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Hadron loops: General theorems and application to charmonium

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In this paper, we develop a formalism for incorporating hadron loops into the quark model. We derive expressions for mass shifts, continuum components, and mixing amplitudes of "quenched" quark model states due to hadron loops, as perturbation series in the valence-continuum coupling Hamiltonian. We prove three general theorems regarding the effects of hadron loops, which show that given certain constraints on the external "bare" quark model states, the valence-continuum coupling, and the hadrons summed in the loops, the following results hold: (1) The loop mass shifts are identical for all states within a given N, L multiplet. (2) These states have the same total open-flavor decay widths. (3) Loop-induced valence configuration mixing vanishes provided that $L_i \neq L_f$ or $S_i \neq S_f$. The charmonium system is used as a numerical case study, with the $^3P_0$ decay model providing the valence-continuum coupling. We evaluate the mass shifts and continuum mixing numerically for all $1S$, $1P$, and $2S$ charmonium valence states due to loops of D, D^*, D_s, and D_s^* meson pairs. We find that the mass shifts are quite large but numerically similar for all the low-lying charmonium states, as suggested by the first theorem. Thus, loop mass shifts may have been "hidden" in the valence quark model by a change of parameters. The two-meson continuum components of the physical charmonium states are also found to be large, creating challenges for the interpretation of the constituent quark model.

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I. INTRODUCTION

The discovery of the narrow charm-strange mesons $D_{s0}^*(2317)^+$ [1] and $D_{s1}(2460)^+$ [2] has given special impetus to the calculation of hadron loop effects, since the loops are often cited as a possible reason for the surprisingly low masses of these mesons. This possibility is supported by the prediction that the corresponding $c\bar{s}$ quark model states have especially strong couplings to the open-flavor decay channels DK and D^*K [3,4]. (For discussions of the importance of hadronic loop effects in this and other contexts, including other heavy-quark mesons, see, for example, Refs. [5–36].)

The subject of valence-continuum couplings is also relevant to the $X(3872)$ seen in $J/\psi\pi^+\pi^-$ [37,38], which may be predominantly a weakly bound $1^{++}DD^*$ molecular state; the size of the $c\bar{c}$ valence component present in this system has important implications for the properties of this state. (See Ref. [39] for a review of these and other recent developments in heavy-flavor hadrons.)

Since the open-flavor decay couplings of hadrons to two-body final states $A \to BC$ are large, one might anticipate that second-order decay loops, in which a hadron virtually decays to a two-body intermediate state and then reforms the original hadron ($A \to BC \to A$), are also important effects. These second-order virtual processes give rise to mass shifts of the bare hadron states and contribute continuum components to the physical hadron state vectors. A careful estimate of these mass shifts is of great interest, since they are usually not included in quark potential models and are only partially present in quenched lattice QCD, and may constitute important "systematic" errors in the results.

In our initial study, we develop a formalism for treating these loops effects, using results from our earlier studies of open-flavor decay amplitudes. In particular, we give results for the loop-induced mass shifts and continuum amplitudes of hadrons, as well as the off-diagonal "spectroscopic mixing amplitudes" induced by hadron loops between different external discrete hadron basis states.

As a numerical application, we consider the charmonium system and evaluate these mass shifts and continuum components for the lighter (narrow) charmonium states that lie below the open charm threshold. Charmonium is especially attractive as a test system for studying loop effects because the low-lying spectrum is clear experimentally, with complete $1S$, $1P$, and $2S$ multiplets, and all eight states in these multiplets are below the open-flavor decay threshold. (This implies that all mass shifts are negative, with no cancellations.) In addition, the charmonium system is only moderately relativistic, and the spectrum is quite well described by quenched potential models and lattice gauge theory. Thus loop effects may be evaluated as (possibly) perturbative corrections to well understood $c\bar{c}$ potential model states, and the results may be contrasted with an unambiguous experimental spectrum.

II. FORMALISM

A. Loop model

To incorporate hadron loop effects in the quark model, we model a physical hadron as a bare valence state $|A\rangle$ augmented by two-hadron continuum components,

$$|\Psi\rangle = |A\rangle + \sum_{BC} \psi_{BC}|BC\rangle. \tag{1}$$

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We assume that the Hamiltonian for this combined system consists of a valence Hamiltonian H_0 (the quark model Hamiltonian) and an interaction H_I which couples the valence and continuum sectors,

$$H = H_0 + H_I. \tag{2}$$

We will evaluate the continuum components of the hadron state and their physical effects as a perturbation series in the valence-continuum coupling H_I. Our starting point for this perturbations series is the set of single valence hadron H_0 eigenstates; a specific valence state is written as $|A(\vec{p}_A)\rangle$, and is assigned an H_0 eigenvalue of $E_A = (M_A^2 + \vec{p}_A^{\,2})^{1/2}$. Since we normally work in the rest frame of the valence hadron, $\vec{p}_A = 0$, this energy eigenvalue is just the rest mass M_A of the bare valence quark model hadron.

The free two-hadron valence states which form our zeroth-order noninteracting continua are written as $|B(\vec{p}_B)C(\vec{p}_C)\rangle$. The valence Hamiltonian H_0 is understood to operate only between the constituents of B and C separately; BC interactions, which are not treated here, would be incorporated in a separate two-hadron interaction Hamiltonian. This BC continuum state has H_0 eigenvalue $E_{BC} = E_B + E_C$ where $E_B = (M_B^2 + \vec{p}_B^{\,2})^{1/2}$ and $E_C = (M_C^2 + \vec{p}_C^{\,2})^{1/2}$. In the A rest frame, we have $\vec{p}_B = -\vec{p}_C \equiv \vec{p}$, and with $p \equiv |\vec{p}|$ the energies are $E_B = (M_B^2 + p^2)^{1/2}$ and $E_C = (M_C^2 + p^2)^{1/2}$.

The matrix elements of the valence-continuum coupling Hamiltonian are of the form

$$\langle BC|H_I|A\rangle = h_{fi}\,\delta(\vec{p}_A - \vec{p}_B - \vec{p}_C). \tag{3}$$

With explicit momentum labels, these rest-frame one- and two-hadron valence states are written as $|A(\vec{0})\rangle$ and $|B(\vec{p})C(-\vec{p}\,)\rangle$, and the coupling matrix element is a function of a single momentum vector, $h_{fi}(\vec{p})$.

B. Mass shifts

The mass shift of a valence hadron A due to its coupling to the BC continuum may be expressed in terms of the coupling matrix element $h_{fi}(\vec{p})$ of Eq. (3) using second-order perturbation theory (for a general discussion, see Ref. [40]). The usual discrete sum $\sum_n$ over intermediate states generalizes to a momentum-space integral over continuum states $|B(\vec{p})C(-\vec{p})\rangle$; the result for a single BC channel is

$$\begin{aligned}\Delta M_A^{(BC)} &= \sum_n \frac{\left|\langle\psi_n|H_I|\psi_{h_0}\rangle\right|^2}{(E_n - E_0)} = -\int d^3p \frac{|h_{fi}|^2}{(E_{BC} - M_A)} \\ &= -\mathcal{P}\int_{M_B+M_C}^{\infty} \frac{dE_{BC}}{(E_{BC} - M_A)} \frac{pE_BE_C}{E_{BC}} \int d\Omega_p |h_{fi}|^2 \\ &\quad - i\pi \left\{ \frac{pE_BE_C}{M_A} \int d\Omega_p |h_{fi}|^2 \right\}\Big|_{E_{BC}=M_A},\end{aligned} \tag{4}$$

where $\mathcal{P}$ is the principal part integral. There is an implicit sum over any intermediate-state polarization labels in the squared Hamiltonian matrix element $|h_{fi}|^2$.

As a check of our central result Eq. (4), note that the imaginary part of the mass shift

$$\mathrm{Im}\big(\Delta M_A^{(BC)}\big) = -\pi \left\{ \frac{pE_BE_C}{M_A} \int d\Omega_p |h_{fi}|^2 \right\}\Big|_{E_{BC}=M_A} \tag{5}$$

should be related to the total decay rate by

$$\Gamma(A \to BC) = -2\mathrm{Im}\big(\Delta M_A^{(BC)}\big). \tag{6}$$

The standard $A \to B + C$ decay rate formula given in Eq. (5) of Ref. [41] is indeed consistent with this relation.

If the initial hadron mass is below the BC threshold ($M_A < M_B + M_C$), we do not encounter a singular energy denominator, and this mass shift is a real, negative definite integral over p,

$$\Delta M_A^{(BC)} = -\int_0^\infty \frac{p^2dp}{(E_{BC} - M_A)} \int d\Omega_p |h_{fi}|^2. \tag{7}$$

If one considers mixing between the valence state A and several continuum BC channels, the total mass shift at this (leading) order in the valence-continuum coupling is the sum of the individual mass shifts due to each channel.

Nonperturbative estimates of the mass shift can be made in the absence of final state interactions by summing iterated bubble diagrams. The result is a full propagator of the form

$$-iG(s) = \frac{1}{(s - M^2 - \Sigma(s))}, \tag{8}$$

where Σ is the one particle irreducible self-energy of the meson in question. The propagator pole yields the meson mass shift and width. Contact to our perturbative, nonrelativistic results can be made by identifying $\sqrt{s}\Gamma(s) = -\mathrm{Im}(\Sigma(s))$ and $2\sqrt{s}\delta M(s) = \mathrm{Re}(\Sigma(s))$, and assuming that the width and mass shift are small relative to the unperturbed meson mass.

We have computed numerical pole positions with iterated loops for the charmonium examples of the next section using this formalism and find rather small differences in mass shifts relative to the single loop approximation [typically $\Delta M(c\bar{c})$ changes by less than 10%].

We have also examined the effect of mixing-induced coupling between states. Thus the denominator of the propagator becomes a matrix, $(s - m_i^2)\delta_{ij} - \Sigma_{ij}(s)$. Solving this equation for the case of the J/ψ coupling to ψ' and ψ'' through DD, D_sD_s, DD^*, $D_sD_s^*$, D^*D^*, and $D_s^*D_s^*$ continua again yields corrections to the one-loop diagonal results that are typically about 10%. We therefore simply present perturbative (one-loop), single-channel mass shifts in the discussion of charmonium.

C. Continuum components

Although mass shifts due to loops may be "hidden" in fitted parameters in quenched approaches, such as m_q or V_0 in potential models and m_q or the coupling in quenched lattice gauge theory, it should nonetheless be possible to identify other, more characteristic effects of the two-meson continuum components. We will require the explicit continuum component wave functions to evaluate their effects on observables. Here we give general results for these wave functions; an example will be considered in the discussion of charmonium.

The valence-continuum coupling H_I induces a continuum component in an initially pure valence state $|A\rangle$. At leading order in h_{fi} this continuum component is given by

$$\sum_{BC} \psi_{BC}|BC\rangle = -(H_0 - M_A)^{-1}\, H_I\, |A\rangle. \tag{9}$$

The momentum-space wave function of the continuum component in a specific channel BC is

$$\phi_{B(\vec{p})C(-\vec{p})} \equiv \phi_{BC}(\vec{p}) = -\frac{h_{fi}(\vec{p})}{(E_{BC}(p) - M_A)}. \tag{10}$$

Using the conventions of Ref. [42], the corresponding real-space wave function in the relative separation $\vec{r} = \vec{r}_B - \vec{r}_C$ is

$$\psi_{BC}(\vec{r}) = \int d^3p\, \phi_{BC}(\vec{p})\frac{e^{i\vec{p}\cdot\vec{r}}}{(2\pi)^{3/2}}. \tag{11}$$

For nonzero spin, this spatial wave function is implicitly summed over the meson orbital and spin magnetic quantum numbers, to give overall states with the J, J_z of meson A.

The norm of this continuum component gives the probability that the physical energy eigenstate is in the two-meson channel BC. This is

$$P_A^{(BC)} = \sum_n \frac{|\langle\psi_n|H_I|\psi_0\rangle|^2}{(E_n - E_0)^2} = \int d^3p\, \frac{|h_{fi}|^2}{(E_{BC} - M_A)^2}. \tag{12}$$

$$= \int_0^\infty \frac{p^2\, dp}{(E_{BC} - M_A)^2} \int d\Omega_p |h_{fi}|^2. \tag{13}$$

D. Spectroscopic mixing

Mixing of discrete "valence" quark model basis states through hadron loops is an interesting effect which may have easily observable consequences. The amplitude a_{fi} to find a discrete basis state $|f\rangle$ in the initially pure valence state $|i\rangle$ as a result of continuum mixing is given by second-order perturbation theory as

$$a_{fi} = \frac{1}{(M_f - M_i)} \sum_{BC} \int d^3p\, \frac{h_{f,BC}(\vec{p})h_{BC,i}(\vec{p})}{(E_{BC}(p) - M_i)}. \tag{14}$$

For an initial valence state within the continuum, this is replaced by a principal part integral, and the amplitude a_{fi} has an imaginary part, analogous to Eq. (4).

Note that this loop-induced mixing amplitude is somewhat counterintuitive, in that it is nonsymmetric in general; that is,

$$|a_{fi}| \neq |a_{if}|. \tag{15}$$

This disagrees with the simple picture of an orthogonal rotation between two basis states often used to describe mixing in the quark model. [Examples include mixing between spin-singlet and spin-triplet axial vector K_1 and D_1 mesons, and between the $|2^3S_1\rangle$ and $|^3D_1\rangle$ charmonium basis states in the $\{\psi'(3686), \psi(3772)\}$ system.] Since this is actually an infinite-dimensional Hilbert space rather than a two-dimensional one, it is of course not necessary that $|a_{fi}| = |a_{if}|$. Instead, the valence state $|i\rangle$ that is closest to the continuum, and hence minimizes the valence-continuum energy denominator $(E_{BC} - M_i)$ in Eq. (14), will tend to experience the largest mixing. This will be illustrated in the next section.

E. Three loop theorems

In the Appendix we show that sums over sets of mesons within the loop under certain conditions gives very simple relations between the mass shifts, strong widths, and configuration mixing amplitudes due to hadron loops. Although these relations are not exactly satisfied in nature, they are sufficiently accurate to be relevant to realistic problems such as the charmonium examples we consider here.

Provided that our conditions are satisfied, one may show that for the states $\{A\}$ in a given N_i, L_i multiplet,

(i) The mass shifts for all states {A} are equal.
(ii) Their strong (open-flavor) total widths are equal.
(iii) The configuration mixing amplitude a_{fi} between any two valence basis states i and f vanishes if $L_i \neq L_f$ or $S_i \neq S_f$.

These conclusions hold to all orders if there are no final state interactions in the continuum channels.

To prove these loop results, we consider a sum over a finite set of intermediate (loop) mesons that runs over all mesons in a given N, L multiplet, taking on all allowed values of spin S and total angular momentum J. Examples of such loop sets include "$S\bar{S}$" $\{(D\bar{D}), (D\bar{D}^*), (D^*\bar{D}), (\bar{D}^*\bar{D}^*)\}$ and "$S\bar{P}$" $\{(D\bar{D}_0^*), (D^*\bar{D}_0^*), (D\bar{D}_1), (D^*\bar{D}_1), (D\bar{D}_1{}'), (D^*\bar{D}_1{}'), (D\bar{D}_2^*), (D^*\bar{D}_2^*)\}$, where we have explicitly indicated antiparticles. The proof assumes that all members of the set of intermediate (loop) mesons which are summed over have the same mass, and for the first two "diagonal" conclusions (equality of mass shifts and total widths) we also require that the external mesons have a common bare mass and radial wave function. In addition, there are general conditions that the valence-continuum coupling must satisfy, which are discussed in the Appendix. These constraints are satisfied by the $^3P_0$ model that is used for illustration in this paper.

The first conclusion suggests how these intrinsically large loop mass shifts can be hidden in the parameters of quenched models; since the largest effect of loops on the spectrum of states in the multiplets we consider is an overall downward mass shift in the multiplet center of gravity, (c.o.g.), this can be approximately parametrized through a change in the quark mass or through a constant in the potential.

As an illustration of this theorem in a specific case, Table I shows the relative mass shifts of all the $1P$ charmonium levels due to their $^3P_0$ couplings to DD, DD^*, and D^*D^* meson loops, assuming that all the bare $1Pc\bar{c}$ masses are identical, the

TABLE I. Relative one-loop mass shifts of $1P$ charmonium states in the equal mass limit.

Bare $c\bar{c}$ state	Relative mass shifts, $\Delta M_i(L_{BC})/\Delta M_{tot}(L_{BC})$					
	$L_{BC}=0$			$L_{BC}=2$		
	DD	DD^*	D^*D^*	DD	DD^*	D^*D^*
1^3P_2	0	0	1	3/20	9/20	2/5
1^3P_1	0	1	0	0	1/4	3/4
1^3P_0	3/4	0	1/4	0	0	1
1^1P_1	0	1/2	1/2	0	1/2	1/2

TABLE II. Valence configuration mixing amplitudes a_{fi} due to loops (DD; DD^*; $D^*D^*\ {}^1P_1$; $D^*D^{*5}P_1$) in the $\{|^3S_1\rangle, |2^3S_1\rangle, |^3D_1\rangle\}$ system. The total a_{fi} is the sum of the individual loop contributions, as indicated. The labels $|I\rangle$ etc. refer to the physical (unnormalized) states one finds due to loop-induced mixing between $|c\bar{c}\rangle$ valence states. (These are perturbative, one-loop results, with parameters as in Table I.) Note the approximate cancellations in $\Delta L \neq 0$ mixing and the nonsymmetric mixing amplitudes.

	$\|I\rangle$	$\|II\rangle$	$\|III\rangle$
$\|^3S_1\rangle$	[1]	$-.013-.011+.000+.006=-.018$	$-.089-.017i+.086-.010-.020=-.033-.017i$
$\|2^3S_1\rangle$	$-.003-.014-.001-.026=-.045$	[1]	$-.572-.138i+.573-.072-.143=-.214-.138i$
$\|^3D_1\rangle$	$+.015-.026+.004+.008=+.001$	$+.340-.469+.063+.126=+.060$	[1]

D and D^* meson masses are identical, and each flavor system has a common radial wave function. Although the individual continuum channels DD, DD^*, and D^*D^* make different contributions to the mass shift of each meson, the *summed* mass shift from all three channels is identical for each of the four $1P$ mesons. Thus, if the mesons are initially degenerate, they remain degenerate after these loop effects are included. This has been noted previously by Tornqvist [34]; the results presented here can be considered an elaboration of this original observation. Furthermore, Tornqvist and Żenczykowski have studied the analogous effect in baryons in Refs. [35,36].

One may also see evidence for the no-loop-mixing result for states with different L (conclusion 3 above) in our charmonium example. In Table II, we show the individual DD, DD^*, and D^*D^* one-loop contributions to the mixing between $|^3S_1\rangle$ $|2^3S_1\rangle$ and $|^3D_1\rangle$ charmonium valence basis states. Note that in some of the disfavored cases, such as $|^3S_1\rangle \to |^3D_1\rangle$, there is an almost complete cancellation of the final $|^3D_1\rangle$ amplitude, due to destructive interference between the DD, DD^*, and D^*D^* loops. This destructive interference between loops is still evident but less complete for mixing between the higher lying states $|2^3S_1\rangle$ and $|^3D_1\rangle$, because they are quite close to the DD threshold; this causes the energy denominators to vary widely between channels, so the mass constraints assumed in the theorem are strongly violated.

III. NUMERICAL RESULTS: APPLICATION TO CHARMONIUM

A. Mass shifts

To illustrate this formalism, we will evaluate the effect of open-charm meson loops on the masses and compositions of $1S$, $1P$, and $2S$ charmonium states. We use the well-established ${}^3P_0$ model [41,43–46] as the valence-continuum coupling Hamiltonian and neglect two-meson interactions. Our general approach is very similar to an earlier study by Heikkila, Ono, and Tornqvist [27], although we find somewhat larger loop effects than reported by this reference.

The ${}^3P_0$ model treats strong decays as being due to a bilinear quark-antiquark pair production interaction Hamiltonian, $H_I = \gamma \sum_q 2m_q \bar{\psi}_q \psi_q$, which is normally evaluated using nonrelativistic quark model matrix elements.

A diagrammatic technique for determining the valence-continuum coupling matrix element h_{fi} between a meson A and a two-meson state BC in the ${}^3P_0$ model is given in Ref. [41]. We use this approach to determine the $\{h_{fi}\}A$-BC valence-continuum matrix elements. Gaussian momentum-space quark model meson wave functions were used, with unequal light, strange, and charm quark masses. For simplicity, a common width parameter β was assumed for all charmonium and open-charm meson wave functions; tests of the overlaps of more realistic Coulomb plus linear plus smeared hyperfine wave functions with Gaussians shows that this is a reasonable "zeroth-order" approximation.

1. J/ψ mass shifts

As a first numerical example, we consider the mass shift of an initial valence J/ψ $c\bar{c}$ state mixing with the DD continuum. The h_{fi} matrix element for the transition $J/\psi \to D(\vec{p})\bar{D}(-\vec{p})$ for a rest J/ψ in polarization state m is given by

$$h_{fi} = \frac{2^3}{3^3}\,\frac{1+3r_n}{1+r_n}\,\frac{\gamma}{\pi^{1/4}\beta^{1/2}}\ \rho e^{-\rho^2/3(1+r_n)^2}\mathrm{Y}_{1m}(\Omega_p), \tag{16}$$

where $r_n = m_n/m_c$ is the light ($n=u,d$) to charm quark mass ratio, $\rho = p/\beta$, β is the simple harmonic oscillator (SHO) meson wave function width parameter (taken to be the same for all mesons in this work), and γ is the dimensionless ${}^3P_0$ pair production amplitude. On substituting this h_{fi} into the mass shift formula Eq. (7), and including a flavor factor of two for neutral and charged DD loops, we find

$$\Delta M_{J/\psi}^{(DD)} = -\frac{2^7}{3^6}\left(\frac{1+3r_n}{1+r_n}\right)^2\frac{\gamma^2\beta}{\pi^{1/2}} \times \int_0^\infty \frac{\rho^4 e^{-2\rho^2/3(1+r_n)^2}d\rho}{2\left(\rho^2+\mu_D^2\right)^{1/2}-\mu_{J/\psi}}, \tag{17}$$

where $\mu \equiv M/\beta$ for each meson.

Numerical evaluation of this integral using $M_{J/\psi} = 3.097$ GeV, $M_D = 1.867$ GeV, $\beta = 0.5$ GeV, $\gamma = 0.35$ (motivated by total widths; see Fig. 2 of Ref. [47]), and $r_n = m_n/m_c = 0.33/1.5$ gives the result

$$\Delta M_{J/\psi}^{(DD)} = -23.1\ \text{MeV}. \tag{18}$$

Using the experimental J/ψ mass as the input bare mass in this manner is of course only appropriate as an estimate of the size of these effects. Since this is, in effect, a renormalization problem, the sum of the (unobservable) bare mass and mass shift should be identified with the experimental J/ψ mass.

Although this DD contribution is a relatively small effect, incorporation of higher $(1S)(1S)$ channels shows that the summed loop mass shifts are quite large. The formulas for the DD, DD^*, and D^*D^* loop integrals in the J/ψ system are identical, but the relative spin-flavor factors of 1:4:7 give a combined mass shift that is an order of magnitude larger than for the DD channel alone. (These 1:4:7 spin-flavor factors were reported earlier by Heikkila *et al.* [27] for loop contributions to mass shifts, and by De Rujula *et al.* [48] and Close [49] for charm production cross sections.) On including all six D, D^*, D_s, and D_s^* pair channels (with $M_D = 1.867$ GeV, $M_{D^*} = 2.008$ GeV, $M_{D_s} = 1.968$ GeV, $M_{D_s^*} = 2.112$ GeV, and $r_s = 0.55/1.5$), we find

$$\sum_{n=1}^{6} \Delta M_{J/\psi}^{(n)} = -457.5\,\text{MeV}. \tag{19}$$

This very large mass shift appears to invalidate the quenched quark model. In the next section, we will see that this scale of mass shift is actually common to all the low-lying charmonium states, and it can therefore be approximately subsumed in a change of parameters (such as the charm quark mass m_c or an overall constant V_0 in the $c\bar{c}$ potential).

2. Mass shifts of other charmonium states

One can understand how such large mass shifts may have been accommodated in pure $c\bar{c}$ quark models by evaluating the mass shifts of the remaining low-lying charmonium states below DD threshold. We again set the bare masses equal to the experimental values to generate this estimate; the values used (in GeV) are $M_{\psi'} = 3.686$, $M_{\eta_c'} = 3.637$, $M_{\chi_2} = 3.556$, $M_{\chi_1} = 3.511$, $M_{\chi_0} = 3.415$, $M_{h_c} = 3.526$, and $M_{\eta_c} = 2.979$, and the other model parameters are as before.

The resulting mass shifts are given in Table III, and evidently are all quite large. Note, however, that they are rather similar, so there is a much smaller scatter about the mean shift; the mean and variance are, respectively, -471 and 49 MeV. The scatter of mass shifts within a multiplet is even smaller; the variance within the $1P$ multiplet, for example, is just 24 MeV. (The similarity of mass shifts within a multiplet was discussed in the previous section and is a consequence of the general nature of the valence-continuum coupling model.)

The large overall shift could be parametrized in a pure $c\bar{c}$ "quenched" potential model through a shift in m_c or through the addition of a large negative constant V_0 to the $c\bar{c}$ potential. One expects that the goodness of fit to the $c\bar{c}$ spectrum is rather insensitive to these modifications.

The J/ψ-η_c and ψ'-η_c' loop-induced mass splitting has been discussed previously by Eichten *et al.* [7]. These authors sum over the same set of intermediate states employed here but use the Cornell decay model for the strong decay interaction. They find a small loop-induced J/ψ-η_c mass splitting of -3.7 MeV and a ψ'-η_c' splitting of -20.9 MeV, bringing their model into good agreement with the experimental ψ'-η_c mass difference. Table III shows that we find a numerically similar ψ'-η_c' splitting of -24 MeV; however, the ground state mass difference due to coupling to the continuum is -34 MeV, indicating that loop effects induce a larger ψ'-η_c' mass difference of approximately $+10$ MeV. This is consistent with the bare model employed here, which finds a substantially smaller bare ψ'-η_c' mass difference [47] than that of Ref. [7].

B. Continuum components

Although the large negative mass shifts may be "hidden" by the choice of m_c or V_0 in potential models and m_c or $a(\beta)$ in quenched LGT, it should nonetheless be possible to identify other observable effects of the two-meson continuum components, since according to Table III their occupation probabilities are comparable to the valence $c\bar{c}$ components. To illustrate this, we will evaluate some of these continuum component wave functions explicitly and consider their effect on some experimentally observed properties of charmonium states.

Recall from Eq. (10) that the continuum component wave function in momentum space, $\phi_{BC}(\vec{p}\,)$, is given by

$$\phi_{BC}(\vec{p}) = -\frac{h_{fi}}{E_{BC}(p) - M_A}. \tag{20}$$

Again specializing to the DD component of the J/ψ as our example, this momentum-space wave function is

$$\phi_{DD}(\vec{p}\,) = \phi_{DD}(p) \mathrm{Y}_{1m}(\Omega_p), \tag{21}$$

TABLE III. Mass shifts (in MeV) and $c\bar{c}$ probabilities for low-lying charmonium states due to couplings to two-meson continua. This one-loop estimate sets the unperturbed bare masses to the experimental values and assumes $^3P_0$ model and SHO wave function parameters $\gamma = 0.35$ and $\beta = 0.5$ GeV and quark mass ratios $r_n = m_n/m_c = 0.33/1.5$ and $r_s = m_s/m_c = 0.55/1.5$.

Bare $c\bar{c}$ state		Mass shifts by channel, ΔM_i (MeV)							$P_{c\bar{c}}$
Multiplet	State	DD	DD^*	D^*D^*	D_sD_s	$D_sD_s^*$	$D_s^*D_s^*$	Total	
$1S$	$J/\psi(1^3S_1)$	-23	-83	-132	-21	-76	-123	-457	0.69
	$\eta_c(1^1S_0)$	0	-114	-105	0	-106	-98	-423	0.73
$2S$	$\psi'(2^3S_1)$	-27	-84	-126	-19	-70	-113	-440	0.51
	$\eta_c'(2^1S_0)$	0	-118	-103	0	-102	-94	-416	0.61
$1P$	$\chi_2(1^3P_2)$	-40	-105	-144	-33	-88	-111	-521	0.49
	$\chi_1(1^3P_1)$	0	-127	-148	0	-90	-130	-496	0.52
	$\chi_0(1^3P_0)$	-57	0	-196	-34	0	-172	-459	0.58
	$h_c(1^1P_1)$	0	-149	-130	0	-118	-107	-504	0.52

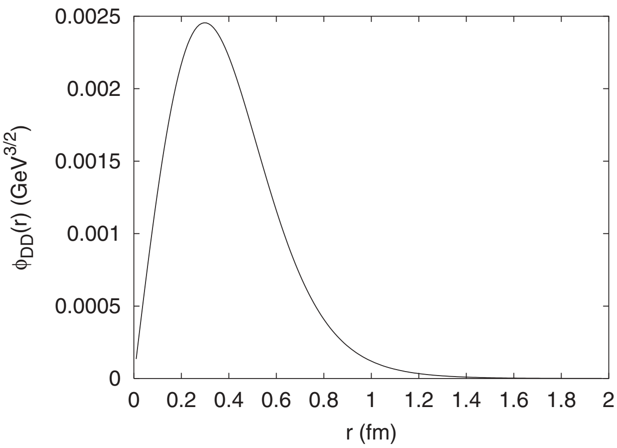

FIG. 1. DD continuum component wave function of the J/ψ.

where

$$\phi_{DD}(p) = -\frac{8}{27}\,\frac{1+3r_n}{1+r_n}\,\frac{\gamma}{\pi^{1/4}\beta^{3/2}}\,\frac{p e^{-p^2/3(1+r_n)^2\beta^2}}{2\left(M_D^2+p^2\right)^{1/2}-M_{J/\psi}}. \tag{22}$$

Note that this component formally diverges as $M_{J/\psi} \to 2M_D$, due to a vanishing energy denominator; this shows that as expected, the largest continuum components arise in valence states that are closest to the continuum. The spatial wave function corresponding to $\phi_{DD}(p)$ is shown in Fig. 1.

C. J/ψ continuum probabilities

The probability of finding the physical J/ψ in the DD continuum [from Eq. (13)] is

$$P_{J/\psi}^{(DD)} = \frac{2^7}{3^6}\left(\frac{1+3r_n}{1+r_n}\right)^2 \frac{\gamma^2}{\pi^{1/2}} \int_0^\infty \frac{\rho^4 e^{-2\rho^2/3(1+r_n)^2} d\rho}{\left[2\left(\rho^2+\mu_D^2\right)^{1/2}-\mu_{J/\psi}\right]^2}. = 0.021. \tag{23}$$

Although this appears to be a reassuringly small correction to the valence quark model description of the J/ψ as a pure $c\bar{c}$ state, when we calculate the probability that the physical state is in any of the $i = 1, \ldots, 6$ meson continuum states DD, D^*D, D^*D^*, D_sD_s, $D_sD_s^*$, $D_s^*D_s^*$, we again find that the summed contribution is quite large. Expressed as the probability that the physical J/ψ is in the valence $c\bar{c}$ state, we find

$$P_{J/\psi}^{(c\bar{c})} = 1 - \sum_{i=1}^{6} P_{J/\psi}^{(i)} = 0.685. \tag{24}$$

Just as was the case for the mass shifts, we find that the continuum components of charmonium states are very large. This represents an interesting challenge in the interpretation of the constituent quark model and quenched QCD, which both neglect meson loops. The main issue is whether such large loop effects can be absorbed into parameter redefinitions when computing observables.

D. Spectroscopic mixing

As noted previously, discrete charmonium levels below the continuum mix at second order in the valence-continuum Hamiltonian H_I through hadron loops, provided that both the initial and final valence states $|i\rangle$ and $|f\rangle$ have nonzero matrix elements to at least one continuum intermediate state $|BC\rangle$.

Here we shall illustrate this effect by calculating the amount of mixing between low-lying 1^{--} states. First we consider the $|^3S_1\rangle$ and $|2^3S_1\rangle c\bar{c}$ basis states, which at leading order are identified with the J/ψ and $\psi(3686)$, respectively. We will give explicit formulas for mixing through DD intermediate states and simply quote numerical results for mixing through higher two-meson continua.

The h_{fi} matrix elements required to evaluate these mixing amplitudes are

$$h_{BC,i}(^3S_1 \to DD) = \frac{2^3}{3^3}\,\frac{1+3r_n}{1+r_n}\,\frac{\gamma}{\pi^{1/4}\beta^{1/2}}\ \rho e^{-\rho^2/3(1+r_n)^2}\Upsilon_{1m}(\Omega_p), \tag{25}$$

$$h_{f,BC}(DD \to 2^3S_1) = \frac{2^{5/2}}{3^{5/2}}\left[1+\frac{2}{9}\frac{1}{1+r_n}-\frac{8}{27}\frac{1+3r_n}{(1+r_n)^3}\rho^2\right] \times \frac{\gamma}{\pi^{1/4}\beta^{1/2}}\rho e^{-\rho^2/3(1+r_n)^2}\Upsilon_{1m}^*(\Omega_p), \tag{26}$$

and

$$h_{f,BC}\left(DD \to {}^3D_1\right) = \frac{2^{11/2}5^{1/2}}{3^{9/2}}\left[\frac{r}{1+r}-\frac{2}{15}\frac{1+3r_n}{(1+r_n)^3}\rho^2\right] \times \frac{\gamma}{\pi^{1/4}\beta^{1/2}}\rho e^{-\rho^2/3(1+r_n)^2}\Upsilon_{1m}^*(\Omega_p). \tag{27}$$

Substitution of these expressions into Eq. (14) and evaluation of the overlap integral gives the $1^3S_1 - 2^3S_1$ mixing amplitudes a_{fi}. Our numerical results, using the same parameters and masses as previously, are given in Table I.

IV. SUMMARY AND CONCLUSIONS

In this paper, we presented a formalism for "unquenching the quark model" through the incorporation of the effects of hadron loops on valence quark model states. We gave expressions for the mass shift, continuum components of the hadron state vector and mixing amplitudes between discrete valence states that follow from hadron loop effects for a given valence-continuum coupling Hamiltonian.

As a numerical example, we applied this formalism to the experimentally well-established light charmonium system, using the $^3P_0$ decay model for the valence-continuum coupling. We evaluated the mass shifts and compositions of the physical charmonium states for all $1S$, $1P$, and $2S$ states using perturbation theory in the valence-continuum coupling; these mass shifts and two-meson components were found to be quite large. Since the mass shifts of the different charmonium levels are numerically rather similar, we speculate that they have been hidden in the choice of m_c or a constant potential shift V_0 in $c\bar{c}$ valence potential models. It remains to be seen whether the two-meson continuum components can be "parametrized away"—it is possible that they lead to important mixing effects between discrete charmonium basis states that may be experimentally observable.

The mixing effects we find using the ${}^3P_0$ decay model as the valence-continuum coupling prove to be quite large for higher mass intermediate continuum states. Although it is possible that these effects can be largely renormalized away, an accurate description of loop effects will probably require the development of a more realistic valence-continuum coupling Hamiltonian than the ${}^3P_0$ model.

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APPENDIX: LOOP THEOREMS

Numerical experiments suggest that although individual loop contributions to physical observables are large, in practice there are often important cancellations or constraints when loop sums over sets of mesons are carried out. This is evident, for example, in the mass shifts in Table III; the individual loop mass shift for a given state varies widely between states, but the total mass shifts when summed over loops are rather similar. One can see that these relations are exact in certain limits. As an example, Table I shows the relative mass shifts of the four P-wave charmonium states in the limit in which they have identical initial masses, and the D and D^* within the loops also have identical masses; although the individual channel mass shifts differ, we find the same total mass shift for each P-wave state on summing over the channels DD, DD^*, and D^*D^*.

A similar result is evident in the loop-induced configuration mixing discussed in the text; the configuration mixing amplitude a_{fi} between initial i and final f meson basis states in the usual N, J, L, S basis is found to be zero if $L_i \neq L_f$, provided that the mesons in the loops have identical masses and we again sum over a complete set of loop meson spin states S_B and S_C. As an example, in this limit this gives a zero mixing amplitude due to loops between any charmonium ${}^3S_1$ and ${}^3D_1$ basis states.

In this Appendix, we give a proof of this mass shift identity and the zero-mixing result for loop sums; these results hold whenever one sums over loops containing a complete set of spin (S) meson states (in a given N, J, L, S multiplet). The proof applies to the ${}^3P_0$ coupling model in particular, but it also holds for a more general class of valence-continuum couplings, specifically to spin-one, factorized, spectator decay models, as discussed by Burns, Close, and Thomas [50]. In this type of model, the valence-continuum coupling proceeds through spin-one $q\bar{q}$ pair production, the initial quarks do not couple to the decay vertex, and the spatial dependence of the decay vertex multiplies the created $q\bar{q}$ spin operator: $\mathcal{O} = \sigma\psi$, where ψ represents the spatial portion of the decay vertex. The proof therefore also applies to the Cornell decay model [8] and a decay model based on the nonrelativistic reduction of the interaction $\int \bar{\psi}\psi(\vec{x})V(\vec{x}-\vec{y})\bar{\psi}\psi(\vec{y})$, but it does not apply to pair production from one gluon exchange (discussed in Ref. [41]).

Given a valence-continuum coupling of this general form, which includes the ${}^3P_0$ used in this paper for numerical examples, one may show that the general $\langle BC|H_I|A\rangle$ matrix element is of the form

$$\begin{aligned}
&\langle J_A[Lj_{BC}]; j_{BC}[j_Bj_C]; j_B[s_B\ell_B]\, j_C[s_C\ell_C]|\boldsymbol{\sigma\psi}|J_A[s_A\ell_A]\rangle \\
&\quad = \sum_{s_{BC}\ell_{BC}L_f} (-)^{\eta}\hat{1}\hat{L}_f\hat{s}_{BC}\hat{\ell}_{BC}\hat{j}_B\hat{j}_C\hat{j}_{BC}\hat{s}_A\hat{s}_B\hat{s}_C\hat{s}_{BC} \\
&\qquad \cdot \langle L_f[L\ell_{BC}];\ell_{BC}[\ell_B\ell_C]||\boldsymbol{\psi}||\ell_A\rangle \\
&\qquad \cdot \begin{Bmatrix} s_B & \ell_B & j_B \\ s_C & \ell_C & j_C \\ s_{BC} & \ell_{BC} & j_{BC} \end{Bmatrix} \begin{Bmatrix} 1/2 & 1/2 & s_B \\ 1/2 & 1/2 & s_C \\ s_A & 1 & s_{BC} \end{Bmatrix} \\
&\qquad \cdot \begin{Bmatrix} s_{BC} & \ell_{BC} & j_{BC} \\ L & j_A & L_f \end{Bmatrix} \begin{Bmatrix} s_{BC} & s_A & 1 \\ \ell_A & L_f & j_A \end{Bmatrix}
\end{aligned} \tag{A1}$$

where $\hat{x} = \sqrt{2x+1}$ and $\eta = L + s_{BC} + \ell_{BC} + L_f + s_B$.

If the expressions for the mass splitting (Eq. (7)) or spectroscopic mixing are summed over intermediate states BC with identical masses, the resulting common energy denominators may be taken outside the sum over channels, and one is left with the expressions

$$\begin{aligned}
\delta m(i) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{m_i - \bar{E}_{BC}(p) + i\epsilon} \\
\times \sum_{BC} |\langle j_i[s_i\ell_i]|\boldsymbol{\sigma\psi}|BC\rangle|^2
\end{aligned} \tag{A2}$$

and

$$a_{fi} = \frac{1}{m_i - m_f}\int \frac{d^3p}{(2\pi)^3}\frac{1}{m_i - \bar{E}_{BC}(p) + i\epsilon}\cdot \tag{A3}$$

$$\times \sum_{BC}\langle j_f[s_f\ell_f]|\boldsymbol{\sigma\psi}|BC\rangle\langle BC|\boldsymbol{\sigma\psi}|j_i[s_i\ell_i]\rangle, \tag{A4}$$

where $\bar{E}_{BC}$ is the common energy of all states in the same multiplet as BC.

The sum over intermediate states simplifies when one considers a subsum over spin multiplets:

$$\sum_{BC} \to \sum_{s_Bs_Cj_Bj_C}; \tag{A5}$$

the angular momenta ℓ_B, ℓ_C, L can remain fixed.

On substituting Eq. (A1) into Eqs.(A2) and (A4), and using the orthogonality relation for $9j$ and $6j$ symbols

$$\sum_{j_{13},j_{24}} \hat{j}_{13}\hat{j}_{24} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & J \end{Bmatrix} \cdot \begin{Bmatrix} j_1 & j_2 & j'_{12} \\ j_3 & j_4 & j'_{34} \\ j_{13} & j_{24} & J \end{Bmatrix} = \frac{\delta(j_{12}, j'_{12})\delta(j_{34}, j'_{34})}{\hat{j}_{12}\hat{j}_{34}}, \quad \text{(A6)}$$

and

$$\sum_{j_{12}} \hat{j}^2_{12} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & J \end{Bmatrix} \cdot \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & J' \end{Bmatrix} = \frac{\delta(J, J')}{\hat{j}}, \quad \text{(A7)}$$

we obtain the sum

$$\frac{\delta_{s_i s_f}\delta_{\ell_i \ell_f}}{2\ell_i + 1} \sum_{\ell_{BC} L_f} |\langle L_f[L\ell_{BC}]; \ell_{BC}[\ell_B \ell_C]||\boldsymbol{\psi}||\ell_i\rangle|^2. \quad \text{(A8)}$$

Since this expression is independent of the initial and final meson spin, we conclude that all mesons in a given (assumed degenerate) spin multiplet receive the same width and mass shift from the sum over all intermediate (loop) mesons in a given spin multiplet. Furthermore, the spectroscopic mixing between mesons of different orbital angular momentum is zero when sums over spin multiplet intermediate states are carried out. (The external meson masses need not be identical to prove this result.) Finally, since these matrix elements drive nonperturbative mixing [see the discussion following Eq. (8)], these conclusions also apply to nonperturbative mixing, in the absence of final state interactions.

Spectroscopic mixing between mesons with differing radial quantum numbers (but identical otherwise) is not zero in general. The size of this mixing is governed by the spatial dependence of the strong decay vertex. Spectroscopic mixing has been studied previously by Geiger and Isgur [29], who considered the closure approximation, in which *all* loop mesons are assumed to be degenerate, not simply those in a spin multiplet. Geiger and Isgur used this approximation to explain the observed weakness of loop-driven OZI (Okubo-Zweig-Iizuka) violation effects. We remark that the closure approximation implies that spectroscopic mixing between states with different radial quantum numbers is zero: under this approximation Eq. (A4) simplifies to

$$a_{fi} = \frac{1}{(m_i - m_f)} \frac{1}{(m_i - \bar{E})} \langle n_f j_f[s_f \ell_f]|\mathcal{O}^2|n_i j_i[s_i \ell_i\rangle$$

$$= \frac{\langle 0|\mathcal{O}^2|0\rangle}{(m_i - m_f)(m_i - \bar{E})} \langle n_f j_f[s_f \ell_f]|n_i j_i[s_i \ell_i]\rangle, \quad \text{(A9)}$$

where the last form follows from the spectator nature of the decay model. Thus, if we impose the equality of all loop meson masses, *all* spectroscopic mixing is zero in the $^3P_0$ model and in a wide range of related decay models.

Finally, the previous discussion remains largely unchanged when considering mixing between initially degenerate states. In this case, one must diagonalize the matrix of second-order matrix elements in the degenerate subspace [51], $\delta H_{ij} = (m_i - m_j)a_{ji}$. Under the conditions of the theorem, off-diagonal matrix element in δH_{ij} are zero when the meson spins or angular momenta differ. Furthermore, the diagonal matrix elements are identical. Thus conclusions concerning mass shifts, widths, and small or zero spectroscopic mixing remain unchanged.

[1] B. Aubert *et al.* (BABAR Collaboration), Phys. Rev. Lett. **90**, 242001 (2003) [arXiv:hep-ex/0304021].
[2] D. Besson *et al.* (CLEO Collaboration), Phys. Rev. D **68**, 032002 (2003) [arXiv:hep-ex/0305100].
[3] S. Godfrey and R. Kokoski, Phys. Rev. D **43**, 1679 (1991).
[4] F. E. Close and E. S. Swanson, Phys. Rev. D **72**, 094004 (2005).
[5] E. S. Swanson, J. Phys. G **31**, 845 (2005).
[6] D. S. Hwang and D. W. Kim, Phys. Lett. **B601**, 137 (2004).
[7] E. J. Eichten, K. Lane, and C. Quigg, Phys. Rev. D **69**, 094019 (2004).
[8] E. Eichten, K. Gottfried, T. Kinoshita, K. D. Lane, and T. M. Yan, Phys. Rev. D **21**, 203 (1980).
[9] E. Eichten, K. Gottfried, T. Kinoshita, K. D. Lane, and T. M. Yan, Phys. Rev. D **17**, 3090 (1978); **21**, 313(E) (1980).
[10] E. Eichten, K. Gottfried, T. Kinoshita, K. D. Lane, and T. M. Yan, Phys. Rev. Lett. **36**, 500 (1976).
[11] G. Rupp and E. van Beveren, Eur. Phys. J. A **31**, 698 (2007).
[12] E. van Beveren and G. Rupp, Phys. Rev. Lett. **97**, 202001 (2006).
[13] E. van Beveren and G. Rupp, arXiv:hep-ph/0605317.
[14] E. van Beveren, J. E. G. N. Costa, F. Kleefeld, and G. Rupp, Phys. Rev. D **74**, 037501 (2006).
[15] E. van Beveren and G. Rupp, Phys. Rev. Lett. **93**, 202001 (2004).
[16] E. van Beveren and G. Rupp, Mod. Phys. Lett. A **19**, 1949 (2004).
[17] E. van Beveren and G. Rupp, arXiv:hep-ph/0312078.
[18] E. van Beveren and G. Rupp, Eur. Phys. J. C **32**, 493 (2004).
[19] E. van Beveren and G. Rupp, Phys. Rev. Lett. **91**, 012003 (2003).
[20] E. van Beveren, G. Rupp, T. A. Rijken, and C. Dullemond, Phys. Rev. D **27**, 1527 (1983).
[21] E. van Beveren, C. Dullemond, and G. Rupp, Phys. Rev. D **21**, 772 (1980); **22**, 787(E) (1980).
[22] C. Hanhart, Yu. S. Kalashnikova, A. E. Kudryavtsev, and A. V. Nefediev, Phys. Rev. D **76,** 034007 (2007).
[23] Yu. S. Kalashnikova, AIP Conf. Proc. **892**, 318 (2007).
[24] Yu. S. Kalashnikova, Phys. Rev. D **72**, 034010 (2005).
[25] C. Amsler and N. A. Tornqvist, Phys. Rep. **389**, 61 (2004).
[26] S. Ono and N. A. Tornqvist, Z. Phys. C **23**, 59 (1984).
[27] K. Heikkila, S. Ono, and N. A. Tornqvist, Phys. Rev. D **29**, 110 (1984); **29**, 2136(E) (1984).
[28] M. R. Pennington and D. J. Wilson, Phys. Rev. D **76**, 077502 (2007).
[29] P. Geiger and N. Isgur, Phys. Rev. D **47**, 5050 (1993).
[30] P. Geiger and N. Isgur, Phys. Rev. Lett. **67**, 1066 (1991).
[31] P. Geiger and N. Isgur, Phys. Rev. D **44**, 799 (1991).
[32] P. Geiger and N. Isgur, Phys. Rev. D **41**, 1595 (1990).
[33] D. Morel and S. Capstick, arXiv:nucl-th/0204014.
[34] N. A. Tornqvist, Ann. Phys. (NY) **123**, 1 (1979); Acta Phys. Pol. B **16**, 503 (1985); **16**, 683(E) (1985).
[35] N. A. Tornqvist and P. Żenczykowski, Phys. Rev. D **29**, 2139 (1984).
[36] P. Żenczykowski, Ann. Phys. (NY) **169**, 453 (1986).

[37] S. K. Choi *et al.* (Belle Collaboration), Phys. Rev. Lett. **91**, 262001 (2003).
[38] D. Acosta *et al.* (CDF II Collaboration), Phys. Rev. Lett. **93**, 072001 (2004).
[39] E. S. Swanson, Phys. Rep. **429**, 243 (2006).
[40] U. Fano, Phys. Rev. **124**, 1866 (1961).
[41] E. S. Ackleh, T. Barnes, and E. S. Swanson, Phys. Rev. D **54**, 6811 (1996).
[42] T. Barnes and E. S. Swanson, Phys. Rev. D **46**, 131 (1992).
[43] L. Micu, Nucl. Phys. **B10**, 521 (1969).
[44] A. Le Yaouanc, L. Oliver, O. Pene, and J. C. Raynal, Phys. Rev. D **8**, 2223 (1973).
[45] A. Le Yaouanc, L. Oliver, O. Pene, and J. C. Raynal, Phys. Lett. **B71**, 397 (1977).
[46] A. Le Yaouanc, L. Oliver, O. Pene, and J. C. Raynal, Phys. Lett. **B72**, 57 (1977).
[47] T. Barnes, S. Godfrey, and E. S. Swanson, Phys. Rev. D **72**, 054026 (2005).
[48] A. De Rujula, H. Georgi, and S. L. Glashow, Phys. Rev. Lett. **37**, 398 (1976).
[49] F. E. Close, Phys. Lett. **B65**, 55 (1976).
[50] T. J. Burns, F. E. Close, and C. E. Thomas, Phys. Rev. D **77**, 034008 (2008).
[51] J. H. van Vleck, Phys. Rev. **33**, 467 (1929).