Comment on "Dynamics of nuclear fluid. VIII. Time-dependent Hartree-Fock approximation from a classical point of view"

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The phase-space paths introduced by Cheuk-Yin Wong in Phys. Rev. C [**25**, 1460 (1982)] and discussed recently in the literature can be used for calculation of the evolution of the Wigner function to the first order in time increments only. The first-order solutions are helpful in the determination the phase-space Green function in the framework of the phase-space path integral method.

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Quantum trajectories exist in the de Broglie-Bohm theory [1–3] and appear in the framework of the deformation quantization [4–9] as the Weyl's symbols of the Heisenberg operators of canonical coordinates and momenta [10–14]. In the de Broglie-Bohm theory, the particle trajectories play an important role in the interpretation of measurements, whereas within the deformation quantization framework quantum trajectories have properties assigned to characteristic lines of first-order partial differential equations (PDE). The evolution equation for the Wigner function is the infinite-order PDE; nevertheless, it be solved with the help of quantum characteristics. The interest in particle trajectories is motivated by the fact that they remain a striking but intuitive feature of transport models in heavy-ion collisions [15–18].

Already some time ago, Wong [19] proposed a new kind of quantum trajectory for solving the evolution problem of the Wigner function. Recently these ideas were discussed in Refs. [20] and [21].

We wish to point out that the phase-space trajectories of Ref. [19] can be used to find the evolution of the Wigner function to the first order in time increments only.

Consider the double Fourier transform of the Wigner function $W(\mathbf{q}, \mathbf{p}, t)$:

$$W(\mathbf{q}, \mathbf{p}, \tau) = \int \frac{d\mathbf{s} d\mathbf{p}'}{(2\pi\hbar)^3} \exp\left(\frac{i}{\hbar}\mathbf{s}(\mathbf{p} - \mathbf{p}')\right) W'(\mathbf{q}, \mathbf{p}', \tau). \quad (1)$$

We restrict ourselves to Hamiltonian functions of the form

$$H(\mathbf{q}, \mathbf{p}) = \tfrac{1}{2}\mathbf{p}^2 + V(\mathbf{q}). \quad (2)$$

Using representation (1), the quantum Liouville equation

$$\frac{\partial}{\partial \tau} W(\mathbf{q}, \mathbf{p}, \tau) = -W(\mathbf{q}, \mathbf{p}, \tau) \wedge H(\mathbf{q}, \mathbf{p}), \quad (3)$$

where $\wedge$ is the Moyal bracket [7–9], can be represented in the form

$$0 = \int \frac{d\mathbf{s} d\mathbf{p}'}{(2\pi\hbar)^3} \exp\left(\frac{i}{\hbar}\mathbf{s}(\mathbf{p} - \mathbf{p}')\right) \left(\frac{\partial}{\partial \tau} + \mathbf{p}\frac{\partial}{\partial \mathbf{q}} \right.$$
$$\left. -\frac{i}{\hbar}\left(V\left(\mathbf{q} + \frac{\mathbf{s}}{2}\right) - V\left(\mathbf{q} - \frac{\mathbf{s}}{2}\right)\right)\right) W'(\mathbf{q}, \mathbf{p}', \tau). \quad (4)$$

It is tempting to require

$$0 \stackrel{?}{=} \left(\frac{\partial}{\partial \tau} + \mathbf{p}\frac{\partial}{\partial \mathbf{q}} - \frac{i}{\hbar}\left(V\left(\mathbf{q} + \frac{\mathbf{s}}{2}\right)\right.\right.$$
$$\left.\left. - V\left(\mathbf{q} - \frac{\mathbf{s}}{2}\right)\right)\right) W'(\mathbf{q}, \mathbf{p}', \tau). \quad (5)$$

This is the first-order PDE. It can be solved using the method of characteristics:

$$W'(\mathbf{q}, \mathbf{p}', \tau) = W'(\mathbf{q} - \mathbf{p}\tau, \mathbf{p}', 0)$$
$$\times \exp\left(\frac{i}{\hbar}\int_0^\tau d\tau'\left(V\left(\mathbf{q} - \mathbf{p}(\tau - \tau') + \frac{\mathbf{s}}{2}\right)\right.\right.$$
$$\left.\left. - V\left(\mathbf{q} - \mathbf{p}(\tau - \tau') - \frac{\mathbf{s}}{2}\right)\right)\right). \quad (6)$$

We observe that Eq. (5) and its solution, Eq. (6), depend on $\mathbf{s}$ and $\mathbf{p}$.

Given the dependence on $\mathbf{s}$ enters into $W'(\mathbf{q}, \mathbf{p}', \tau)$, equation $W(\mathbf{q}, \mathbf{p}, \tau) \equiv W'(\mathbf{q}, \mathbf{p}, \tau)$ does not hold anymore; however, Eq. (4) is still valid. The appearance of $\mathbf{s}$ in $W'(\mathbf{q}, \mathbf{p}', \tau)$ does not violate the equivalence of Eqs. (3) and (4). However, given the dependence on $\mathbf{p}$ enters $W'(\mathbf{q}, \mathbf{p}', \tau)$, Eq. (4) becomes distinct from Eq. (3).

There is, therefore, the obvious shift in the definition of W' from Eq. (1) to Eq. (5), because W' depends in Eq. (5) on a parameter. This parameter, $\mathbf{p}$, is not strictly related to the same notation in Eq. (1) on which W depends but W' does not. The arguments leading to Eqs. (5) and (6) are thereby not founded.

On the other hand, suppose that Eq. (6) holds nevertheless. Consider the evolution of a one-dimensional oscillator with a potential $V(q) = \frac{1}{2}q^2$. The Wigner function of the ground state has the form

$$W(q, p) = 2\exp(-q^2/\hbar - p^2/\hbar). \quad (7)$$

Applying Eqs. (1) and (6), we obtain

$$W(q, p, \tau) \stackrel{??}{=} 2\exp(-(q + p\tau)^2/\hbar - (p - q\tau - p\tau^2/2)^2/\hbar). \quad (8)$$

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The Wigner function of stationary states does not depend on time. This particular contradiction proves that Eqs. (5) and (6) are incorrect for finite τ.

Equation (8) holds, however, to order $O(\tau)$. One can verify, indeed, that Eq. (6) upon the double Fourier transform (1) gives the Wigner function correct to order $O(\tau)$.

In Ref. [21] a numerical example is considered to compare the exact evolution of the Wigner function with the Wigner function calculated by the infinitesimal step-by-step evolution according to Eq. (6). A discrete form of the phase-space path integral is calculated per se. Equation (6) in its infinitesimal form leads to the representation of the path integral discussed by Leaf [22] and Marinov [23]. The first-order solution is therefore useful to obtain the correct numerical results [21].

The role of trajectories entering the phase-space path integral is distinct from the role of characteristics in phase space. Particle trajectories of the de Broglie-Bohm theory depend on quantum states. To find them one has to know the wave function. The phase-space trajectories appearing as the Weyl's symbols of the Heisenberg operators of canonical coordinates and momenta are currently the only known objects that can be treated as quantum characteristics. They constitute the complete information on the evolution of quantum systems and can be used to find the evolution of the Wigner function for finite time intervals.

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