Nonextensive hydrodynamics for relativistic heavy-ion collisions

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The nonextensive one-dimensional version of a hydrodynamic model for multiparticle production processes is proposed and discussed. It is based on nonextensive statistics assumed in the form proposed by Tsallis and characterized by a nonextensivity parameter q. In this formulation, the parameter q describes some specific form of local equilibrium that is characteristic of the nonextensive thermodynamics and replaces the local thermal equilibrium assumption of the usual hydrodynamic models. We argue that there is correspondence between the perfect nonextensive hydrodynamics and the usual dissipative hydrodynamics. It leads to a simple expression for dissipative entropy current and allows predictions of the ratio of bulk and shear viscosities to entropy density, ζ/s and η/s, to be made.

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I. INTRODUCTION

Multiparticle production experiments are our main source of information on multiparticle production processes in which the initial kinetic energy of two projectiles is to a large extent converted into a multitude of observed secondaries. This is especially true in the case of heavy-ion collisions in which one expects the formation of a new hadronic state of matter, the quark-gluon plasma (QGP) [1]. Such processes call for some form of statistical approach, which is usually based on the Boltzmann-Gibbs (BG) statistics. On the other hand, in the case of a multiparticle production process, conditions leading to BG statistics are satisfied only approximately at best. This is because, among other things, hadronizing systems experience strong intrinsic fluctuations and long-range correlations [2,3], which can be interpreted as signals of some dynamical, nonequilibrium effects showing up (as, for example, the flow phenomenon or decay of resonances, see Ref. [4]). It is therefore difficult to expect the occurrence of the usual (local) thermal equilibrium; instead, one has some kind of stationary state. It turns out that these phenomena can be incorporated, at least to some extent and without going into deeper dynamical considerations concerning the sources of such fluctuations, in the formalism of the nonextensive statistics (which we shall apply here in the manner proposed by Tsallis [5]) in the form of a more general equilibrium summarily described by a single parameter q [6–9]. This parameter characterizes the corresponding Tsallis entropy S_q, which in such approach replaces the usual BG entropy to which it converges for $q \to 1$. Because such systems are in general nonextensive, the parameter q is usually called a *nonextensivity parameter*.

Such an approach has been successfully applied to multiparticle production processes, either by using nonextensive versions of the respective distribution functions [10–16], or by deriving such distributions from the appropriately modified nonextensive version of the Boltzmann transport equation [6,7,9,17,18]. In both cases, this amounts to replacing the usual exponential factors by their q-exponential equivalents,

$$P_{\rm BG}(E) = \exp(-E/T) \Longrightarrow P_q(E) = \exp_q(-E/T)$$
$$= [1-(1-q)E/T]^{1/(1-q)}. \qquad (1)$$

Notice that $P_{\rm BG}(E) = P_{q=1}(E)$. In all these applications, one finds that $q > 1$. It represents the effect of some intrinsic fluctuations existing in the hadronizing system and revealing themselves as fluctuations of its temperature or of the mean multiplicity of secondaries.[1] Generically one has that [2]

$$q = 1 + \frac{\langle (1/T)^2\rangle}{\langle (1/T)\rangle^2}, \qquad (2)$$

where, depending on the kind of process considered, T can be replaced by some other variable [13–15]. Because different observables are sensitive to different kinds of fluctuations, it is natural to expect that they are described by different values of the parameter q [14]. For example, single-particle, one-dimensional distributions in longitudinal phase-space, such as dN/dy, are most sensitive to fluctuations of the mean multiplicity $\langle n\rangle$ of secondaries [13] and are described by the nonextensivity parameter $q = q_{\rm L}$ of the order of $q_{\rm L} - 1 \sim 0.1$–0.2.[2] However, distributions in transverse momenta, dN/dp_T,

[1]The situation in which $q < 1$ was analyzed in Ref. [19]. As discussed in Ref. [3], interpretation in terms of fluctuations is not clear now. Instead, it was shown that in this case the first role of the parameter q is to restrict the allowed phase space, and $q < 1$ reflects the fact that only a fraction K (called *inelasticity*) of the initially available energy is used for the production of secondaries, the remaining $1 - K$ part is to be found in the so-called leading particles. As a result, one gets the $\exp_{q<1}(-X)$ distribution with X limited to the $X > 1/(1-q)$ region only.

[2]Actually, these are precisely the same fluctuations that lead to the negative binomial form of the observed multiplicity distributions, $P(n;\langle n\rangle;k)$, with its characteristic parameter k given by $k = 1/(q_{\rm L} - 1)$ [13] (in this case, one can also speak of fluctuations in the so-called *partition temperature* $T_{\rm pt} = E/\langle n\rangle$, see Ref. [20].

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which are believed to probe the local thermal equilibrium of the hadronizing system (assuming that such a phenomenon indeed occurs) and serve as a source of information on the temperature T of the hadronizing system, are very sensitive to fluctuations of this temperature represented by the nonextensivity parameter $q = q_T$ [15]; changes as small as $q_T - 1 \sim 0.01$–0.05 already substantially affect the resultant p_T spectra.[3] Notice that data indicate that $q_{\rm L} - 1 \gg q_T - 1$, i.e., fluctuations governed by $q_{\rm L}$ are dominant, and therefore for the whole system, $q \sim q_{\rm L}$ [14].

Among statistical approaches to the multiparticle production processes, a specially important role is played by hydrodynamic models (for the most recent reviews, see Ref. [21]), which so far are all based on the BG statistics. The existing general nonextensive version of fluid dynamics discussed in Ref. [22] is not suitable for applications to multiparticle production processes (among other things because of its noncovariant formulation). We would like to fill this gap and present a fully covariant hydrodynamic model based on q statistics which can be applied to multiparticle production processes, especially to relativistic heavy-ion collisions. Because of the exploratory character of our paper, we limit ourselves to the $(1 + 1)$ dimensional case only and confront our results with rapidity and transverse momenta distributions obtained recently at the BNL Relativistic Heavy Ion Collider (RHIC), leaving the most detailed studies of all aspects of available experimental data for future investigations.

The hydrodynamic model of multiparticle production means, in fact, a number of separate problems connected with the consecutive steps of the collision process: the choice of initial conditions summarizing the preparatory stage of collision (it should end in some form of local thermal equilibrium), the choice of equation of state (EOS) of the quark-gluon and/or hadronic matter being equilibrated, further hydrodynamic evolution of this matter assumed to form a kind of fluid, and the final conversion of this fluid into observed secondaries. Because dynamical factors underlying each step are different, the resulting fluctuation patterns can also differ, presumably leading to the parameter q changing during the collision process. However, in the present study, we shall restrict ourselves to only the case of the nonextensive parameter q remaining the same in the whole collision process.[4]

Recently, renewed interest in dissipative hydrodynamic models [23–32] has been prompted by the apparent success of hydrodynamic models in describing data obtained at RHIC [33–35] and by the recent calculations of transport coefficients of a strongly interacting quark-gluon system using the anti-de-Sitter space/conformal field theory (AdS/CFT) correspondence [36]. The questions addressed are whether and under what circumstances dissipative hydrodynamics is really needed and how it should be applied. The reason is that formulations of the relativistic hydrodynamic equations for dissipative fluids suffer from ambiguities in the form they are written [27], the unphysical instability of the equilibrium state in the first-order theory [28], and the loss of causality in the first-order equation approach [31], to mention a few. In this work, we argue that there is a link between dissipative hydrodynamics (d hydrodynamics) and the nonextensive hydrodynamics (q hydrodynamics) we are proposing, which we call *nonextensive/dissipative correspondence* (NexDC). In particular, in Sec. V we demonstrate that it is possible to write some of the corresponding transport coefficients of the produced matter (believed to be QGP) as (implicit) functions of the nonextensivity parameter q. The merit of using the NexDC is that one can formulate and solve the q hydrodynamic equations of perfect nonextensive hydrodynamics (or perfect q hydrodynamics) in a way that is analogous to that for the usual perfect hydrodynamics, which seems to be *a priori* a much easier task. Although this does not fully solve the problems of d hydrodynamics, nevertheless it allows us to extend the usual perfect fluid approach (using only one new parameter q) well beyond its usual limits, namely, toward the regions reserved for the dissipative approach only.

The paper is organized as follows. We start in Sec. II with a short reminder of the nonextensive version of kinetic theory from which nonextensive hydrodynamics is derived in Sec. III. Section IV contains examples of comparisons with experimental data, whereas in Sec. V we discuss the possible physical meaning of the proposed q hydrodynamics. We end with Sec. VI, which contains our conclusions and summary. Some specialized topics and derivations are presented in Appendixes A–D.

II. RELATIVISTIC NONEXTENSIVE KINETIC THEORY

Following Ref. [17], we start with the nonextensive version of the Boltzmann equation [the metric used is: $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$],

$$p^\mu \partial_\mu f_q^q(x, p) = C_q(x, p), \tag{3a}$$

$$C_q(x, p) = \frac{1}{2}\int \frac{d^3p_1}{p_1^0}\frac{d^3p'}{p'^0}\frac{d^3p'_1}{p_1'^0} \times \{h_q[f'_q, f'_{q1}]W(p', p'_1|p, p_1) - h_q[f_q, f_{q1}]W(p, p_1|p', p'_1)\}. \tag{3b}$$

Here $f_q(x, p)$ and $C_q(x, p)$ are q versions of the, respectively, corresponding phase-space distribution function and the q collision term in which $W(p, p_1|p', p'_1)$ is the transition rate between the two-particle state with initial four-momenta p and p_1 and some final state with four-momenta p' and p'_1,

[3]In this case, q measures the fluctuation of temperature T as given by the specific heat parameter C, and $q_T - 1 = C$ [2,15]. As such, it should be inversely proportional to the volume of the interaction region; this effect is indeed observed [15].

[4]Notice that our analysis of multiparticle production using a q hydrodynamic model differs substantially from previous applications of q statistics presented in Refs. [10–18], because now the local thermal equilibrium (in its q version) is superimposed on the longitudinal flow. It is therefore not clear *a priori* which q should enter at which step of the collision process. We plan to discuss this subject elsewhere.

whereas $h_q[f_q, f_{q1}]$ is the correlation function related to the presence of two particles in the same space-time position x but with different four-momenta p and p_1, respectively. Notice two distinct features of Eq. (3a): (i) it applies to $f_q^q[=(f_q)^q]$ rather than to f_q itself, and (ii) in C_q one assumes a new, q generalized, version of the Boltzmann molecular chaos hypothesis [6–9,17,37] according to which

$$h_q[f_q, f_{q1}] = \exp_q[\ln_q f_q + \ln_q f_{q1}], \tag{4}$$

where $\exp_q(X) = [1+(1-q)X]^{1/(1-q)}$ and $\ln_q(X) = [X^{(1-q)} - 1]/(1-q)$. Equation (4) is our central point; it amounts to assuming that instead of the strict (local) equilibrium, *a kind of stationary state is being formed, which also includes some interactions* (see Refs. [6–9]).

With such correlation functions one finds that divergence of the entropy current, which we define as

$$s_q^\mu(x) \equiv -k_B \int \frac{d^3p}{(2\pi\hbar)^3}\frac{p^\mu}{p^0} \times \left\{ f_q^q(x,p)\ln_q f_q(x,p) - f_q(x,p)\right\}, \tag{5}$$

is always positive at any space-time point,

$$\partial_\mu s_q^\mu(x) \geqslant 0. \tag{6}$$

(This fact is equivalent to demonstrating the validity of the relativistic local H theorem when using this q entropy current).

To obtain the explicit form of the distribution functions $f_q(x,p)$, we proceed now as in Ref. [17]. At first, using the momentum conservation condition in two-particle collisions, $p^\mu + p_1^\mu = p'^\mu + p_1'^\mu$, we form the collision invariant

$$F[\psi] = \int \frac{d^3p}{p^0}\psi(x,p)C_q(x,p) \equiv 0, \tag{7}$$

where $\psi(x,p) = a(x) + b_\mu(x)p^\mu$ with arbitrary functions $a(x)$ and $b_\mu(x)$. We assume here that the correlation function h_q is symmetric and positive, $h_q[f,f_1] = h_q[f_1,f] \geqslant 0$, and that a detailed balance holds, i.e., $W(p,p_1|p',p_1') = W(p',p_1'|p,p_1)$. For $a(x) \equiv 0$ and $b_\mu(x) =$ constant, one gets the q version of the local energy-momentum conservation [17], that is,

$$\partial_\nu \mathcal{T}_q^{\mu\nu}(x) = 0, \tag{8}$$

with a nonextensive energy-momentum tensor defined by

$$\mathcal{T}_q^{\mu\nu}(x) \equiv \frac{1}{(2\pi\hbar)^3}\int \frac{d^3p}{p^0}p^\mu p^\nu f_q^q(x,p). \tag{9}$$

At the same time for $a(x) =$ constant and $b_\mu(x) \equiv 0$, one gets [17]

$$\partial_\mu \int \frac{d^3p}{(2\pi\hbar)^3}\frac{p^\mu}{p^0} f_q^q(x,p) = 0, \tag{10}$$

which implies that ($d\Omega$ stands for the corresponding phase-space volume element)

$$\frac{d}{dt}\int d\Omega f^q(x,p) = 0, \tag{11}$$

i.e., that the normalization $Z_q \equiv \int d\Omega f_q^q(x,p)$ is conserved as well.[5] Since the divergence of the q entropy current can also be expressed via the collision invariant,

$$\partial_\mu s_q^\mu = \frac{1}{(2\pi\hbar)^3}F[\ln_q f_q(x,p)], \tag{12}$$

demanding that $\partial_\mu s_q^\mu(x) \equiv 0$, one finally obtains

$$f_q(x,p) = [1 + (1-q)(a(x) + b_\mu(x)p^\mu)]^{1/(1-q)}, \tag{13}$$

which represents the distribution function in a stationary state. Setting $a(x) = 0$ and $b^\mu(x) = -u_q^\mu(x)/k_B T_q(x)$ [where $T_q(x)$ is the temperature function[6]] one obtains the well-known (unnormalized) Tsallis distribution function

$$f_q(x,p) = \left[1 - (1-q)\frac{p_\mu u_q^\mu}{k_B T_q(x)}\right]^{1/(1-q)} \equiv \exp_q\left[-\frac{p_\mu u_q^\mu(x)}{k_B T_q(x)}\right], \tag{14}$$

where $u_q^\mu(x)$ should be regarded as a hydrodynamic flow four-vector (hereafter we use the convention that $\hbar = k_B = c = 1$).

We shall now assume that the q-modified energy-momentum tensor $\mathcal{T}_q^{\mu\nu}$ can be decomposed in the usual way in terms of the q-modified energy density and pressure, ε_q and P_q, by using the q-modified flow u_q^μ [such that for $q \to 1$, it becomes the usual hydrodynamic flow u^μ, and in the rest frame of the fluid, $u_q^\mu = (1,0,0,0)$], that is,

$$\mathcal{T}_q^{\mu\nu} = (\varepsilon_q + P_q)u_q^\mu u_q^\nu - P_q g^{\mu\nu} \tag{15a}$$

$$= \varepsilon_q u_q^\mu u_q^\nu - P_q \Delta_q^{\mu\nu}, \tag{15b}$$

where $\Delta_q^{\mu\nu} \equiv g^{\mu\nu} - u_q^\mu u_q^\nu$. Denoting $e \equiv p^0/T, z \equiv m/T$, with g being the degeneracy factor depending on the type of particles composing our fluid, one gets that in its rest frame (or in q equilibrium)

$$\varepsilon_q \equiv u_{q\mu}\mathcal{T}_q^{\mu\nu}u_{q\nu} = \frac{gT_q^4}{2\pi^2}\int de\sqrt{e^2 - z^2}e^2\,[1-(1-q)e]^{q/(1-q)}, \tag{16a}$$

$$P_q \equiv -\frac{1}{3}\mathcal{T}_q^{\mu\nu}\Delta_{q\mu\nu} = \frac{gT_q^4}{2\pi^2}\int de\sqrt{e^2-z^2}e\,[1-(1-q)e]^{1/(1-q)}, \tag{16b}$$

[5]Notice that the normalization Z_q and (unnormalized) energy density $\mathcal{T}_q^{\mu\nu}$ are conserved independently. Therefore our further consideration will be concentrated only on getting $f_q(x,p)$. However, the q-dependent normalization is important when analyzing particle distribution functions, because it couples the widths of distributions (as given by f_q) with their heights (as given by Z_q, cf. Ref. [13]).

[6]One should be aware that there is still an ongoing discussion on the meaning of the temperature in nonextensive systems. However, the small values of the parameter q deduced from data allow us to argue that, to first approximation, T can be regarded as the hadronizing temperature in such a system. One must only remember that in general what we study here is not so much the state of equilibrium but rather some kind of stationary state. For a thorough discussion of the temperature of nonextensive systems, see Ref. [38].

$$s_q \equiv s_q^\mu u_{q\mu} = \frac{gT_q^3}{2\pi^2}\int de\sqrt{e^2-z^2}e\{e[1-(1-q)e]^{q/(1-q)} + [1-(1-q)e]^{1/(1-q)}\}. \tag{16c}$$

(Notice that for $q<1$, the integration range is limited to $z \leqslant \epsilon \leqslant 1/(1-q)$ in order to keep the integrand positive.) It is straightforward to check that in the baryon-free case, to which we shall limit ourselves here,

$$T_q s_q = \varepsilon_q + P_q, \tag{17a}$$

and

$$\frac{dP_q}{dT_q} = s_q, \tag{17b}$$

i.e., that the usual thermodynamic relations also hold for the q-modified quantities.

III. THE NONEXTENSIVE (q) HYDRODYNAMIC MODEL

A. Equations of nonextensive (q) flow

Our starting point in formulating the q hydrodynamic model is Eq. (8) with the energy-momentum tensor $\mathcal{T}_q^{\mu\nu}$ given by Eq. (9). Because of the q version of thermodynamic relations in Eq. (17), in our case Eq. (8) also implies conservation of q entropy, that is,

$$\partial_\mu s_q^\mu = 0, \tag{18}$$

with $s_q^\mu(x)$ defined by Eq. (5), which can also be written as

$$s_q^\mu(x) = s_q(x)u_q^\mu(x). \tag{19}$$

Therefore, we have only one general equation which, when written using general coordinates and covariant derivatives,[7] takes the form

$$\mathcal{T}_{q;\mu}^{\mu\nu} = \left[(\varepsilon_q + P_q)u_q^\mu u_q^\nu - P_q g^{\mu\nu}\right]_{;\mu} = 0. \tag{20}$$

This means that we are dealing here with *perfect* q hydrodynamics.

Before proceeding further, some specific points of q hydrodynamics not mentioned in the general derivation presented in Sec. II must be kept in mind. At first, notice that whereas in the usual perfect hydrodynamics (based on BG statistics) entropy is conserved in hydrodynamic evolution both locally and globally, in the nonextensive approach it is conserved only locally, cf. Eq. (18). The total entropy of the whole *expanding* system is not conserved, because for two volumes $V_{1,2}$ one finds that

$$S_q^{(V_1)} + S_q^{(V_2)} \neq S_q^{(V_1\oplus V_2)}, \tag{21}$$

where $S_q^{(V)}$ are the corresponding total entropies. Although, strictly speaking, the hydrodynamic model does not require global entropy conservation but only its local conservation, the above feature of q hydrodynamics should be always remembered (the consequences of this fact will be discussed in more detail in Sec. V). The second point concerns the causality problem. To guarantee that hydrodynamics makes sense, there should exist some spacial scale L such that volume L^3 contains enough particles composing our fluid. However, when there are some fluctuations and/or correlations with some typical correlation length l, for which we expect that $l > L$, one has to use nonextensive entropy $S_q^{(L^3)}$ [cf. Eq. (21)] and its locally defined density $s_q(x) = S_q^{(L^3)}/L^3$. When formulating the corresponding q hydrodynamics, one takes, as usual, the limit $L \to 0$, in which case explicit dependence on the scale L vanishes whereas the correlation length leaves its imprint as parameter q. In this sense, perfect q hydrodynamics can be considered as preserving causality, and nonextensivity q is then related with the correlation length l. One can argue that very roughly $q \sim l/L_{\rm eff} \geqslant 1$, where $L_{\rm eff}$ is some effective spacial scale of the q hydrodynamics. Note here that if the correlation length l is compatible with the scale $L_{\rm eff}$, i.e., $l \approx L_{\rm eff}$, one recovers the condition of the usual local thermal equilibrium, and in this case the q hydrodynamics reduces to the usual (BG) hydrodynamics.

Let us now continue our presentation. When contracted with the velocity $u_{q\nu}$ or with the projection tensor $\Delta_{q\lambda\nu} \equiv g_{\lambda\nu} - u_{q\lambda}u_{q\nu}$, it leads to the following two equations:

$$u_q^\mu\partial_\mu\varepsilon_q + (\varepsilon_q + P_q)u_{q;\mu}^\mu - P_q u_{q\nu}g_{;\mu}^{\mu\nu} = 0, \tag{22}$$

$$(\varepsilon_q + P_q)u_q^\mu\Delta_{q\lambda\nu}u_{q;\mu}^\nu - \Delta_{q\lambda\nu}\partial^\nu P_q - P_q\Delta_{q\lambda\nu}g_{;\mu}^{\mu\nu} = 0. \tag{23}$$

These are the equations to be solved now for the $(1+1)$ dimensional case. We shall assume longitudinal expansion only and introduce proper time τ and the space-time rapidity η:

$$\tau \equiv \sqrt{t^2 - z^2}, \tag{24a}$$

$$\eta \equiv \frac{1}{2}\ln\frac{t+z}{t-z}. \tag{24b}$$

The corresponding metric tensor in this $(\tau-\eta)$ space is $g^{\mu\nu} = \mathrm{diag}(1, -\frac{1}{\tau^2})$. The corresponding four-velocity of our fluid can be expressed by the local fluid rapidity $\alpha_q(x)$ as

$$u_q^\mu(x) = [\cosh(\alpha_q - \eta), \frac{1}{\tau}\sinh(\alpha_q - \eta)]. \tag{25}$$

In this case, Eq. (22) reduces to [here $v_q \equiv \tanh(\alpha_q - \eta)$]

$$\frac{\partial\varepsilon_q}{\partial\tau} + \frac{v_q}{\tau}\frac{\partial\varepsilon_q}{\partial\eta} + (\varepsilon_q + P_q)\left\{v_q\frac{\partial\alpha_q}{\partial\tau} + \frac{1}{\tau}\frac{\partial\alpha_q}{\partial\eta}\right\} = 0, \tag{26}$$

whereas Eq. (23) reduces to the q generalized relativistic Euler equation (cf. Appendix A for details):

$$(\varepsilon_q + P_q)\left\{\frac{\partial\alpha_q}{\partial\tau} + \frac{v_q}{\tau}\frac{\partial\alpha_q}{\partial\eta}\right\} + v_q\frac{\partial P_q}{\partial\tau} + \frac{1}{\tau}\frac{\partial P_q}{\partial\eta} = 0. \tag{27}$$

[7]The covariant derivatives of the vector u^μ and tensor $g^{\mu\nu}$ are defined by using the Christoffel symbol $\Gamma_{\lambda\mu}^\nu \equiv \frac{1}{2}g^{\nu\sigma}(\partial_\mu g_{\sigma\lambda} + \partial_\lambda g_{\sigma\mu} - \partial_\sigma g_{\lambda\mu})$ and are equal to $u_{;\mu}^\nu = \partial_\mu u^\nu + \Gamma_{\lambda\mu}^\nu u^\lambda$ and $g_{;\mu}^{\mu\nu} = \partial_\mu g^{\mu\nu} + \Gamma_{\sigma\mu}^\mu g^{\sigma\nu} + \Gamma_{\sigma\mu}^\nu g^{\mu\sigma}$, respectively.

To solve these equations, one needs additional input in terms of EOS, $P_q = P_q(\varepsilon_q)$, and the choice of boundary conditions, which we set as $v_q = 0$ at $\eta = 0$ (because of the symmetry $\alpha \equiv 0$). At $\eta = 0$, Eqs. (22) and (23) reduce to

$$\frac{\partial \varepsilon_q}{\partial \tau} = -\frac{\varepsilon_q + P_q}{\tau} \frac{\partial \alpha}{\partial \eta}\bigg|_{\eta=0}, \tag{28a}$$

$$\frac{\partial P_q}{\partial \eta}\bigg|_{\eta=0} = 0. \tag{28b}$$

B. Nonextensive equation of state: *q*-EOS

The next important ingredient of any hydrodynamic model is an equation of state (EOS) defining a relation between the pressure and the energy density, which depends on the properties of the hadronic matter under consideration. In this work, we shall only work with an EOS for the relativistic pion gas (with $m_\pi = 0.14$ GeV) without considering different phases of hadronic matter as in Ref. [39]. The pressure P_q and the energy density ε_q can be connected in the form of EOS, $P_q(\varepsilon_q)$, using Eqs. (16a) and (16b). However, differently than in the usual cases of $q = 1$, the additional freedom represented by the nonextensivity parameter q makes $P_q = P_q(\varepsilon_q)$ ambiguous, and one has to additionally specify the possible variations of the parameter q during the evolution process. In what follows, we shall assume that the parameter q remains fixed during the the whole evolution of our hadronic fluid. We therefore get different EOSs for different (but fixed) values of the parameter q, examples of which are shown in Fig. 1. It displays the ratio P_q/ε_q as a function of energy density ε_q for different values of $q = 1.0$, 1.1, and 1.2; the temperature T_q was varied in the range 0.1–500 MeV. It turns out that the q dependence is confined only to the very low energy density region (supporting therefore the previous results on this matter in Refs. [17,40]). In the region of interest, $\varepsilon_q \sim 0.1$–5.0 GeV/fm^3, the changes are very small and rapidly vanish with increasing ε.

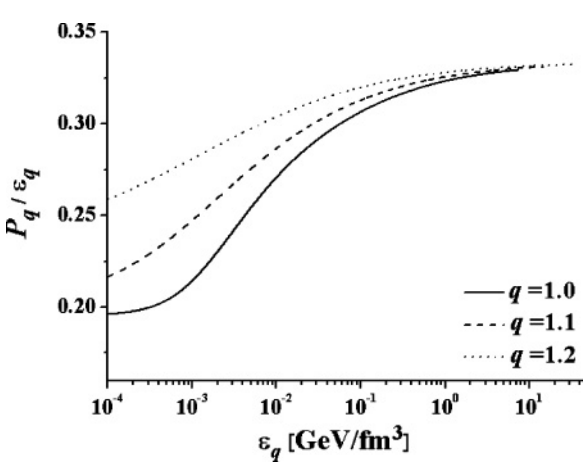

FIG. 1. EOS for the relativistic nonextensive pionic gas ($m = 0.14$ GeV). P_q/ε_q is plotted as a function of energy density ε_q for different nonextensivity parameters q. Notice that the q dependence of the EOS shows mainly at low and very low energy densities.

C. Nonextensive (*q*) initial conditions

To solve equations of $(1+1)$ q hydrodynamics, one has to decide on the initial conditions from which the hydrodynamic expansion starts. They must contain some form of the local thermal equilibrium, which we assume is established during the collision process. According to recent estimations, this can happen very rapidly, already in the first 1 fm of expansion, if caused by some violent, nonperturbative mechanisms operating at this stage.[8] It is thus natural to expect that there must also exist some intrinsic fluctuations already present in this preparatory stage of the collision process which, according to our philosophy, should be accounted for by the same q statistical approach as that used to form the q hydrodynamics. Following Refs. [39,43], we shall use Gaussian initial conditions interpolating between two extreme situations, the one described by the Bjorken scaling type model [44] and the other corresponding to the Landau model [45], but we shall modify them accordingly by changing $\exp(X)$ to $\exp_q(X)$ [as we saw in Eq. (4), it reduces to the usual Gaussian of Ref. [39] for $q = 1$]. As in Ref. [39], initial conditions are imposed for the energy density ε_q expressed as a function of rapidity η:

$$\varepsilon_q(\tau_0, \eta) = \varepsilon^{(\mathrm{in})} \exp_q\left[-\frac{\eta^2}{2\sigma^2}\right]. \tag{29}$$

In what follows we shall also require that the q-fluid and space-time rapidities coincide at τ_0, that is,

$$\alpha_q(\tau_0, \eta) = \eta. \tag{30}$$

In all calculations presented in this paper, we shall assume for simplicity that ε_q and α_q are independent of the transverse coordinate. However, the remaining two parameters, $\varepsilon^{(\mathrm{in})}$ and σ, are not independent because one has to reproduce the total energy E_{tot} allocated to the fluid which is fixed by the conditions of the experiment, i.e.,

$$E_{\mathrm{tot}} = \pi A_T^2 \tau_0 \int d\eta \varepsilon_q(\tau_0, \eta), \tag{31}$$

where A_T is the transverse size of the fluid, τ_0 is the initial proper time τ when the fluid starts to expand. The E_{tot} can be obtained knowing the mean number of participating nucleons N_{part} and the total energy loss per participating nucleon ΔE, that is,[9]

$$E_{\mathrm{tot}} = N_{\mathrm{part}} \Delta E. \tag{32}$$

The possible initial conditions vary therefore between two extremal situations (cf., Fig. 2):

[8]See, for example, the review in Ref. [41] and references therein. We mention at this point that the so-called kinematic thermalization used here, in which equilibration of energies is due to the collisions, has been recently contrasted with the so-called stochastic thermalization based on the process of erasing of memory of the initial state resulting in a state of maximal entropy and coinciding with the above thermal equilibrium state, see Ref. [42] and references therein.

[9]Notice that, in principle, both the N_{part} and the ΔE are also fluctuating quantities, but we shall not consider these fluctuations here. One can argue that they are to some extent accounted for by the nonextensive version of initial conditions considered here.

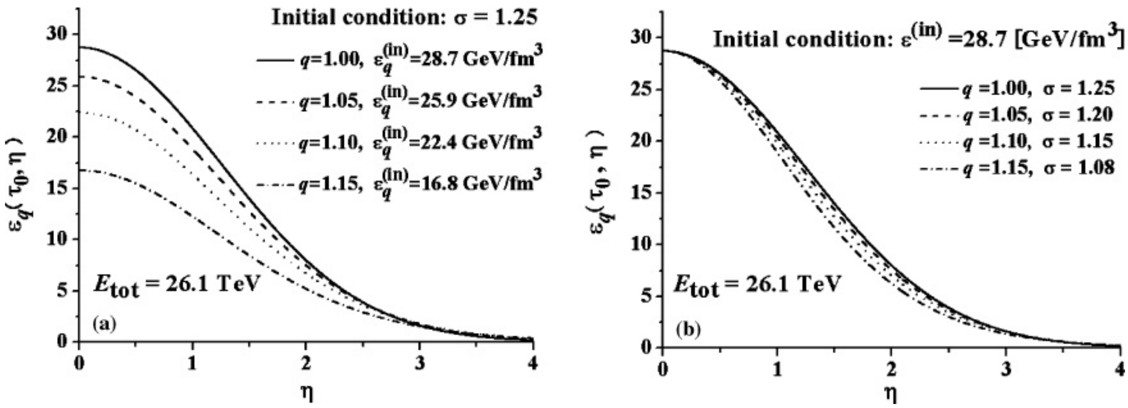

FIG. 2. Examples of two different types of initial conditions: (a) type (i) with fixed $\sigma = 1.25$ and $\varepsilon^{(\mathrm{in})}$ varying according to Eq. (31), and (b) type (ii) with fixed $\varepsilon^{(\mathrm{in})} = 28.7$ GeV/fm^3 [this value corresponds to the $q = 1$ situation in (a)] and σ varying according to Eq. (31).

(i) The width σ in Eq. (29) is assumed to be fixed and kept constant, but its distribution varies by changing $\varepsilon^{(\mathrm{in})}$ to reproduce the fixed total energy E_{tot} when the nonextensive parameter q changes.

(ii) The maximum energy density $\varepsilon^{(\mathrm{in})}$ in Eq. (29) is assumed to be fixed and kept constant, but its distribution varies by changing σ to reproduce the fixed total energy E_{tot} when the nonextensive parameter q changes.

We would like to stress at this point that such q-dependent initial conditions introduce a completely new element to hydrodynamic models, not discussed previously. The real situation will interpolate in an *a priori* unknown manner between these two extremes; therefore, in what follows, we shall restrict ourself only to them. As one can see in Fig. 2, whereas the first extreme introduces sizable q dependence, the second one leads to only minor effects. In both cases, increasing the value of q results, as expected [13], in the enhancement of tails for large values of η. Following Ref. [39], our calculations were performed for Au+Au collisions at $\sqrt{s_{\mathrm{NN}}} = 200$ GeV, using results reported by the BRAHMS experiment [46], with $E_{\mathrm{tot}} = 26.1$ TeV, $N_{\mathrm{part}} = 357$, $\Delta E = 73 \pm 6$ GeV and with $A_T = 6.5$ fm, $\tau_0 = 1.0$ fm, see Table I. In Fig. 3 are shown the initial conditions for the energy density $\varepsilon_q(\tau_0, \eta)$, entropy density $s_q(\tau_0, \eta)$, and temperature $T_q(\tau_0, \eta)$ in the case of $\sigma = 1.25$ fixed [panels (a)–(c)] and $\varepsilon^{(\mathrm{in})} = 27.8$ GeV/fm^3 fixed [panels (d)–(f)] and reproducing the initial energy $E_{\mathrm{tot}} = 26.1$ TeV. We start with ε_q [panels (a) and (d) of Fig. 3] which is given by Eq. (29), use it to solve Eq. (16a) and find $T_q(\tau_0)$ [panels (b) and (e) of Fig. 3], and eventually obtain $s_q(\tau_0, \eta)$ using these results and Eq. (16c) [panels (c) and (f) of Fig. 3].

D. Examples of q hydrodynamic evolution of different thermodynamic quantities

Let us now demonstrate examples of q hydrodynamic evolution of different thermodynamic quantities and of the fluid rapidity. Calculations were performed using the method presented in Appendix B. In Fig. 4 we present the evolution of the energy density ε_q, temperature T_q, and entropy density s_q using the initial conditions discussed in Sec. III C (the exact values of relevant parameters for both types of initial conditions are listed in Table II). One can see that the initial

TABLE I. Parameters of the initial conditions used in Fig. 2. The initial temperature $T_{\mathrm{in}} \equiv T(\tau_0, \eta = 0)$ is shown for two types of EOS: for the relativistic nonextensive pion gas for some selected values of $q \geqslant 1$ and for the usual BG pion gas with $q = 1$.

Initial condition: $\sigma = 1.25$ fixed										
E_{tot}	$\varepsilon^{(\mathrm{in})}$	σ				EOS	T_{in} (GeV)			
(TeV)	(GeV/fm^3)	$q = 1.00$	$q = 1.05$	$q = 1.10$	$q = 1.15$		$q = 1.00$	$q = 1.05$	$q = 1.10$	$q = 1.15$
26.1	27.8		1.20	1.15	1.08	nonex.π gas		0.648	0.591	0.531
		1.25				BG π gas	0.702			
Initial condition: $\varepsilon^{(\mathrm{in})} = 28.7$ GeV/fm^3 fixed										
E_{tot}	σ	$\varepsilon^{(\mathrm{in})}$(GeV/fm^3)				EOS	T_{in} (GeV)			
(TeV)		$q = 1.00$	$q = 1.05$	$q = 1.10$	$q = 1.15$		$q = 1.00$	$q = 1.05$	$q = 1.10$	$q = 1.15$
26.1	1.25		25.9	22.4	16.8	nonex.π gas		0.631	0.556	0.464
		27.8				BG π gas	0.702			

TABLE II. Values of energy density ε_q, entropy density s_q, and temperature T_q at $\eta = 0.0$ and $\eta = 3.0$ and at $\tau = 5.0$ and 25.0 fm for $q = 1.0$ and $q = 1.1$ for two extremal types of initial conditions: with fixed $\sigma = 1.25$ (upper panel) and with fixed $\varepsilon^{\text{in}} = 27.8$ GeV/fm^3 (lower panel).

Initial condition: $\sigma = 1.25$ fixed																			
		ε_q (GeV/fm^3)						T_q (GeV)						s_q (1/fm^3)					
	τ	1 fm		5 fm		25 fm		1 fm		5 fm		25 fm		1 fm		5 fm		25 fm	
	η	0.0	3.0	0.0	3.0	0.0	3.0	0.0	3.0	0.0	3.0	0.0	3.0	0.0	3.0	0.0	3.0	0.0	3.0
q	1.00	28.7	1.61	2.80	0.293	0.249	0.060	0.702	0.343	0.393	0.225	0.216	0.153	54.5	6.24	9.46	1.72	1.52	0.513
	1.10	22.4	1.78	2.19	0.285	0.196	0.053	0.556	0.296	0.311	0.189	0.171	0.124	53.7	8.00	9.34	2.01	1.51	0.553
Initial condition: $\varepsilon^{(\text{in})} = 28.7$ GeV/fm^3 fixed																			
		ε_q (GeV/fm^3)						T_q (GeV)						s_q (1/fm^3)					
	τ	1 fm		5 fm		25 fm		1 fm		5 fm		25 fm		1 fm		5 fm		25 fm	
	η	0.0	3.0	0.0	3.0	0.0	3.0	0.0	3.0	0.0	3.0	0.0	3.0	0.0	3.0	0.0	3.0	0.0	3.0
q	1.00	28.7	1.61	2.80	0.293	0.249	0.060	0.702	0.343	0.393	0.225	0.216	0.153	54.5	6.24	9.46	1.72	1.52	0.513
	1.10	28.7	1.53	2.72	0.272	0.235	0.057	0.591	0.285	0.329	0.186	0.179	0.127	64.7	7.14	11.0	1.94	1.73	0.587

functional forms of $\varepsilon_q(\tau, \eta)$, $T_q(\tau, \eta)$, and $s_q(\tau, \eta)$ generally follow their original Gaussian shapes assumed at the initial time τ_0 for $q = 1.0$, 1.05, and 1.1. On the other hand, during the whole hydrodynamic evolution, both the energy density ε_q and the temperature T_q calculated for $q > 1$ are smaller than those for $q = 1$ for $\tau > \tau_0 = 1$ fm (see Table II). That is even true for the initial condition with fixed $\sigma = 1.25$, for which the initial energy density $\varepsilon_{q=1.1}(\tau_0, \eta = 3) > \varepsilon_{q=1.0}(\tau_0, \eta = 3)$, in which case, after the q hydrodynamic evolution is completed, one observes that $\varepsilon_{q=1} > \varepsilon_{q>1}$. The same trend is also observed for the temperature, i.e., $T_{q=1} > T_{q>1}$ for all τ and η. However, the corresponding entropy density s_q evolves differently: for both types of initial conditions and any η, inequality relations between $s_{q=1.1}(\tau, \eta)$ and $s_{q=1.0}(\tau, \eta)$ given at initial $\tau = \tau_0$ are preserved during hydrodynamic evolution. In what concerns the fluid rapidity α_q, it is always set to be equal to $\alpha_q(\tau_0, \eta) \equiv \eta$ at $\tau = \tau_0$. However, the pressure gradient, which is characteristic to Gaussian-type initial conditions applied here, accelerates the fluid; therefore α_q evolves with time τ (actually, this is true even for $q = 1$), see Fig. 5. In these figures the fluid rapidity α_q (actually its deviation from the rapidity η, $\alpha_q - \eta$) is shown as a function of τ and the corresponding energy density ε_q. Notice that $\alpha_q - \eta \equiv 0$ at τ_0 for the whole η space (i.e., for all regions of the ε_q). As shown in Fig. 5, the fluid rapidity α_q grows during the hydrodynamic expansion from its initial value $\alpha(\tau_0, \eta) = \eta$. One can observe that α_q for $q > 1$ is decelerated compared to the usual hydrodynamic expansion (i.e., $\alpha_{q=1} > \alpha_{q>1}$).

To summarize this part: one observes that nonextensive fluid (q fluid with $q > 1$) evolves more slowly than the ideal fluid (with $q = 1$).

E. Freeze-out surface and single-particle spectra

We now present examples of single-particle spectra emerging from our approach. We shall follow the simplest possibility,

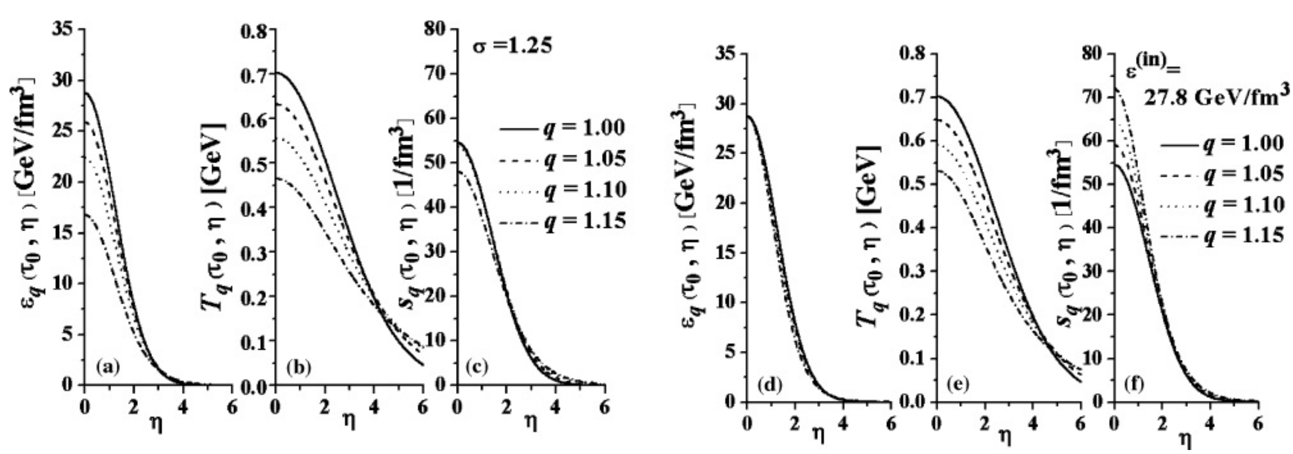

FIG. 3. Dependence of initial conditions on parameter q. The initial conditions for the energy density $\varepsilon_q(\tau_0, \eta)$ [(a) and (d)], the corresponding temperature $T_q(\tau_0, \eta)$ [(b) and (e)], and the entropy density $s_q(\tau_0, \eta)$ [(c) and (f)] are plotted for different values of q and for the two types of initial conditions. In both cases, we assume that $E_{\text{tot}} = 26.1$ TeV for all q, whereas τ_0 in Eq. (29) is put equal to 1.0 fm. Temperatures T_q for different values of q displayed on (b) and (e) are determined by solving Eq. (16a), whereas the entropy densities s_q displayed on (c) and (f) are obtained from Eq. (16c) using the values of T_q displayed in (b) and (e).

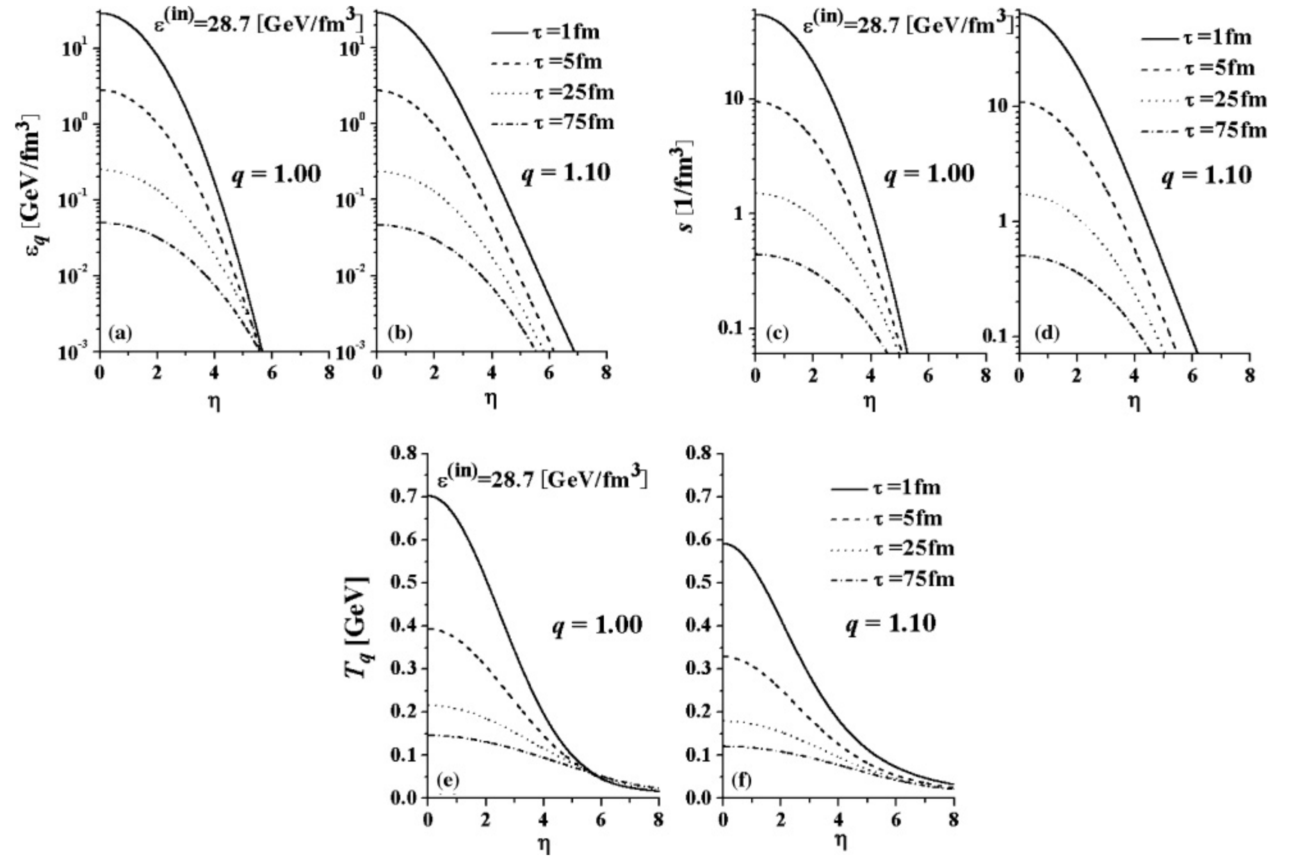

FIG. 4. Profiles of energy density ε_q [(a) and (b)], entropy density s_q [(c) and (d)], and temperature T_q [(e) and (f)] as functions of η calculated for different proper times τ and different nonextensivity parameters q by using type (ii) initial conditions. We obtained similar profiles for the type (i) initial condition case.

in which they are expressed as an integral of the phase-space particle density over a freeze-out surface $\Sigma_{\rm f}$ [47],

$$E\frac{d^3N}{dp^3} = \frac{d^3N}{m_T\, dm_T dy d\phi} = \frac{g}{(2\pi)^3}\int_{\Sigma_{\rm f}} d\sigma_\mu(x)p^\mu f_{\rm eq}(x,p). \tag{33}$$

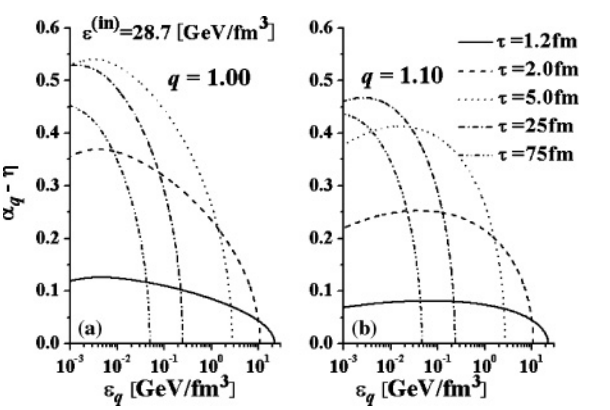

FIG. 5. Evolution of fluid rapidity α_q (presented as $\alpha_q - \eta$) as a function of the energy density ε_q for different values of τ and for different q with (a) fixed $\varepsilon^{(\rm in)} = 28.7$ GeV/fm^3 and (b) fixed $\sigma = 1.25$ initial conditions.

In the $\tau - \eta$ metric, the surface element of $\Sigma_{\rm f}$ is given by

$$d\sigma_\mu = (d\sigma_\tau, d\sigma_\eta) = A_T\tau d\eta\left(1, -\frac{n_\eta}{n_\tau}\right), \tag{34}$$

where n_μ is the normal covariant vector of the isothermals,

$$n_\mu = (n_\tau, n_\eta) = \left(-\frac{\partial T}{\partial\tau}, -\frac{\partial T}{\partial\eta}\right), \tag{35}$$

and A_T is the transverse area of the generated fluid. In all examples of applications to Au+Au collisions discussed in this paper, we use $A_T = 6.5$ fm. The momentum of the produced particle in the $\tau - \eta$ metric is given by

$$p^\mu = \left[m_T\cosh(y-\eta), \frac{1}{\tau}m_T\sinh(y-\eta)\right], \tag{36}$$

where y is the observed rapidity (after freeze-out). Using these expressions, the single-particle density is given by

$$E\frac{d^3N}{dp^3} = \frac{d^3N}{m_T\, dm_T\, dy} = \frac{gA_T^2}{4\pi}\int d\eta\tau_{\rm f}(\eta)\left[m_T\cosh(y-\eta) - \frac{1}{\tau}\frac{n_\eta(\eta)}{n_\tau(\eta)}m_T\sinh(y-\eta)\right]f_q(y,\eta), \tag{37}$$

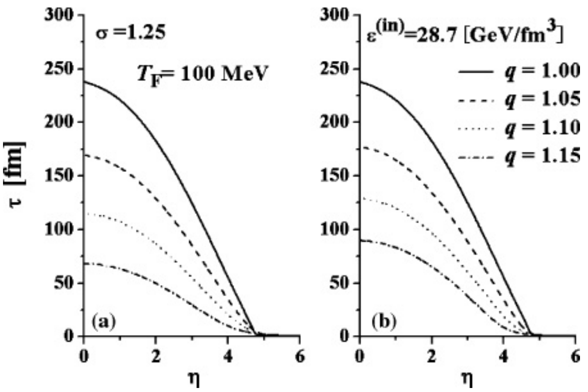

FIG. 6. Examples of freeze-out surfaces of constant temperature $T_F = 100$ MeV calculated for different values of parameter q and for different initial conditions: (a) with constant $\sigma = 1.25$ and (b) with constant energy density $\varepsilon_F = 9.82 \times 10^{-3}$ GeV/fm^3.

TABLE III. Values of the freeze-out temperatures T_F (in MeV) for the different freeze-out (F.O.) conditions used and different values of q investigated.

q	1.00	1.05	1.10	1.15
F.O. = T_F fixing	100	100	100	100
F.O. = ε_F fixing	100	91.8	83.2	74.3
F.O. = s_F fixing	100	89.3	78.5	67.4

where

$$f_q(y,\eta) = \left[1-(1-q)\frac{m_T\cosh(y-\eta)}{T_F}\right]^{\frac{1}{1-q}}, \qquad (38)$$

and T_F is the freeze-out temperature, which is given by the corresponding freeze-out energy density ε_F. In principle, the freeze-out surface can be defined either as the surface of constant temperature T_F, as the surface of constant energy density ε_F, or, finally, as the surface of constant entropy density s_F (cf. Table III). They all coincide in the usual extensive case ($q = 1$). In Fig. 6 we show as an example freeze-out surfaces (calculated for different values of parameter q and for different initial conditions) for $T_F = 100$ MeV. One observes quite a strong q dependence of the freeze-out surface characteristics on these parameters. These dependencies are much weaker when calculated for the surface of constant energy density and even weaker for constant entropy density (not shown here explicitly). Note that values of T_F corresponding to freeze-out conditions set by fixing ε_F or s_F now depend on the parameter q (see Table III).

In Fig. 7 we show examples of single-particle rapidity and transverse momentum spectra calculated for both types of initial conditions using $T_F = 100$ MeV. Note that different types of the freeze-out surface used are connected with using different sets of parameter (q, T_F), cf. Table III). Both distributions are sensitive to q; however, in the case of dN/dy, this dependence is almost entirely due to the q dependence of the initial entropy density in the central region observed in Figs. 3(c) and 3(f) and practically vanishes in the case of normalized distributions calculated for the constant ε_F freeze-out surface, as seen in Figs. 8(a) and 8(b). This is because of the observed q dependence of the corresponding total multiplicities and is connected with the increase of the entropy observed in nonextensive processes, see Fig. 8(c). We shall discuss this point in more detail in Sec. V. The weak residual q dependence observed in this case can be attributed to the (apparently very weak) effects of the EOS and freeze-out surface. As for the p_T spectra shown there for different initial conditions and freeze-out surfaces, one observes a very strong dependence on q, which changes the slope of p_T considerably. It is interesting to note that, as seen in Fig. 7, the p_T distributions apparently are sensitive to neither to the type of initial conditions nor the freeze-out surfaces used.

In the p_T distributions, the slope depends on both q and T_F, and increasing T_F while keeping constant q gives a similar effect as increasing q at fixed T_F. On the whole, one observes the tendency that transverse expansion as measured by these distributions gets stronger with increasing nonextensivity, i.e., with increasing q.

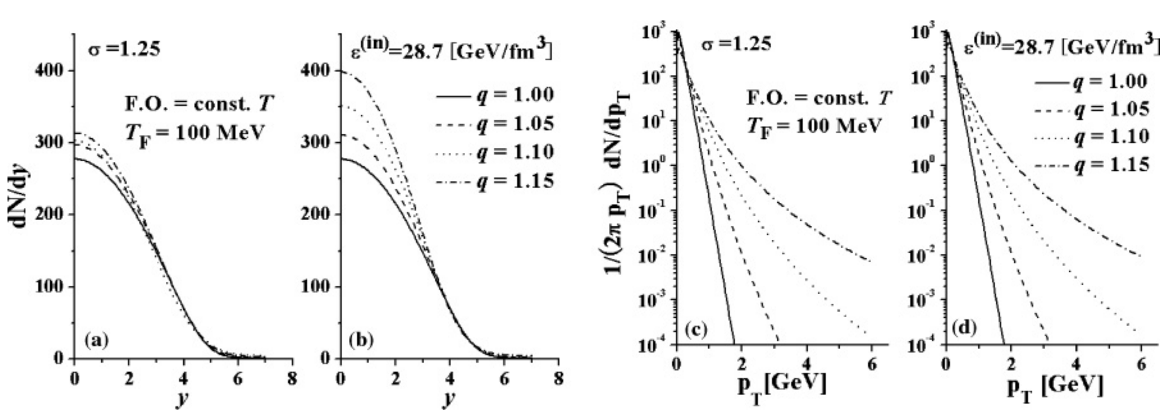

FIG. 7. dN/dy and p_T spectra obtained from the q hydrodynamic evolution with constant T_F =100 MeV and for different values of parameter q and for both types of initial conditions. Rapidity spectra are obtained by integrating Eq. (37) over $p_T \in (0, 6.0)$ GeV/c, whereas p_T spectra are obtained by integrating Eq. (37) over $|y| \leqslant 0.5$.

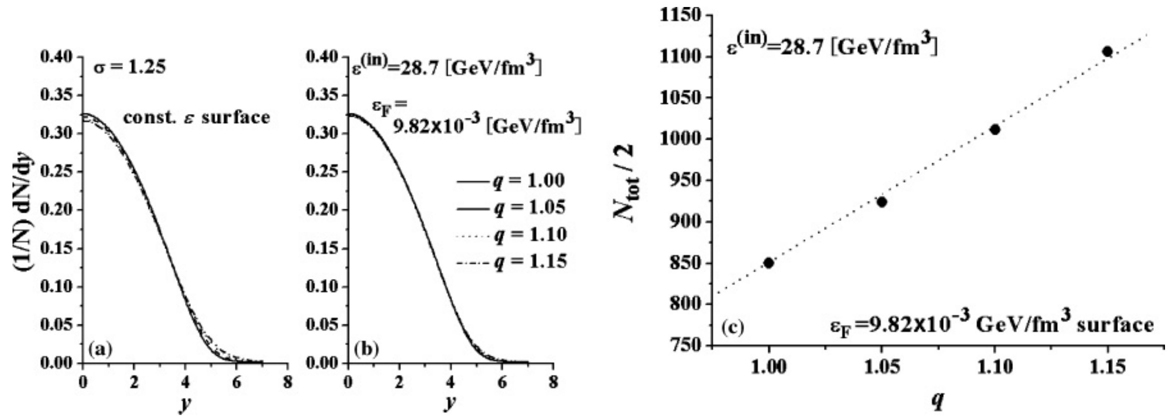

FIG. 8. Normalized rapidity distributions $1/N\, dN/dy$ (defined as in Fig. 7) calculated for constant energy density freeze-out surface $\varepsilon_F = 9.82 \times 10^{-3}$ GeV/fm^3 for different values of q using initial conditions with (a) fixed $\sigma = 1.25$ and (b) fixed $\varepsilon^{(\mathrm{in})} = 28.7$ GeV/fm^3. Notice that there is only very weak q dependence confined to small and large regions of rapidity y. (c) q dependence of the total multiplicity N_{tot} obtained from the q hydrodynamic evolution with fixed $\varepsilon^{(\mathrm{in})} = 28.7$ GeV/fm^3 initial condition and $\varepsilon_F = 9.82 \times 10^{-3}$ GeV/fm^3 freeze-out condition. The total multiplicity N_{tot} increases linearly with q.

IV. COMPARISON WITH EXPERIMENTAL DATA

We shall now confront our approach with experimental data. Because of the still explanatory character of our work, we limit ourselves to a comparison with only some selected rapidity and p_T distributions. At this stage, no attempts for exact fits have been made. They must wait for a more detailed version, which, for example, would account for the possible changes of the nonextensivity parameter q during the collision process as mentioned in Sec. I. The same remarks apply to the potentially promising analysis of anisotropic flow or particle interferometry (for example, in the way it was done in Refs. [33–35]), which we postpone until the $(1 + 2)$ dimensional version of our approach accounting for expansion in transverse directions is available). Because, as shown in Sec. III E, the most sensitive for q dependence are p_T distributions, we start with them and show in Fig. 9 that data from Ref. [48] prefer $q = 1.08$ and $T_F = 100$ MeV. (We attribute the visible discrepancy at largest values of p_T to the contamination from quark jets which carry large momentum in the initial stage of nuclear collisions and which are not accounted for in q hydrodynamic model.) With these values of q and T_F, the data provided by Ref. [49] for dN/dy distributions and by Ref. [48] for p_T were compared with predictions of different initial conditions characterized by $\varepsilon^{(\mathrm{in})}$, see Fig. 10. As one can see, the q hydrodynamic model with the q-Gaussian initial condition can reproduce reasonably well both the rapidity and transverse momentum distribution data simultaneously. It should be stressed at this point that with the parameter $q > 1$, which according to the general philosophy of the nonextensive approach accounts for all possible intrinsic fluctuations in the system [2,6,9], our model also accounts for the possible presence of resonances [12,15] which therefore, to avoid double counting, should not be added independently. It must be noticed that in the present version we do not, in fact, account for the possible creation of a QGP phase. For this, one should use a more elaborate version of EOS than discussed here in Sec. III B. Nevertheless, one can say that a simple q hydrodynamic model reproduces experimental data reasonably well using $\varepsilon^{(\mathrm{in})} = 19.0$–$22.3$ GeV/fm^3 ($\sigma = 1.28$–1.32), $T_F = 100$–120 MeV, and $q = 1.07$–1.08.

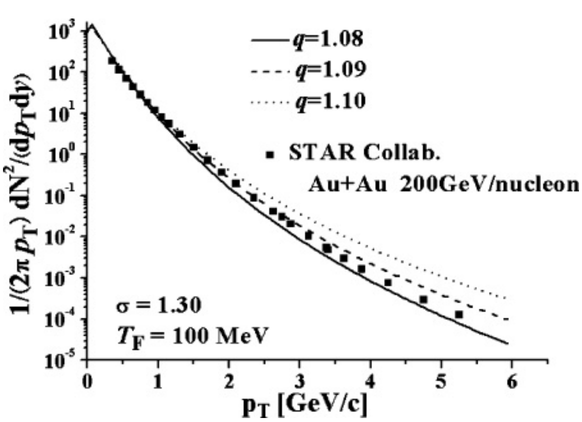

FIG. 9. Comparison of q hydrodynamic model results with experimental data observed by the STAR Collaboration [48] performed using $\sigma = 1.30$ and $T_F = 100$ MeV for $q = 1.08$, 1.09, and 1.1 (with corresponding values of $\varepsilon^{(\mathrm{in})} = 21.2$, 205, and 19.7 GeV/fm^3). The best agreement is obtained for $q = 1.08$.

V. DISCUSSION: CAN PERFECT *q* HYDRODYNAMICS MIMIC *d* HYDRODYNAMICS?

A. Nonextensive/dissipative correspondence: Formulation

Our starting point is the observation made at the end of Sec. III D that a q fluid evolves more slowly than an ideal fluid. To this, one can add the observation from Sec. III E that transverse expansion measured by the behavior of p_T spectra is much stronger in a q fluid. Those are precisely the features observed in viscous fluids (cf., for example, Ref. [35]). Let us then treat these observation seriously and look more closely for the possible connections between q fluid and viscous fluid apparently emerging from our q hydrodynamic model.

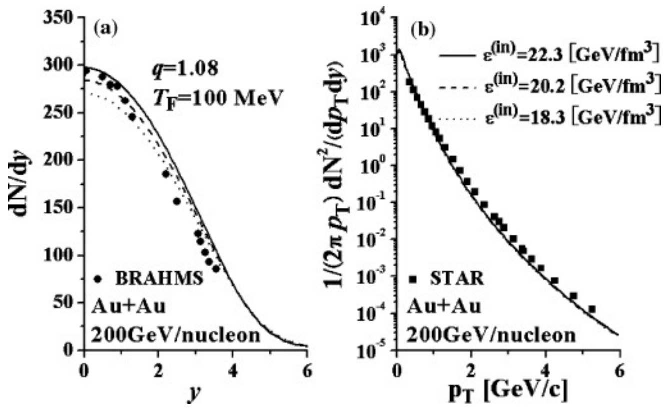

FIG. 10. Comparison of the q hydrodynamic model results with experimental data on rapidity [49] and p_T [48] distributions calculated for $q = 1.084$ (fixed) and $T_F = 100$ MeV, as in Fig. 9, but for different (Gaussian) initial conditions parametrized by $\varepsilon^{(\text{in})}$.

Let us start with recalling the possible physical meaning of perfect q hydrodynamics. It originated from the modified Boltzmann kinetic equation (3a) in which a new, q-generalized version of the Boltzmann molecular chaos hypothesis [6–9, 17,37] is used in the form of Eq. (4). It can be introduced in different ways,[10] but effectively it always amounts to postulating a new kind of equilibrium, which includes some interactions and in which some stationary state is formed [8] summarily characterized by parameter q. In our case, it leads to Eq. (20), which is formally identical to a perfect hydrodynamic equation but with all the usual ingredients replaced by their q counterparts ["perfect" means here that there is nothing on the right-hand side of Eq. (20)]. It is natural to ask how Eq. (20) would look when written in terms of the usual perfect hydrodynamic (with $q = 1$) and some reminder depending on the parameter q. As we have seen, in general, q differs only slightly from unity, $q - 1 \ll 1$; therefore, it is tempting to simply expand Eq. (20) in the small parameter $|q - 1|$ [11,50]. However, as shown in Appendix C, in such case one faces some unsurmountable problems, because terms multiplying $|q - 1|$ are not small enough in the whole of phase space. We shall therefore follow a more general approach.

All our results presented above come from the Eq. (20), which is the equation for perfect q hydrodynamics. Notice that nonextensivity affects not only the thermodynamical quantities such as energy density ε and pressure P but also the flow velocity field $u^\mu(x)$, that is,

$$\varepsilon(T) \to \varepsilon_q(T_q) \equiv \varepsilon(T_q) + \Delta\varepsilon_q(T_q),$$
$$P(T) \to P_q(T_q) \equiv P(T_q) + \Delta P_q(T_q),$$
$$u^\mu(x) \to u^\mu_q(x) \equiv u^\mu(x) + \delta u^\mu_q(x),$$

where $u^\mu(x)$ is formally a solution of the equation which has the form of the dissipative hydrodynamic equation [23–32]

$$\left[\tilde{\varepsilon} u^\mu u^\nu - \tilde{P}\Delta^{\mu\nu} + 2W^{(\mu}u^{\nu)} + \pi^{\mu\nu}\right]_{;\mu} = 0. \tag{39}$$

The notation used is

$$\tilde{\varepsilon} = \varepsilon_q + 3\Pi, \quad \tilde{P} = P_q + \Pi, \tag{40a}$$
$$W^\mu = w_q[1+\gamma]\Delta^\mu_\lambda \delta u^\lambda_q, \tag{40b}$$
$$\pi^{\mu\nu} = \frac{W^\mu W^\nu}{w_q[1+\gamma]^2} + \Pi\Delta^{\mu\nu} = w_q \delta u^{\langle\mu}_q \delta u^{\nu\rangle}_q, \tag{40c}$$

where $\tilde{\varepsilon}$ is energy density, $\tilde{P}$ pressure, W^μ energy or heat flow vector, and $\pi^{\mu\nu}$ the (symmetric and traceless) shear pressure tensor, and where

$$w_q \equiv \varepsilon_q + P_q, \tag{41}$$
$$\gamma \equiv u_\mu \delta u^\mu_q = -\tfrac{1}{2}\delta u_{q\mu}\delta u^\mu_q, \tag{42}$$

and

$$A^{(\mu}B^{\nu)} \equiv \tfrac{1}{2}(A^\mu B^\nu + A^\nu B^\mu),$$
$$a^{\langle\mu}b^{\nu\rangle} \equiv \left[\tfrac{1}{2}\left(\Delta^\mu_\lambda\Delta^\nu_\sigma + \Delta^\mu_\sigma\Delta^\nu_\lambda\right) - \tfrac{1}{3}\Delta^{\mu\nu}\Delta_{\lambda\sigma}\right]a^\lambda b^\sigma,$$

whereas

$$\Pi \equiv \tfrac{1}{3}w_q\left[\gamma^2 + 2\gamma\right]. \tag{43}$$

This last quantity can be regarded as a bulk pressure to be used below.

Now comes the crucial point of our argument. To proceed further, we shall assume that there exists some temperature T and velocity field δu^μ_q satisfying the following relations:

$$P(T) = P_q(T_q), \tag{44a}$$
$$\varepsilon(T) = \varepsilon_q(T_q) + 3\Pi \tag{44b}$$

(ε and P are energy density and pressure defined in the usual Boltzmann-Gibbs statistics, i.e., for $q = 1$). In this case, one can transform Eq. (39) into

$$[\varepsilon(T)u^\mu u^\nu - (P(T)+\Pi)\Delta^{\mu\nu} + 2W^{(\mu}u^{\nu)} + \pi^{\mu\nu}]_{;\mu} = 0, \tag{45}$$

which has the familiar form of the usual *d hydrodynamic* equation. This means that perfect q hydrodynamics represented by Eq. (20) can be regarded as being formally *equivalent* to some form of d hydrodynamics as represented by Eq. (45). We shall call this observation the NexDC correspondence [and, respectively, we shall call Eq. (44) with Eq. (40b) and (40c) the NexDC relations]. This observation can be traced back to the fact of generic nonconservation of global entropy in nonextensive systems, cf. Eq. (21), visualized in Fig. 8 as an increase of the multiplicity with increasing q.

[10]For example, in Ref. [9] it was a random distortions of energy and momentum conservation caused by the surrounding system which resulted in emergence of some nonextensive equilibrium. In Refs. [6, 7] the two-body energy composition is replaced by generalized energy sum $h(E_1, E_2)$ (assumed to be associative) which is not necessarily the simple addition and which contains contributions stemming from the pair interaction. It turns out that under quite general assumptions about the function h, the division of the total energy among free particles can be done. Different forms of function h lead to different forms of entropy formula, among which one encounters the known Tsallis form as well. The origin of this kind of thinking can be traced to the analysis of the q-Hagedorn model in Ref. [50].

B. Nonextensive/dissipative correspondence: Consequences

We shall now present shortly the most specific immediate consequences of NexDC correspondence: the entropy production and estimations of the corresponding transport coefficients.

1. Entropy production in q hydrodynamics

Let us start with the observation that Eq. (43) and NexDC relations (44) lead following the form of q enthalpy,

$$\varepsilon_q(T_q) + P_q(T_q) = \frac{\varepsilon(T) + P(T)}{[1+\gamma]^2}, \tag{46}$$

which can be also used in the definition of γ because $w \equiv Ts = \varepsilon + P$ and $1/(\gamma+1) = \sqrt{1 - 3\Pi/w}$ (s is the entropy density in the usual Boltzmann-Gibbs statistics). Notice that in NexDC, one has

$$W^\mu W_\mu = -3\Pi w, \tag{47a}$$

$$\pi^{\mu\nu} W_\nu = -2\Pi W^\mu, \tag{47b}$$

$$\pi_{\mu\nu}\pi^{\mu\nu} = 6\Pi^2. \tag{47c}$$

Suppose now that we define a *true equilibrium state* as a state with $q = 1$, i.e., with no residual correlations between fluid elements and no intrinsic fluctuations present, with energy momentum tensor

$$\mathcal{T}^{\mu\nu}_{\rm eq} \equiv \mathcal{T}^{\mu\nu} = \varepsilon(T) u^\mu u^\nu - P(T)\Delta^{\mu\nu}, \tag{48}$$

and with equilibrium distribution given by the usual Boltzmann distribution,

$$f_{\rm eq}(x,p) = \exp\left[-\frac{p^\mu u_\mu(x)}{k_{\rm B} T(x)}\right]. \tag{49}$$

In this case, the state characterized by $f_q(x,p)$ given by Eq. (14) must be regarded as some stationary state existing *near equilibrium*. Therefore, because we expect that $|q-1|$ is small, we can define a *near equilibrium state* by the correlation function h_q in Eq. (4) for which the energy momentum tensor is $\mathcal{T}^{\mu\nu}_q \equiv (\varepsilon_q + P_q) u^\mu_q u^\nu_q - P_q g^{\mu\nu}$; cf., Eq. (15). Therefore, we can write

$$\mathcal{T}^{\mu\nu}_q = \mathcal{T}^{\mu\nu}_{\rm eq} + \delta\mathcal{T}^{\mu\nu}, \tag{50}$$

where

$$\delta\mathcal{T}^{\mu\nu} = -\Pi\Delta^{\mu\nu} + W^\mu u^\nu + W^\nu u^\mu + \pi^{\mu\nu}. \tag{51}$$

Using now Eq. (44), we obtain the relation

$$\gamma = \sqrt{1 + \delta\epsilon_q} - 1, \tag{52a}$$

where

$$\delta\epsilon_q \equiv \frac{\varepsilon(T) - \varepsilon_q(T_q)}{\varepsilon_q(T_q) + P_q(T_q)}, \tag{52b}$$

which connects the velocity field u [solution of the dissipative hydrodynamics given by Eq. (45)] with the velocity field u_q [solution of the q hydrodynamics given by Eq. (20)].

In the $(1+1)$ dimensional case discussed here, one can always parametrize these velocity fields by using the respective fluid rapidities α_q and α, $u^\mu_q(x) = [\cosh(\alpha_q - \eta), \frac{1}{\tau}\sinh(\alpha_q - \eta)]$ and $u^\mu(x) = [\cosh(\alpha - \eta), \frac{1}{\tau}\sinh(\alpha - \eta)]$. Because $\gamma = u_\mu \delta u^\mu_q = \cosh(\alpha_q - \alpha) - 1$, one has

$$\cosh(\alpha_q - \alpha) = \sqrt{1 + \delta\epsilon_q}, \tag{53}$$

which provides us with a connection between u and u_q. From Eq. (53), one obtains finally

$$\alpha = \alpha_q - \log\left(\epsilon_q + \sqrt{1 + \delta\epsilon_q}\right). \tag{54}$$

We abandon here another solution of Eq. (53), namely, that $\alpha = \alpha_q + \log(\epsilon_q + \sqrt{1 + \delta\epsilon_q})$, because it leads to the entropy reduction; i.e., for it, $[su^\mu]_{;\mu} < 0$ for $q > 1$. Taking the covariant derivative of Eq. (50) and multiplying it by u_ν we obtain

$$u_\nu \mathcal{T}^{\mu\nu}_{q;\mu} = T[su_\mu]_{;\mu} + u_\nu \delta\mathcal{T}^{\mu\nu}_{;\mu} = 0. \tag{55}$$

Therefore, although in ideal q hydrodynamics the q entropy is *conserved*, i.e., $[s_q u^\mu_q]_{;\mu} = 0$, we can rewrite it in the form corresponding to dissipative fluid with *entropy production*, i.e.,

$$[su^\mu]_{;\mu} = -\frac{u_\nu}{T}\delta\mathcal{T}^{\mu\nu}_{;\mu}. \tag{56}$$

To illustrate this, we show in Fig. 11 the expected entropy production as given by Eq. (56). Notice that $su^\mu_{;\mu} > 0$ for the large η region at any τ (but especially for the early stage of the hydrodynamic evolution). It supports therefore a dissipative

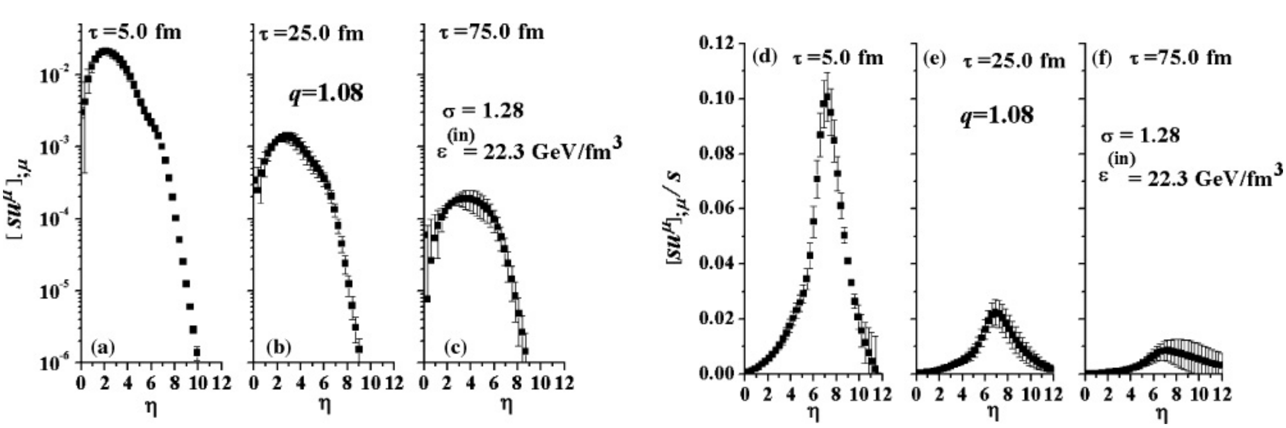

FIG. 11. Evolution of the entropy production $[su^\mu]_{;\mu}$ and ratio $[su^\mu]_{;\mu}/s$ as functions of η for different values of τ for $q = 1.08$ with $\varepsilon^{(\rm in)} = 22.3$ GeV/fm^3 (or $\sigma = 1.28$). The error bar is estimated by the value of $[s_q u^\mu_q]_\mu$ obtained in the numerical calculation, which should be zero in an analytical calculation.

analogy of the q hydrodynamics mentioned before and leads us to the very interesting conclusion that the equilibrium state generated in high-energy heavy-ion collisions may in fact be the q equilibrium state which can be regarded as some stationary state near the usual (i.e., $q = 1$) equilibrium state and which contains also some dissipative phenomena.

2. *Calculation of transport coefficients from q hydrodynamics*

There are different formulations of d hydrodynamics [23–32]. In what follows, we shall choose for comparison only the second-order theory of dissipative fluids (in particular as given by Refs. [24,25]) leaving investigations of other approaches from nonextensive perspectives for the future. As is known, this theory does not violate causality (at least not the global causality over the distant scale given by the relaxation time); on the other hand, it contains now some dissipative fluxes such as heat conductivity and bulk and shear viscosities. We shall now see to what extent these transport coefficients can be calculated in q hydrodynamics.

To this end, let us start by considering more closely the respective entropies. Dissipation is connected with the production of entropy, and in Refs. [24,25] the most general off-equilibrium four-entropy current σ^μ can be written as

$$\sigma^\mu = P(T)\beta^\mu + \beta_\nu\left(\mathcal{T}^{\mu\nu}_{\rm eq} + \delta\mathcal{T}^{\mu\nu}\right) + Q^\mu, \tag{57}$$

where $\beta^\mu \equiv u^\mu/T$ and $Q^\mu = Q^\mu(\delta\mathcal{T}^{\mu\nu})$ is some function that characterizes the off-equilibrium state. In the case of the q entropy current of Eq. (5), the NexDC conjecture [i.e., Eqs. (40b) and (46)] leads to the following off-equilibrium state:

$$Q^\mu = Q^\mu_\chi \equiv \chi\left[su^\mu + \frac{W^\mu}{T}\right], \tag{58}$$

with $\chi \equiv \frac{T}{T_q}\sqrt{1-\frac{3\Pi}{w}} - 1$, which results in

$$\sigma^\mu_q\left(\equiv s^\mu_q\right) = su^\mu + \frac{W^\mu}{T} + \chi\left\{su^\mu + \frac{W^\mu}{T}\right\}. \tag{59}$$

Notice that because of the strict q entropy conservation assumed here, using $Q^\mu = Q^\mu_\chi$ one always gets $\sigma^\mu_{q;\mu} = 0$. This means that although there is no production of q entropy, there is production of the usual entropy; i.e., our q system is really *dissipative* in the usual meaning of this word.

Let us now be more specific and use the most general algebraic form of Q^μ, calculated up to the second order in the dissipative flux, as given by [25]

$$Q^\mu_{\rm 2nd} = \frac{\left[-\beta_0\Pi^2 + \beta_1 W_\nu W^\nu - \beta_2\pi_{\nu\lambda}\pi^{\nu\lambda}\right]}{2T}u^\mu - \frac{\alpha_0\Pi W^\mu}{T} + \frac{\alpha_1\pi^{\mu\nu}W_\nu}{T}. \tag{60}$$

Here $\beta_{i=1,2,3}$ are the corresponding thermodynamic coefficients for the, respectively, scalar, vector, and tensor dissipative contributions to the entropy current, whereas $\alpha_{i=0,1}$ are the corresponding viscous/heat coupling coefficients. The Π is the bulk pressure defined before in Eq. (43).[11] In the NexDC, one has

$$Q^\mu_{\rm 2nd} \to \Gamma_{\rm 2nd}su^\mu + \Upsilon_{\rm 1st}\frac{W^\mu}{T}, \tag{61}$$

where

$$\Gamma_{\rm 2nd} \equiv -\frac{3\beta_1}{2}\Pi - \frac{(\beta_0 + 6\beta_2)}{2w}\Pi^2, \tag{62a}$$

$$\Upsilon_{\rm 1st} \equiv -(\alpha_0 + 2\alpha_1)\Pi. \tag{62b}$$

Q^μ can then be expressed by polynomials in the bulk pressure Π defined by Eq. (43). It is then natural to expect that the most general entropy current in the NexDC approach has the form

$$Q^\mu_{\rm full} = \Gamma(\Pi)su^\mu + \Upsilon(\Pi)\frac{W^\mu}{T}, \tag{63}$$

where Γ, Υ are (in general infinite) series in powers of the bulk pressure Π. In this sense, the $Q^\mu_{\rm full}$ can be regarded as the *full-order dissipative current*.

In general, one has entropy production/reduction, $\sigma^\mu_{;\mu} \neq 0$; however, when $\Gamma(\Pi) = \Upsilon(\Pi) = \chi$, one has $\sigma^\mu_{\chi;\mu} = 0$, so one can write the full-order dissipative entropy current as

$$Q^\mu_{\rm full} = (\chi + \xi)su^\mu + (\chi - \xi)\frac{W^\mu}{T}, \tag{64}$$

where Γ and Υ are determined by $\chi \equiv (\Gamma + \Upsilon)/2$ and $\xi \equiv (\Gamma\text{-}\Upsilon)/2$. From two solutions for (Γ, Υ),

$$\frac{\Gamma}{2} \equiv \frac{T}{T_q}\left(\sqrt{1-\frac{3\Pi}{w}} - 1\right), \quad \frac{\Upsilon}{2} \equiv \frac{T - T_q}{T_q}, \tag{65a}$$

or

$$\frac{\Gamma}{2} \equiv \frac{T - T_q}{T_q}, \quad \frac{\Upsilon}{2} \equiv \frac{T}{T_q}\left(\sqrt{1-\frac{3\Pi}{w}} - 1\right), \tag{65b}$$

only Eq. (65a) is acceptable because only for it $u_\mu Q^\mu_{\rm full} \leqslant 0$ [i.e., entropy is maximal in the equilibrium [25], because $(T - T_q)/T_q$ is always positive for $q \geqslant 1$]. In this way, we finally arrive at the following possible expression for the full-order dissipative entropy current in the NexDC approach:

$$\sigma^\mu_{\rm full} \equiv su^\mu + \frac{W^\mu}{T} - \frac{2T}{T_q}\left[1 - \sqrt{1-\frac{3\Pi}{w}}\right]su^\mu + \frac{2(T - T_q)}{T_q}\frac{W^\mu}{T}. \tag{66}$$

Limiting ourselves to situations when $T/T_q \approx 1$ and neglecting terms higher than $\mathcal{O}(3\Pi/w)^2$, we obtain

$$Q^\mu_{\rm full} \approx \left[-\left(\frac{3\Pi}{w}\right) - \frac{1}{4}\left(\frac{3\Pi}{w}\right)^2\right]su^\mu. \tag{67}$$

[11] Notice that whereas the time evolution of Π is controlled by q hydrodynamics (via the respective time dependencies of ε_q, P_q, and x), its form is determined by the assumed constraints which must ensure that the local entropy production in the standard second-order hydrodynamic theory [24,25] is never negative.

Comparing now Eqs. (62) and (67), one gets[12]

$$\beta_1 = \frac{2}{w}, \quad \beta_0 + 6\beta_2 = \frac{9}{2w}, \quad \alpha_0 + 2\alpha_1 = 0. \tag{68}$$

Since in the Israel-Stewart theory [24] the relaxation time τ is proportional to thermodynamic coefficients $\beta_{0,1,2}$, it is natural to assume that in our NexDC case, $\tau \propto 1/w$; i.e., it is proportional to the inverse of the enthalpy (notice that for the classical Boltzmann gas of massless particles, one obtains $\beta_2 = 3/w$ [25,32]).

We shall now derive the bulk and shear viscosities emerging from the NexDC approach. Let us start with the observation that the local entropy production by the full-order entropy current Eq. (66) can be also written as

$$\sigma^{\mu}_{\text{full};\mu} = [(1+\chi)\Phi^{\mu}]_{;\mu} + [\xi\Psi^{\mu}]_{;\mu}, \tag{69}$$

where $\Phi^{\mu} = su^{\mu} + \frac{W^{\mu}}{T}$ and $\Psi^{\mu} = su^{\mu} - \frac{W^{\mu}}{T}$. Because the conservation of q entropy, $\sigma^{\mu}_{q;\mu} = 0$, is equivalent to $[(1+\chi)\Phi^{\mu}]_{;\mu} = 0$, using Eq. (46) one gets

$$\Psi^{\mu} = -\frac{W^{\nu}W_{\nu}}{3\Pi T}u^{\mu} + \frac{W_{\nu}}{2\Pi T}\pi^{\mu\nu}, \tag{70}$$

and [see Appendix D for details of derivation of Eqs. (71) and (73)]

$$\sigma^{\mu}_{\text{full};\mu} = -\frac{\Pi}{T}(wu^{\mu}X_{\mu}) - \frac{W^{\mu}}{T}\tilde{Y}_{\mu} + \frac{\pi^{\mu\nu}}{T}Z_{\mu\nu}, \tag{71}$$

where

$$X_{\mu} = -\frac{\xi}{\Pi}\left[\frac{\partial_{\mu}\Pi}{\Pi} + \frac{\partial_{\mu}T}{T} + \frac{\partial_{\mu}\xi}{\xi}\right], \tag{72a}$$

$$Y_{\mu} = \frac{\xi}{\Pi}\left[\frac{2}{3}u^{\nu}W_{\mu;\nu} + \frac{1}{3}W_{\mu}u^{\nu}_{;\nu} - \frac{1}{2}\pi^{\nu}_{\mu;\nu}\right], \tag{72b}$$

$$Z_{\mu\nu} = \frac{\xi}{\Pi}\left[\frac{1}{2}W_{\nu;\mu}\right], \tag{72c}$$

and

$$\tilde{Y}_{\mu} = Y_{\mu} - \Pi X_{\mu}, \quad \tilde{Z}_{\mu\nu} \equiv Z_{\mu\nu} + \frac{\tilde{Y}_{\mu}W_{\nu}}{2\Pi}. \tag{72d}$$

One can now use Eq. (47) to eliminate the term proportional to heat flow $\frac{W^{\mu}}{T}$. In this way, one avoids the explicit contribution to entropy production from the heat flow $\frac{W^{\mu}}{T}$, which is present in Eq. (71) when one discusses a baryon-free fluid, in which case the necessity to use the Landau frame would appear. As one can see, Eq. (69) is covariant and therefore it does not depend on the frame used. After that, one obtains

$$\sigma^{\mu}_{\text{full};\mu} = -\frac{\Pi}{T}(wu^{\mu}X_{\mu}) + \frac{\pi^{\mu\nu}}{T}\tilde{Z}_{\mu\nu}. \tag{73}$$

[12]The fact that we obtained nonzero coefficients $\beta_{i=1,2,3}$ and couplings $\alpha_{i=0,1}$ for dissipative flux of Eq. (60) as found in Eq. (68) means that d hydrodynamics obtained via NexDC from q hydrodynamics accounts for all second-order terms. One may conclude that it seems that such d hydrodynamics with full-order entropy current has global causality (over a distance scale given by relaxation time). However, the question of whether NexDC violates causality remains so far unsettled.

Equation (73) can be now used to find the bulk and shear viscosities from $\sigma^{\mu}_{\text{full};\mu}$ given by Eq. (69). The positive transport coefficients, bulk viscosity ζ, and shear viscosity η can be estimated by writing the entropy production $\sigma^{\mu}_{\text{full};\mu}$ as

$$\sigma^{\mu}_{\text{full};\mu} = \frac{\Pi^2}{\zeta T} + \frac{\pi^{\mu\nu}\pi_{\mu\nu}}{2\eta T} \geqslant 0 \tag{74}$$

and using Eq. (47). We arrive then at the sum rule connecting transport coefficients (expressed as ratios of bulk and shear viscosities over the entropy density s),

$$\frac{1}{\zeta/s} + \frac{3}{\eta/s} = \frac{w\sigma^{\mu}_{\text{full};\mu}}{\Pi^2}. \tag{75}$$

This is as far as we can go. The heat conductivity, as shown above, can be expressed by two other transport coefficients for which we have only one equation in the form of sum rule (75). To proceed any further and to disentangle Eq. (75), one has to add some additional input. Suppose then that we are interested in the extremal situation, when total entropy is generated by action of shear viscosity only. In this case, one can rewrite Eq. (73) as

$$\sigma^{\mu}_{\text{full};\mu} = \frac{\pi^{\mu\nu}}{T}\left[-\frac{\pi_{\mu\nu}}{6\Pi}(wu^{\lambda}X_{\lambda}) + \tilde{Z}_{\mu\nu}\right], \tag{76}$$

resulting in

$$\frac{\eta}{s} = \frac{\gamma(\gamma+2)}{3(\gamma+1)^2}\left[\frac{\pi^{\mu\nu}}{\Pi}\frac{\tilde{Z}_{\mu\nu}}{T} - su^{\lambda}X_{\lambda}\right]^{-1}. \tag{77}$$

Note that Eq. (77) allows all values of η/s, in particular that $\eta/s < \frac{1}{4\pi}$, what violates the limit obtained from AdS/CFT correspondence that $\eta/s \geqslant 1/4\pi$ [36]. To impose this limit, we shall now use Eq. (75). This can be done only in the region where the right-hand side of Eq. (77) is smaller than (or equal to) $1/4\pi$ (in which case, we put $\eta/s = 1/4\pi$); otherwise, because of our earlier assumed limitation, we put $\zeta/s = 0$ and use Eq. (77) to evaluate η/s. The corresponding results for ζ/s and η/s are shown in Figs. 12(a) and 12(b), respectively. Notice that when the right-hand side of Eq. (77) approaches $1/4\pi$, the ζ/s given by Eq. (75) approaches infinity. All curves presented in Fig. 12 were calculated for Au+Au collisions

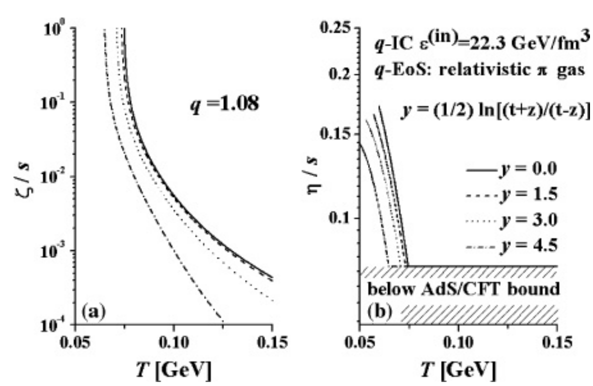

FIG. 12. NexDC predictions for the ratios of (a) bulk and (b) shear viscosities over the entropy density, i.e., ζ/s and η/s, respectively, as functions of temperature T and calculated for a number of space-time rapidities y [as in Eq. (24b) using the q hydrodynamic model].

with an identical set of parameters as the best fit presented in Fig. 10 above.

VI. SUMMARY AND CONCLUSIONS

We have presented a nonextensive version of the hydrodynamic model for multiparticle production processes, the q hydrodynamic model, which is based on the nonextensive statistics represented by Tsallis entropy and indexed by the nonextensivity parameter q. In doing so, we have followed the usual approach originating in the appropriate kinetic equations formulated in nonextensive form in Ref. [17]. We have found the nonextensive entropy current which satisfies not only the nonextensive H theorem, Eq. (5), but also the q version of thermodynamic relations, Eq. (17). The $(1+1)$ dimensional q hydrodynamics with the q-Gaussian initial condition and the q-EOS can reasonably reproduce the single-particle spectra observed at RHIC energy for $q = 1.07$–1.08 for $T_F = 100$–120 MeV if quark jet contributions to p_T spectra are small, i.e., up to a transverse momentum range around $p_T \leqslant 6.0$ GeV/c. We also found a possible correspondence between the q hydrodynamics and the usual $(q = 1)$ d hydrodynamics (NexDC) as provided by Eq. (45) with NexDC relations Eq. (44). Based on this correspondence, we have evaluated entropy production in relativistic heavy-ion collisions at RHIC energy using the results of our perfect q hydrodynamics (understood as an approach without any q viscosity effects added). The fact that the data comparison reveals that $q > 1$ indicates that, indeed, some dynamic factors are present, the detailed form of which has not yet been disclosed but which summarily can be accounted for by the nonextensive approach and which action is summarized by the parameter $q - 1$.

Regarding the obtained p_T dependence, our formula continues the attempts that have been made to interpret power-law spectra as a new kind of equilibrium phenomena for the whole p_T range, pushing the usual interpretation via the onset of “hard” collisions (imposed on the “soft” ones) to really high values of p_T (cf., Refs. [8,11,15] and references therein). In such an approach, there is no characteristic scale at which the transition from soft (or locally thermalized) to hard (or unthermalized) dynamics occurs, which appears in the conventional descriptions using viscous hydrodynamics as, for example, in Refs. [34,51].

One of the results of our investigation is that fluctuations in the initial conditions seem to be the most important part of the hydrodynamic model, which by using Tsallis statistics, attempts to account for any possible fluctuations in some general, model-independent way. This is quite a reasonable result because at the initial stage, our system consists of a relatively small number of degrees of freedom and is therefore more sensitive to any fluctuations. On the contrary, at freez-eout this number is much bigger, and the system only weakly responds to any fluctuations. This finding agrees nicely with a recent analysis [52] of the elliptical flow performed by using a hydrodynamic approach that attempts to account for fluctuations (without, however, using q statistics). On the other hand, however, it should be remembered that the analysis presented here is considerably simplified by using the same nonextensivity parameter q at all stages of the collision process. There is therefore room for improvements which will facilitate comparison with data. One can argue that the intrinsic fluctuations existing in different stages of the collision process are of different (albeit connected) dynamical origin, and therefore parameters q for the initial conditions, for the EOS, and, finally, for the hydrodynamic expansion should be allowed to have different values (and should be also different for the longitudinal and transverse dynamics). The other problem would be how to connect our q parameter expressing fluctuations with fluctuations in all momentum observables as seen when analyzing a nonideal liquid as was done, for example, in Ref. [53]. We plan to address these problems elsewhere. In any case, similar to the fact that the concept of ideal fluid is never realized in nature [35] (the bound of $\eta/s \geqslant 1/4\pi$ found in Ref. [36] being a strong argument supporting this), the $q = 1$ case should be replaced by investigations of the q fluid with $q > 1$.

In this context, the natural question arises concerning the deeper physical meaning of the q hydrodynamic proposed here. The most important observation discussed in Sec. V is the apparent correspondence found between the perfect q hydrodynamics and the usual d hydrodynamics, which we call the NexDC correspondence. It allows calculation of transport coefficients of viscous fluid in terms of parameters of q (ideal) fluid, i.e., essentially as dependent on a single parameter which, as it was already stressed many times, represents summary effect of many possible dynamical factors, without entering into dynamical details (i.e., in a purely phenomenological way). The detailed discussion of the NexDC phenomenon is, however, outside the scope of the present paper, and we plan to address is elsewhere.

We close by remarking that hydrodynamics can also be derived using information theory with its method of maximization of information entropy under some specific constraints [54]. It is therefore plausible that our results could also be derived using a nonextensive version of information theory (in the same way as the work in Refs. [13,55] can be regarded as a nonextensive generalization of the information theory approach to single-particle distributions obtained in the multiparticle production processes proposed in Ref. [56]). We shall not pursue this possibility here.

ACKNOWLEDGMENTS

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APPENDIX A: DERIVATION OF EQUATIONS (26) AND (27)

Consider some details of the $(1+1)$ dimensional relativistic hydrodynamics under the assumption that one ignores the transverse expansion of the fluid. With $g_{\mu\nu} = \mathrm{diag}(1, -\tau^2)$ and four fluid velocity $u_\mu = [\cosh(\alpha_q - \eta), -\tau\sinh(\alpha_q - \eta)]$, the projection matrix is equal to

$$\Delta_{q\mu\nu} \equiv g_{\mu\nu} - u_{q\mu}u_{q\nu} = \begin{bmatrix} -\sinh^2(\alpha_q - \eta) & \tau\cosh(\alpha_q - \eta)\sinh(\alpha_q - \eta) \\ \tau\cosh(\alpha_q - \eta)\sinh(\alpha_q - \eta) & -\tau^2\cosh^2(\alpha_q - \eta) \end{bmatrix}. \tag{A1}$$

The nonvanishing components of Christoffel symbols are $\Gamma^\eta_{\tau\eta} = \Gamma^\eta_{\eta\tau} = 1/\tau$ and $\Gamma^\tau_{\eta\eta} = \tau$; therefore, the covariant derivative of fluid velocity, which is defined by $u^\mu_{;\nu} = \partial_\nu u_\mu + \Gamma^\mu_{\lambda\nu}u^\lambda$, has the form

$$u^\tau_{q;\tau} = \sinh(\alpha_q - \eta)\frac{\partial\alpha_q}{\partial\tau}, u^\eta_{q;\tau} = \frac{1}{\tau}\cosh(\alpha_q - \eta)\frac{\partial\alpha_q}{\partial\tau},$$
$$u^\tau_{q;\eta} = \sinh(\alpha_q - \eta)\frac{\partial\alpha_q}{\partial\eta}, u^\eta_{q;\eta} = \frac{1}{\tau}\cosh(\alpha_q - \eta)\frac{\partial\alpha_q}{\partial\eta}.$$

Using these expressions, one obtains

$$u^\mu_q\Delta_{q\tau\nu}u^\nu_{q;\mu} = \cosh(\alpha_q - \eta)\sinh(\alpha_q - \eta)\frac{\partial\alpha_q}{\partial\tau} + \frac{1}{\tau}\sinh^2(\alpha_q - \eta)\frac{\partial\alpha_q}{\partial\eta},$$
$$u^\mu_q\Delta_{q\eta\nu}u^\nu_{q;\mu} = -\tau\cosh^2(\alpha_q - \eta)\frac{\partial\alpha_q}{\partial\tau} - \cosh(\alpha_q - \eta)\sinh(\alpha_q - \eta)\frac{\partial\alpha_q}{\partial\eta}.$$

Because $g^{\mu\nu}_{;\nu} = 0$ for $\mu, \nu = \tau$ and η, Eq. (23) is reduced to the following two equations:

$$(\varepsilon_q + P_q)\left\{u^\mu_q\Delta_{q\tau\nu}u^\nu_{q;\mu}\right\} - \Delta_{q\tau\tau}\frac{\partial P_q}{\partial\tau} + \frac{1}{\tau^2}\Delta_{q\tau\eta}\frac{\partial P_q}{\partial\eta} = 0, \tag{A2}$$

$$(\varepsilon_q + P_q)\left\{u^\mu_q\Delta_{q\eta\nu}u^\nu_{q;\mu}\right\} - \Delta_{q\eta\tau}\frac{\partial P_q}{\partial\tau} + \frac{1}{\tau^2}\Delta_{q\eta\eta}\frac{\partial P_q}{\partial\eta} = 0. \tag{A3}$$

Equations (A2) and (A3) are equivalent; therefore, one has only one equation,

$$(\varepsilon_q + P_q)\left\{\frac{\partial\alpha_q}{\partial\tau} + \frac{\tanh(\alpha_q - \eta)}{\tau}\frac{\partial\alpha_q}{\partial\eta}\right\} + \tanh(\alpha_q - \eta)\frac{\partial P_q}{\partial\tau} + \frac{1}{\tau}\frac{\partial P_q}{\partial\eta} = 0, \tag{A4}$$

which is Eq. (27). Since the four-divergence of the fluid velocity $u^\mu_{q;\mu}$ is given by

$$u^\mu_{q;\mu} = \sinh(\alpha_q - \eta)\frac{\partial\alpha_q}{\partial\tau} + \frac{1}{\tau}\cosh(\alpha_q - \eta)\frac{\partial\alpha_a}{\partial\eta},$$

the Eq. (23) can be written as

$$\cosh(\alpha - \eta)\frac{\partial\varepsilon_q}{\partial\tau} + \frac{\sinh(\alpha_q - \eta)}{\tau}\frac{\partial\varepsilon_q}{\partial\eta} + (\varepsilon_q + P_q)\left\{\sinh(\alpha_q - \eta)\frac{\partial\alpha_q}{\partial\tau} + \frac{1}{\tau}\cosh(\alpha_q - \eta)\frac{\partial\alpha_a}{\partial\eta}\right\} = 0,$$

leading immediately to

$$\frac{\partial\varepsilon_q}{\partial\tau} + \frac{\tanh(\alpha_q - \eta)}{\tau}\frac{\partial\varepsilon_q}{\partial\eta} + (\varepsilon_q + P_q)\left\{\tanh(\alpha_q - \eta)\frac{\partial\alpha_q}{\partial\tau} + \frac{1}{\tau}\frac{\partial\alpha_a}{\partial\eta}\right\} = 0, \tag{A5}$$

which is Eq. (26).

APPENDIX B: NUMERICAL METHOD USED

For the purpose of numerical calculations, we express Eqs. (26) and (27) in the form of the finite difference equations

$$A^{(n)}_{1(j)}\left\{\frac{\varepsilon^{(n+1)}_{q(j)} - \frac{1}{2}\left[\varepsilon^{(n)}_{q(j+1)} + \varepsilon^{(n)}_{q(j-1)}\right]}{\Delta\tau}\right\} + A^{(n)}_{2(j)}\left\{\frac{\varepsilon^{(n)}_{q(j+1)} - \varepsilon^{(n)}_{q(j-1)}}{2\Delta\eta}\right\}, + A^{(n)}_{3(j)}\left\{\frac{\alpha^{(n+1)}_{q(j)} - \frac{1}{2}\left[\alpha^{(n)}_{q(j+1)} + \alpha^{(n)}_{q(j-1)}\right]}{\Delta\tau}\right\} + A^{(n)}_{4(j)}\left\{\frac{\alpha^{(n)}_{q(j+1)} - \alpha^{(n)}_{q(j-1)}}{2\Delta\eta}\right\} = 0 \tag{B1}$$

and

$$B^{(n)}_{1(j)}\left\{\frac{P^{(n+1)}_{q(j)} - \frac{1}{2}\left[P^{(n)}_{q(j+1)} + P^{(n)}_{q(j-1)}\right]}{\Delta\tau}\right\} + B^{(n)}_{2(j)}\left\{\frac{P^{(n)}_{q(j+1)} - P^{(n)}_{q(j-1)}}{2\Delta\eta}\right\}, + B^{(n)}_{3(j)}\left\{\frac{\alpha^{(n+1)}_{q(j)} - \frac{1}{2}\left[\alpha^{(n)}_{q(j+1)} + \alpha^{(n)}_{q(j-1)}\right]}{\Delta\tau}\right\} + B^{(n)}_{4(j)}\left\{\frac{\alpha^{(n)}_{q(j+1)} - \alpha^{(n)}_{q(j-1)}}{2\Delta\eta}\right\} = 0. \tag{B2}$$

The subscript (j) and superscript (n) represent the corresponding grid number in the η and τ space with grid spacings $\Delta\eta$ and $\Delta\tau$, respectively, i.e., with $\eta_j = j\Delta\eta$ and $\tau_n = \tau_0 + n\Delta\tau$. The coefficients appearing in the above equations are defined as

$$\begin{aligned} A^{(n)}_{1(j)} &\equiv 1, & B^{(n)}_{1(j)} &\equiv v^{(n)}_{q(j)},\\ A^{(n)}_{2(j)} &\equiv \left[v^{(n)}_{q(j)}\right]/\tau_n, & B^{(n)}_{2(j)} &\equiv 1/\tau_n,\\ A^{(n)}_{3(j)} &\equiv \left(\varepsilon^{(n)}_{q(j)} + P^{(n)}_{q(j)}\right)\left[v^{(n)}_{q(j)}\right], & B^{(n)}_{3(j)} &\equiv \left(\varepsilon^{(n)}_{q(j)} + P^{(n)}_{q(j)}\right),\\ A^{(n)}_{4(j)} &\equiv \left(\varepsilon^{(n)}_{q(j)} + P^{(n)}_{q(j)}\right)/\tau_n, & B^{(n)}_{4(j)} &\equiv \left(\varepsilon^{(n)}_{q(j)} + P^{(n)}_{q(j)}\right)\left[v^{(n)}_{q(j)}\right]/\tau_n. \end{aligned} \tag{B3}$$

Introducing now the notation

$$c^{2(n)}_{s(j)} \equiv \frac{P^{(n)}_{q(j)}}{\varepsilon^{(n)}_{q(j)}}, \tag{B4}$$

where $c^{2(n)}_{s(j)}$ is a function of $\varepsilon^{(n)}_{q(j)}$ [due to the equation of state $P_q(\varepsilon_q)$], one can rewrite these two equations in the form

$$\begin{aligned} &\left[A^{(n)}_{1(j)}\right]\varepsilon^{(n+1)}_{q(j)} + \left[A^{(n)}_{3(j)}\right]\alpha^{(n+1)}_{q(j)} - \frac{1}{2}\left[A^{(n)}_{1(j)} - A^{(n)}_{2(j)}\frac{\Delta\tau}{\Delta\eta}\right]\varepsilon^{(n)}_{q(j+1)}\\ &\quad - \frac{1}{2}\left[A^{(n)}_{1(j)} + A^{(n)}_{2(j)}\frac{\Delta\tau}{\Delta\eta}\right]\varepsilon^{(n)}_{q(j-1)}\\ &\quad - \frac{1}{2}\left[A^{(n)}_{3(j)} - A^{(n)}_{4(j)}\frac{\Delta\tau}{\Delta\eta}\right]\alpha^{(n)}_{q(j+1)}\\ &\quad - \frac{1}{2}\left[A^{(n)}_{3(j)} + A^{(n)}_{4(j)}\frac{\Delta\tau}{\Delta\eta}\right]\alpha^{(n)}_{q(j-1)} = 0, \end{aligned} \tag{B5}$$

and

$$\begin{aligned} &c^{2(n+1)}_{s(j)}\left[B^{(n)}_{1(j)}\right]\varepsilon^{(n+1)}_{q(j)} + \left[B^{(n)}_{3(j)}\right]\alpha^{(n+1)}_{q(j)}\\ &\quad - \frac{c^{2(n)}_{s(j+1)}}{2}\left[B^{(n)}_{1(j)} - B^{(n)}_{2(j)}\frac{\Delta\tau}{\Delta\eta}\right]\varepsilon^{(n)}_{q(j+1)}\\ &\quad - \frac{c^{2(n)}_{s(j-1)}}{2}\left[B^{(n)}_{1(j)} + B^{(n)}_{2(j)}\frac{\Delta\tau}{\Delta\eta}\right]\varepsilon^{(n)}_{q(j-1)}\\ &\quad - \frac{1}{2}\left[B^{(n)}_{3(j)} - B^{(n)}_{4(j)}\frac{\Delta\tau}{\Delta\eta}\right]\alpha^{(n)}_{q(j+1)}\\ &\quad - \frac{1}{2}\left[B^{(n)}_{3(j)} + B^{(n)}_{4(j)}\frac{\Delta\tau}{\Delta\eta}\right]\alpha^{(n)}_{q(j-1)} = 0. \end{aligned} \tag{B6}$$

Eliminating $\alpha^{(n+1)}_{q(j)}$ from the above two equations, one obtains

$$\begin{aligned} &F^{(n)}_{10(j)}\varepsilon^{(n+1)}_{q(j)} + F^{(n)}_{1+(j)}\varepsilon^{(n)}_{q(j+1)} + F^{(n)}_{0-(j)}\varepsilon^{(n)}_{q(j-1)} + G^{(n)}_{0+(j)}\alpha^{(n)}_{q(j+1)}\\ &\quad + G^{(n)}_{0-(j)}\alpha^{(n)}_{q(j-1)} = 0, \end{aligned} \tag{B7}$$

where

$$\begin{aligned} F^{(n)}_{10(j)} &\equiv 1 - c^{2(n+1)}_{s(j)}\left[v^{(n)}_{q(j)}\right]^2,\\ F^{(n)}_{0+(j)} &\equiv -\frac{1}{2}\left(1 - c^{2(n)}_{s(j+1)}\left[v^{(n)}_{q(j)}\right]^2\right)\\ &\quad + \frac{1}{2}\left(1 - c^{2(n)}_{s(j+1)}\right)\frac{\left[v^{(n)}_{q(j)}\right]}{\tau_n}\frac{\Delta\tau}{\Delta\eta},\\ F^{(n)}_{0-(j)} &\equiv -\frac{1}{2}\left(1 - c^{2(n)}_{s(j-1)}\left[v^{(n)}_{q(j)}\right]^2\right)\\ &\quad - \frac{1}{2}\left(1 - c^{2(n)}_{s(j-1)}\right)\frac{\left[v^{(n)}_{q(j)}\right]}{\tau_n}\frac{\Delta\tau}{\Delta\eta},\\ G^{(n)}_{0+(j)} &\equiv +\frac{1}{2}\left(1 - \left[v^{(n)}_{q(j)}\right]^2\right)\left(\varepsilon^{(n)}_{q(j)} + P^{(n)}_{q(j)}\right)\frac{1}{\tau_n}\frac{\Delta\tau}{\Delta\eta},\\ G^{(n)}_{0-(j)} &\equiv -\frac{1}{2}\left(1 - \left[v^{(n)}_{q(j)}\right]^2\right)\left(\varepsilon^{(n)}_{q(j)} + P^{(n)}_{q(j)}\right)\frac{1}{\tau_n}\frac{\Delta\tau}{\Delta\eta}. \end{aligned}$$

One can now find $\varepsilon^{(n+1)}_{q(j)}$ by solving the nonlinear Eq. (B7). For $v^{(n)}_{q(j)} = 0$ (i.e., for the scaling case where $\eta = \alpha$), one obtains

$$\varepsilon^{(n+1)}_{q(j)} - \varepsilon^{(n)}_{q(j)} + \frac{\varepsilon^{(n)}_{q(j)} + P^{(n)}_{q(j)}}{\tau_n}\Delta\tau = 0, \tag{B8}$$

where relations $\alpha^{(n)}_{q(j+1)} - \alpha^{(n)}_{q(j-1)} = 2\Delta\eta$ and $\frac{1}{2}[\varepsilon^{(n)}_{q(j+1)} + \varepsilon^{(n)}_{q(j-1)}] = \varepsilon^{(n)}_{q(j)}$ were used. After finding $\varepsilon^{(n+1)}_{q(j)}$, one can find $\alpha^{(n+1)}_{q(j)}$ by using the recurrence formula

$$\begin{aligned} \alpha^{(n+1)}_{q(j)} &= \frac{1}{2}\left[\alpha^{(n)}_{q(j+1)} + \alpha^{(n)}_{q(j-1)}\right]\\ &\quad - \frac{1}{2}\left[\alpha^{(n)}_{q(j+1)} - \alpha^{(n)}_{q(j-1)}\right]\frac{v^{(n)}_{q(j)}}{\tau_n}\frac{\Delta\tau}{\Delta\eta}\\ &\quad + \frac{v_{q(j)}}{2}\left[\frac{c^{2(n)}_{s(j+1)}}{1 + c^{2(n)}_{s(j)}}\frac{\varepsilon^{(n)}_{q(j+1)}}{\varepsilon^{(n)}_{q(j)}} + \frac{c^{2(n)}_{s(j-1)}}{1 + c^{2(n)}_{s(j)}}\frac{\varepsilon^{(n)}_{q(j-1)}}{\varepsilon^{(n)}_{q(j)}}\right.\\ &\quad \left. - 2\frac{c^{2(n+1)}_{s(j)}}{1 + c^{2(n)}_{s(j)}}\frac{\varepsilon^{(n+1)}_{q(j)}}{\varepsilon^{(n)}_{q(j)}}\right]\\ &\quad - \frac{1}{2}\left[\frac{c^{2(n)}_{s(j+1)}}{1 + c^{2(n)}_{s(j)}}\frac{\varepsilon^{(n)}_{q(j+1)}}{\varepsilon^{(n)}_{q(j)}} - \frac{c^{2(n)}_{s(j-1)}}{1 + c^{2(n)}_{s(j)}}\frac{\varepsilon^{(n)}_{q(j-1)}}{\varepsilon^{(n)}_{q(j)}}\right]\frac{1}{\tau_n}\frac{\Delta\tau}{\Delta\eta}. \end{aligned} \tag{B9}$$

APPENDIX C: INADEQUACY OF THE SIMPLE EXPANSION IN $|q-1|$

From previous experience in applying q statistics to multiparticle production [10–15], we know that $|q-1|<1$. It seems then natural to argue that (see, for example, Refs. [11,50]) one could simply expand $f_q(x,p)=[1-(1-q)\frac{p^\mu u_\mu}{k_B T(x)}]^{1/(1-q)}$ from Eq. (14) in $z=1-q$, retaining only terms linear in z, and obtain

$$f_q(x,p)=f(z)=[1-zA]^{\frac{1}{z}}\equiv\left[\frac{1}{z}\ln(1-zA)\right] \simeq f(z=0)+z\frac{df(z)}{dz}\Big|_{z=0}, \tag{C1}$$

Here, $A=A(x,p)=\frac{p^\mu u_\mu(x)}{k_B T}$ and the arguments (x,p) are suppressed for clarity.

However, such expansion can only be performed under some conditions, which we shall clarify in what follows. It is straightforward to show that to get the first step of the expansion,

$$f(z)\simeq\exp\left[-A\left(1+\frac{A}{2}z+\cdots\right)\right] =\exp[-A]\exp\left[-\frac{A^2}{2}z-\cdots\right], \tag{C2}$$

it is necessary that

$$zA(x,p)<1. \tag{C3}$$

The second step needed is to additionally expand the exponent, and this requires

$$zA^2(x,p)<2. \tag{C4}$$

When this is satisfied, one finally gets $f_q(x,p)$ in terms of $f_{q=1}(x,p)$ only, i.e.,

$$f_q(x,p)\simeq f_{q=1}(x,p)+(1-q)\left[1-\frac{A^2(x,p)}{2}\right]f_{q=1}(x,p). \tag{C5}$$

At first this procedure looks very promising because using it one gets

$$\mathcal{T}_q^{\mu\nu}\equiv\mathcal{T}_{q=1}^{\mu\nu}+(q-1)\tau_q^{\mu\nu}, \tag{C6}$$

where $\mathcal{T}_{q=1}$ is the usual energy-momentum tensor for the equilibrium of the Boltzmann-Gibbs statistics, i.e., the one usually used when describing an ideal fluid,

$$\mathcal{T}_{q=1}^{\mu\nu}\equiv\frac{g}{(2\pi)^3}\int\frac{d^3p}{p^0}p^\mu p^\nu\exp\left(-\frac{pu}{T}\right), \tag{C7}$$

whereas the nonextensive correction tensor $\tau_q^{\mu\nu}$ is given by

$$\tau_q^{\mu\nu}\equiv\frac{g}{(2\pi)^3}\int\frac{d^3p}{p^0}p^\mu p^\nu \times\exp\left(-\frac{pu}{T}\right)\left[-\left(\frac{pu}{T}\right)+\frac{1}{2}\left(\frac{pu}{T}\right)^2\right]. \tag{C8}$$

However, in our case, condition (C4) would impose too severe constraints on the allowed q and the region of phase space, p and x, considered, thus rendering this approximation rather unpractical for our purposes.

APPENDIX D: DERIVATION OF EQUATIONS (71) AND η/s, EQUATION (73).

The entropy production is given by

$$\sigma^\mu_{\text{full};\mu}=[\xi\Psi^\mu]_{;\mu}=\frac{\partial_\mu\xi}{\xi}\xi\Psi^\mu+\xi\Psi^\mu_{;\mu}, \tag{D1}$$

where

$$\Psi^\mu=-\frac{W^\nu W_\nu}{3\Pi T}u^\mu+\frac{W_\nu}{2\Pi T}\pi^{\mu\nu} =\frac{-1}{6\Pi T}\{2W^\nu W_\nu u^\mu-3W_\nu\pi^{\mu\nu}\}. \tag{D2}$$

Then,

$$\Psi^\mu_{;\mu}=\left(\frac{\partial_\mu\Pi}{\Pi}+\frac{\partial_\mu T}{T}\right)\Psi^\mu+\frac{-1}{6\Pi T}\psi^\mu_{;\mu}, \tag{D3}$$

where $\psi^\mu\equiv 2W^\nu W_\nu u^\mu-3W_\nu\pi^{\mu\nu}$. The $\psi^\mu_{;\mu}$ is explicitly written as

$$\psi^\mu_{;\mu}=\{2W^\nu W_\nu u^\mu-3W_\nu\pi^{\mu\nu}\}_{;\mu} =\left[4W_{\nu;\mu}u^\mu+2W_\nu u^\mu_{;\mu}-3\pi^\mu_{\nu;\mu}\right]W^\nu+[-3W_{\nu;\mu}]\pi^{\mu\nu}, \tag{D4}$$

and

$$\frac{-1}{6\Pi T}\psi^\mu_{;\mu}=\frac{-1}{\Pi}\left[\frac{2}{3}W_{\nu;\mu}u^\mu+\frac{1}{3}W_\nu u^\mu_{;\mu}-\frac{1}{2}\pi^\mu_{\nu;\mu}\right]\frac{W^\nu}{T} +\frac{-1}{\Pi}\left[-\frac{1}{2}W_{\nu;\mu}\right]\frac{\pi^{\mu\nu}}{T}. \tag{D5}$$

Hence we obtain that

$$\begin{aligned}\xi\Psi^\mu_{;\mu}&=\xi\left(\frac{\partial_\mu\Pi}{\Pi}+\frac{\partial_\mu T}{T}\right)\Psi^\mu\\&-\frac{\xi}{\Pi}\left[\frac{2}{3}W_{\nu;\mu}u^\mu+\frac{1}{3}W_\nu u^\mu_{;\mu}-\frac{1}{2}\pi^\mu_{\nu;\mu}\right]\frac{W^\nu}{T}\\&-\frac{\xi}{\Pi}\left[-\frac{1}{2}W_{\nu;\mu}\right]\frac{\pi^{\mu\nu}}{T}\\&=\xi\left(\frac{\partial_\mu\Pi}{\Pi}+\frac{\partial_\mu T}{T}\right)\Psi^\mu-Y_\nu\frac{W^\nu}{T}+Z_{\mu\nu}\frac{\pi^{\mu\nu}}{T}.\end{aligned} \tag{D6}$$

Finally one arrives at

$$\begin{aligned}\sigma^\mu_{\text{full};\mu}&=[\xi\Psi^\mu]_{;\mu}\\&=\frac{\xi}{\Pi}\left(\frac{\partial_\mu\xi}{\xi}+\frac{\partial_\mu\Pi}{\Pi}+\frac{\partial_\mu T}{T}\right)\Pi\Psi^\mu\\&-Y_\nu\frac{W^\nu}{T}+Z_{\mu\nu}\frac{\pi^{\mu\nu}}{T}\\&=-X_\mu T\Psi^\mu\frac{\Pi}{T}-Y_\nu\frac{W^\nu}{T}+Z_{\mu\nu}\frac{\pi^{\mu\nu}}{T}\end{aligned}$$

$$= -X_\mu (wu^\mu)\frac{\Pi}{T} - \tilde{Y}_\nu \frac{W^\nu}{T} + Z_{\mu\nu}\frac{\pi^{\mu\nu}}{T}. \quad \text{(D7)}$$

This is Eq. (71). Using now $\pi^{\mu\nu}W_\nu = -2\Pi W^\mu$, one gets

$$\sigma^\mu_{\text{full};\mu} = -X_\mu (wu^\mu)\frac{\Pi}{T} - \tilde{Y}_\nu \frac{W^\nu}{T} + Z_{\mu\nu}\frac{\pi^{\mu\nu}}{T}$$

$$= -X_\mu (wu^\mu)\frac{\Pi}{T} + \left[\frac{\tilde{Y}_\nu W_\nu}{2\Pi T} + Z_{\mu\nu}\right]\frac{\pi^{\mu\nu}}{T}$$

$$= -X_\mu (wu^\mu)\frac{\Pi}{T} + \tilde{Z}_{\mu\nu}\frac{\pi^{\mu\nu}}{T}, \quad \text{(D8)}$$

which is Eq. (73).

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