Critical view of WKB decay widths

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A detailed comparison of the expressions for the decay widths obtained within the semiclassical WKB approximation using different approaches to the tunneling problem is performed. The differences between the available improved formulas for tunneling near the top and the bottom of the barrier are investigated. Though the simple WKB method gives the right order of magnitude of the decay widths, a small number of parameters are often fitted. The need to perform the fitting procedure remaining consistently within the WKB framework is emphasized in the context of the fission model based calculations. Calculations for the decay widths of some recently found superheavy nuclei using microscopic alpha-nucleus potentials are presented to demonstrate the importance of a consistent WKB calculation. The half-lives are found to be sensitive to the density dependence of the nucleon-nucleon interaction and the implementation of the Bohr-Sommerfeld quantization condition inherent in the WKB approach.

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I. INTRODUCTION

The Wentzel-Kramers-Brillouin or the WKB approximation [1,2], sometimes also known as the BWK [3], the semiclassical approximation or the phase integral method [4,5] has been widely used in the evaluation of the half-lives of radioactive nuclei. It was evident from the historical papers of Gamow [6] and Condon and Gurney [7] that one could treat the alpha decay of nuclei in terms of the tunneling of a preformed α-particle confined to the interior of the nucleus, through the Coulomb potential barrier of the alpha-nucleus system. The WKB approximation which is really applicable when a problem can be reduced to a one-dimensional one was found suitable to evaluate the barrier penetration probabilities and the decay width in general was defined as a product of the frequency of collisions of the α with the barrier (the so-called assault frequency) and the penetration probability. The objective of the present work is to critically examine the decay widths obtained within some entirely different approaches to the tunneling problem, however, all working within a WKB framework. To be specific, we examine four approaches; the first a two potential approach (TPA) [8], another a path integral method with Jost functions [4], a third one using comparison equations to obtain improved WKB formulas [9], and a fourth one involving a super asymmetric fission model (SAFM) [10]. It is gratifying to know that all four approaches indeed lead to the same formulas for the WKB decay widths. A WKB expression for the vibrational energy, E_ν, emerges from the above comparisons. For the first three approaches, the vibrational energy (and hence the assault frequency) can be simply evaluated from the potential and the tunneling particle energy. In the SAFM however, the equation for E_ν turns out to be a transcendental equation. However, this equation is often neglected by the SAFM calculations in literature and a fit to the vibrational energies is performed, leaving the calculation somewhat incompatible with the WKB framework.

Besides this, the widths calculated within the SAFM often neglect the Bohr-Sommerfeld condition and the Langer modification which are essential ingredients of the WKB framework. The Langer modification is a necessary transformation while going from the one dimensional problem with x ranging from $-\infty \rightarrow \infty$ to the radial one-dimensional tunneling problem with r ranging from $0 \rightarrow \infty$. To demonstrate the importance of performing a completely consistent WKB calculation of widths, we perform a realistic calculation of the alpha decay widths of super heavy nuclei which are a topic of current interest.

With most radioactive decays occurring away from the extremes of the potential barrier, the standard WKB is found to be a reasonable approximation for such calculations. However, for specific cases, where the tunneling can occur near the top or the bottom of the barrier, the validity of this approach becomes questionable. Some attempts at obtaining improved formulas which can be used near the extremes of the barrier do exist [4,9] and will be investigated by applying to realistic examples in the present work.

The alpha-daughter nucleus interaction in the present work is described using folding model potentials with a realistic nucleon-nucleon (NN) interaction [11]. Such a model has been shown to be quite successful in predicting half-lives of unstable nuclei [12]. We perform calculations with the added ingredient of a density dependent NN interaction and find the results and fitted parameters sensitive to this input. In the next section, we perform a comparison of the different WKB approaches as mentioned above. In Sec. III, we briefly present the relevant formulas of the potentials used. The results are discussed in Sec. IV.

II. THE QUASICLASSICAL ALPHA TUNNELING PROBLEM

Starting with the description of a radioactive nucleus as a cluster of its daughter nucleus and an alpha particle, it is by now standard practice to study the alpha decay of nuclei as a tunneling of the α through the potential barrier of the alpha-daughter nucleus system. Though most quasiclassical

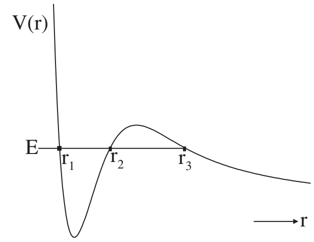

FIG. 1. Typical potential for an alpha-nucleus tunneling problem.

approaches agree on the proportionality of the decay width to an exponential factor, namely, $\Gamma \propto e^{-2G}$, where G is the famous Gamow factor, the details of the calculations vary depending on the approach used. Typically, one considers the tunneling of the α through an r-space potential of the form

$$V(r) = V_n(r) + V_c(r) + \frac{\hbar^2 (l+1/2)^2}{\mu r^2}, \tag{1}$$

where $V_n(r)$ and $V_c(r)$ are the nuclear and Coulomb parts of the α-nucleus (daughter) potential, r the distance between the centres of mass of the daughter nucleus and alpha, and μ their reduced mass. The last term represents the Langer modified centrifugal barrier [13]. With the WKB being valid for one-dimensional problems, the above modification from $l(l+1) \to (l+1/2)^2$ is essential to ensure the correct behavior of the WKB scattered radial wave function near the origin as well as the validity of the connection formulas used [14]. Another requisite for the correct use of the WKB method is the Bohr-Sommerfeld quantization condition, which for an alpha with energy E is given as

$$\int_{r_1}^{r_2} K(r)\, dr = (n + 1/2)\pi, \tag{2}$$

where $K(r) = \sqrt{\frac{2\mu}{\hbar^2}(|V(r) - E)|}$, n is the number of nodes of the quasibound wave function of α-nucleus relative motion and r_1 and r_2 which are solutions of $V(r) = E$, are the classical turning points (as shown in Fig. 1). We shall now examine the decay widths obtained within different WKB based approaches to the above problem.

A. The two potential approach

Starting with a typical potential as in Fig. 1, the authors in [8] consider the tunneling problem of a metastable (quasistationary) state and obtain a perturbative expansion for the decay width of the metastable state. The potential is split into two parts, $V(r) = U(r) + W(r)$, where the authors first consider an unperturbed bound wave function $\Phi_0(r)$, which is an eigenstate of the Hamiltonian, $H_0 = -(\hbar^2/2\mu)\nabla^2 + U(r)$. $W(r)$ is a perturbation and when it is switched on, $\Phi_0(r)$ is not an eigenfunction of $H = H_0 + W(r)$, but one rather has a wave packet which is expressed as an expansion in terms of $\Phi_0(r)$ and the continuum wave functions, $\Phi_k(r)$. Once the perturbative expansion for the width is obtained in terms of the wave functions, retaining the first term and expressing the wave function using the semiclassical WKB approximation, the width is given as

$$\Gamma_{\rm TPA}(E) = \frac{\hbar^2}{2\mu}\left[\int_{r_1}^{r_2} \frac{dr}{k(r)}\right]^{-1} e^{-2\int_{r_2}^{r_3} k(r) dr}, \tag{3}$$

where, $k(r) = \sqrt{\frac{2\mu}{\hbar^2}(|V(r) - E|)}$. The factor in front of the exponential arises from the normalization of the bound state wave function in the region between r_1 and r_2. Indeed, this factor is related to the so-called 'assault frequency' of the tunneling particle at the barrier. Expressing the time interval, Δt, for the particle traversing a distance, Δr as

$$\Delta t = \frac{\Delta r}{v(r)} = \frac{\mu \Delta r}{\hbar k(r)}, \tag{4}$$

the assault frequency ν can be written as the inverse of the time required to traverse the distance back and forth between the turning points r_1 and r_2 as [4]

$$\nu = T^{-1} = \frac{\hbar}{2\mu}\left[\int_{r_1}^{r_2} \frac{dr}{\sqrt{\frac{2\mu}{\hbar^2}(|V(r) - E|)}}\right]^{-1}. \tag{5}$$

Thus, $\Gamma_{\rm TPA}(E) = \hbar \nu e^{-2W}$, where, $W = \int_{r_2}^{r_3} k(r)\, dr$. It is interesting to note that the above definition of the period T (which is twice the time spent between the turning points r_1 and r_2), is formally similar to that of "traversal time in tunneling" as defined by Büttiker and Landauer [15]. They defined the time for a particle traversing the barrier and hence obtained a similar definition as that for $T/2$ above, but within the limits r_2 to r_3.

B. Improved WKB widths

The width obtained in the two potential approach is in accord with the widths obtained in [4] and [9] for energies well away from the top or the bottom of the barrier. For example, in [9], considering a double humped (DH) barrier in one dimension (with cartesian coordinates) and using the method of comparison equations, the decay width for all energies except near the very top or the very bottom of the barrier was found to be

$$\Gamma_{\rm DH}(E) = 2\hbar\, \nu_{\rm DH} \ln[1 + e^{-2W}] \simeq 2\hbar\, \nu_{\rm DH} e^{-2W}, \tag{6}$$

where, $\nu_{\rm DH}$ is the assault frequency in the case of a double humped barrier. Replacing $\nu_{\rm DH} = \nu/2$, for small values of the exponent, $\Gamma_{\rm DH}(E)$ indeed reduces exactly to $\Gamma_{\rm TPA}(E)$.

1. At the base of the barrier

The width for energies of the tunneling particle near the bottom of the well was found in [9] to be

$$\Gamma_{\rm DH}^{\rm low}(E_n) = 2\hbar \nu_{\rm DH} \ln[1 + \alpha^{-1} e^{-2W}] \simeq \alpha^{-1} \Gamma_{\rm DH}(E_n), \tag{7}$$

where, $\alpha^{-1} = (1/n!)\sqrt{2\pi}[(n+1/2)/e]^{n+1/2}$. The relation obtained in [8] using the TPA is somewhat different, namely,

$$\Gamma_{\rm TPA}^{\rm low}(E_n) = \frac{\sqrt{2}}{\pi}(\Gamma(1/4))^2 (n+1/2)^{1/2}\, \Gamma_{\rm TPA}(E_n). \tag{8}$$

It is mentioned in [8] that the above result coincides with that in [16] obtained using complex-time path integral methods

[17]. The prefactors in front of the exponential factors in the expression for the widths in [16,17] depend on characteristic parameters of instanton trajectories, however, the above modification in Eq. (8) does agree in form with that obtained in [16]. Though the TPA approach and that of comparison equations in [9] do give the same expressions for the widths away from the extremes of the barrier [$\Gamma_{\rm TPA}(E) = \Gamma_{\rm DH}(E)$ was seen above], they do not seem to agree on the situation at low energies. Equations (7) and (8) seem to give very different results. For example, for $n = 0, 1$, respectively, one finds that $\Gamma^{\rm low}_{\rm DH}(E) = 1.075\Gamma_{\rm DH}(E)$ and $\Gamma^{\rm low}_{\rm DH}(E) = 1.027\Gamma_{\rm DH}(E)$, whereas, $\Gamma^{\rm low}_{\rm TPA}(E) = 4.184\Gamma_{\rm TPA}(E)$ and $\Gamma^{\rm low}_{\rm TPA}(E) = 7.247\Gamma_{\rm TPA}(E)$, respectively. In [18], one can find yet another expression for tunneling near the bottom of the barrier such that the tunneling probability at the base of the barrier vanishes exactly.

2. At the top of the barrier

In [9], the decay width within the WKB approximation at the top of the barrier was also evaluated and found to be

$$\Gamma^{\rm top}_{\rm DH}(E) = \frac{2\hbar \ln[1 + e^{-2W}]}{\left[T_{\rm DH} - 2\hbar \frac{d\phi}{dE}\right]}, \qquad (9)$$

where, $T_{\rm DH}$ is the period for moving back and forth between the two humps and $\phi = arg\Gamma(\frac{1}{2} - iW/\pi) + \frac{W}{\pi}[\ln(W/\pi) - 1]$. A very similar formula was also found in [4], however, with a difference of a sign in the denominator. Having derived the expressions for the Jost function of a radial barrier transmission problem (with one hump) by the path integral method, the authors obtain the following expression for evaluating the width at the top of the barrier:

$$\Gamma^{\rm top}_{\rm Fr\ddot{o}man}(E) = \frac{2\hbar[(1 + e^{-2W})^{1/4} - (1 + e^{-2W})^{-1/4}]}{\left[T - 2\hbar \frac{d\sigma}{dE}\right]}. \qquad (10)$$

The numerators in Eqs. (9) and (10) for energies near the top of the barrier are almost equal, i.e., $2\hbar[(1 + e^{-2W})^{1/4} - (1 + e^{-2W})^{-1/4}] \simeq \hbar \ln[1 + e^{-2W}]$. With the period, $T_{\rm DH}$ being twice as much as T and the factor $\phi = -2\sigma$, Eqs. (9) and (10) indeed agree but up to a sign in the denominator. In Eq. (5.7) of [4], however, one can see that there exists a choice for the sign appearing in front of $d\sigma/dE$ and the authors choose the negative sign without any particular justification. We shall later notice with a realistic example near the top of the barrier, that it is indeed the choice of a positive sign in the denominator which improves the WKB width estimate. The choice of a positive sign also brings Eq. (10) in agreement with Eq. (9).

We now move on to discuss one more approach which is popularly used in literature and has off late been often used for the evaluation of the half-lives of superheavy nuclei.

C. Fission model approach

An approach where the alpha decay is considered as a very asymmetric fission process was introduced about two decades ago by Poenaru and co-workers [10]. This model, also known as the super asymmetric fission model (SAFM) has recently been implemented extensively for the evaluation of the WKB widths of superheavy nuclei [19,20]. The decay width of a metastable state in the SAFM within the WKB framework is given as

$$\Gamma_{\rm SAFM}(E) = \nu P = \frac{E_\nu}{\pi}(1 + e^{2K})^{-1}, \qquad (11)$$

where, P is the probability of penetration through the potential barrier, E_ν is the "vibrational energy," $K = \int_{r_2}^{r_3} \kappa(r)\,dr$ with $\kappa(r) = \sqrt{(2\mu/\hbar^2)(V(r) - Q')}$, and $Q' = E + E_\nu$. The Gamow factor with K over here differs from the W occurring in the equations so far as obtained in [4,8,9] due the energy E of the tunneling particle being replaced by $E + E_\nu$. Replacing the assault frequency ν from Eq. (5) into the above Eq. (11), we indeed recover an expression similar to that of $\Gamma_{\rm TPA}(E)$, with the $k(r)$ replaced here by $\kappa(r)$. The SAFM width is

$$\Gamma_{\rm SAFM}(E) = \frac{\hbar^2}{2\mu}\left[\int_{r_1}^{r_2} \frac{dr}{\kappa(r)}\right]^{-1} e^{-2\int_{r_2}^{r_3} \kappa(r)\,dr}, \qquad (12)$$

where we have approximated $(1 + e^{2K})^{-1} \simeq e^{-2K}$ for sufficiently large K. In fact, if we start with the definition of $E_\nu = (1/2)\hbar\omega = (1/2)\hbar\,(2\pi\nu)$ (as defined in the SAFM based works [10,19,20]) and use Eq. (5) for $\nu = T^{-1}$, we obtain a theoretical relation for E_ν, namely,

$$E_\nu = \frac{\hbar^2 \pi}{2\mu}\left[\int_{r_1}^{r_2} \frac{dr}{\sqrt{(2\mu/\hbar^2)(V(r) - E - E_\nu)}}\right]^{-1}. \qquad (13)$$

One could have of course inferred the above Eq. (13) simply by the comparison of Eqs. (11) and (12). Provided the potential is known, for a given energy E of the tunneling particle, Eq. (13) is a transcendental equation for the vibrational energy E_ν.

Comparing the expressions for the widths, $\Gamma_{\rm TPA}$, $\Gamma_{\rm DH}$, and $\Gamma_{\rm Fr\ddot{o}man}$, away from the extremes of the barrier (taken in the limit of large W and negligible $d\sigma/dE$), one can see that indeed

$$\Gamma_{\rm TPA}(E) = \Gamma_{\rm DH}(E) = \Gamma_{\rm Fr\ddot{o}man}(E) = \frac{\hbar^2}{2\mu}\left[\int_{r_1}^{r_2} \frac{dr}{k(r)}\right]^{-1} e^{-2\int_{r_2}^{r_3} k(r)\,dr}. \qquad (14)$$

$\Gamma_{\rm SAFM}$ agrees exactly in form with the above formulas for widths. The only difference lies in the replacement of E by $E + E_\nu$ as mentioned before. It is interesting to note that even though the prefactor in front of the exponential in $\Gamma_{\rm TPA}$ arises due to the normalization of the WKB wave function, it agrees exactly with the prefactors in $\Gamma_{\rm DH}$ and $\Gamma_{\rm Fr\ddot{o}man}$ where it arises due to the replacement of the assault frequency as in Eq. (5).

Coming back to Eq. (13), one notices that for a given potential, $V(r)$, in a particular decay problem with a given Q-value, E_ν can be determined by resolving Eq. (13). However, starting from the pioneering works of Poenaru and co-workers until some recent ones, E_ν is fitted to reproduce the half-lives under consideration. There is no mention in these works of the fitted value being consistent with Eq. (13). Such a fitting procedure performed without a consistency check with Eq. (13) would be somewhat ambiguous and outside the spirit of a proper WKB calculation. Without the condition (13), E_ν

becomes simply a parameter to compensate for the mismatch of the theoretical width with experiment. The interpretation of E_ν as a vibrational energy and its addition to the Q value ($E = Q$) giving, $Q' = Q + E_\nu$, is then not justified. The above point will be clarified with realistic examples of the calculation of half-lives of superheavy nuclei in Sec. IV. In the next section, we briefly describe the potentials used for the calculations of the present work.

III. THE ALPHA NUCLEUS POTENTIAL

With the objective of the present work being a critical examination of the various semiclassical methods used for the evaluation of alpha decay half-lives, we perform calculations using different available inputs in literature. The potential in Eq. (1) is written using a double-folding model with realistic nucleon-nucleon interactions as given in [11]. The folded nuclear potential is written as

$$V_n(r) = \lambda \int d\mathbf{r}_1\, d\mathbf{r}_2\, \rho_\alpha(\mathbf{r}_1)\, \rho_d(\mathbf{r}_2)\, v(\mathbf{r}_{12} = \mathbf{r} + \mathbf{r}_2 - \mathbf{r}_1, E), \tag{15}$$

where ρ_α and ρ_d are the densities of the alpha and the daughter nucleus in a decay and $v(\mathbf{r}_{12}, E)$ is the nucleon-nucleon interaction. $|\mathbf{r}_{12}|$ is the distance between a nucleon in the alpha and a nucleon in the daughter nucleus. $v(\mathbf{r}_{12}, E)$ is written using the M3Y nucleon-nucleon (NN) interaction as in [11] as

$$\begin{aligned} v(\mathbf{r}_{12}, E) &= 7999\frac{\exp(-4|\mathbf{r}_{12}|)}{4|\mathbf{r}_{12}|} - 2134\frac{\exp(-2.5|\mathbf{r}_{12}|)}{2.5|\mathbf{r}_{12}|} \\ &\quad + J_{00}\delta(\mathbf{r}_{12}), \\ J_{00} &= -276(1 - 0.005E_\alpha/A_\alpha). \end{aligned} \tag{16}$$

The alpha particle density is given using a standard Gaussian form, namely,

$$\rho_\alpha(r) = 0.4229\exp(-0.7024\, r^2) \tag{17}$$

and the daughter nucleus density is taken to be

$$\rho_d(r) = \frac{\rho_0}{1 + \exp\left(\frac{r-c}{a}\right)}, \tag{18}$$

where ρ_0 is obtained by normalizing $\rho_d(r)$ to the number of nucleons A_d and the constants are given as $c = 1.07A_d^{1/3}$ fm and $a = 0.54$ fm. Equation (15) involves a six-dimensional integral. However, the numerical evaluation becomes simpler if one works in momentum space as shown in [11]. The constant λ is determined by imposing the Bohr-Sommerfeld quantization condition (2) using the above potential. The number of nodes are reexpressed as $n = (G - l)/2$, where G is a global quantum number obtained from fits to data [12,21] and l is the orbital angular momentum quantum number. We shall perform calculations with two possible fitted values of G [21], namely, 22 and 24. The Coulomb potential is obtained using a similar double folding procedure with the matter densities of the alpha and the daughter replaced by their respective charge density distributions ρ_α^c and ρ_d^c. Thus, double folding the proton proton coulomb potential,

$$V_c(r) = \int d\mathbf{r}_1\, d\mathbf{r}_2\, \rho_\alpha^c(\mathbf{r}_1)\rho_d^c(\mathbf{r}_2)\frac{e^2}{|\mathbf{r}_{12}|}. \tag{19}$$

The charge distributions are taken to have a similar form as the matter distributions, except for the fact that they are normalized to the number of protons in the alpha and the daughter.

One could further improvise the double folding potential by taking into account the density dependence of the NN interaction $v(\mathbf{r}_{12})$. For example, in [22] a reasonably good description of elastic alpha-nucleus scattering data was obtained by assuming a factorized form of the density dependence as follows:

$$\tilde{v}(\mathbf{r}_{12}, \rho_\alpha, \rho_d, E) = C v(\mathbf{r}_{12}, E) f(\rho_\alpha, E) f(\rho_d, E), \tag{20}$$

where, $f(\rho_X, E) = 1 - \beta\rho_X^{2/3}$, with X being either the α or d. The parameters, C and β were found to be energy independent and $C = 1.3$ and $\beta = 1.01$ fm^2 for the range of analysed data between alpha particle energies of 100 MeV and 172 MeV. Due to the lack of much information, for the case of super heavy nuclei, C was chosen to be unity and $\beta = 1.6$ fm^2 [19]. We note here that even if the potential is improvised with the density dependence of the NN interaction included, the Bohr-Sommerfeld condition should still be satisfied in the WKB framework. As we shall see later, the normalization λ in Eq. (15) is different from unity even for the above interaction.

In order to test the applicability of the formulas (9) and (10) at the top of the barrier, we shall examine the case of the $l = 2$, ^{8}Be resonance which decays 100% into two alphas. The nuclear potential for the $\alpha - \alpha$ case is taken to be [11]

$$V_n^{\alpha\alpha}(r) = -122.6225\exp(-0.22\, r^2)\,\text{MeV}. \tag{21}$$

Since the aim of the present work is to make a comparative study and not fit parameters to match the theoretical widths with the experimental ones, the alpha particle preformation probability for all the calculations in this work has been taken to be unity.

IV. RESULTS AND DISCUSSIONS

The objective of the calculations performed for the superheavy nuclei is to test the sensitivity of the results to (i) the implementation of the Bohr-Sommerfeld quantization condition (2) which fixes the strength of the potential λ in Eq. (15), (ii) the 'fitted' global quantum number G appearing in Eq. (2), (iii) the density dependence of the nucleon-nucleon ($DD - NN$) interaction, and finally (iv) to verify if the fitted vibrational energies used in literature are consistent with the theoretical Eq. (13). Some recent works [19] on superheavy elements in literature neglect (i), (ii), and (iv) from above. In what follows, we shall see that the exact values of the decay widths of the nuclei considered do depend strongly on the ingredients (i), (ii), and (iv) and hence any conclusions drawn in works which neglect these aspects of the WKB framework would have to be treated with caution.

TABLE I. Half-lives in ms, for $G = 22$.

Parent nucleus	λ[a]	$t_{\frac{1}{2}}$(ms)[a]	λ[b]	$t_{\frac{1}{2}}$(ms)[b]	$t_{\frac{1}{2}}(ms)$[c]	$t_{\frac{1}{2}}(ms)$[d]
$^{271}_{106}$Sg$Q = 8.67$ MeV	0.644	17638.6	2.095	2794.5	213197.8	288000
$^{275}_{108}$Hs$Q = 9.44$ MeV	0.639	356.3	2.080	56.65	4077.1	150
$^{273}_{110}$Ds$Q = 11.368$ MeV	0.638	0.021	2.072	0.0034	0.205	0.17
$^{274}_{111}$Rg$Q = 11.36$ MeV	0.639	0.0439	2.076	0.0071	0.46	6.4
^{277}112$Q = 11.3$ MeV	0.637	0.117	2.072	0.019	1.253	0.69
^{286}114$Q = 10.35$ MeV	0.634	102.07	2.067	15.61	1248.3	400
^{293}116$Q = 10.67$ MeV	0.627	57.583	2.049	8.701	679.6	53
^{294}118$Q = 11.81$ MeV	0.625	0.416	2.042	0.064	4.594	1.8

[a]Free NN.
[b]Density dependent NN.
[c]Density dependent NN with $\lambda = 1$.
[d]Experimental value.

A. Sensitivity of superheavy nuclear half-lives to $DD - NN$ and the Bohr-Sommerfeld condition

In Tables I and II, the half-lives $t_{1/2} = \ln(2)/\Gamma(Q)$, of some currently discovered superheavy nuclei are shown for two different choices of the global quantum number G of the Bohr-Sommerfeld quantization condition. Since all the decays discussed in Tables I and II take place at energies $E = Q$ which are away from the extremes of the barrier, we use $\Gamma(Q) = \Gamma_{\rm TPA}(Q)$ which in turn is the same as evaluating $\Gamma_{\rm DH}(Q)$ or $\Gamma_{\rm Fröman}(Q)$ as shown in Eq. (14). Theoretically, the Q value of the decay is defined as the difference of the masses of the parent nucleus and the sum of the masses of the alpha and the daughter nucleus ($Q = M_{\rm parent} - M_\alpha - M_d$). We shall however use the Q deduced from the measured α-particle energies, E_α, by applying a standard recoil correction as suggested by Perlman and Rasmussen [23] and frequently used in literature. With Z_p and A_p being the charge and mass numbers respectively of the parent nucleus,

$$Q = \frac{A_p}{A_p - 4}E_\alpha + \left(65.3\, Z_p^{7/5} - 80.0\, Z_p^{2/5}\right)10^{-6}\,\text{MeV}. \quad (22)$$

The results obtained using Eq. (16) for the nucleon-nucleon (NN) interaction are labeled as "free-NN" in Tables I and II. The density dependent NN interaction ($DD - NN$) calculations use Eq. (20) instead of $v(\mathbf{r}_{12})$ in Eq. (15) and are also shown in the tables. One can see that the introduction of density dependence in the NN interaction reduces the lifetimes $t_{1/2}$ by an order of magnitude as compared to the free NN results. Neglecting the BS condition (i.e., $\lambda = 1$) however, 'increases' the $DD - NN t_{1/2}$ by two orders of magnitude as compared to the proper $DD - NN$ calculation using the BS condition with λ around 2. Any conclusions based on calculations neglecting the BS condition can hence be quite misleading. In an ideal case, when the potential $V_n(r)$ is known exactly for a particular system, one would expect λ in Eq. (15) which gets fixed by the BS condition to be unity. One can however see that using the "free NN" interaction, the value of λ ranges around $0.6 - 0.7$, while for the $DD - NN$ case it is in the range of 2–2.3. It is however interesting to note that the value of λ hardly depends on the mass or atomic number of the parent nucleus for the considered range of nuclei. The parameter C in Eq. (20) was in fact chosen to be unity due

TABLE II. Half-lives in ms, for $G = 24$.

Parent nucleus	λ[a]	$t_{\frac{1}{2}}$(ms)[a]	λ[b]	$t_{\frac{1}{2}}$(ms)[b]	$t_{\frac{1}{2}}(ms)$[c]	$t_{\frac{1}{2}}(ms)$[d]
$^{271}_{106}$Sg$Q = 8.67$ MeV	0.720	10511.8	2.337	1680.8	213197.8	288000
$^{275}_{108}$Hs$Q = 9.44$ MeV	0.715	213.1	2.320	34.185	4077.1	150
$^{273}_{110}$Ds$Q = 11.37$ MeV	0.713	0.012	2.312	0.0021	0.205	0.17
$^{274}_{111}$Rg$Q = 11.36$ MeV	0.714	0.027	2.316	0.0044	0.46	6.4
^{277}112$Q = 11.3$ MeV	0.712	0.0702	2.311	0.0114	1.253	0.69
^{286}114$Q = 10.35$ MeV	0.707	60.57	2.302	9.355	1248.3	400
^{293}116$Q = 10.67$ MeV	0.699	34.102	2.280	5.213	679.6	53
^{294}118$Q = 11.81$ MeV	0.697	0.248	2.272	0.038	4.594	1.8

[a]Free NN.
[b]Density dependent NN.
[c]Density dependent NN with $\lambda = 1$.
[d]Experimental value.

TABLE III. Comparison of the vibrational energies and corresponding assault frequencies obtained from fitted values, Eq. (23) and Eq. (24).

Parent nucleus	$E_{\nu,\text{fit}}$ (MeV)	$E_{\nu,\text{test}}$ (MeV)	$E_{\nu,\text{TPA}}$ (MeV)	$\nu_{\text{fit}} = (\frac{2E_{\nu,\text{fit}}}{h})$ $(10^{21}s^{-1})$	$\nu_{\text{test}} = (\frac{2E_{\nu,\text{test}}}{h})$ $(10^{21}s^{-1})$	$\nu_{\text{TPA}} = (\frac{2E_{\nu,\text{TPA}}}{h})$ $(10^{21}s^{-1})$
$^{271}_{106}$Sg	0.786	4.130	6.043	0.380	1.998	2.923
$^{275}_{108}$Hs	0.856	4.257	5.987	0.414	2.059	2.896
$^{273}_{110}$Ds	1.031	4.226	5.903	0.499	2.044	2.856
$^{274}_{111}$Rg	0.871	4.220	5.917	0.421	2.041	2.863
^{277}112	1.025	4.210	5.913	0.496	2.036	2.861
^{286}114	1.082	4.184	5.949	0.523	2.023	2.878
^{293}116	0.968	4.153	5.910	0.468	2.010	2.859
^{294}118	1.234	4.127	5.856	0.597	1.996	2.833

to lack of information. One could rather choose $C = 2$ and $C = 2.3$ leading to a λ close to 1 for the $DD - NN$ cases in Tables I and II, respectively.

Comparing the numbers in Tables I and II, one observes that the half-lives are reduced when increasing the value of the global quantum number from $G = 22$ to $G = 24$. With the primary objective of the work being a comparison of the different WKB approaches, the orbital quantum number l was taken to be zero. Note however, that the introduction of the Langer modification, namely, $l(l+1) \to (l+1/2)^2$ for the radial one-dimensional WKB problem introduces an additional turning point near the origin, even for the $l = 0$ case [14]. This detail has also been missed out in some works [19].

In passing, we note that in some of the cases like the decay, $^{286}114 \to \alpha + ^{282}112$ for example, the lifetime of the daughter nucleus, $t_{1/2}(^{282}112) = 0.5$ ms, is much smaller than that of the parent, $t_{1/2}(^{286}114) = 160$ ms (implying that the daughter in the cluster decays before the parent can decay). The application of quantum tunneling (which assumes a preformed cluster) to such a problem would be somewhat ambiguous with the daughter decaying faster; however, one could also argue that the daughter inside the cluster does not decay as fast as the free one and the picture is still valid. Indeed, in recent literature the tunneling picture is used without considerations of the lifetimes of the parent and the daughter nuclei.

B. Assault frequencies and vibration energies

We shall now examine the evaluation of the widths using Γ_{SAFM} of Eq. (12). As mentioned before, the only difference in the evaluation of Γ_{SAFM} as compared to Γ_{TPA} is in the replacement of Q by $Q + E_\nu$. The zero point vibration energies, E_ν, are usually taken from fits [24] and are given for the superheavy case [19] as, $E_\nu = 0.1045Q$ for even-even, $E_\nu = 0.0962Q$ for odd-Z–even-N, $E_\nu = 0.0907Q$ for even-Z–odd-N, and $E_\nu = 0.0767Q$ for odd-Z–odd-N parent nuclei. With an average Q value for the superheavy nuclei around $Q = 10$, one could say that E_ν would be of the order of 1 MeV. Such values are however not consistent with Eq. (13). If for example, we provide the above values of E_ν from fits as an input for the right hand side of Eq. (13), the outcome (which in principle must be E_ν itself) turns out to be a much larger energy. For the eight nuclei considered in Tables I and II, we evaluated the right hand side of Eq. (13) providing E_ν as an input. To be precise, we evaluated

$$E_{\nu,\text{test}} = \frac{\hbar^2\pi}{2\mu}\left[\int_{r_1}^{r_2} \frac{dr}{\sqrt{(2\mu/\hbar^2)(V(r) - Q - E_{\nu,\text{fit}})}}\right]^{-1} \quad (23)$$

using the fitted values of E_ν, defined as $E_{\nu,\text{fit}}$ in the equation above. The $E_{\nu,\text{test}}$'s are not the same as $E_{\nu,\text{fit}}$'s [as should have been the case due to Eq. (13)] and are listed in Table III. The calculations were done with the density dependent NN interaction in the nuclear potential and with the value of $\lambda = 1$ to perform a comparison with the works which use the SAFM. The vibration energy is in fact related to the assault frequency at the barrier as $E_\nu = (1/2)\hbar(2\pi\nu)$. In Table III we also list the assault frequencies corresponding to the fitted E_ν's as used in the SAFM models and to the calculated $E_{\nu,\text{test}}$ values. For comparison, we present the assault frequencies appearing in the widths, Γ_{TPA}, Γ_{DH} and $\Gamma_{\text{Fröman}}$ which are the same in all the three cases [see Eq. (5)] and label them as ν_{TPA}. The corresponding $E_{\nu,\text{TPA}}$ is given as

$$E_{\nu,TPA} = \frac{\hbar^2\pi}{2\mu}\left[\int_{r_1}^{r_2} \frac{dr}{\sqrt{(2\mu/\hbar^2)(V(r) - Q)}}\right]^{-1}. \quad (24)$$

The above calculation of $E_{\nu,TPA}$ is performed for a density dependent NN interaction and including the BS condition. The assault frequencies, ν_{TPA} are of the order of $10^{21}s^{-1}$ which is more like the standard result expected for alpha particle tunneling [25].

C. Improved WKB formula at the top of the barrier

Finally, we discuss the result of the application of the improved formulas (9) and (10) for the decay taking place with an energy close to the top of the barrier. Such examples are indeed difficult to find among the alpha decay of nuclei. A suitable one is the decay of the ^{8}Be (2^+) level at 3.03 MeV above its ground state. The experimental width of this state is 1.513 MeV with 100% α decay. Using the analytical nuclear potential (21), the folded Coulomb potential and the usual formula (3) for the WKB decay width Γ_{TPA} for regions away from the barrier gives a theoretical width of 1.232 MeV for this level. However, with the barrier height being 3.27 MeV and the Q value of the decay $^8\text{Be} \to \alpha + \alpha$ being 3.122 MeV, the use of the standard WKB formula (3) is not recommendable. Instead, if we use Eq. (10) but with a positive sign in the denominator as explained below Eq. (10), the width $\Gamma^{\text{top}}_{\text{Fröman}}(Q) = 1.535$ MeV and is closer to the experimental value of 1.513 MeV. Using the formula of [9], $\Gamma^{\text{top}}_{\text{DH}}(Q) = 1.53$ MeV which is again close to the experimental number as well as consistent with $\Gamma^{\text{top}}_{\text{Fröman}}(Q)$. If we use the expression of [4] as it is in Eq. (10) with a minus sign in the denominator, the width turns out to be 0.63 MeV which indeed worsens the result of 1.232 MeV, obtained with the standard WKB formula and also disagrees with the $\Gamma_{DH}^{top}(Q) = 1.53$ MeV. Since the authors had a choice of the sign in [4] and chose the negative sign without any particular argument, we guess that the choice should have rather been the opposite and the expression should be read with a $T + 2\hbar\, d\sigma/dE$ in the denominator.

V. CONCLUSIONS

To summarize, we first performed a survey of the available WKB decay width formulas in literature, which were obtained using different models and approaches. After having noted the similarities as well as differences in the various approaches, we applied them to the calculation of the half-lives of superheavy nuclei which form a topic of current interest. The motivation to apply for the case of superheavy nuclei was also to emphasize the need for performing calculations remaining consistently within the spirit of the WKB approximation. Following are the main observations of the present work:

(i) The decay widths of the super heavy nuclei are sensitive to the input of density dependence in the nucleon-nucleon interaction of the nucleons in the α particle and the daughter nucleus. Since the half-lives can reduce by an order of magnitude as compared to the results with a free NN interaction, any conclusions drawn in such works regarding the angular momentum, 'l' values, become model dependent.

(ii) Conclusions obtained in some recent fission model based calculations of superheavy nuclei neglect the Bohr-Sommerfeld condition which amounts to discarding the semiclassical nature inherent to the WKB approach. We find once again that the half-lives can change by orders of magnitude by neglecting this condition.

(iii) The results as in (ii) above, often seem to be in agreement with data (see $t^c_{1/2}$ in Tables I and II). However, in comparing theory with experiment one should not get tempted to choose an inconsistent approach for the sake of obtaining agreement as is often done in the SAFM calculations.

(iv) The assault frequencies appearing in the fission model calculations are shown to disagree with three other approaches existing in literature. These frequencies and hence the vibrational energies which are fitted in the fission model based calculations are in principle inconsistent with the formulas obtained from the standard WKB method.

(v) Improved formulas for the decay at the top of the barrier are compared and applied to a realistic example. We suggest the flip of a sign in the denominator of the expression (10) obtained in the work of Drukarev, Fröman, and Fröman [4]. Such a sign flip brings the results in agreement with experiments as well as with Eq. (9) for the width from another work [9]. The sign flip is consistent with the theory in [4], since at some point in their derivation one encounters a choice of signs.

To compare theoretical WKB widths with experiment, it is mandatory to perform a consistent calculation, taking into account carefully the details like the Langer modification, Bohr-Sommerfeld condition and a theoretically derived vibrational energy. We conclude by mentioning that any attempt to extract physical information from fitted parameters (such as the alpha cluster preformation probabilities or the unknown angular momenta of superheavy nuclei) while calculating the half-lives within the WKB approximation should bear in mind the limitations introduced in the model due to the sensitivities mentioned above.

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