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Number of spin I states for bosons

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We study number of spin I states for bosons in this article. We extend Talmi's recursion formulas for number of states with given spin I to boson systems, and we prove empirical formulas for five bosons by using these recursions. We obtain number of states with given spin I and F spin for three and four bosons by using sum rules of six-j and nine-j symbols. We also present empirical formulas of states for d bosons with given spin I and $F = F_{\max} - 1$ and $F_{\max} - 2$.

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I. INTRODUCTION

Recently there have been many efforts to obtain simple formulas of enumerating number of states with given spin. In Ref. [1], Ginocchio and Haxton obtained a simple formula of spin-zero states for four particles. In Ref. [2] Zamick and Escuderos gave a much simpler proof for dimension of spin-zero states of the j^4 configuration. In Ref. [3], two of present authors, Zhao and Arima, obtained empirical formulas for given spin I states with particle number $n = 3$ and 4 and some for $n = 5$. In Ref. [4], Talmi developed recursion relations for $n, n-1$, and $n-2$ fermions and proved results of Ref. [3] for three fermions. In Ref. [5], we found a simple correspondence between number of given spin states of fermions and that of bosons and proved results of Ref. [3] for $n = 4$ by using reduction rule of d bosons. In Ref. [6], Zamick and Escuderos derived an interesting relation between dimension for isospin $T = 0$ and spin I states and that for isospin $T = 2$ and spin I states. In Ref. [7], formulas of dimension with given spin and isospin for three and four nucleons are derived by using sum rules of six-j and nine-j symbols of Refs. [8,9]. However, most studies concentrated on fermions, it is therefore interesting to study boson systems as well. The purpose of this article is to present formulas for spin-l bosons that have not been extensively discussed in previous studies.

This article is organized as follows. In Sec. II we extend Talmi's recursions to boson systems and apply it to prove empirical results for $n = 5$ in Ref. [3]. In Sec. III we present number of states with given spin I and isospin F for three and four bosons, by using sum rules of six-j and nine-j symbols derived in Ref. [8]. In Sec. IV we study number of d states with given spin and F spin, introduced by Arima in Ref. [10]. Although we do not know how to prove them, the significance here is that d bosons are basic building blocks of the IBM, and F spin is a very relevant and well-conserved quantum number for medium and heavy nuclei. In Sec. V we summarize this article. In Appendix we present number of spin I states for d bosons with an odd number. Those with an even number correspond to those of four fermions and bosons, and those of d bosons with an odd number correspond to fictitious systems (but mathematically useful), according to our earlier articles [5,9].

II. NUMBER OF STATES FOR FIVE BOSONS

In this section, we use notations of Talmi's article [4] for bosons. We denote z-axis projection of total spin I of n spin-l bosons by $M = m_1 + m_2 + \cdots + m_n$ and the number of states with given M in the l^n configuration by $N(M, l, n)$. The number of states with given value of I in the l^n configuration will be denoted $D(I, l, n)$.

Similar to Talmi's procedure of Ref. [4], there are states where $m_1 < l$ and $m_n \geqslant -l$. For $m_n > -l$, the number of states with given M is $N(M, l-1, n)$, and for $m_n = -l$, one should consider $m_{n-1} \geqslant -l$. In the case of $m_{n-1} > -l$, the number of states with given M projection equals that of $(n-1)$ bosons with z-axis projection $M + l$, which is $N(M + l, l-1, n-1)$; and for $m_{n-1} = -l$, one again considers $m_{n-2} \geqslant -l$. In the case of $m_{n-2} > -l$, the number of l^n states with projection M is given by $N(M + 2l, l - 1, n-2)$; for $m_{n-2} = -l$, one should continue to consider

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$m_{n-3} \geqslant -l, \cdots$. For $m_2 > -l$, the number of states for two bosons is $N[M+(n-2)l, l-1, 2]$, and for $m_2 = -l$ this number is $N[M+(n-1)l, l-1, 1]$.

Then for $m_1 < l$, the sum of number of states with z-axis projection M for n spin-l bosons is given by

$$\begin{aligned} &N(M, l-1, n) + N(M+l, l-1, n-1) \\ &\quad + N(M+2l, l-1, n-2) + \cdots \\ &\quad + N[M+(n-2)l, l-1, 2] \\ &\quad + N[M+(n-1)l, l-1, 1]; \end{aligned}$$

and for $m_1 = l$, the sum of number of states with z-axis projection M for n spin-l bosons is given by

$$\begin{aligned} &N(M, l, n-2) + N(M-l, l-1, n-1) \\ &\quad + N(M-2l, l-1, n-2) + \cdots \\ &\quad + N[M-(n-2)l, l-1, 2] \\ &\quad + N[M-(n-1)l, l-1, 1]. \end{aligned}$$

The total number of states with given M of the l^n configuration is

$$\begin{aligned} N(M, l, n) = {} & N(M, l-1, n) + N(M+l, l-1, n-1) \\ & + N(M+2l, l-1, n-2) + \cdots \\ & + N[M+(n-2)l, l-1, 2] \\ & + N[M+(n-1)l, l-1, 1] \\ & + N(M, l, n-2) + N(M-l, l-1, n-1) \\ & + N(M-2l, l-1, n-2) + \cdots \\ & + N[M-(n-2)l, l-1, 2] \\ & + N[M-(n-1)l, l-1, 1]. \end{aligned} \tag{1}$$

If $M \geqslant 0$, one has

$$D(I, l, n) = N(M = I, l, n) - N(M = I+1, l, n). \tag{2}$$

If $M < 0$ and $M + 1 \leqslant 0$

$$\begin{aligned} D(I-1, l, n) = {} & N[M = -(I-1), l, n] \\ & - N(M = -I, l, n). \end{aligned} \tag{3}$$

One has following recursion relations for bosons. For $I \leqslant l-1$,

$$\begin{aligned} D(I, l, n) = {} & D(I, l, n-2) + D(I, l-1, n) \\ & + D(I+l, l-1, n-1) \\ & + D(I+2l, l-1, n-2) + \cdots \\ & + D[I+(n-2)l, l-1, 2] \\ & + D[I+(n-1)l, l-1, 1] \\ & - D(l-1-I, l-1, n-1) \\ & - D(2l-1-I, l-1, n-2) - \cdots \\ & - D[(n-2)l-1-I, l-1, 2] \\ & - D[(n-1)l-1-I, l-1, 1]. \end{aligned} \tag{4}$$

For $I = 1$ and $n = 5$, we obtain

$$\begin{aligned} D(1, l, 5) = {} & D(1, l, 3) + D(1, l-1, 5) \\ & + D(l+1, l-1, 4) + D(2l+1, l-1, 3) \\ & - D(l-2, l-1, 4) - D(2l-2, l-1, 3). \end{aligned} \tag{5}$$

In Ref. [3], an empirical formula for $I = 1$ and $n = 5$ was given by $D(1, l, 5) = (Q+1)(Q+1+q)$, where

$$\begin{cases} Q = \left[\frac{l}{4}\right], & q = (l \bmod 4 - 1)/2, \quad \text{if } l \bmod 2 = 1, \\ Q = \left[\frac{l-3}{4}\right], & q = [(l-3) \bmod 4 - 1]/2, \quad \text{if } l \bmod 2 = 0, \end{cases}$$

and [ ] means to take the largest integer not exceeding the value inside.

Now we prove the formula of $D(1, l, 5)$ by induction with respect to l, namely we assume that it holds for spin $l-1$ bosons and prove it holds also for spin l bosons (it was shown to hold for lower spins up to $l = 99$ in Ref. [3]). For convenience, we first take cases with even $l = 6k$ (k is an odd number here; cases with even k can be shown similarly). Cases of other even $l = 6k+2$ and $6k+4$ can be solved in the same way. We also note without details that one can repeat this process while proving the formula of $D(1, l, 5)$ in Ref. [3] for odd l and that the formula of $I = 0$ and $n = 5$ for spin-l bosons can be proved via the same procedure.

Using Eq. (1) of Ref. [3], we obtain

$$\begin{aligned} D(1, 6k, 3) &= 0, \\ D(12k+1, 6k-1, 3) &= k, \\ D(12k-2, 6k-1, 3) &= k. \end{aligned} \tag{6}$$

Using Eq. (5) of Ref. [3], we obtain

$$\begin{aligned} D(6k+1, 6k-1, 4) &= 3k^2 - k + 3\left[\frac{k}{2}\right]^2 + 4\left[\frac{k}{2}\right] + 1, \\ D(6k-2, 6k-1, 4) &= 3k^2 - k + 3\left[\frac{k}{2}\right]^2 + 7\left[\frac{k}{2}\right] + 3. \end{aligned} \tag{7}$$

Here we used following identities: for odd k, $[\frac{6k-1}{3}] = 2k-1$, $[\frac{k-1}{2}] = [\frac{k}{2}]$, $[\frac{6k+2}{4}] = 3[\frac{k}{2}] + 2$, $(6k-1) \bmod 3 = 2$, and $(k-1) \bmod 2 = 0$. According to our assumption,

$$\begin{aligned} D(1, 6k-1, 5) &= \left(\left[\frac{6k-1}{4}\right] + 1\right)\left(\left[\frac{6k-1}{4}\right] + 1 + [((6k-1) \bmod 4 - 1)/2]\right) \\ &= \left(3\left[\frac{k}{2}\right] + 2\right)^2. \end{aligned} \tag{8}$$

Here we note that $[\frac{6k-1}{4}] = 3[\frac{k}{2}] + 1$, $(6k-1) \bmod 4 = 1$.

Substituting Eqs. (6)–(8) into Eq. (5), we obtain that

$$D(1, 6k, 5) = \left(3\left[\frac{k}{2}\right] + 1\right)\left(3\left[\frac{k}{2}\right] + 2\right). \tag{9}$$

For odd k, $3[\frac{k}{2}] = [\frac{6k-3}{4}] = [\frac{l-3}{4}]$, $[(6k-3) \bmod 4 - 1]/2 = [(l-3) \bmod 4 - 1]/2 = 1$, we obtain

$$\begin{aligned} D(1, 6k, 5) = {} & \left(\left[\frac{l-3}{4}\right] + 1\right) \\ & \times \left(\left[\frac{l-3}{4}\right] + 1 + [(l-3) \bmod 4 - 1]/2\right). \end{aligned} \tag{10}$$

This is indeed identical to $D(1, l, 5)$ result of Ref. [3]. We shall not go to cases with $l = 6k+2$ (or $l = 6k+4$) and k is odd or cases with $l = 6k+1$ (or $l = 6k+3$, $l = 6k+5$) but point out the procedure is exactly the same as above.

By using correspondence of dimension for bosons and fermions given in Ref. [5] and Talmi's recursion formulas of dimension for fermions in a single-j shell, we can obtain following recursion formula for bosons. Let $l = j + \frac{n-1}{2}$, and when $I <= l + \frac{n-3}{2}$, one has

$$\begin{aligned} D(I,l,n) &= D(I,l-1,n) \\ &\quad + D\left(I + l + \frac{n-1}{2}, l - \frac{1}{2}, n-1\right) \\ &\quad + D(I,l,n-2) \\ &\quad - D\left(l + \frac{n-3}{2} - I, l - \frac{1}{2}, n-1\right) \end{aligned} \tag{11}$$

for bosons with spin l. Here $D(I + l + \frac{n-1}{2}, l - \frac{1}{2}, n-1)$ and $D(l + \frac{n-3}{2} - I, l - \frac{1}{2}, n-1)$ are dimensions of bosons with a half integer spin (fictitious bosons), which equal to dimensions of d bosons with odd number of particles.

III. NUMBER OF STATES WITH GIVEN SPIN AND F SPIN FOR BOSONS IN A SINGLE-l SHELL

F spin, similar to the isotopic spin in the nuclear shell model, was introduced into the neutron-proton interacting boson model in Ref. [10] to classify the symmetries of proton and neutron boson configurations. Proton bosons and neutron bosons can be considered as having an intrinsic quantity, called F spin, with $F = 1/2$ and $F_z = 1/2$ (proton boson) or $-1/2$ (neutron boson). F spin was found to be an approximately good quantum number in low-lying states. It is therefore interesting to study number of states with given spin and F spin for bosons.

In this section we apply the method of Ref. [7], in which we obtained number of states with given spin and isospin for nucleons in a single-j orbit to obtain number of states with given spin and F spin for three and four spin-l bosons.

We first discuss the case of four spin-l bosons. Similarly to Eq. (2) of Ref. [7], we obtain that the trace of H_{IF} matrix is given by summing

$$\begin{aligned} &\langle 0|[A^{(JF_2)}A^{(KF_2')}]^{(IF)}_{MM_F}[A^{(JF_2)\dagger}A^{(KF_2')\dagger}]^{(IF)}_{MM_F}|0\rangle \\ &= 1 + (-)^{I+F}\delta_{JK} \\ &\quad + 4(2J+1)(2K+1)(2F_2+1)(2F_2'+1) \\ &\quad \times \begin{Bmatrix} l & l & J \\ l & l & K \\ J & K & I \end{Bmatrix} \begin{Bmatrix} 1/2 & 1/2 & F_2 \\ 1/2 & 1/2 & F_2' \\ F_2 & F_2' & F \end{Bmatrix} \end{aligned} \tag{12}$$

over K, F_2, and F_2'. Here $F_2 (F_2')$ and F are F spin for two and four bosons, respectively. Similar to Eq. (3) of Ref. [7], one sees

$$\begin{aligned} &\sum_J \sum_\alpha \langle j^4 \alpha I F|H_J|j^4 \alpha I F\rangle \\ &= \sum_{JKF_2F_2'} \langle 0|[A^{(JF_2)}A^{(KF_2')}]^{(IF)}_{MM_F}[A^{(JF_2)\dagger}A^{KF_2'\dagger}]^{(IF)}_{MM_F}|0\rangle \\ &= 6D(I,l,4,F). \end{aligned} \tag{13}$$

$D(I,l,n,F)$ refer to number of states l^n bosons with given spin I and F spin.

The same as D_{IT} with $T = T_{\max}$ in Ref. [9], $D(I,l,n,F)$ with $F = F_{\max}$ here must equals D_I of Refs. [3,5], and we shall not discuss this case in the present article.

For convenience we define

$$S_I(l^4, \text{condition } X \text{ on } J \text{ and } K) = \sum_X \begin{Bmatrix} l & l & J \\ l & l & K \\ J & K & I \end{Bmatrix}. \tag{14}$$

Now we discuss the case of $n = 4$ and $F = 1$. Here (F_2, F_2') can take the following values: (1,0), (0,1), (1,1). Because of the symmetry of the wave functions of bosons, corresponding requirements for (J,K) are (J = even, K = odd), (J = odd, K = even), or (J = even, K = even). Thus we obtain

$$\begin{aligned} &6D(I,l,4,1) \\ &= \sum_{\text{even } J \text{ even } K} \Bigg[1 - (-)^I\delta_{JK} \\ &\quad + 36(2K+1)(2J+1) \begin{Bmatrix} l & l & J \\ l & l & K \\ J & K & I \end{Bmatrix} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ 1 & 1 & 1 \end{Bmatrix}\Bigg] \\ &\quad + \sum_{\text{odd } J \text{ even } K} \Bigg[1 - (-)^I\delta_{JK} \\ &\quad + 12(2J+1)(2K+1) \begin{Bmatrix} l & l & J \\ l & l & K \\ J & K & I \end{Bmatrix} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 1 \end{Bmatrix}\Bigg] \\ &\quad + \sum_{\text{even } J \text{ odd } K} \Bigg[1 - (-)^I\delta_{JK} \\ &\quad + 12(2J+1)(2K+1) \begin{Bmatrix} l & l & J \\ l & l & K \\ J & K & I \end{Bmatrix} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 1 & 1 \end{Bmatrix}\Bigg] \\ &= \sum_{\text{even } J \text{ even } K} [1 - (-)^I\delta_{JK}] \\ &\quad + 2 \sum_{\substack{JKI \text{ forms a triangle} \\ \text{even } J \text{ odd } K}} 1 + S(l^4, \text{even } J \text{ odd } K) \end{aligned} \tag{15}$$

for $F = 1$.

When $n = 4$ and $F = 1$, $I_{\max}$ equals $4l$. For $I \geqslant 2l$, let us define $I = I_{\max} - 2I_0 - 1$ for odd I and $I_{\max} - 2I_0 - 2$ for even I. Using (33) of Ref. [8], we obtain

$$D(I,l,4,1) = \left(\left[\frac{I_0}{2}\right] + 1\right)\left\{\left[\frac{I_0}{2}\right] + 1 + (I_0 \bmod 2)\right\}. \tag{16}$$

Now we come to case with $n = 4$ and $I \leqslant 2l - 1$. We define $I_0 = (I - 1)/2$ for odd I, and obtain

$$D(\{I, l, 4, 1\}) = (I_0 + 1)\left(l + \frac{1}{2}\right) - \left(1 + 4\left[\frac{I_0}{2}\right] + 6\left[\frac{I_0}{2}\right]^2 + (I_0 \bmod 2)\left(6\left[\frac{I_0}{2}\right] + 3\right)\right)\Big/2; \quad (17)$$

we define $I_0 = I/2$ for even I, and obtain

$$D(\{I, l, 4, 1\}) = (I_0 + 1)\left(l + \frac{1}{2}\right) - (l - I_0) - \left(1 + 4\left[\frac{I_0}{2}\right] + 6\left[\frac{I_0}{2}\right]^2 + (I_0 \bmod 2)\left(6\left[\frac{I_0}{2}\right] + 3\right)\right)\Big/2. \quad (18)$$

Next we discuss the case of $F = 0$. Here (F_2, F_2') can take (1,1) and (0,0). Their corresponding requirements for (J, K) are (J = even, K = even) or (J = odd, K = odd). We obtain

$$\begin{aligned} 6D(I, l, 4, 0) &= \sum_{\text{even } J \text{ even } K} \left(1 + (-)^I \delta_{JK} + 36(2J+1)(2K+1) \begin{Bmatrix} l & l & J \\ l & l & K \\ J & K & I \end{Bmatrix} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ 1 & 1 & 0 \end{Bmatrix}\right) \\ &\quad + \sum_{\text{odd } J \text{ odd } K} \left(1 + (-)^I \delta_{JK} + 4(2J+1)(2K+1) \begin{Bmatrix} l & l & J \\ l & l & K \\ J & K & I \end{Bmatrix} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{Bmatrix}\right) \\ &= \sum_{\text{even } J \text{ even } K} [1 + (-)^I \delta_{JK}] + \sum_{\text{odd } J \text{ odd } K} [1 + (-)^I \delta_{JK}] \\ &\quad + \frac{1}{2} S(l^4, \text{odd } J \text{ odd } K) - \frac{1}{2} S(l^4, \text{even } J \text{ even } K). \quad (19) \end{aligned}$$

For $F = 0$ with $I \geqslant 2l$, we use Eqs. (25), (29), and (30) of Ref. [8], and obtain for even I,

$$D(I, l, 4, 0) = \left(\left[\frac{I_0}{3}\right] + 1\right)\left(\frac{3}{2}\left[\frac{I_0}{3}\right] + 1 + I_0 \bmod 3\right), \quad (20)$$

where $I_0 = (I_{\max} - I)/2 - 1$; for odd I with $I \geqslant 2l$, we have $D(I, l, 4, 0) = D(I + 3, l, 4, 0)$ and $D(4n - 1, l, 4, 0) = 0$.

For $F = 0$ with $I \leqslant 2l$, we define $I = 6k + \kappa$, $L = [\frac{l}{3}] - k$, $m = l \bmod 3$. We use Eqs. (23)–(25), (29), and (30) of Ref. [8], and obtain

$$\begin{aligned} \kappa = 0 \quad & D(I, l, 4, 0) = (6k + 2)L + (2k + 1)m + \tfrac{3}{2}k(k + 3) - 3k; \\ \kappa = 1 \quad & D(I, l, 4, 0) = 2k(l + 3/2) - \tfrac{1}{2}k(9k + 1) - 3k; \\ \kappa = 2 \quad & D(I, l, 4, 0) = (6k + 4)L + (2k + 1)m + \tfrac{1}{2}(k + 1)(3k + 4) - (3k + 2); \\ \kappa = 3 \quad & D(I, l, 4, 0) = (6k + 2)L + (2k + 1)m + \tfrac{3}{2}k(k + 1) - (3k + 1); \\ \kappa = 4 \quad & D(I, l, 4, 0) = \tfrac{1}{2}(k + 1)(9k + 4) - 2(k + 1)l; \\ \kappa = 5 \quad & D(I, l, 4, 0) = (6k + 4)L + (2k + 1)m + \tfrac{1}{2}k(3k + 1) - (3k + 2). \end{aligned} \quad (21)$$

We notice that

$$\begin{aligned} D(6k, l, 4, 0) - D(6k + 3, l, 4, 0) &= 3k + 1; \\ D(6k + 2, l, 4, 0) - D(6k + 5, l, 4, 0) &= 3k + 2; \\ D(6k + 4, l, 4, 0) - D6k + 7, l, 4,) &= 3(k + 1). \end{aligned} \quad (22)$$

They are in the same form as Eqs. (16), (19), and (22) of Ref. [7], which addressed dimension with four nucleons in a single-j orbit.

We now come to cases with $n = 3$ and $F = 1/2$. Similarly, we obtain

$$\begin{aligned} 3D(I, l, 3, 1/2) &= \sum_{\text{even } J} \left[1 - 6(2J+1) \begin{Bmatrix} l & l & J \\ l & I & J \end{Bmatrix} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{Bmatrix}\right] + \sum_{\text{odd } J} \left[1 - 2(2J+1) \begin{Bmatrix} l & l & J \\ l & I & J \end{Bmatrix} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{Bmatrix}\right] \\ &= \sum_{\text{even } J} \left[1 - (2J+1) \begin{Bmatrix} l & l & J \\ l & I & J \end{Bmatrix}\right] + \sum_{\text{odd } J} \left[1 + (2J+1) \begin{Bmatrix} l & l & J \\ l & I & J \end{Bmatrix}\right]. \quad (23) \end{aligned}$$

Substituting A(4) and A(11) of Ref. [9] into Eq. (23), we obtain $D(I, l, 3, 1/2)$ for three bosons with $F = 1/2$:

$$\begin{aligned} I \leqslant l D(I, l, 3, 1/2) &= I - \left[\frac{I}{3}\right] \\ I \geqslant l D(I, l, 3, 1/2) &= \left[\frac{I_{\max} - I + 2}{3}\right]. \end{aligned} \quad (24)$$

IV. NUMBER OF STATES OF *d* BOSONS WITH GIVEN SPIN *I* AND *F* SPIN

In most IBM calculations, one usually uses sd bosons. Thus it is interesting to study number of states for d bosons with given spin and F spin, denoted by $D(I, F, n)$. $F_{\max} = n/2$ for n bosons. The case of $D(I, F_{\max}, n)$ can be obtained by SU(5) reduction rule. Formulas of $D(I, F_{\max}, n)$ was already given in Ref. [5] when n is even. The results with odd n and $F = n/2$ will be given in Appendix. In this section, we present our empirical formulas with $F = F_{\max} - 1$ and $F = F_{\max} - 2$. We shall not go to cases of lower F states, because there has been no observation of $F = F_{\max} - 3$ excitations so far.

We first come to $F = F_{\max} - 1$ and classify our results into two cases: (1) $I \leqslant n$ and (2) $I \geqslant n$. Here $n \geqslant 3$. For $I \leqslant n = 3k$,

$$D(2I_0, F_{\max}-1, n) = \left[\frac{k-1}{2}\right] + 2kI_0 - I_0^2 + \left[\frac{I_0+2}{3}\right],$$
$$D(2I_0+1, F_{\max}-1, n) = \left[\frac{k-1}{2}\right] + 2\left[\frac{k}{2}\right] - 1 + 2kI_0 - I_0^2 + \left[\frac{I_0+2}{3}\right] - 3\left[\frac{I_0}{3}\right] + 2\delta_{I_0 \bmod 3,0}; \tag{25}$$

For $I \leqslant n = 3k+1$,

$$D(2I_0, F_{\max}-1, n) = \left[\frac{k}{2}\right] + 2kI_0 - I_0(I_0-1),$$
$$D(2I_0+1, F_{\max}-1, n) = k + \left[\frac{k+1}{2}\right] + 2kI_0 - I_0^2; \tag{26}$$

For $I \leqslant n = 3k+2$,

$$D(2I_0, F_{\max}-1, n) = \left[\frac{k+1}{2}\right] + 2kI_0 - (I_0-1)^2 - \left[\frac{I_0-1}{3}\right]$$
$$D(2I_0+1, F_{\max}-1, n) = \left[\frac{k+1}{2}\right] + 2\left[\frac{k}{2}\right] + 1 - (I_0-1)^2 + 2kI_0 - \left[\frac{I_0-1}{3}\right] - 3\left[\frac{I_0}{3}\right] - 2\delta_{(I_0+1) \bmod 3,0}. \tag{27}$$

For $I > n$,

$$D(I_{\max}-2I_0, F_{\max}-1, n) = 3\left[\frac{I_0}{3}\right]\left(\left[\frac{I_0}{3}\right]+1\right) - \left[\frac{I_0}{3}\right] + \left(2\left[\frac{I_0}{3}\right]+1\right)(I_0 \bmod 3) + \delta_{(I_0+1) \bmod 3,0},$$
$$D(I_{\max}-2I_0-3, F_{\max}-1, n) = D(I_{\max}-2I_0, F_{\max}-1, n) + I_0 + 2. \tag{28}$$

We now come to $F = F_{\max} - 2$. Here $n \geqslant 3$. For $I \leqslant n = 3k$

$$D(2I_0, F_{\max}-2, n) = 2k + 5kI_0 + I_0 - \frac{5I_0(I_0+1)}{2} - \delta_{k \bmod 2,1} - 3 + 2\delta_{I,0} + 2\delta_{I,n} + \delta_{I,n-1},$$
$$D(2I_0+1, F_{\max}-2, n) = 3k + 5kI_0 - I_0 - \frac{5I_0(I_0+1)}{2} - \delta_{k \bmod 2,1} - 2\delta_{k \bmod 2,0} - 3 + \delta_{I,1} + 2\delta_{I,n} + \delta_{I,n-1}; \tag{29}$$

For $I \leqslant n = 3k+1$,

$$D(2I_0, F_{\max}-2, n) = 5kI_0 - \frac{5I_0(I_0-1)}{2} - \left[\frac{I_0-1}{3}\right] + 2k + 2\delta_{I,0} - \delta_{k \bmod 2,0} - 2I_0 - 3 + 2\delta_{I,n} + \delta_{I,n-1},$$
$$D(2I_0+1, F_{\max}-2, n) = 5kI_0 - \frac{5I_0(I_0+1)}{2} + 6\left[\frac{k-1}{2}\right] + I_0 - 2 - \left[\frac{I_0-2}{3}\right] + \delta_{k \bmod 2,0} + \delta_{I,1} + 2\delta_{I,n} + \delta_{I,n-1}; \tag{30}$$

For $I \leqslant n = 3k+2$,

$$D(2I_0, F_{\max}-2, n) = 5kI_0 - \frac{5I_0(I_0-1)}{2} + \left[\frac{I_0-1}{3}\right] + 2k - I_0 - 1 - \delta_{k \bmod 2,1} + 2\delta_{I,0} + 2\delta_{I,n} + \delta_{I,n-1},$$
$$D(I = 2I_0+1, F_{\max}-2, n) = 5kI_0 - \frac{5I_0(I_0-1)}{2} + \left[\frac{I_0+1}{3}\right] + 3k - 3I_0 - 2 - \delta_{k \bmod 2,0} + \delta_{I,1} + 2\delta_{I,n} + \delta_{I,n-1}; \tag{31}$$

For $I \geqslant n$

$$D(I_{\max}-6k, F_{\max}-2, n) = 12(k-1) + \tfrac{15}{2}(k-1)(k-2) + 10k - 3 - \delta_{I_{\max}-6k,n} - \delta_{I_{\max}-6k,n+1},$$
$$D(I_{\max}-6k-1, F_{\max}-2, n) = 12(k-1) + \tfrac{15}{2}(k-1)(k-2) + 11k - 3 - \delta_{I_{\max}-6k-1,n} - \delta_{I_{\max}-6k-1,n+1},$$
$$D(I_{\max}-6k-2, F_{\max}-2, n) = 12(k-1) + \tfrac{15}{2}(k-1)(k-2) + 15k - 2 - \delta_{I_{\max}-6k-2,n} - \delta_{I_{\max}-6k-2,n+1},$$
$$D(I_{\max}-6k-3, F_{\max}-2, n) = 12(k-1) + \tfrac{15}{2}(k-1)(k-2) + 16k - 2 - \delta_{I_{\max}-6k-3,n} - \delta_{I_{\max}-6k-3,n+1},$$
$$D(I_{\max}-6k-4, F_{\max}-2, n) = 17k + \tfrac{15}{2}k(k-1) + 3 - \delta_{I_{\max}-6k-4,n} - \delta_{I_{\max}-6k-4,n+1},$$
$$D(I_{\max}-6k-5, F_{\max}-2, n) = 18k + \tfrac{15}{2}k(k-1) + 4 - \delta_{I_{\max}-6k-5,n} - \delta_{I_{\max}-6k-5,n+1}. \tag{32}$$

V. SUMMARY AND DISCUSSION

To summarize, in this article we studied number of spin-I boson states for l^n configurations [denoted by $D(I, l, n)$]. First, we extended Talmi's recursion relations to bosons and proved number of states with $I = 1$ and $n = 5$, which was constructed empirically in Ref. [3]. The same procedure is readily used to prove other formulas for bosons. Second, we derived number of states for three and four spin-l bosons with total angular momentum I and F spin, by using sum-rules of six-j and nine-j symbols given in Ref. [9]. Third, we empirically constructed formulas of number of states for d bosons with total angular momentum I and $F = F_{\max} - 1$ and $F_{\max} - 2$. These results are interesting because F spin is an approximately good quantum number and very relevant in structure of medium and heavy nuclei.

In Appendix, we presented in this article formulas of dimension for d bosons with an odd particle number.

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APPENDIX: NUMBER OF SPIN I STATES FOR d BOSONS WITH AN ODD PARTICLE NUMBER

$D(I)$ of d bosons with an even particle number was enumerated in Ref. [5] by the reduction from SU$(n+1)$ to SO(3). In this Appendix we present formulas for d bosons with an odd particle number.

We define $K = [\frac{I_0}{6}]$, $\kappa = I_0 \bmod 6$, $\theta(x) = 1$ if $x > 0$ or zero otherwise. For $I \leqslant n$ and $n = 6k+1$,

$$D_{I=2I_0} = (I_0+1)k - [9K^2 - K + 3K\kappa + (2\kappa-5)\theta(2\kappa-5)] + \left[\frac{I_0+3}{6}\right] + \delta_{\kappa,1} - \delta_{\kappa,3};$$

for $I \leqslant n$ and $n = 6k+3$,

$$D_{I=2I_0} = (I_0+1)k - [9K^2 - K + 3K\kappa + (2\kappa-5)\theta(2\kappa-5)] + 2\left[\frac{I_0+3}{6}\right] + \left[\frac{I_0+4}{6}\right] + \delta_{\kappa,0} + \delta_{\kappa,1} - \delta_{\kappa,3} - \delta_{\kappa,4};$$

for $I \leqslant n$ and $n = 6k+5$,

$$D_{I=2I_0} = (I_0+1)(k+1) - [9K^2 - K + 3K\kappa + (2\kappa-5)\theta(2\kappa-5)] - \left[\frac{I_0+4}{6}\right];$$

or $I \leqslant n$ and I is odd,

$$D_{I=2I_0} - D_{I=2I_0+3} = \left[\frac{I_0+1}{2}\right]. \quad \text{(A1)}$$

For $I \geqslant n$, D_I for odd particle number is equal to that for even particle number:

$$D_{I=I_{\max}-2I_0} = D_{I=I_{\max}-2I_0-3} = 3\left[\frac{I_0}{6}\right]\left(\left[\frac{I_0}{6}\right]+1\right) - \left[\frac{I_0}{6}\right] + \left(\left[\frac{I_0}{6}\right]+1\right)[(I_0 \bmod 6)+1] + \delta_{I_0 \bmod 6,0} - 1, \quad \text{(A2)}$$

where $I_{\max} = 2n$.

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