

**Local approximations for polarization potentials**

W. H. Z. Cárdenas

*Instituto de Física, Universidade de São Paulo, C. P. 66318, 05389-970, São Paulo, Brazil*

M. S. Hussein

*Instituto de Física, Universidade de São Paulo, C. P. 66318, 05389-970, São Paulo, Brazil and**Max-Planck-Institut für Physik komplexer Systeme, Nöthnitzer Straße 38, D-01187 Dresden, Germany*

L. F. Canto

*Instituto de Física, Universidade Federal do Rio de Janeiro, C. P. 68528, 21941-972 Rio de Janeiro, Brazil*

J. Lubian

*Instituto de Física, Universidade Federal Fluminense, Av. Litorânea S-N, 24210-340 Niterói, RJ, Brazil*

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We discuss the derivation of an equivalent polarization potential independent of angular momentum  $l$  for use in the optical Schrödinger equation that describes the elastic scattering of heavy ions. Three different methods are used for this purpose. Application of our theory to the low energy scattering of light heavy-ion systems at near-barrier energies is made. It is found that the notion of an  $l$ -independent polarization potential has some validity but cannot be a good substitute for the  $l$ -dependent local equivalent Feshbach polarization potential.

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**I. INTRODUCTION**

The coupled-channels (CC) method is the most powerful tool for studying nuclear reactions. However, it becomes extremely complicated when it is necessary to include a large number of channels in the calculation. In situations where one is only interested in a detailed description of a single channel, e.g., in elastic scattering, one can resort to the polarization potential approach. It consists of deriving a potential, to be added to the Hamiltonian of the elastic channel which leads to the same elastic wave function as that obtained by solution of the CC equations. The natural framework for deriving this potential is the Feshbach formalism [1]. Although the exact derivation of the polarization potential may be as difficult as solving the CC problem, in some situations it is possible to find good approximations for it.

Several authors have derived approximate polarization potentials for collisions at near-barrier energies [2–4] and energies well above the barrier [5]. A serious drawback of these potentials is that they are nonlocal and dependent on both the energy  $E$  and the angular momentum  $l$ . Although any nonlocal potential can be replaced by a trivially equivalent local one, the latter presents poles and has an artificial dependence on the quantum numbers of the elastic wave function.

For practical purposes, it is convenient to have a local polarization potential independent of  $l$ , so that it could be used in standard computer codes. Several approaches have been proposed to achieve this goal. In a recent paper, Lubian and Nunes [6] tested the validity of the approach of Thompson *et al.* [7], in the case of breakup coupling, which is very important in collisions of weakly bound projectiles [8]. In the present paper, we extend this study to other procedures to derive  $l$ -independent potentials in the case of light heavy-ion scattering at near-barrier energies. For simplicity, we represent

the continuum by an effective bound channel, restricting ourselves to two channels. Although this is not an appropriate description of the continuum, it is suitable for the qualitative purposes of the present work, since this effective state also leads to a long-range polarization potential.

This paper is organized as follows. In Sec. II we give a brief description of the polarization potential, according to the Feshbach formalism. In Sec. III we discuss different prescriptions suggested to derive  $l$ -independent local potentials without poles. In Sec. IV we apply these prescriptions to  $^{11}\text{Li}+^{12}\text{C}$  scattering, taken to exemplify a case involving the coupling of the elastic channel to a very low  $Q$  value “inelastic” channel, and compare the results with the corresponding ones obtained with the CC method. Finally, in Sec. V we summarize the conclusions of this work.

**II. FESHBACH'S FORMALISM FOR POLARIZATION POTENTIALS**

Let us consider a scattering situation described in terms of the collision degrees of freedom (projectile-target separation vector), designated as  $\mathbf{r}$ , and a set of intrinsic degrees of freedom, represented by  $\xi$ . The scattering wave function  $\Psi^{(+)}(\mathbf{r}, \xi)$  satisfies a Schrödinger equation with the total Hamiltonian

$$\mathbb{H} = H + h + \mathbb{V}.$$

Above,  $H$  depends only on the collision degrees of freedom,  $h$  acts only on the intrinsic space, and  $\mathbb{V}$  couples collision with intrinsic degrees of freedom. It is convenient to perform the channel expansion of the scattering state in terms of the

eigenstates of  $h$ , which satisfy the equation

$$h |n\rangle = \varepsilon_n |n\rangle,$$

in the form

$$|\Psi^{(+)}\rangle = \sum_{\alpha=0}^n |\psi_{\alpha}^{(+)}\rangle |\alpha\rangle. \quad (1)$$

The intrinsic space can be divided into two complementary parts by the action of the projectors

$$P = |0\rangle\langle 0|; \quad Q = \sum_{\alpha=1}^n |\alpha\rangle\langle \alpha|. \quad (2)$$

Above,  $|0\rangle$  stands for the ground state of  $h$ , and we take  $\varepsilon_0 = 0$ . These projectors have the properties

$$P^2 = P, \quad Q^2 = Q, \quad QP = PQ = 0, \quad \text{and } P + Q = 1. \quad (3)$$

Acting with these projectors on the scattering state, one gets

$$P|\Psi^{(+)}\rangle = |\psi_0^{(+)}\rangle |0\rangle \equiv |\Psi_P\rangle, \quad (4)$$

$$Q|\Psi^{(+)}\rangle = \sum_{\alpha=1}^n |\psi_{\alpha}^{(+)}\rangle |\alpha\rangle \equiv |\Psi_Q\rangle, \quad (5)$$

$$|\Psi^{(+)}\rangle = |\Psi_P\rangle + |\Psi_Q\rangle. \quad (6)$$

Applying these projectors to the Schrödinger equation, using Eq. (6) and rearranging the terms, one obtains

$$[E - H]|\Psi_P\rangle = P\nabla Q|\Psi_Q\rangle. \quad (7)$$

$$[E - Q\mathbb{H}Q]|\Psi_Q\rangle = Q\nabla P|\Psi_P\rangle. \quad (8)$$

Note that  $PHQ = QHP = 0$ .

Equation (7) can be used to define the polarization potential operator, that is,

$$P\nabla Q|\Psi_Q\rangle = U^{\text{pol}}|\Psi_P\rangle. \quad (9)$$

This is the basis of the calculations of Refs. [6,9], as they employ a numerical solution of the CC equations and make use of the above definition. To derive an explicit expression for the polarization potential operator, one first derives an expression for the projected state  $|\Psi_Q\rangle$  by multiplying Eq. (8) from the left with the Green's function

$$\mathbb{G}_{QQ}^{(+)} = \frac{1}{E - Q\mathbb{H}Q + i\varepsilon}. \quad (10)$$

The result is

$$|\Psi_Q\rangle = \mathbb{G}_{QQ}^{(+)} Q\nabla P|\Psi_P\rangle. \quad (11)$$

Inserting this equation into Eq. (7), we get

$$[E - \mathbb{H}_{\text{eff}}]|\Psi_P\rangle = 0, \quad (12)$$

where  $\mathbb{H}_{\text{eff}}$  is the effective Hamiltonian

$$\mathbb{H}_{\text{eff}} = PHP + P\nabla Q\mathbb{G}_{QQ}^{(+)} Q\nabla P. \quad (13)$$

Taking the scalar product of Eq. (12) with  $\langle 0|$ , using the explicit form of  $P$  [Eq. (2)], and replacing  $\langle 0|\Psi^{(+)}\rangle = |\psi_0\rangle$ , we obtain the Schrödinger equation for the elastic wave function in the space of the collision degree of freedom

$$(K + U^{\text{opt}} + U^{\text{pol}})|\psi_0\rangle = E|\psi_0\rangle. \quad (14)$$

Above,  $K$  is the kinetic energy operation; and the optical and polarization potentials, respectively, are

$$U^{\text{opt}} = \langle 0|H - K|0\rangle, \quad (15)$$

$$U^{\text{pol}} = \langle 0|\nabla Q\mathbb{G}_{QQ}^{(+)} Q\nabla|0\rangle. \quad (16)$$

The above form for the polarization potential is numerically identical to the one defined by Eq. (9).

As a consequence of the analytical structure of Green's operator [the presence of  $i\varepsilon$  in Eq. (10)], we can immediately write

$$\text{Im}\{\mathbb{G}_{QQ}^{(+)}\} = -\pi\delta(E - Q\mathbb{H}Q), \quad (17)$$

while the real part is

$$\text{Re}\{\mathbb{G}_{QQ}^{(+)}\} = \mathcal{P} \left\{ \frac{1}{E - Q\mathbb{H}Q} \right\}. \quad (18)$$

This last equation can be rewritten as

$$\text{Re}\{\mathbb{G}_{QQ}^{(+)}\} = -\mathcal{P} \int dz \frac{\delta(z - Q\mathbb{H}Q)}{z - E},$$

which, with the help of Eq. (17), gives the desired result

$$\text{Re}\{\mathbb{G}_{QQ}^{(+)}(E)\} = \frac{1}{\pi} \mathcal{P} \int dz \frac{\text{Im}\{\mathbb{G}_{QQ}^{(+)}(z)\}}{z - E}. \quad (19)$$

When Eqs. (17) and (19) are used for Green's function in Eq. (10), we obtain the operator form of the dispersion relation,

$$\text{Re}\{U^{\text{pol}}(E)\} = \frac{1}{\pi} \mathcal{P} \int dz \frac{\text{Im}\{U(z)\}}{z - E}. \quad (20)$$

One would expect the calculated polarization potential to satisfy the above relation. Since one usually resorts to several approximations to derive a local  $l$ -independent polarization potential, the above relation may eventually be broken. However, it does supply an important check on the numerical evaluation of the polarization potential, just as unitarity and the  $S$  matrix do.

For practical purposes, it is convenient to write Eq. (14) in the coordinate representation. While the optical potential is generally taken to be local and energy dependent (owing to the effect of exchange nonlocality, removed to obtain a nondispersive local equivalent potential [10]), the nonlocality of  $\mathbb{G}_{QQ}^{(+)}$  leads to a nonlocal polarization potential. One obtains the equation

$$[K + U^{\text{opt}}(\mathbf{r})]\psi_0(\mathbf{r}) + \int U^{\text{pol}}(\mathbf{r}, \mathbf{r}')\psi_0(\mathbf{r}') d^3\mathbf{r}' = E\psi_0(\mathbf{r}), \quad (21)$$

with

$$U^{\text{pol}}(\mathbf{r}, \mathbf{r}') = \langle \mathbf{r}|[(\langle 0|\nabla Q\mathbb{G}_{QQ}^{(+)} Q\nabla|0\rangle)]|\mathbf{r}'\rangle. \quad (22)$$

For some applications, one replaces  $U^{\text{pol}}(\mathbf{r}, \mathbf{r}')$  by the trivially equivalent local potential,

$$\bar{U}^{\text{pol}}(\mathbf{r}) = \frac{1}{\psi_0(\mathbf{r})} \int d^3\mathbf{r}' U^{\text{pol}}(\mathbf{r}, \mathbf{r}')\psi_0(\mathbf{r}'). \quad (23)$$

This potential has some undesirable features. First, it has poles where  $\psi_0(\mathbf{r})$  vanishes. Second, this procedure introduces

artificial dependences on the quantum numbers of  $\psi_0$ . These points will be discussed in more detail in the next section.

### III. EXIT DOORWAY MODEL OF THE POLARIZATION POTENTIAL

We should mention that our general expression for  $U^{\text{pol}}(\mathbf{r}, \mathbf{r}')$  in Eq. (22) can account for the most general coupled-channels situation. In particular, in the continuum discretized coupled-channels (CDCC) calculation discussed in Ref. [6], the breakup continuum is discretized into an orthonormalized set of bins, which would then span the  $Q$  space. The polarization potential for this case, after writing for  $Q$ ,

$$Q = \sum_b |b\rangle\langle b|, \quad (24)$$

where  $|b\rangle$  designates the  $b$ th bin, and assuming that the coupling is local in the  $\mathbf{r}$  space, becomes

$$U^{\text{pol}}(\mathbf{r}, \mathbf{r}') = \sum_{bb'} \mathcal{F}_b(\mathbf{r}) \mathbb{G}_{bb'}^{(+)}(\mathbf{r}, \mathbf{r}') \mathcal{F}_{b'}(\mathbf{r}'), \quad (25)$$

where

$$\mathbb{G}_{bb'}^{(+)}(\mathbf{r}, \mathbf{r}') = \langle \mathbf{r} | \left[ |b\rangle \frac{1}{E - Q\mathbb{H}Q + i\epsilon} \langle b'| \right] | \mathbf{r}' \rangle, \quad (26)$$

and

$$\mathcal{F}_b(\mathbf{r}) = \langle 0 | \mathbb{V}(\mathbf{r}) | b \rangle, \quad (27)$$

with an analogous expression for  $\mathcal{F}_{b'}(\mathbf{r}')$ . If we neglect continuum-continuum coupling and the width of the bins, Green's function becomes diagonal, and Eq. (25) becomes

$$U^{\text{pol}}(\mathbf{r}, \mathbf{r}') = \sum_b \mathcal{F}_b(\mathbf{r}) \mathbb{G}_b^{(+)}(\mathbf{r}, \mathbf{r}') \mathcal{F}_b(\mathbf{r}'). \quad (28)$$

Clearly, both expressions for  $U^{\text{pol}}$  above are highly nonlocal by construction. In addition, the inclusion of continuum-continuum couplings produces more nonlocal effects [6]. It is easier to deal with a  $U^{\text{pol}}$  having the form of Eq. (28). How do we find an equivalent no-continuum-continuum-coupling polarization potential? To answer this question, we rely on our recent work on the excitation of giant resonances in heavy-ion reactions. One usually excites a given state, which itself is coupled to many other excited states. Using the exit doorway idea [9], namely, the excitation of these other states from the ground state proceeds from the exit doorway(s), one is bound to attach a width to the exit doorways. Labeling the doorway states by  $d$ , Green's function takes the diagonal form

$$U^{\text{pol}}(\mathbf{r}, \mathbf{r}') = \sum_d \mathcal{F}_d(\mathbf{r}) \mathbb{G}_d^{(+)}(\mathbf{r}, \mathbf{r}') \mathcal{F}_d(\mathbf{r}'). \quad (29)$$

The energies

$$\epsilon_d = \langle d | H | d \rangle$$

appearing in

$$\mathbb{G}_d^{(+)} = \frac{1}{E - \epsilon_d - \langle d | H | d \rangle + i\epsilon} \quad (30)$$

are complex. The width of any given  $d$  state measures the strength of the continuum-continuum coupling. The above expression for  $U^{\text{pol}}(\mathbf{r}, \mathbf{r}')$  with complex  $\epsilon_d$  should be a faithful representation the full Green's function *with* continuum-continuum coupling. In the following, we consider a much simpler two-channel case to discuss the notion of equivalent  $l$ -independent polarization potential, and we leave the discussion of the continuum-continuum case to a future publication. We mention at this point that an example of the use of the exit doorway is in the case of deuteron scattering. Such an exit doorway treatment of the elastic breakup of the deuteron has been reported earlier in Refs. [11,12], following a hitherto not-so-well-known treatment of the photodisintegration of the deuteron advanced by Schwinger [13]. In a future publication, we will employ the formalism above to treat the breakup coupling effect on the elastic scattering of loosely bound nuclei. We only mention here that the most important new feature which the exit doorway brings in is the complex  $Q$  value and the complex form factor.

### IV. LOCAL, $l$ -INDEPENDENT POLARIZATION POTENTIALS

One frequently performs angular momentum projections in the Schrödinger equation. For the simple case of a scalar Hamiltonian with scalar coupling, one makes the expansions

$$\psi_0^{(+)}(\mathbf{r}) = \sum_l \frac{u_l(kr)}{r} Y_{lm}(\hat{\mathbf{r}}), \quad (31a)$$

$$U^{\text{pol}}(\mathbf{r}, \mathbf{r}') = \frac{1}{rr'} \sum_l Y_{lm}(\hat{\mathbf{r}}) U_l^{\text{pol}}(r, r') Y_{lm}(\hat{\mathbf{r}}'). \quad (31b)$$

The angular momentum projected version of Eq. (23) is the local equivalent polarization potential

$$U_l^{\text{pol}}(r) = \frac{1}{u_l(kr)} \int dr' U_l^{\text{pol}}(r, r') u_l(kr'). \quad (32)$$

This potential should be included in the Hamiltonian for optical model calculations of the elastic radial wave function  $u_l(r)$ . In this way, one needs a different polarization potential for each partial wave.

The polarization potential of Eq. (32) is not very useful, since it requires the knowledge of the exact radial wave function in the elastic channel,  $u_l(kr)$ . Determining  $u_l(kr)$  is as hard as solving the original CC equations. A rather widely used approximation for the  $l$ -dependent local polarization potential consists of replacing  $u_l(kr)$  by the optical radial wave function,  $w_l(kr)$ , which is the solution of the partial-wave projected Schrödinger equation with the polarization potential switched off. This potential, denoted by  $\bar{U}_l$ , is

$$\bar{U}_l(r) = \frac{1}{w_l(kr)} \int dr' U_l^{\text{pol}}(r, r') w_l(kr'). \quad (33)$$

A further inconvenience of trivially local equivalent potentials is that they have poles wherever the radial wave function vanishes. We next discuss three different prescriptions for obtaining  $l$ -independent polarization potentials free of poles.

### A. Prescription of Thompson *et al.*

Thompson *et al.* [7] proposed the following definition of an  $l$ -independent version of the polarization potential:

$$U^T(r) = \frac{\sum_l (2l+1) T_l |u_l(kr)|^2 U_l^{\text{pol}}(r)}{\sum_l (2l+1) T_l |u_l(kr)|^2}, \quad (34)$$

where  $T_l$  is the transmission coefficient in the elastic channel for the  $l$ th partial wave. The above definition of  $U^T$  guarantees that no poles remain in the polarization potential, which arise from the presence of  $u_l(r)$  in Eq. (34). Furthermore, owing to the presence of  $T_l$  in the  $l$  sum, only values of  $l$  where  $T_l$  is close to unity will contribute. Of course the probability density  $|u_l(r)|^2$  is small for small values of  $l$  because of absorption. Thus, in the prescription of Thompson *et al.*, Eq. (34) should contain contributions of  $l$  in the vicinity of the grazing one. We doubt that this is guaranteed always, since there are interference effects in  $|u_l(r)|^2$  that may end up allowing the contribution of small  $l$  as well. This prescription was recently used in Ref. [6] in the context of the CDCC calculation of breakup and elastic scattering of  $^8\text{B}$ .

### B. Modified form of the prescription of Thompson *et al.*

We now introduce a slightly modified version of the Thompson prescription, by substituting  $T_l$  by its derivative with respect to  $l$ , namely,

$$U^{\text{MT}}(r) = \frac{\sum_l (2l+1) (dT_l/dl) |u_l(kr)|^2 U_l^{\text{pol}}(r)}{\sum_l (2l+1) (dT_l/dl) |u_l(kr)|^2}. \quad (35)$$

This modified Thompson prescription (MT) guarantees the contribution of the  $l$  values around the grazing one, regardless of the behavior of  $|u_l(r)|^2$ . One physical motivation for choosing  $dT_l/dl$  instead of  $T_l$  is that the distorted-wave Born approximation amplitudes of nonelastic processes in the adiabatic limit do behave as the  $l$  derivative of the elastic  $S$ -matrix elements, which enters into the definition of  $T_l = 1 - |S_l|^2$ .

### C. Semiclassical prescription

This prescription relies on the semiclassical idea that the orbital angular momentum, if treated classically, should be related to  $r$  and energy  $E$  through the definition of the classical turning point  $r_t$ , namely,

$$\frac{\hbar^2}{2\mu r_t^2} l(l+1) + U(r_t) = E, \quad (36)$$

where  $U$  is the real part of the optical potential containing the nuclear and Coulomb pieces. It is clear that for a given collision energy,  $r_t \equiv r_t(l)$  is a function of  $l$ . The prescription consists of identifying  $r \equiv r_t(l)$  in order to build the  $l$ -independent polarization potential,

$$U^{\text{SC}}(r) = \bar{U}_l^{\text{pol}}(r_t(l)). \quad (37)$$

Using Eq. (37), one gets the potential  $U^{\text{SC}}$  at a discrete set of  $r$  values (one for each partial wave); and interpolating between these points, one obtains a continuous function.

All three  $l$ -independent potentials discussed above contain a further energy dependence besides the one that arises from Green's function in the polarization potential. This extra energy dependence is nondispersive and thus could render the applicability of the dispersion relation questionable. We shall verify this point in the following section.

In all these prescriptions, the starting point is the trivially equivalent local potential obtained from angular momentum projected versions of Eqs. (22) and (23) or from the solution of the CC equations used in Eq. (9). This will be shown in detail in the next section.

## V. APPLICATION TO LIGHT HEAVY-ION SCATTERING WITH LOW $Q$ VALUE

We have chosen the system  $^{11}\text{Li} + ^{12}\text{C}$  to test the different local approximations for the polarization potential discussed in the previous sections. We have decided not to consider the coupling to the continuum explicitly, but rather to consider the coupling to a single bound effective channel with a very small  $Q$  value. The justification for considering this schematic application is that our purpose in this paper is to compare different approximations for the polarization potential and not to perform quantitative calculations of elastic or fusion cross sections.

The two CC equations used in our calculation can be easily read out from Eqs. (7) and (8). We denote the two intrinsic states by  $|0\rangle$  (ground state) and  $|1\rangle$  (excited state). For simplicity, the numerical calculations are performed in the sudden limit. In this limit, the excitation energy of the continuum states are neglected ( $\varepsilon_0 = \varepsilon_1 = 0$ ). The projectors are  $P = |0\rangle\langle 0|$  and  $Q = |1\rangle\langle 1|$  and, after taking the optical potentials in the two channels to be the same, Eqs. (7) and (8) become

$$[E - K - U^{\text{opt}}(\mathbf{r})]\Psi_0(\mathbf{r}) = \mathcal{F}(\mathbf{r})\Psi_1(\mathbf{r}), \quad (38)$$

$$[E - K - U^{\text{opt}}(\mathbf{r})]\Psi_1(\mathbf{r}) = \mathcal{F}(\mathbf{r})\Psi_0(\mathbf{r}). \quad (39)$$

Above,  $\mathcal{F}(\mathbf{r})$  is the complex and symmetric form factor

$$\mathcal{F}(\mathbf{r}) = \int d^3\mathbf{x} \varphi_0(\mathbf{x}) \nabla(\mathbf{r}, \mathbf{x}) \varphi_1(\mathbf{x}), \quad (40)$$

evaluated with the coupling interaction

$$\nabla(\mathbf{r}, \mathbf{x}) = U_{f_1 T}(\mathbf{r}_{f_1 T}) + U_{f_2 T}(\mathbf{r}_{f_2 T}) - U^{\text{opt}}(\mathbf{r}).$$

Since the intrinsic states are orthogonal and  $U^{\text{opt}}(\mathbf{r})$  does not depend on  $\mathbf{x}$ , Eq. (40) reduces to

$$\mathcal{F}(\mathbf{r}) = \int d^3\mathbf{x} \varphi_0(\mathbf{x}) [U_{f_1 T}(\mathbf{r}_{f_1 T}) + U_{f_2 T}(\mathbf{r}_{f_2 T})] \varphi_1(\mathbf{x}). \quad (41)$$

For the present application, we assume that  $^{11}\text{Li}$  breaks up into two fragments. The first,  $f_1$ , corresponds to a neutron pair, which we treat as a single particle (dineutron). The other,  $f_2$ , is the  $^9\text{Li}$  core. In this way, the coordinates appearing in Eqs. (40) and (41) are

$$\begin{aligned} \mathbf{r}_{f_1 T} &= \mathbf{r} + \gamma_1 \mathbf{x}, & \gamma_1 &= \frac{9}{11}, \\ \mathbf{r}_{f_2 T} &= \mathbf{r} + \gamma_2 \mathbf{x}, & \gamma_2 &= -\frac{2}{11}, \end{aligned}$$

and the projectile-target separation vector  $\mathbf{r}$ .  $U_{f_1T}(\mathbf{r}_{f_1T})$ ,  $U_{f_2T}(\mathbf{r}_{f_2T})$ , and  $U^{\text{opt}}(\mathbf{r})$  are the corresponding interactions. Of course  $\mathcal{F}(\mathbf{r})$  can be evaluated exactly numerically once these potentials are given, and the single-particle wave functions of the halo neutron in the ground and in the excited state,  $\varphi_0(\mathbf{x}) = \langle \mathbf{x} | 0 \rangle$  and  $\varphi_1(\mathbf{x}) = \langle \mathbf{x} | 1 \rangle$ , are used to evaluate the integral in Eq. (40). For our purposes, we parametrize  $\mathcal{F}(\mathbf{r})$  as [14]

$$\mathcal{F}(\mathbf{r}) = \mathcal{F}_0 \exp\left[-\frac{r}{\gamma_1 \alpha}\right], \quad (42)$$

with

$$\alpha = \frac{\hbar}{\sqrt{2\mu_{1-2}B}}. \quad (43)$$

Above,  $\mu_{1-2}$  is the reduced mass of the fragments inside the projectile and  $B$  is the breakup threshold. For  $^{11}\text{Li}$ ,  $\mu_{1-2} = 18m_0/11$ ,  $B = 0.376$  MeV, and one gets  $\alpha = 5.83$  fm. For the purpose of simplicity, we take  $\mathcal{F}_0$  to be real. We also ignore the Coulomb coupling altogether. A qualitative justification for the approximation of Eq. (42) is given in the Appendix.

The optical potentials can be written as

$$U^{\text{opt}}(\mathbf{r}) = U_C(r) + U_N(r),$$

where

$$U_C(r) = \begin{cases} \frac{Z_P Z_T e^2}{2R_c} \left(3 - \frac{r^2}{R_c^2}\right), & \text{for } r < R_c, \\ \frac{Z_P Z_T e^2}{r}, & \text{for } r \geq R_c, \end{cases}$$

and

$$U_N(r) = \frac{-V_0}{1 + \exp[(r - R_r)/a_r]} + \frac{-W_0}{1 + \exp[(r - R_i)/a_i]}.$$

Above,  $R_C = r_{0C}(A_P^{1/3} + A_T^{1/3})$ ,  $R_r = r_{0r}(A_P^{1/3} + A_T^{1/3})$ , and  $R_i = r_{0i}(A_P^{1/3} + A_T^{1/3})$ . In the above equations, we use typical values for the parameters:

$$\begin{aligned} \mathcal{F}_0 &= 3.0 \text{ MeV}, & r_{0C} &= 1.4 \text{ fm}, \\ V_0 &= 60 \text{ MeV}, & W_0 &= 60 \text{ MeV}, \\ r_{0r} &= 1.25 \text{ fm}, & r_{0i} &= 1.00 \text{ fm}, \\ a_r &= 0.60 \text{ fm}, & a_i &= 0.60 \text{ fm}. \end{aligned} \quad (44)$$

The reduction of these equations to obtain the polarization potential proceeds as in the previous section, and we get for the Schrödinger equation for  $\psi_0(\mathbf{r})$  the following:

$$[E - K - U^{\text{opt}}(r) - U^{\text{pol}}(\mathbf{r})]\psi_0(\mathbf{r}) = 0, \quad (45)$$

with

$$U^{\text{pol}}(\mathbf{r}, \mathbf{r}') = \mathcal{F}(\mathbf{r}) G_1^{(+)}(\mathbf{r}, \mathbf{r}') \mathcal{F}(\mathbf{r}'). \quad (46)$$

To perform numerical calculations, it is convenient to carry out the usual partial-wave expansions. Since we are using scalar form factors, the partial-wave projected polarization potential is given by

$$U_l^{\text{pol}}(r, r') = \mathcal{F}(r) g_{1,l}^{(+)}(r, r') \mathcal{F}(r'). \quad (47)$$

The partial-wave projected optical Green's function in channel 1 can be written as [15]

$$g_{1,l}^{(+)}(r, r') = -\frac{2\mu}{\hbar^2 k_1} e^{-i\delta_l} w_l(k_1 r_{<}) \mathcal{H}_l^{(+)}(k_1 r_{>}). \quad (48)$$

Above,  $\mu$  is the reduced mass of the projectile-target system,  $k_1 = \sqrt{2\mu E_1}/\hbar$ ,  $r_{<}(r_{>})$  is the smaller (larger) of the radial separations  $r$  and  $r'$ ,  $w_l(k_1 r)$  is the regular solution of the optical Schrödinger equation [partial-wave projected Eq. (39) setting  $\mathcal{F}(\mathbf{r}) = 0$ ] with the asymptotic form

$$w_l(k_1 r \rightarrow \infty) = \frac{i}{2} [H_l^{(-)}(k_1 r) - \bar{S}_l H_l^{(-)}(k_1 r)], \quad (49)$$

and  $\delta_l$  is the nuclear phase shift. Here,  $H_l^{(-)}(H_l^{(+)})$  is the Coulomb wave function with ingoing (outgoing) boundary condition, and  $\bar{S}_l = \exp(2i\delta_l)$  is the nuclear  $S$  matrix at the  $l$ th partial wave. In Eq. (48),  $\mathcal{H}_l^{(+)}(k_1 r)$  is the solution of the same optical Schrödinger equation, but with a different asymptotic behavior. At large separations, it is the outgoing wave

$$\mathcal{H}_l^{(+)}(k_1 r \rightarrow \infty) = e^{i\delta_l} H_l^{(+)}(k_1 r_\infty).$$

To determine  $\mathcal{H}_l^{(+)}$  at finite projectile-target separations, the radial equation must be numerically integrated inward, starting from a large  $r$  value where the above asymptotic form is valid. This can be easily achieved using a negative mesh step in any conventional code for numerical integration.

The dispersion relation is clearly satisfied by Eq. (47). But is it satisfied by the  $l$ -dependent potential  $\bar{U}_l$  [inserting Eq. (47) into Eq. (32)]? We recall that the expression used to find  $\bar{U}_l$  is Eq. (33). One would expect the dispersion relation to be satisfied by  $\bar{U}_l$  only if  $u_l(kr)$  is real, which is certainly not the case. However, a remnant of the dispersion relation should still be seen in  $\bar{U}_l$  as has been shown over and over again in the study of the threshold anomaly [16]. Notwithstanding the above reservations, we will give below an account of our calculation of the equivalent  $l$ -independent potential.

In Fig. 1 we show polarization potentials at three collision energies: (a)  $E = 2$  MeV, (b)  $E = 4$  MeV, and (c)  $E = 6$  MeV. We present results obtained using the Thompson (T) prescription [Eq. (34)], the modified Thompson (MT) one [Eq. (35)], and the semiclassical (SC) prescription [Eq. (37)]. In the cases of the T and the MT potentials, we evaluate the radial wave functions and the transmission coefficients solving the CC equations and then carry out the  $l$ -averages of Eqs. (34) and (35). Also shown are the approximate  $l$ -dependent potential  $\bar{U}_l$  (Eq. (33)) for  $l = 0$  and  $l = 10$  and the optical potential employed in our two coupled channels. The singularities of  $\bar{U}_l$ , arising from the nodes of the radial wave functions, are treated by the technique developed in Ref. [17]. We see clearly that whereas the  $\bar{U}_{l=0}$  is quite oscillatory, the T, MT and SC ones seem to behave smoothly in so far as the real part of the potential is concerned. Oscillations in  $\bar{U}_l$  occur near the nodes of  $w_l$ . Since this wave function is complex and their real and imaginary parts do not vanish simultaneously, the polarization potential remains finite.

In Fig. 2 we show the results for the elastic scattering angular distributions at  $E = 2, 4$ , and 6 MeV. The coupled-channels results are shown as the full circles. None of the

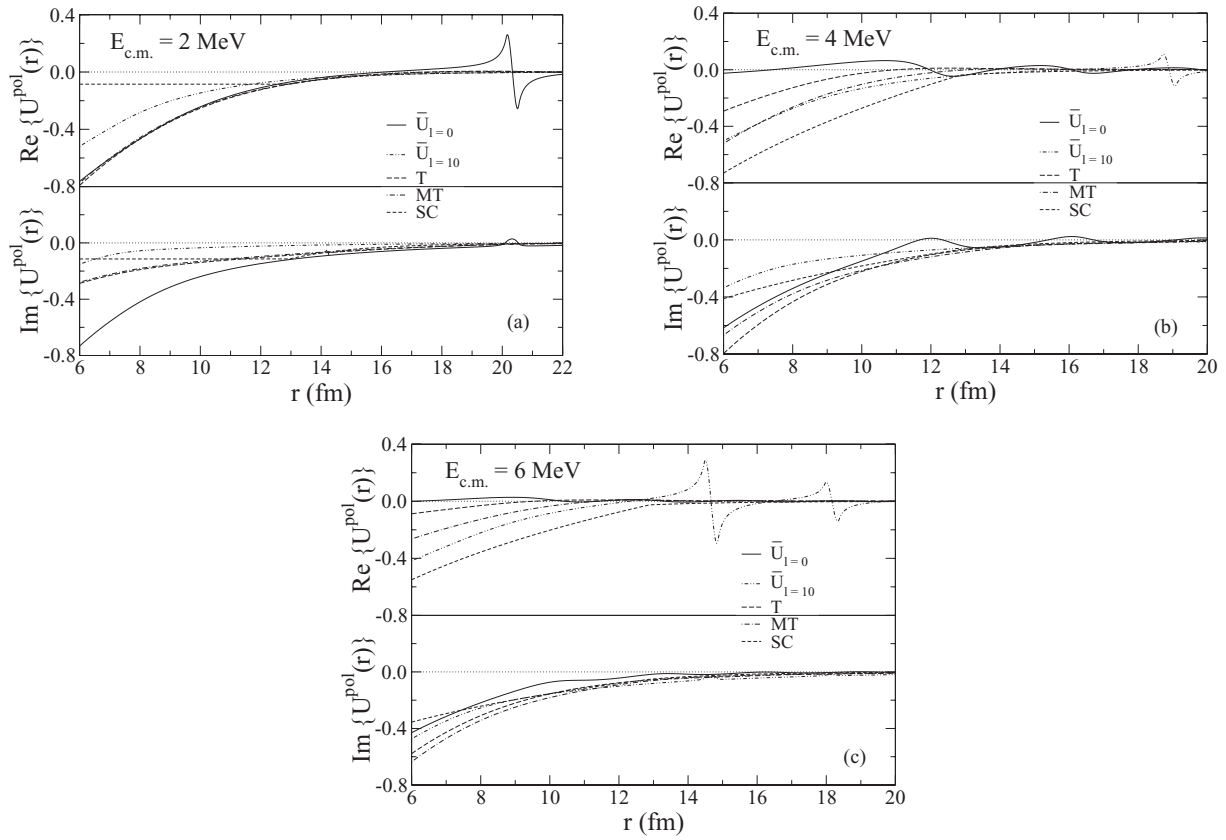


FIG. 1. Real and imaginary parts of the polarization potentials (in MeV) at three collision energies for the system  $^{11}\text{Li}+^{12}\text{C}$ . Results were calculated using the Thompson (T), modified Thompson (MT), and semiclassical (SC) recipes, and the  $l$ -dependent potentials for  $l = 0$  and  $l = 10$ . The coupling strength  $\mathcal{F}_0$  was taken to be 3 MeV.

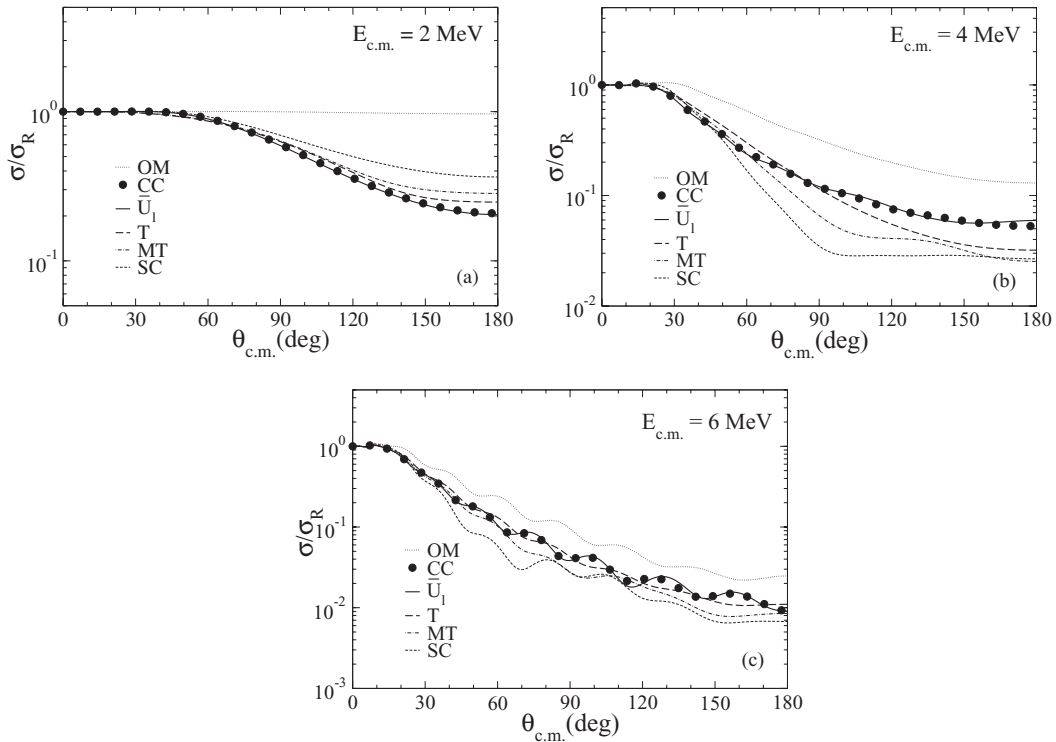


FIG. 2. Ratio of the elastic angular distribution to Rutherford calculated with CC (coupled channels), OM (optical model, no coupling), and with the  $l$ -dependent polarization potential,  $\bar{U}_l$ . The other symbols are the same as in Fig. 1.

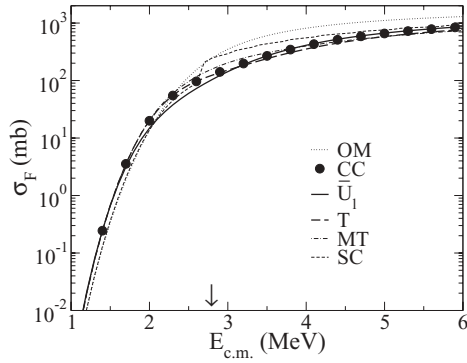


FIG. 3. Fusion excitation function for the same cases as in Fig. 2.

$l$ -independent polarization potentials seem to work very well. However, the T potential is better than the others. It is close to the CC results, except for the collision energy of 6 MeV. In that case, the angular distribution obtained with this potential oscillates out of phase with respect to the corresponding CC results. On the other hand, the approximate  $l$ -dependent potential  $\bar{U}_l$  reproduces accurately the CC results at the three collision energies.

In Fig. 3 we show our results for the fusion cross section excitation function, obtained by the relation

$$\sigma_F(E) = \frac{k}{E} \langle \psi_0 | -\text{Im}\{U^{\text{opt}}\} | \psi_0 \rangle.$$

The wave function  $\psi_0$  was calculated solving the Schrödinger equation for the elastic channel, including the optical and each of the above-discussed polarization potentials. Here, both the  $\bar{U}_l$  and T potentials seem to work rather well over the energy range considered. The SC overshoots in the barrier region, but otherwise it reproduces the CC calculation. The MT seems to be quite off; however, on the average it works well too.

Finally, to be sure about the consistency of our calculation, we checked the dispersion relation. We fixed  $r = 12$  fm and plotted in Fig. 4 the resulting behavior of the real and imaginary parts of the polarization potential vs  $E$ . It is clear that the

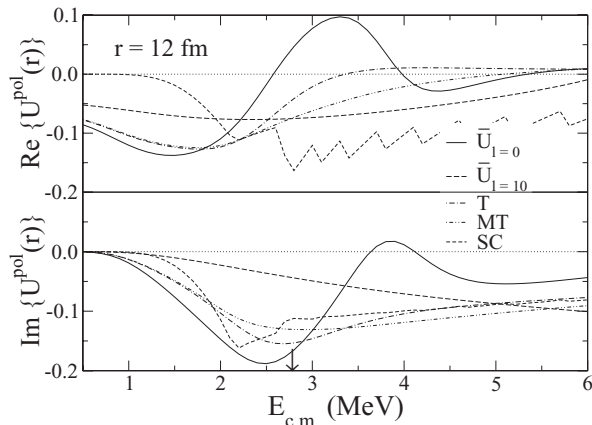


FIG. 4. Energy dependence of the real and imaginary parts of the polarization potential (in MeV), indicating the trend dictated by the dispersion relation.

$\bar{U}_l$  for  $l = 0$  shows the general trend of what one would expect from the dispersion relation: maxima in the real part accompanied by sharp variations in the imaginary part. Similar behavior is found for the T potential. Note the unphysical sharp oscillations of the SC potentials. They arise from the fact that this potential can only be evaluated over a sparse mesh of  $r$  values, corresponding to the turning points at each angular momentum. Thus the T and  $\bar{U}_l$  potentials exhibit the threshold anomaly [16]. In particular, the T potential shows a maximum in the imaginary part at  $E = 3.9$  MeV, whereas the real part shows a maximum at  $E = 3.5$  MeV. This behavior of the T potential is in line with the breakup threshold anomaly of Ref. [18].

## VI. CONCLUSIONS

In this work, we have investigated  $l$ -independent polarization potentials in a schematic two-channel model, in which the range of the polarization potentials is equivalent to that associated with the breakup channel. We used several prescriptions to derive the  $l$ -independent polarization potential and found that none could reproduce satisfactorily the results of coupled-channels calculations. This conclusion is consistent with that of realistic CDCC calculations [6] using the Thompson prescription. On the other hand, we have shown that an approximate  $l$ -dependent potential, obtained using the unperturbed Green's function in Feshbach theory, gave reasonable descriptions of elastic angular distribution at energies above the barrier.

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## APPENDIX

We give below, a qualitative justification of the adopted parametrization of  $\mathcal{F}(r)$ . Since the diffusivity of the potentials is very small as compared to  $\alpha$ ,  $R_1$ , and  $R_2$ , we set  $a_r \simeq 0$ . In this way, the Woods-Saxon potentials take the forms of the step functions,

$$U_{fT}(r_{fT}) \simeq -V_0 \Theta(|\mathbf{r} + \gamma \mathbf{x}| - R_T), \quad (\text{A1})$$

where  $f = f_1$  or  $f_2$ , and  $\gamma = \gamma_1$  or  $\gamma_2$ .

At large projectile-target separations, that is,  $r > R_P + R_T$ , only  $x > R_P$  contributes to the integral of Eq. (41). Since we are neglecting angular momenta, the ground state wave function in this range depends only on the radial and has an exponential form. That is,

$$\varphi_0(x) \propto \exp\left(-\frac{x}{\alpha}\right),$$

with  $\alpha$  given by Eq. (43). Since we are adopting the sudden approximation, the wave number of the state  $\varphi_1$  is supposed to be vanishingly small. Therefore, this wave function is constant within the integral. The contribution from the potential to the form factor takes the form

$$\mathcal{F}(r) \propto \int_0^\infty dx \exp\left(-\frac{x}{\alpha}\right) \int_{-1}^1 \Theta(|\mathbf{r} + \gamma \mathbf{x}| - R_T) dt, \quad (\text{A2})$$

where  $t$  stands for the cosine of the angle between the  $\mathbf{x}$  and  $\mathbf{r}$  vectors. The main contributions to this integral comes from  $\gamma \mathbf{x}$  antiparallel to  $\mathbf{r}$ . We then replace

$$\Theta(|\mathbf{r} + \gamma \mathbf{x}| - R_T) \rightarrow \Theta(r - |\gamma|x - R_T)$$

and assume that the integration over  $t$  does not depend strongly on  $r$ , leading only to a renormalization of the strength of the form factor. The integral of Eq. (A2) then becomes

$$\mathcal{F}(r) \propto \int_{(r-R_T)/|\gamma|}^\infty dx \exp\left(-\frac{x}{\alpha}\right) \propto \exp\left[-\frac{r}{|\gamma|\alpha}\right].$$

Of course, the form factor will be dominated by the contribution from the dineutron, which has a longer range ( $|\gamma_1||\gamma_2|$ ). Denoting by  $\mathcal{F}_0$  the constant of proportionality, we can write

$$\mathcal{F}(r) \simeq \mathcal{F}_0 \exp\left(-\frac{r}{\gamma_1\alpha}\right),$$

which is adopted in the present calculation.

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