q'} \int_0^{2\pi} d\varphi_{q'} \times \sum_{\alpha'} T_{\alpha\alpha'}(p,\tilde{\pi},x_{p\tilde{\pi}};\epsilon)\psi^{\alpha'}(\pi\,q'\,x_{\pi q'}), \quad (21)$$

where

$$T_{\alpha\alpha'}(p,\tilde{\pi},x_{p\tilde{\pi}};\epsilon) = \sum_{\gamma,\gamma'} g_{\alpha\gamma}\, g_{\gamma'\alpha'}\, \delta_{m_{s_3} m'_{s_1}}\, \delta_{m_{t_3} m'_{t_1}} \times t_{a\, m_{s_1} m_{s_2}\, m_{t_1} m_{t_2}}^{m'_{s_2} m'_{s_3}\, m'_{t_2} m'_{t_3}}(p,\tilde{\pi},x_{p\tilde{\pi}};\epsilon), \quad (22)$$

$$\begin{aligned}
x_{pq} &= x_p x_q + \sqrt{1-x_p^2}\sqrt{1-x_q^2}\sin(\phi_p-\phi_q),\\
x_{pq'} &= x_p x_{q'} + \sqrt{1-x_p^2}\sqrt{1-x_{q'}^2}\sin(\phi_p-\phi_{q'}),\\
x_{qq'} &= x_q x_{q'} + \sqrt{1-x_q^2}\sqrt{1-x_{q'}^2}\sin(\phi_q-\phi_{q'}),\\
\tilde{\pi} &= \sqrt{\frac{1}{4}q^2+q'^2+qq'x_{qq'}},\\
x_{p\tilde{\pi}} &= \frac{\frac{1}{2}qx_{pq}+q'x_{pq'}}{\tilde{\pi}},\\
\pi &= \sqrt{q^2+\frac{1}{4}q'^2+qq'x_{qq'}},\\
x_{\pi q'} &= \frac{qx_{qq'}+\frac{1}{2}q'}{\pi}.
\end{aligned} \quad (23)$$

In a standard PW approach, Eq. (21) is replaced by a set of an infinite number of coupled two-dimensional integral equations for the amplitudes with the kernels containing relatively complicated geometrical expressions:

$$\psi^{\alpha}(p\,q) = \frac{1}{E-\frac{p^2}{m}-\frac{3q^2}{4m}} \int_0^{\infty} dq' q'^2 \int_{-1}^{+1} dx_{q'} \times \sum_{l''_{12},\alpha'} \frac{t_{l_{12}l''_{12}}^{s_{12}j_{12}t_{12}}(p,\tilde{\pi};\epsilon)}{\tilde{\pi}^{l''_{12}}} G_{\alpha\alpha'}(q,q',x_{q'}) \frac{\psi^{\alpha'}(\pi\,q')}{\pi^{l''_{12}}}, \quad (24)$$

where, as is shown in Fig. 1, the spin space as well as the isospin parts of the basis states in the PW decomposition are $|\alpha\rangle \equiv |((l_{12}s_{12})j_{12}(l_3s_3)j_3)JM_J(t_{12}t_3)TM_T\rangle$. $G_{\alpha\alpha'}(q,q',x_{q'})$ is composed of Legendre functions, powers of q and q', and purely complicated geometrical quantities like Clebsch-Gordan coefficients and $6j$ symbols. The comparison of Eqs. (21) and (24) shows that new 3D formalism avoids the highly involved angular-momentum algebra occurring for the permutations and additionally it will be more efficient especially for the three-body forces [30].

B. The 3N wave function

The representation of the total wave function, Eq. (2), with respect to the basis states that have been introduced in Eq. (3), reads as follows:

$$\begin{aligned}
\langle \mathbf{p}\mathbf{q}\,\alpha|\Psi\rangle &= \langle \mathbf{p}\mathbf{q}\,\alpha|(1+P)|\psi\rangle\\
&= \langle \mathbf{p}\mathbf{q}\,\alpha\,|\psi\rangle + \langle \mathbf{p}\mathbf{q}\,\alpha\,|P_{12}P_{23}|\psi\rangle\\
&\quad + \langle \mathbf{p}\mathbf{q}\,\alpha|P_{13}P_{23}|\psi\rangle,
\end{aligned} \quad (25)$$

where the first Faddeev component

$$\langle \mathbf{p}\mathbf{q}\,\alpha|\psi\rangle \equiv {}_3\langle \mathbf{p}\mathbf{q}\,\alpha\,|\psi\rangle \equiv \psi^{\alpha}(\mathbf{p},\mathbf{q}) \equiv \psi^{\alpha}(p\,q\,x_{pq}), \quad (26)$$

is given explicitly as a three-dimensional integral equation, Eq. (21). Here the subscript 3 of the bra basis states stands for the three-body subsystem $(12,3)$, which as matter of convenience, is called subsystem 3. For the second and third components we need to evaluate the action of the cyclic and the anticyclic permutation operators $P_{12}P_{23}$ and $P_{13}P_{23}$ on the first component as:

$$\begin{aligned}
\langle \mathbf{p}\mathbf{q}\,\alpha|P_{12}P_{23}|\psi\rangle &\equiv {}_3\langle \mathbf{p}\,\mathbf{q}\,\alpha|P_{12}P_{23}|\psi\rangle\\
&= \sum_{\alpha'}\int d^3p' \int d^3q'_3 \langle \mathbf{p}\,\mathbf{q}\alpha|P_{12}P_{23}|\mathbf{p}'\mathbf{q}'\alpha'\rangle_3\\
&\quad \times {}_3\langle \mathbf{p}'\,\mathbf{q}'\alpha'|\psi\rangle\\
&= \sum_{\alpha'}\int d^3p' \int d^3q'_3 \langle \mathbf{p}\,\mathbf{q}\alpha|\mathbf{p}'\,\mathbf{q}'\alpha'\rangle_1\\
&\quad \times {}_3\langle \mathbf{p}'\,\mathbf{q}'\,\alpha'|\psi\rangle,
\end{aligned} \quad (27)$$

$$\begin{aligned}
\langle \mathbf{p}\mathbf{q}\,\alpha|P_{13}P_{23}|\psi\rangle &\equiv {}_3\langle \mathbf{p}\,\mathbf{q}\,\alpha|P_{13}P_{23}|\psi\rangle\\
&= \sum_{\alpha'}\int d^3p' \int d^3q'_3 \langle \mathbf{p}\,\mathbf{q}\alpha|P_{13}P_{23}|\mathbf{p}'\,\mathbf{q}'\alpha'\rangle_3\\
&\quad \times {}_3\langle \mathbf{p}'\,\mathbf{q}'\alpha'|\psi\rangle\\
&= \sum_{\alpha'}\int d^3p' \int d^3q'_3 \langle \mathbf{p}\,\mathbf{q}\alpha|\mathbf{p}'\,\mathbf{q}'\alpha'\rangle_2\\
&\quad \times {}_3\langle \mathbf{p}'\,\mathbf{q}'\alpha'\,|\psi\rangle,
\end{aligned}$$

the space as well as the spin-isospin parts of the coordinate transformations ${}_3\langle|\rangle_1$ and ${}_3\langle|\rangle_2$ can be evaluated as:

$$\begin{aligned}
{}_3\langle \mathbf{p}\,\mathbf{q}\alpha|\mathbf{p}'\,\mathbf{q}'\alpha'\rangle_1 &= {}_3\langle \mathbf{p}\,\mathbf{q}|\mathbf{p}'\,\mathbf{q}'\rangle_1\, {}_3\langle \alpha|\alpha'\rangle_1\\
&= \delta^3\left(\mathbf{p}'+\frac{1}{2}\mathbf{p}+\frac{3}{4}\mathbf{q}\right)\delta^3\left(\mathbf{q}'-\mathbf{p}+\frac{1}{2}\mathbf{q}\right)\\
&\quad \times \delta_{M_S M'_S}\delta_{SS'}\,\delta_{M_T M'_T}\delta_{TT'}\,C_S^*(\alpha_S,s'_{23})\\
&\quad \times C_T^*(\alpha_T,t'_{23}),
\end{aligned} \quad (28)$$

$$\begin{aligned}
{}_3\langle \mathbf{p}\,\mathbf{q}\alpha|\mathbf{p}'\,\mathbf{q}'\alpha'\rangle_2 &= {}_3\langle \mathbf{p}\,\mathbf{q}|\mathbf{p}'\,\mathbf{q}'\rangle_2\, {}_3\langle \alpha|\alpha'\rangle_2\\
&= \delta^3\left(\mathbf{p}'+\frac{1}{2}\mathbf{p}-\frac{3}{4}\mathbf{q}\right)\delta^3\left(\mathbf{q}'+\mathbf{p}+\frac{1}{2}\mathbf{q}\right)\\
&\quad \times \delta_{M_S M'_S}\delta_{SS'}\,\delta_{M_T M'_T}\delta_{TT'}\,C_S^{**}(\alpha_S,s'_{31})\\
&\quad \times C_T^{**}(\alpha_T,t'_{31}),
\end{aligned} \quad (29)$$

where the spin coefficients C_S^* and C_S^{**} are given as:

$$C_S^*(\alpha_S, s'_{23}) = (-)^{s'_{23}+2s_1+s_2+s_3} \begin{Bmatrix} s_1 & s_2 & s_{12} \\ s_3 & S & s'_{23} \end{Bmatrix},$$
$$C_S^{**}(\alpha_S, s'_{31}) = (-)^{s'_{31}+2s_2+s_3+s_1} \begin{Bmatrix} s_1 & s_2 & s_{12} \\ s_3 & S & s'_{31} \end{Bmatrix}, \quad (30)$$

and the isospin coefficients C_T^* and C_T^{**} are similar to the corresponding spin coefficients. By these considerations we obtain the second and third Faddeev components as:

$$\begin{aligned}
\langle \mathbf{pq}\,\alpha | P_{12} P_{23} | \psi \rangle &= \sum_{s'_{23}, t'_{23}} C_S^*(\alpha_S, s'_{23}) C_T^*(\alpha_T, t'_{23}) \\
&\quad \times \psi^{\alpha^*}\left(-\frac{1}{2}\mathbf{p} - \frac{3}{4}\mathbf{q}, \mathbf{p} - \frac{1}{2}\mathbf{q}\right) \\
&\equiv \sum_{s'_{23}, t'_{23}} C_S^*(\alpha_S, s'_{23})\, C_T^*(\alpha_T, t'_{23}) \\
&\quad \times \psi^{\alpha^*}(\pi_1\, \pi_2\, x_{\pi_1\pi_2}), \\
\langle \mathbf{p}\,\mathbf{q}\,\alpha | P_{13} P_{23} | \psi \rangle &= \sum_{s'_{31}, t'_{31}} C_S^{**}(\alpha_S, s'_{31}) C_T^{**}(\alpha_T, t'_{31}) \\
&\quad \times \psi^{\alpha^{**}}\left(-\frac{1}{2}\mathbf{p} + \frac{3}{4}\mathbf{q}, -\mathbf{p} - \frac{1}{2}\mathbf{q}\right) \\
&\equiv \sum_{s'_{31}, t'_{31}} C_S^{**}(\alpha_S, s'_{31})\, C_T^{**}(\alpha_T, t'_{31}) \\
&\quad \times \psi^{\alpha^{**}}(\Pi_1\, \Pi_2\, x_{\Pi_1\Pi_2}), \qquad (31)
\end{aligned}$$

where

$$\begin{aligned}
|\alpha^*\rangle &= \left| \left(s'_{23}\, \frac{1}{2}\right) S M_S \left(t'_{23}\, \frac{1}{2}\right) T M_T \right\rangle, \\
\pi_1 &= \sqrt{\frac{1}{4}p^2 + \frac{9}{16}q^2 + \frac{3}{4}pq\, x_{pq}}, \\
\pi_2 &= \sqrt{p^2 + \frac{1}{4}q^2 - pq x_{pq}}, \\
x_{\pi_1\pi_2} &= \frac{1}{\pi_1\pi_2}\left(-\frac{1}{2}p^2 + \frac{3}{8}q^2 - \frac{1}{2}pq x_{pq}\right), \qquad (32) \\
|\alpha^{**}\rangle &= \left| \left(s'_{31}\, \frac{1}{2}\right) S M_S \left(t'_{31}\, \frac{1}{2}\right) T M_T \right\rangle, \\
\Pi_1 &= \sqrt{\frac{1}{4}p^2 + \frac{9}{16}q^2 - \frac{3}{4}pq\, x_{pq}}, \\
\Pi_2 &= \sqrt{p^2 + \frac{1}{4}q^2 + pq x_{pq}}, \\
x_{\Pi_1\Pi_2} &= \frac{1}{\Pi_1\Pi_2}\left(\frac{1}{2}p^2 - \frac{3}{8}q^2 - \frac{1}{2}pq x_{pq}\right). \qquad (33)
\end{aligned}$$

C. Comparison of coupled Faddeev equations in both 3D and PW schemes

In this section we discuss the number of coupled equations in both 3D and PW approaches. In a standard PW approach the infinite set of coupled integral equations, given in Eq. (24), is truncated in the actual calculations at sufficiently high values of the angular-momentum quantum numbers. If one assumes that the NN t matrix acts only in very few partial waves then the number of the coupled equations are correspondingly small. As shown in Table I, if $NN\,t$ matrix acts up to $j_{12}^{\max} = 1, 2, 3, 4$, and 5, then the number of channels will be 5, 18, 26, 34, and 42. This is while the total isospin is restricted to $T = \frac{1}{2}$ [31].

TABLE I. The number of PW channels which compose the Triton wave function when the NNt matrix acts up to different total two-nucleon angular momenta $j_{12}^{\max}$. Total isospin is restricted to $T = \frac{1}{2}$. The number of channels for $j_{12}^{\max} = 1$, namely $N_\alpha = 5$, is related to only positive-parity states.

$j_{12}^{\max}$	1	2	3	4	5
N_α	5	18	26	34	42

In Table II we list all the spin-isospin states which compose the $3N$, i.e., ^{3}H and ^{3}He, wave function and consequently in Tables III and IV we present the number of spin-isospin states for the $3N$ bound states as well as the number of coupled

TABLE II. Quantum numbers of the spin-isospin states that compose ^{3}H or ^{3}He wave function.

Channel	$(s_{12}\,\frac{1}{2})\,S\,M_S$	$(t_{12}\,\frac{1}{2})\,T\,M_T$	$(S-T)$
1	$(0\,\frac{1}{2})\,\frac{1}{2}\,\frac{+1}{2}$	$(0\,\frac{1}{2})\,\frac{1}{2}\,\frac{+1}{2}/\frac{-1}{2}$	$(\frac{1}{2}-\frac{1}{2})$
2	$(0\,\frac{1}{2})\,\frac{1}{2}\,\frac{-1}{2}$	$(0\,\frac{1}{2})\,\frac{1}{2}\,\frac{+1}{2}/\frac{-1}{2}$	$(\frac{1}{2}-\frac{1}{2})$
3	$(1\,\frac{1}{2})\,\frac{1}{2}\,\frac{+1}{2}$	$(0\,\frac{1}{2})\,\frac{1}{2}\,\frac{+1}{2}/\frac{-1}{2}$	$(\frac{1}{2}-\frac{1}{2})$
4	$(1\,\frac{1}{2})\,\frac{1}{2}\,\frac{-1}{2}$	$(0\,\frac{1}{2})\,\frac{1}{2}\,\frac{+1}{2}/\frac{-1}{2}$	$(\frac{1}{2}-\frac{1}{2})$
5	$(0\,\frac{1}{2})\,\frac{1}{2}\,\frac{+1}{2}$	$(1\,\frac{1}{2})\,\frac{1}{2}\,\frac{+1}{2}/\frac{-1}{2}$	$(\frac{1}{2}-\frac{1}{2})$
6	$(0\,\frac{1}{2})\,\frac{1}{2}\,\frac{-1}{2}$	$(1\,\frac{1}{2})\,\frac{1}{2}\,\frac{+1}{2}/\frac{-1}{2}$	$(\frac{1}{2}-\frac{1}{2})$
7	$(1\,\frac{1}{2})\,\frac{1}{2}\,\frac{+1}{2}$	$(1\,\frac{1}{2})\,\frac{1}{2}\,\frac{+1}{2}/\frac{-1}{2}$	$(\frac{1}{2}-\frac{1}{2})$
8	$(1\,\frac{1}{2})\,\frac{1}{2}\,\frac{-1}{2}$	$(1\,\frac{1}{2})\,\frac{1}{2}\,\frac{+1}{2}/\frac{-1}{2}$	$(\frac{1}{2}-\frac{1}{2})$
9	$(0\,\frac{1}{2})\,\frac{1}{2}\,\frac{+1}{2}$	$(1\,\frac{1}{2})\,\frac{3}{2}\,\frac{+1}{2}/\frac{-1}{2}$	$(\frac{1}{2}-\frac{3}{2})$
10	$(0\,\frac{1}{2})\,\frac{1}{2}\,\frac{-1}{2}$	$(1\,\frac{1}{2})\,\frac{3}{2}\,\frac{+1}{2}/\frac{-1}{2}$	$(\frac{1}{2}-\frac{3}{2})$
11	$(1\,\frac{1}{2})\,\frac{1}{2}\,\frac{+1}{2}$	$(1\,\frac{1}{2})\,\frac{3}{2}\,\frac{+1}{2}/\frac{-1}{2}$	$(\frac{1}{2}-\frac{3}{2})$
12	$(1\,\frac{1}{2})\,\frac{1}{2}\,\frac{-1}{2}$	$(1\,\frac{1}{2})\,\frac{3}{2}\,\frac{+1}{2}/\frac{-1}{2}$	$(\frac{1}{2}-\frac{3}{2})$
13	$(1\,\frac{1}{2})\,\frac{3}{2}\,\frac{+3}{2}$	$(0\,\frac{1}{2})\,\frac{1}{2}\,\frac{+1}{2}/\frac{-1}{2}$	$(\frac{3}{2}-\frac{1}{2})$
14	$(1\,\frac{1}{2})\,\frac{3}{2}\,\frac{+1}{2}$	$(0\,\frac{1}{2})\,\frac{1}{2}\,\frac{+1}{2}/\frac{-1}{2}$	$(\frac{3}{2}-\frac{1}{2})$
15	$(1\,\frac{1}{2})\,\frac{3}{2}\,\frac{-1}{2}$	$(0\,\frac{1}{2})\,\frac{1}{2}\,\frac{+1}{2}/\frac{-1}{2}$	$(\frac{3}{2}-\frac{1}{2})$
16	$(1\,\frac{1}{2})\,\frac{3}{2}\,\frac{-3}{2}$	$(0\,\frac{1}{2})\,\frac{1}{2}\,\frac{+1}{2}/\frac{-1}{2}$	$(\frac{3}{2}-\frac{1}{2})$
17	$(1\,\frac{1}{2})\,\frac{3}{2}\,\frac{+3}{2}$	$(1\,\frac{1}{2})\,\frac{1}{2}\,\frac{+1}{2}/\frac{-1}{2}$	$(\frac{3}{2}-\frac{1}{2})$
18	$(1\,\frac{1}{2})\,\frac{3}{2}\,\frac{+1}{2}$	$(1\,\frac{1}{2})\,\frac{1}{2}\,\frac{+1}{2}/\frac{-1}{2}$	$(\frac{3}{2}-\frac{1}{2})$
19	$(1\,\frac{1}{2})\,\frac{3}{2}\,\frac{-1}{2}$	$(1\,\frac{1}{2})\,\frac{1}{2}\,\frac{+1}{2}/\frac{-1}{2}$	$(\frac{3}{2}-\frac{1}{2})$
20	$(1\,\frac{1}{2})\,\frac{3}{2}\,\frac{-3}{2}$	$(1\,\frac{1}{2})\,\frac{1}{2}\,\frac{+1}{2}/\frac{-1}{2}$	$(\frac{3}{2}-\frac{1}{2})$
21	$(1\,\frac{1}{2})\,\frac{3}{2}\,\frac{+3}{2}$	$(1\,\frac{1}{2})\,\frac{3}{2}\,\frac{+1}{2}/\frac{-1}{2}$	$(\frac{3}{2}-\frac{3}{2})$
22	$(1\,\frac{1}{2})\,\frac{3}{2}\,\frac{+1}{2}$	$(1\,\frac{1}{2})\,\frac{3}{2}\,\frac{+1}{2}/\frac{-1}{2}$	$(\frac{3}{2}-\frac{3}{2})$
23	$(1\,\frac{1}{2})\,\frac{3}{2}\,\frac{-1}{2}$	$(1\,\frac{1}{2})\,\frac{3}{2}\,\frac{+1}{2}/\frac{-1}{2}$	$(\frac{3}{2}-\frac{3}{2})$
24	$(1\,\frac{1}{2})\,\frac{3}{2}\,\frac{-3}{2}$	$(1\,\frac{1}{2})\,\frac{3}{2}\,\frac{+1}{2}/\frac{-1}{2}$	$(\frac{3}{2}-\frac{3}{2})$

TABLE III. The number of spin-isospin states for $3N$ bound sates, i.e., ^{3}H and ^{3}He, in a realistic 3D formalism. N_S and N_T are the number of spin and isospin states, respectively.

$(s_{12}\,\frac{1}{2})S\,M_S$	$S=\frac{1}{2}$	$S=\frac{3}{2}$	$S=\frac{1}{2},\frac{3}{2}$	$(t_{12}\,\frac{1}{2})T\,M_T$	$T=\frac{1}{2}$	$T=\frac{3}{2}$	$T=\frac{1}{2},\frac{3}{2}$
$(0\,\frac{1}{2})\frac{1}{2}\,\frac{\pm1}{2}$	2	0	2+0	$(0\,\frac{1}{2})\frac{1}{2}\,\frac{+1}{2}/\frac{-1}{2}$	1	0	1+0
$(1\,\frac{1}{2})\frac{1}{2}\,\frac{\pm1}{2}$	2	0	2+0	$(1\,\frac{1}{2})\frac{1}{2}\,\frac{+1}{2}/\frac{-1}{2}$	1	0	1+0
$(1\,\frac{1}{2})\frac{3}{2}\,\frac{\pm1}{2}\,\frac{\pm3}{2}$	0	4	0+4	$(1\,\frac{1}{2})\frac{3}{2}\,\frac{+1}{2}/\frac{-1}{2}$	0	1	0+1
N_S	4	4	8	N_T	2	1	3

Faddeev equations in realistic 3D formalism presented in this article. It is clear that $M_T=\frac{+1}{2}$ refers to ^{3}He and $M_T=\frac{-1}{2}$ refers to ^{3}H. Because the angular-momentum quantum numbers, i.e., l_{12}, l_3, do not appear explicitly in our formalism, the number of coupled equations that are fixed according to the spin-isospin states are strongly reduced. This is an indication that the present formalism automatically considers all partial waves without any truncation on the space part. Considering the spin-isospin degrees of freedom for both ^{3}H and ^{3}He states yields the same number of coupled equations and it leads to 8, 12, 16, and 24 coupled equations for different combinations of the total spin-isospin states $S-T$: $(\frac{1}{2}-\frac{1}{2})$, $(\frac{1}{2}-\frac{3}{2}^*)$, $(\frac{3}{2}^*-\frac{1}{2})$ and $(\frac{3}{2}^*-\frac{3}{2}^*)$, respectively. The star superscript indicates all the spin or isospin states that we have taken into account up to a specific value. It is clear that in the 3D formalism, e.g., for a fully charge dependent calculation, there are only 24 coupled equations, whereas in the PW approach after truncation of the Hilbert space to $T=\frac{1}{2}$ there are 42 coupled equations. Therefore our 3D formalism leads to a small number of coupled equations in comparison with the very large number of coupled equations in the truncated PW formalism. However, it should be mentioned that our formulation leads to coupled equations in three variables for the amplitudes, whereas the PW formulation after truncation leads to a finite number of coupled equations in two variables for the amplitudes. So the 3D formulation leads to a lesser number of coupled integral equations in three dimensions and the PW formulations leads to more coupled integral equations in two dimensions. Thus, the price for the smaller number of equations is the higher dimensionality of the integral equations. In other words, algebraic simplification is achieved by a more involved numerical scheme.

TABLE IV. The number of coupled Faddeev equations for the $3N$ bound state, i.e., ^{3}H and ^{3}He, in a realistic 3D formalism according to the spin-isospin states $(S-T)$. $N=N_S\times N_T$ is the total number of coupled Faddeev equations. The star superscript indicates all the spin or isospin states that one can take into account up to a specific value.

$(S-T)$	$(\frac{1}{2}-\frac{1}{2})$	$(\frac{1}{2}-\frac{3}{2}^*)$	$(\frac{3}{2}^*-\frac{1}{2})$	$(\frac{3}{2}^*-\frac{3}{2}^*)$
N_S	4	4	8	8
N_T	2	3	2	3
N	8	12	16	24

III. NUMERICAL RESULTS FOR ^{3}H

A. Triton binding energy

To be able to test our realistic 3D formalism for the $3N$ bound state we solve the three-dimensional Faddeev integral equations, Eq. (21). We calculate the triton binding energy by solving eight coupled Faddeev equations for $(\frac{1}{2}-\frac{1}{2})$ spin-isospin states and compare our results with the other PW results. In this respect, we use Bonn one-boson-exchange (OBE) potential in the parametrization of Bonn-B [31] and in an operator form that can be incorporated in the 3D formalism [29]. In the numerical treatment, the dependence of Faddeev components to the continuous momentum and the angle variables, should be replaced by a dependence on certain discrete values. For this purpose we use the Gaussian quadrature grid points.

The coupled Faddeev equations represent a set of three-dimensional homogenous integral equations, which after discreatization turns into a huge matrix eigenvalue equation. The huge matrix eigenvalue equation requires an iterative solution method. We use a Lanczos-like scheme that is proved to be very efficient for nuclear few-body problems [32]. The momentum variables have to cover the interval $[0,\infty]$. In practice we limit the intervals to suitable cutoffs and their values are chosen large enough to achieve cut-off independence. The functional behavior of the kernel of eigenvalue equation is determined by the anti-symmetrized two-body t matrix. We also solve the Lippman-Schwinger equation for the fully off-shell two-body t matrix in an approach based on a helicity representation directly as a function of the Jacobi vector variables (see Appendix B). For antisymmetrized two-body t matrix calculations 40 grid points for the Jacobi momentum variables, 32 grid points for the spherical angle variables, and 20 grid points for the polar angle variables have been used, respectively. Because the coupled integral equations require a very large number of interpolations, we use the cubic Hermitian splines of Ref. [33] for its accuracy and high computational speed.

In Table V we show the convergence of the triton binding energy as function of the number of the grid points for Bonn-B potential in the 3D approach. As demonstrated in this table, the calculation of triton binding energy converges to a value of $E_t=-8.152$ MeV. The results of the Faddeev equations with different PW based methods are presented in Table VI to compare them with our calculations. The overall agreement is quite satisfactory. As we can see from this comparison our result provides the same accuracy while the numerical procedure is actually easier to implement.

TABLE V. The calculated binding energy E_t of the three-dimensional Faddeev integral equations as function of the number of the grid points in the Jacobi momenta $N_{\rm jac}$ and the spherical angles $N_{\rm sph}$. The number of the grid points in polar angles is 20. The calculations are based on the Bonn-B potential.

$N_{\rm jac}$	$N_{\rm sph}$	E_t (MeV)
32	20	−8.154
32	24	−8.153
36	20	−8.153
36	24	−8.152
40	20	−8.152
40	24	−8.152

B. Expectation value of the Hamiltonian operator

In this section we investigate the numerical stability of the presented algorithm and the 3D formalism of the Faddeev equations. With the binding energy E_t and the Faddeev component $|\psi\rangle$ available, we are able to calculate the total wave function $|\Psi\rangle$ from Eq. (2) by considering the choice of coordinate system that is used in representation of Eq. (21). So we can evaluate the expectation value of the Hamiltonian operator H and compare this value to the previously calculated binding energy of the eigenvalue equation. Explicitly we evaluate the following expression:

$$\begin{aligned}\langle\Psi|H|\Psi\rangle &= \langle\Psi|H_0|\Psi\rangle + \langle\Psi|V|\Psi\rangle \\ &= 3\,\langle\psi|H_0|\Psi\rangle + 3\,\langle\Psi|V_{12}|\Psi\rangle, \end{aligned} \tag{34}$$

TABLE VI. A list of triton binding energy calculations ordered according to $j_{12}^{\max}$ by different authors using slightly different numerical methods. All results for binding energies are related to the total isospin $T = \frac{1}{2}$.

$j_{12}^{\max}$	Ref.	E_t (MeV)
1		
	[34]	−8.14
	[35,36]	−8.17
	[37]	−8.165
	[7,38]	−8.16
2		
	[39,40]	−8.088
	[41]	−8.100
	[39]	−8.101
	[37]	−8.103
3		
	[42]	−8.14
4		
	[31,43]	−8.13
	[7,44]	−8.14

where

$$\begin{aligned}\langle\psi|H_0|\Psi\rangle &= \sum_\alpha \int d^3p \int d^3q \sum_{\alpha'} \int d^3p' \int d^3q' \\ &\quad \times \langle\,\psi|\mathbf{p}\,\mathbf{q}\alpha\rangle\langle\mathbf{p}\,\mathbf{q}\alpha|H_0|\mathbf{p}'\,\mathbf{q}'\alpha'\rangle\langle\mathbf{p}'\,\mathbf{q}'\alpha'|\Psi\rangle \\ &= \int_0^\infty dp p^2 \int_0^\infty dq q^2 \left(\frac{p^2}{m} + \frac{3q^2}{4m}\right) \int_{-1}^{+1} dx_p \\ &\quad \times \int_0^{2\pi} d\varphi_p \int_{-1}^{+1} dx_q \int_0^{2\pi} d\varphi_q \\ &\quad \times \sum_\alpha \psi^\alpha(p\,q\,x_{pq})\,\Psi^\alpha(p\,q\,x_{pq}), \end{aligned} \tag{35}$$

$$\begin{aligned}\langle\Psi|V_{12}|\Psi\rangle &= \sum_\alpha \int d^3p \int d^3q \sum_{\alpha'} \int d^3p' \int d^3q' \langle\Psi|\mathbf{p}\,\mathbf{q}\alpha\rangle \\ &\quad \times \langle\mathbf{p}\,\mathbf{q}\alpha|V_{12}|\mathbf{p}'\,\mathbf{q}'\,\alpha'\rangle\langle\mathbf{p}'\,\mathbf{q}'\alpha'|\Psi\rangle. \end{aligned} \tag{36}$$

As is well known, the rotational, parity, and time-reversal invariance restricts any NN potential V_{12} to be formed of six independent terms [45] as

$$V_{12}(\mathbf{p},\mathbf{p}') = \langle\mathbf{p}|V_{12}|\mathbf{p}'\rangle = \sum_{i=1}^{6} v_i(p,\,p',\,x_{pp'})\,W_i, \tag{37}$$

here $v_i(p,\,p',\,x_{pp'})$ are scalar spin-independent functions, which depend on the magnitudes of the Jacobi momenta $\mathbf{p}$, $\mathbf{p}'$ and the angle between them, $x_{pp'} \equiv \hat{\mathbf{p}}.\hat{\mathbf{p}}'$, and W_i $(i = 1$ to $6)$ are operators to the spin states of the two-nucleon such that

$$\begin{aligned} &V_{12 m_{s_1} m_{s_2} m'_{s_1} m'_{s_2}}(\mathbf{p},\mathbf{p}') \\ &\quad = \langle\mathbf{p} m_{s_1} m_{s_2}|V_{12}|\mathbf{p}' m'_{s_1} m'_{s_2}\rangle \\ &\quad = \sum_{i=1}^{6} v_i(p,\,p',\,x_{pp'})\,\langle m_{s_1} m_{s_2}|W_i|m'_{s_1} m'_{s_2}\rangle, \end{aligned} \tag{38}$$

so the matrix elements of NN potential can be evaluated as:

$$\langle\mathbf{p}\,\mathbf{q}\alpha|V_{12}|\mathbf{p}'\,\mathbf{q}'\alpha'\rangle = \delta^3(\mathbf{q}-\mathbf{q}')\langle\,\alpha_T|V_{12}^T|\alpha'_T\rangle\langle\mathbf{p}\alpha_S|V_{12}|\mathbf{p}'\alpha'_S\rangle, \tag{39}$$

where V_{12}^T is the isospin part of the potential, it is unity for the isospin-independent terms and $\tau_1.\tau_2$ for the isospin-dependent terms. So it can be easily evaluated as

$$\langle\alpha_T|V_{12}^T|\alpha'_T\rangle = \mathbf{T}\delta_{\alpha_T\alpha'_T},$$

$$\mathbf{T} = \begin{cases} 1, & \text{isospin-independent terms;} \\ 2t_{12}^2 - 3, & \text{isospin-dependent terms.} \end{cases} \tag{40}$$

The spin-space part of the potential can be evaluated as:

$$\begin{aligned}\langle\mathbf{p}\alpha_S|V_{12}|\mathbf{p}'\,\alpha'_S\rangle &= \sum_{\gamma_S}\sum_{\gamma'_S}\langle\alpha_S|\gamma_S\rangle\langle\gamma'_S|\alpha'_S\rangle\langle\mathbf{p}\gamma_S|V_{12}|\mathbf{p}'\gamma'_S\rangle \\ &= \sum_{\gamma_S}\sum_{\gamma'_S} g^S_{\alpha\gamma}\,g^S_{\alpha'\gamma'}\,\langle\mathbf{p}\gamma_S|V_{12}|\mathbf{p}'\gamma'_S\rangle \end{aligned}$$

$$= \sum_{\gamma_S} \sum_{\gamma'_S} g^S_{\alpha\gamma} g^S_{\alpha'\gamma'} \delta_{m_{s_3} m'_{s_3}} \times \left\langle \mathbf{p} m_{s_1} m_{s_2} \middle| V_{12} \middle| \mathbf{p}' m'_{s_1} m'_{s_2} \right\rangle = \sum_{\gamma_S} \sum_{\gamma'_S} g^S_{\alpha\gamma} g^S_{\alpha'\gamma'} \delta_{m_{s_3} m'_{s_3}} \times V_{12 m_{s_1} m_{s_2} m'_{s_1} m'_{s_2}}(\mathbf{p}, \mathbf{p}'), \quad (41)$$

inserting Eqs. (40) and (41) into Eq. (39) yields:

$$\langle \mathbf{p}\,\mathbf{q}\alpha | V_{12} | \mathbf{p}'\,\mathbf{q}'\alpha' \rangle = \delta^3(\mathbf{q} - \mathbf{q}') \mathbf{T} \delta_{\alpha_T \alpha'_T} \sum_{\gamma_S} \sum_{\gamma'_S} g^S_{\alpha\gamma} g^S_{\alpha'\gamma'} \delta_{m_{s_3} m'_{s_3}} \times V_{12 m_{s_1} m_{s_2} m'_{s_1} m'_{s_2}}(\mathbf{p}, \mathbf{p}'), \quad (42)$$

by these considerations the expectation value of the NN potential, Eq. (36), can be rewritten as:

$$\begin{aligned} \langle \Psi | V_{12} | \Psi \rangle = & \sum_{\alpha} \sum_{\alpha'} \mathbf{T} \delta_{\alpha_T \alpha'_T} \sum_{\gamma_S} \sum_{\gamma'_S} g^S_{\alpha\gamma} g^S_{\alpha'\gamma'} \delta_{m_{s_3} m'_{s_3}} \\ & \times \int_0^\infty dp p^2 \int_{-1}^{+1} dx_p \int_0^{2\pi} d\varphi_p \int_0^\infty dp' p'^2 \\ & \times \int_{-1}^{+1} dx'_p \int_0^{2\pi} d\varphi'_p V_{12 m_{s_1} m_{s_2} m'_{s_1} m'_{s_2}}(p, p', x_{pp'}) \\ & \times \int_0^\infty dq q^2 \int_{-1}^{+1} dx_q \int_0^{2\pi} d\varphi_q \\ & \times \Psi^\alpha(p\,q\,x_{pq})\, \Psi^{\alpha'}(p'\,q\,x_{p'q}), \end{aligned} \quad (43)$$

where

$$x_{pp'} \equiv \hat{\mathbf{p}}.\hat{\mathbf{p}}' = x_p x_{p'} + \sqrt{1 - x_p^2}\sqrt{1 - x_{p'}^2} \sin(\phi_p - \phi_{p'})$$

and

$$x_{p'q} \equiv \hat{\mathbf{p}}'.\hat{\mathbf{q}} = x_{p'} x_q + \sqrt{1 - x_{p'}^2}\sqrt{1 - x_q^2} \sin(\phi_{p'} - \phi_q).$$

The expectation values of the kinetic energy $\langle H_0 \rangle$, the two-body interaction $\langle V \rangle$, and the Hamiltonian operator $\langle H \rangle$ are listed in Table VII for Bonn-B potential calculated in the 3D scheme as a function of the number of the grid points in the Jacobi momenta N_{jac} and the spherical angles N_{sph}. In the same table, the triton binding energies calculated in the 3D scheme are also shown to compare with the expectation values of the Hamiltonian operator. One can see that the energy expectation value and the eigenvalue energies E_t agree with good accuracy.

TABLE VII. The expectation values of the kinetic energy $\langle H_0 \rangle$, the NN interaction $\langle V \rangle$, and the Hamiltonian operator $\langle H \rangle$ calculated in the 3D scheme as a function of the number of the grid points in the Jacobi momenta N_{jac} and the spherical angles N_{sph} for the triton. The number of the grid points in polar angles is 20. The calculations are based on the Bonn-B potential. Additionally the expectation values of the Hamiltonian operator are compared with the triton binding energy results from the three-dimensional Faddeev integral equations. All energies are given in MeV.

N_{jac}	N_{sph}	$\langle H_0 \rangle$	$\langle V \rangle$	$\langle H \rangle$	E_t
32	20	+39.222	−47.356	−8.134	−8.154
32	24	+39.222	−47.356	−8.134	−8.154
36	20	+39.222	−47.357	−8.135	−8.153
36	24	+39.222	−47.357	−8.135	−8.152
40	20	+39.223	−47.358	−8.135	−8.152
40	24	+39.223	−47.358	−8.135	−8.152

IV. SUMMARY AND OUTLOOK

In this article we have introduced the three-dimensional Faddeev integral equations for the calculation of the triton binding energy with the spin-isospin dependent potential. In comparison with the PW approach, as is commonly used, this direct approach has greater advantages. The pertinent results can be summarized as follows:

(i) The 3D formalism leads only to a strictly finite number of coupled three-dimensional integral equations to be solved, whereas in the PW case after truncation one has a set of finite number of coupled equations with kernels containing relatively complicated geometrical expressions. So the 3D formalism avoids the highly involved angular-momentum algebra occurring for the permutations and also automatically consider all the partial waves without any truncation on the space part. However, the 3D formulation leads to a lesser number of coupled integral equations in three dimensions and the PW formulations leads to more coupled integral equations in two dimensions.

(ii) Our result for the triton binding energy with Bonn-B potential is in good agreement with the pervious values calculated with the standard PW approach. The stability of present algorithm and the 3D formalism of Faddeev components as presented in this article have been achieved with the calculation of the expectation value of the Hamiltonian operator and we have reached to a resonable agreement between the obtained energy eigenvalue and expectation value of the Hamiltonian operator. The $3N$ bound state calculations with AV18 potential is also potentially valuable and the numerical results with this potential will be reported in the future.

(iii) We predict that the incorporation of three-nucleon force probably will be less cumbersome in a realistic 3D approach. This is very promising and nourishes our hope that four-nucleon bound state formulation and calculations with realistic two and three-nucleon forces in a realistic 3D approach will be more easily implemented than the traditional partial wave based method.

The calculations of three-nucleon bound state, with the phenomenological Tucson-Melbourne (TM) 2π exchange three-nucleon potential, and the formulation of the four-nucleon bound state is currently underway and they will be reported before long [30].

ACKNOWLEDGMENTS

We thank W. Glöckle, Ch. Elster, and I. Fachruddin for using their helicity formalism related to the two-body t matrix.

This work was supported by the research council of the University of Tehran.

APPENDIX A: $g_{\gamma\alpha}$ CLEBSCH-GORDAN COEFFICIENTS

In the usual coupling scheme, for the three identical particles with spin $\frac{1}{2}$, to completely classify the states of definite total spin the quantum numbers

$$|\gamma_S\rangle \equiv \left|s_1 m_{s_1} s_2 m_{s_2} s_3 m_{s_3}\right\rangle \equiv \left|m_{s_1}\, m_{s_2} m_{s_3}\right\rangle, \tag{A1}$$

are replaced by the set

$$|\alpha_S\rangle \equiv \left|[(s_1\, s_2) s_{12}\, s_3] S M_S\right\rangle \equiv \left|\left(s_{12}\, \frac{1}{2}\right) S M_S\right\rangle. \tag{A2}$$

The $3N$ basis states $|\alpha_S\rangle$ can be obtained from free $3N$ basis states $|\gamma_S\rangle$ as:

$$\begin{cases} \left|\left(1\,\frac{1}{2}\right)\frac{3}{2}+\frac{3}{2}\right\rangle \equiv |\uparrow\uparrow\uparrow\rangle \\ \left|\left(1\,\frac{1}{2}\right)\frac{3}{2}+\frac{1}{2}\right\rangle \equiv \frac{1}{\sqrt{3}}\{|\downarrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle\} \\ \left|\left(1\,\frac{1}{2}\right)\frac{3}{2}-\frac{1}{2}\right\rangle \equiv \frac{1}{\sqrt{3}}\{|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle\} \\ \left|\left(1\,\frac{1}{2}\right)\frac{3}{2}-\frac{3}{2}\right\rangle \equiv |\downarrow\downarrow\downarrow\rangle \\ \left|\left(1\,\frac{1}{2}\right)\frac{1}{2}+\frac{1}{2}\right\rangle \equiv \frac{1}{\sqrt{6}}\{|\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle - 2|\downarrow\uparrow\uparrow\rangle\} \\ \left|\left(1\,\frac{1}{2}\right)\frac{1}{2}-\frac{1}{2}\right\rangle \equiv \frac{1}{\sqrt{6}}\{|\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle - 2|\uparrow\downarrow\downarrow\rangle\} \\ \left|\left(0\,\frac{1}{2}\right)\frac{1}{2}+\frac{1}{2}\right\rangle \equiv \frac{1}{\sqrt{2}}\{|\uparrow\uparrow\downarrow\rangle - |\uparrow\downarrow\uparrow\rangle\} \\ \left|\left(0\,\frac{1}{2}\right)\frac{1}{2}-\frac{1}{2}\right\rangle \equiv \frac{1}{\sqrt{2}}\{|\downarrow\uparrow\downarrow\rangle - |\downarrow\downarrow\uparrow\rangle\}. \end{cases} \tag{A3}$$

If one considers all total spin states, i.e., $S=\frac{1}{2}$ and $S=\frac{3}{2}$, the relevant Clebsch-Gordan coefficients $g^S_{\gamma\alpha}$ are $1, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}, -\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$. As indicated in Sec. II C the isospin states are similar to the spin states, but the third component of total isospins is restricted to $M_T=\frac{+1}{2}$ for ^{3}He and $M_T=\frac{-1}{2}$ for ^{3}H. Thus for a fully charge-dependent calculation the necessary isospin coefficients $g^T_{\gamma\alpha}$ are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}, -\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$. Because in our calculations for the triton binding energy we consider only the total spin-isospin states $(S-T)=(\frac{1}{2}-\frac{1}{2})$, therefore we only use the following Clebsch-Gordan coefficients $\frac{1}{\sqrt{6}}, -\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$.

APPENDIX B: ANTISYMMETRIZED NNT-MATRIX AND CONNECTION TO HELICITY REPRESENTATION

In our formulation of the $3N$ bound state, we need the physical representation of NNt-matrix or matrix elements ${}_a\langle \mathbf{p} m_{s_1} m_{s_2} m_{t_1} m_{t_2}|t(\varepsilon)|\mathbf{p}' m'_{s_1} m'_{s_2} m'_{t_1} m'_{t_2}\rangle_a$. The connection of these matrix elements to those in the momentum-helicity basis is given in Ref. [46], here we prepare this connection according to the notation to be used in our work. First, we introduce the momentum-helicity basis states for the total spin s_{12} and the relative momentum $\mathbf{p}$ of the two nucleons as:

$$|\mathbf{p};\hat{\mathbf{p}} s_{12}\lambda\rangle, \tag{B1}$$

where λ is the eigenvalue of the helicity operator $\mathbf{s_{12}}.\hat{\mathbf{p}}$. By introducing parity operator P and the two-nucleon isospin states $|t_{12} m_{t_{12}}\rangle$, the antisymmetrized two-nucleon basis states are given as:

$$|\mathbf{p};\hat{\mathbf{p}} s_{12}\lambda; t_{12}\rangle^{\pi a} \equiv \frac{1}{\sqrt{2}}(1-\eta_\pi(-)^{s_{12}+t_{12}})|t_{12}\rangle|\mathbf{p};\hat{\mathbf{p}} s_{12}\lambda\rangle_\pi, \tag{B2}$$

with the parity eigenvalues $\eta_\pi=\pm 1$ and eigenstates $|\mathbf{p};\hat{\mathbf{p}} s_{12}\lambda\rangle_\pi = \frac{1}{\sqrt{2}}(1+\eta_\pi P)|\mathbf{p};\hat{\mathbf{p}} s_{12}\lambda\rangle$. Based on these basis states the NNt-matrix element is defined as:

$$t^{\pi s_{12} t_{12}}_{\lambda\lambda'}(\mathbf{p},\mathbf{p}';\varepsilon) \equiv {}^{\pi a}\langle \mathbf{p};\hat{\mathbf{p}} s_{12}\lambda; t_{12}|t(\varepsilon)|\mathbf{p}';\hat{\mathbf{p}}' s_{12}\lambda'; t_{12}\rangle^{\pi a}. \tag{B3}$$

As shown in Ref. [46], the selection of $\mathbf{p}'$ parallel to the z axis allows, together with the properties of the potential, that the angular dependencies of the NNt-matrix elements can be simplified as:

$$\begin{aligned} t^{\pi s_{12} t_{12}}_{\lambda\lambda'}(\mathbf{p},\mathbf{p}';\varepsilon) &= e^{-i\lambda\Omega_{pp'}}\, t^{\pi s_{12} t_{12}}_{\lambda\lambda'}(p\hat{\mathbf{n}}_{pp'}, p'\hat{\mathbf{z}};\varepsilon) \\ &= e^{-i\lambda\Omega_{pp'}}\, e^{i\lambda'\phi_{pp'}}\, t^{\pi s_{12} t_{12}}_{\lambda\lambda'}(p,\, p',\cos\theta_{pp'};\varepsilon) \\ &\equiv e^{i(\lambda'\phi_{pp'}-\lambda\Omega_{pp'})} t^{\pi s_{12} t_{12}}_{\lambda\lambda'}(p,\, p',\cos\theta_{pp'};\varepsilon), \end{aligned} \tag{B4}$$

the direction $\hat{\mathbf{n}}_{pp'}$ can be determined by the spherical and polar angles $\vartheta_{pp'}$ and $\varphi_{pp'}$, where

$$\begin{aligned} \cos\theta_{pp'} &= \cos\theta_p\cos\theta_{p'} + \sin\theta_p\sin\theta_{p'}\cos(\phi_p-\phi_{p'}), \\ \sin\theta_{pp'} e^{i\varphi_{pp'}} &= -\cos\theta_p\sin\theta_{p'} + \sin\theta_p\cos\theta_{p'}\cos(\phi_p-\phi_{p'}) \\ &\quad + i\sin\theta_p\sin(\phi_p-\phi_{p'}), \end{aligned} \tag{B5}$$

and the exponential factor $e^{i(\lambda'\phi_{pp'}-\lambda\Omega)}$ is calculated as:

$$\begin{aligned} e^{i\lambda\Omega_{pp'}} &= \frac{\sum_{N=-s_{12}}^{s_{12}} D^{s_{12}}_{N\lambda}(\phi_p\theta_p 0) D^{s_{12}}_{N\lambda'}(\phi_{p'}\theta_{p'} 0)}{D^{s_{12}}_{\lambda'\lambda}(\phi_{pp'}\theta_{pp'}\,0)}, \\ e^{i(\lambda'\phi_{pp'}-\lambda\Omega_{pp'})} &= \frac{\sum_{N=-s_{12}}^{s_{12}} e^{iN(\phi_p-\phi_{p'})} d^{s_{12}}_{N\lambda}(\theta_p) d^{s_{12}}_{N\lambda'}(\theta_{p'})}{d^{s_{12}}_{\lambda'\lambda}(\theta_{pp'})}. \end{aligned} \tag{B6}$$

In the above expressions, $D^{s_{12}}_{N\lambda}(\phi_p\,\theta_p 0)$ are the Wigner D functions and $d^{s_{12}}_{\lambda'\lambda}(\theta)$ are rotation matrices [47]. Finally the connection of the t-matrix elements ${}_a\langle \mathbf{p} m_{s_1} m_{s_2} m_{t_1} m_{t_2}|t(\varepsilon)|\mathbf{p}' m'_{s_1} m'_{s_2} m'_{t_1} m'_{t_2}\rangle_a$ to those in the momentum-helicity basis, namely $t^{\pi s_{12} t_{12}}_{\lambda\lambda'}(\mathbf{p},\mathbf{p}';\varepsilon)$, is given as:

$$\begin{aligned} &{}_a\langle \mathbf{p} m_{s_1} m_{s_2} m_{t_1} m_{t_2}|t(\varepsilon)|\mathbf{p}' m'_{s_1} m'_{s_2} m'_{t_1} m'_{t_2}\rangle_a \\ &= \frac{1}{4}\delta_{m_{t_1}+m_{t_2},m'_{t_1}+m'_{t_2}} e^{-i(\lambda_0\phi_p-\lambda'_0\phi_{p'})} \sum_{s_{12}\pi t_{12}} (1-\eta_\pi(-)^{s_{12}+t_{12}}) \\ &\quad \times C\left(\frac{1}{2}\frac{1}{2} t_{12}; m_{t_1} m_{t_2}\right) C\left(\frac{1}{2}\frac{1}{2} t'_{12}; m'_{t_1} m'_{t_2}\right) \\ &\quad \times C\left(\frac{1}{2}\frac{1}{2} s_{12}; m_{s_1} m_{s_2}\right) C\left(\frac{1}{2}\frac{1}{2} s'_{12}; m'_{s_1} m'_{s_2}\right) \\ &\quad \times \sum_{\lambda\lambda'} d^{s_{12}}_{\lambda_0\lambda}(\theta_p) d^{s_{12}}_{\lambda'_0\lambda'}(\theta_{p'}) t^{\pi s_{12} t_{12}}_{\lambda\lambda'}(\mathbf{p},\mathbf{p}';\varepsilon). \end{aligned} \tag{B7}$$

It should be mentioned that $t^{\pi s_{12} t_{12}}_{\lambda\lambda'}(p,\, p',\cos\theta_{pp'};\varepsilon)$ obeys a set of coupled Lippman-Schwinger equations that for $(S=0)$ is a single equation but for $(S=1)$ is a set of two

coupled equations (Ref. [29]). So the matrix elements of the antisymmetrized NNt matrix, which explicitly appears in Eq. (22), is functionally the same as Eq. (B7) and can be obtained as:

$$
\begin{aligned}
&t_{a\,m_{s_1}m_{s_2}m_{t_1}m_{t_2}}^{m'_{s_2}m'_{s_3}m'_{t_2}m'_{t_3}}(p,\tilde{\pi},x_{p\tilde{\pi}};\epsilon)\\
&\quad\equiv_a \left\langle \mathbf{p}\, m_{s_1}m_{s_2}m_{t_1}m_{t_2}\middle|t(\varepsilon)\middle|\pi\, m'_{s_1}m'_{s_2}m'_{t_1}m'_{t_2}\right\rangle_a\\
&\quad= \frac{1}{4}\delta_{m_{t_1}+m_{t_2},m'_{t_1}+m'_{t_2}}\, e^{-i(\lambda_0\phi_p-\lambda'_0\phi_{\tilde{\pi}})}\sum_{s_{12}\pi t_{12}}(1-\eta_\pi(-)^{s_{12}+t_{12}})\\
&\qquad\times C\left(\frac{1}{2}\frac{1}{2}t_{12};m_{t_1}m_{t_2}\right)C\left(\frac{1}{2}\frac{1}{2}t'_{12};m'_{t_1}m'_{t_2}\right)\\
&\qquad\times C\left(\frac{1}{2}\frac{1}{2}s_{12};m_{s_1}m_{s_2}\right)C\left(\frac{1}{2}\frac{1}{2}s'_{12};m'_{s_1}m'_{s_2}\right)\\
&\qquad\times\sum_{\lambda\lambda'}d^{s_{12}}_{\lambda_0\lambda}(\theta_p)d^{s_{12}}_{\lambda'_0\lambda'}(\theta_{\tilde{\pi}})t^{\pi s_{12}t_{12}}_{\lambda\lambda'}(\mathbf{p},\tilde{\pi};\varepsilon),
\end{aligned}
\tag{B8}
$$

with the same variables as Eqs. (B5) and (B6) with $\tilde{\pi}, \theta_{\tilde{\pi}}, \phi_{\tilde{\pi}}$ instead of $p', \theta_{p'}, \phi_{p'}$.

[1] E. Hiyama, M. Kamimura, T. Motoba, T. Yamada, and Y. Yamamoto, Phys. Rev. Lett. **85**, 270 (2000).
[2] J. Usukura, Y. Suzuki, and K. Varga, Phys. Rev. B **59**, 5652 (1999).
[3] M. Viviani, A. Kievsky, and S. Rosati, Phys. Rev. C **71**, 024006 (2005).
[4] R. B. Viringa , S. C. Pieper, J. Carlson, and V. R. Pandharipande, Phys. Rev. C **62**, 014001 (2000).
[5] P. Navrátil, J. P. Vary, and B. R. Barrett, Phys. Rev. C **62**, 054311 (2000).
[6] N. Barnea, W. Leidemann, and G. Orlandini, Phys. Rev. C **67**, 054003 (2003).
[7] F. Sammarruca, D. P. Xu, and R. Machleidt, Phys. Rev. C **46**, 1636 (1992).
[8] Y. Song and R. Machleidt, AIP Conf. Proc. **334**, 455 (1995).
[9] A. Stadler, J. Adam, Jr., H. Henning, and P. U. Sauer Phys. Rev. C **51**, 2896 (1995).
[10] A. Nogga, D. Hüber, H. Kamada, and W. Glöckle, Phys. Lett. **B409**, 19 (1997).
[11] A. Stadler and F. Gross Phys. Rev. Lett. **78**, 26 (1997).
[12] I. Fachruddin, W. Glöckle, Ch. Elster, A. Nogga, Phys. Rev. C **69**, 064002 (2004).
[13] C. R. Chen, G. L. Payne, J. L. Friar, and B. F. Gibson, Phys. Rev. Lett. **55**, 374 (1985).
[14] C. R. Chen, G. L. Payne, J. L. Friar, and B. F. Gibson, Phys. Rev. C **33**, 1740 (1986).
[15] J. L. Friar, B. F. Gibson, and G. L. Payne, Phys. Rev. C **36**, 1138 (1987).
[16] N. W. Schellingerhout, L. P. Kok, and G. D. Bosveld, Phys. Rev. A **40**, 5568 (1989).
[17] N. W. Schellingerhout, J. J. Schut, and L. P. Kok, Phys. Rev. C **46**, 1192 (1992).
[18] J. L. Friar, G. L. Payne, V. G. J. Stoks, and J. J. de Swart, Phys. Lett. **B311**, 4 (1993).
[19] P. F. Bedaque, H.-W. Hammer, and U. van Kolck Nucl. Phys. **A676**, 357 (2000).
[20] E. Epelbaum, H. Kamada, A. Nogga, H. Witala, W. Glockle, and Ulf-G. Meissner, Phys. Rev. Lett. **86**, 4787 (2001).
[21] E. Epelbaum, A. Nogga, W. Glockle, H. Kamada, Ulf-G. Meissner, and H. Witala, Phys. Rev. C **66**, 064001 (2002).
[22] L. Platter, H.-W. Hammer, and U.-G. Meißner, Phys. Lett. **B607**, 254 (2005).
[23] Ch. Elster, W. Schadow, A. Nogga, and W. Glöckle, Few-Body Syst. **27**, 83 (1999).
[24] H. Liu, Ch. Elster, W. Glöckle, Few-Body Syst. **33**, 241 (2003).
[25] M. R. Hadizadeh and S. Bayegan, *Proceedings of the 3rd Asia-Pacific Conference, Nakhon Ratchasima, Thailand, July 2005* (World Scientific, Singapore, 2007), p. 16.
[26] M. R. Hadizadeh and S. Bayegan, Few-Body Syst. **40**, 171 (2007).
[27] M. R. Hadizadeh and S. Bayegan, Eur. Phys. J. A **36**, 201 (2008).
[28] S. Bayegan, M. R. Hadizadeh, and M. Harzchi, *to appear in Few Body Systems*. arXiv:0711.4036.
[29] I. Fachruddin, Ch. Elster, and W. Glöckle, Phys. Rev. C **62**, 044002 (2000).
[30] M. R. Hadizadeh and S. Bayegan (in preparation).
[31] R. Machleidt, Adv. Nucl. Phys. **19**, 189 (1989).
[32] A. Stadler, W. Glöckle, and P. U. Sauer, Phys. Rev. C **44**, 2319 (1991).
[33] D. Hüber, H. Witala, A. Nogga, W. Glöckle, and H. Kamada, Few-Body Syst. **22**, 107 (1997).
[34] R. A. Brandenburg, G. S. Chulick, R. Machleidt, A. Picklesimer, and R. M. Thaler, Phys. Rev. C **37**, 1245 (1988).
[35] B. Juliá-Díaz, J. Haidenbauer, A. Valcarce, and F. Fernández, Phys. Rev. C **65**, 034001 (2002).
[36] A. Valcarce, H. Garcilazo, F. Fernández, and P. González, Rep. Prog. Phys. **68**, 965 (2005).
[37] W. Schadow, W. Sandhas, J. Haidenbauer, and A. Nogga, Few-Body Syst. **28**, 241 (2000).
[38] J. Haidenbauer and K. Holinde, Phys. Rev. C **53**, R25 (1996).
[39] W. Schadow and W. Sandhas, Nucl. Phys. **A631**, 588c (1998).
[40] W. Schadow, O. Nohadani, and W. Sandhas, Phys. Rev. C **63**, 044006 (2001).
[41] J. Adam, Jr., M. T. Peña, and A. Stadler, Phys. Rev. C **69**, 034008 (2004).
[42] W. Glöckle and H. Kamada, Phys. Rev. Lett. **71**, 971 (1993).
[43] G. Q. Li, R. Machleidt, and R. Brockmann, Phys. Rev. C **45**, 2782 (1992).
[44] H. Witala, W. Glöckle, and H. Kamada, Phys. Rev. C **43**, 1619 (1991).
[45] L. Wolfenstein, Phys. Rev. **96**, 1654 (1954).
[46] I. Fachruddin, Ch. Elster, and W. Glöckle, Phys. Rev. C **68**, 054003 (2003).
[47] M. E. Rose, *Elementary Theory of Angular Momentum* (Wiley, New York, 1957).