Light nuclei as quantized Skyrmions

Olga V. Manko,* Nicholas S. Manton,† and Stephen W. Wood‡
DAMTP, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, United Kingdom
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We consider the rigid-body quantization of Skyrmions with topological charges 1 to 8, as approximated by the rational map ansatz. Novel general expressions for the elements of the inertia tensors, in terms of the approximating rational map, are presented and are used to determine the kinetic energy contribution to the total energy of the ground and excited states of the quantized Skyrmions. Our results are compared to the experimentally determined energy levels of the corresponding nuclei, and the energies and spins of a few as yet unobserved states are predicted.

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I. INTRODUCTION

The Skyrme model [1] is a nonlinear effective theory of mesons, specifically pions. Its nonlinearity allows for the existence of topological soliton solutions, labeled by an integer-valued topological charge, B. A quantized Skyrmion of topological charge B is interpreted as a nucleus with baryon number B.

The $B = 1$ Skyrmion was first quantized by Adkins, Nappi, and Witten [2] and subsequently by Adkins and Nappi [3], taking account of the positive pion mass. They provided the first calibration of the Skyrme model, by fitting the model to the proton and delta masses. The toroidal $B = 2$ Skyrmion was quantized in Refs. [4,5], and the energies corresponding to the ground state, representing the deuteron, and excited states were calculated. This analysis was extended in Ref. [6], to allow the two single Skyrmions to separate in the most attractive channel. This led to a more accurate determination of the mean charge radius, as the deuteron is rather loosely bound.

The interpretation of the nuclei helium-3 and hydrogen-3 (triton) as quantized states of the $B = 3$ Skyrmion was considered by Carson [7], and the spins and energies were calculated. This analysis was extended in Ref. [8] by a computation of the static electroweak properties of the quantized Skyrmion.

In Ref. [9], the $B = 4$ Skyrmion was semiclassically quantized, and the ground state (corresponding to the α particle) and first excited state were determined, and their energies were calculated. The results, though novel, involved consideration of selected vibrational modes, as well as rigid-body motion, and are difficult to generalize to higher baryon numbers. Here we will consider the $B = 4$ case from a different perspective, which may easily be generalized to higher baryon numbers and enables us to compute the excitation energies of further excited states.

Further results on the allowed spin and isospin states of quantized Skyrmions for B up to 8 and beyond have been obtained by Irwin [10] and taken further by Krusch [11]. However, in this work, there were no estimates of the energies of the states.

It is not easy to assess the qualitative success of the Skyrme model, as calibrated by Adkins and Nappi, just from the results for $B \leqslant 4$. The nuclei have the correct spin and isospin quantum numbers, but on the whole the ground states represent nuclei that are too small and too tightly bound. This traditional parameter choice was too reliant on its fit to the mass of the delta resonance, for which the rigid-body approximation is especially poor, because it does not allow the Skyrmion to deform as it spins [12], nor to radiate pions. Nuclei with $B = 2$ or $B = 3$ have no excited states, experimentally, so Skyrmion excited states based on rigid-body quantization are not meaningful, and one expects them to break up into individual nucleons if further degrees of freedom are included. Rigid-body quantization should be much more reliable for Skyrmions of larger B, since the angular velocities are much smaller for a given angular momentum, and the excitation energies are smaller.

There have been a number of developments that make it worthwhile to reconsider these results on quantized Skyrmions, and it is also possible to extend them to the range of baryon numbers $1 \leqslant B \leqslant 8$. First, it has been noted that a reparametrization of the Skyrme model is desirable to achieve a better fit to nuclear sizes and related quantities such as moments of inertia [13]. The Skyrme length scale should be roughly doubled, and consequently the dimensionless pion mass parameter also doubled (to keep the physical pion mass fixed). Doubling the pion mass parameter has little effect on the qualitative character of classical Skyrmion solutions up to $B = 7$, but for $B \geqslant 8$ there is a clear difference [14]. The stable solutions are no longer the hollow polyhedra found earlier for B up to 22 and beyond, but instead they are more dense structures closer to what one expects for nuclei. In particular, for B a multiple of 4, there are stable solutions that look like bound states of two or more of the cubically symmetric $B = 4$ Skyrmions [15]. We shall analyze in the following the quantum states of the $B = 8$ Skyrmion, which is made up of two $B = 4$ cubes, and compare with the states of nuclei with $B = 8$, including beryllium-8. This reparametrization is at the expense of exact fits to the nucleon and delta in the $B = 1$ sector but, as already mentioned, these are suspect.

*O.V.Manko@damtp.cam.ac.uk
†N.S.Manton@damtp.cam.ac.uk
‡S.W.Wood@damtp.cam.ac.uk

Another development is a better understanding of the quantization rules for Skyrmions, the so-called Finkelstein-Rubinstein (FR) constraints [16], which encode the requirement that a quantized $B = 1$ Skyrmion is a spin $\frac{1}{2}$ fermion. The FR constraints combine the symmetry of a Skyrmion, for any value of B, with the topology of the Skyrme model, to constrain the spins and isospins of quantum states. Here the rational map ansatz is relevant [17]. It gives a separation of variables between the angular and radial dependence of the Skyrme field. True solutions do not exactly exhibit this separation, but they do so approximately. The ansatz gives a simple closed formula for the angular dependence of a Skyrme field, and rotational symmetries are easier to find than if one just has a numerical Skyrmion solution. The optimized rational map ansatz gives good approximations to true solutions up to $B = 7$ (and far beyond for zero or small pion mass). Even if it is a poor approximation, it can still be helpful in the numerical search for true solutions, and more importantly here, it is helpful in determining the effect of the FR constraints. Krusch has recently found a simple formula for determining the crucial signs that occur in the FR constraints [11]. This formula requires knowledge of the rational map approximating the Skyrmion.

The rational map ansatz allows a further simplification, valid to the extent that a Skyrmion is well approximated by the ansatz. Kopeliovich noted that the moments of inertia (both rotational and isorotational) of a Skyrme field described by the ansatz are rather simpler than for a general Skyrme field [18], since the effect of a rotation is just to rotate the map, leaving the radial profile function invariant. We simplify Kopeliovich's formulas further, taking advantage of the complex analytic character of a rational map, and obtain formulas for the 36 components of the spin/isospin inertia tensor. These can be accurately evaluated, and it is easy to recognize if certain components vanish because of symmetry. Using these moments of inertia, we estimate anew the energies of ground and excited states of quantized Skyrmions over the range of baryon numbers $1 \leqslant B \leqslant 4$, and for the first time those in the range $5 \leqslant B \leqslant 8$. The quantization is based on the established method of rigid-body quantization of rotations and isospin rotations. Particularly interesting for us are the states of the $B = 6$ Skyrmion, because our reparametrization of the Skyrme model [13] was based on the mass and charge radius of the lithium-6 nucleus. Also interesting are the states for $B = 8$, because the double cube $B = 8$ Skyrmion has not previously been quantized.

A problem for the Skyrme model that emerged in the work of Irwin [10] is that the spin states of the $B = 5$ and $B = 7$ Skyrmions disagree with those of the corresponding nuclei in their ground states. It has been suggested more than once (see, e.g., Ref. [19]) that it might be appropriate, for these baryon numbers, to quantize a deformed Skyrmion with different symmetry. This would make sense, especially if the allowed spins were thereby reduced, making the spin energy smaller. The smaller spin energy might more than compensate for the increased classical energy of the deformed Skyrmion. In this paper we are able to quantitatively assess this idea. For $B = 7$ it looks reasonable. A ground state with the correct spin $\frac{3}{2}$ for the lithium-7/beryllium-7 isodoublet can be obtained, and the previously found spin $\frac{7}{2}$ state can be interpreted as the observed, relatively low-lying second excited state. The spin $\frac{1}{2}$ first excited state is still problematic, however. For $B = 5$ the situation is less satisfactory.

In the next section we review the Skyrme model and briefly describe the recent reparametrization of the Skyrme model using the lithium-6 nucleus [13]. Although this is very important, we show that a reparametrization alone cannot solve all the problems of the Skyrme model. In Sec. III we describe the rational map ansatz for Skyrmions. Section IV deals with the quantization of Skyrmions, which proceeds by parametrizing time-dependent solutions through collective coordinates. Here, we recall how the model is fermionically quantized by the imposition of FR constraints. In Sec. V we present expressions for the inertia tensors that appear in the formula for the kinetic energy operator, in terms of the approximating rational map. Sections VI–XIII deal with the energy levels of quantized Skyrmions of baryon numbers 1 to 8, respectively. In Sec. XIV we provide a conclusion.

II. THE SKYRME MODEL

The Skyrme model is defined in terms of an SU(2)-valued scalar, the Skyrme field [1,20]. We call the topological soliton solutions that it admits *Skyrmions*. It is a low-energy effective theory of QCD, becoming exact as the number of quark colors, N_c, becomes large [21,22]. Recent work inspired by string theory and the AdS/CFT correspondence gives further credence to the idea that, at large N_c, baryons and nuclei are described by some variant of the Skyrme model [23].

The Lagrangian density is given by

$$\mathcal{L} = \frac{F_\pi^2}{16}\,\mathrm{Tr}\,\partial_\mu U \partial^\mu U^{-1} + \frac{1}{32e^2}\,\mathrm{Tr}\,[\partial_\mu U U^{-1}, \partial_\nu U U^{-1}] \times [\partial^\mu U U^{-1}, \partial^\nu U U^{-1}] + \frac{1}{8} m_\pi^2 F_\pi^2\,\mathrm{Tr}\,(U - 1_2), \quad (1)$$

where $U(t, \mathbf{x})$ is the Skyrme field, F_π is the pion decay constant, e is a dimensionless parameter, and m_π is the pion mass.

Using energy and length units of $F_\pi/4e$ and $2/eF_\pi$, respectively, we may express the Lagrangian as follows:

$$L = \int \left\{ -\frac{1}{2}\,\mathrm{Tr}\,(R_\mu R^\mu) + \frac{1}{16}\,\mathrm{Tr}\,([R_\mu, R_\nu][R^\mu, R^\nu]) + m^2\,\mathrm{Tr}\,(U - 1_2) \right\} d^3x, \quad (2)$$

where we have introduced the $\mathfrak{su}(2)$-valued current $R_\mu = (\partial_\mu U)U^{-1}$ and defined the dimensionless pion mass parameter $m = 2m_\pi/eF_\pi$.

Field configurations of finite energy must satisfy the boundary condition $U \to 1_2$ as $|\mathbf{x}| \to \infty$. This compactifies $\mathbb{R}^3$ to a 3-sphere of infinite size, and so topologically $U: S^3 \to S^3$ at a fixed time. Field configurations U therefore lie in topological sectors labeled by their topological degree

$$B = \int B_0(\mathbf{x}) d^3x, \quad (3)$$

where

$$B_\mu(x) = \frac{1}{24\pi^2}\,\epsilon_{\mu\nu\alpha\beta}\,\mathrm{Tr}\partial^\nu U U^{-1}\partial^\alpha U U^{-1}\partial^\beta U U^{-1}. \quad (4)$$

The degree B, which takes integer values, is identified with the baryon number. We refer to B_0 as the baryon density.

The kinetic part of the Lagrangian L is

$$T = \int \left\{ -\frac{1}{2}\,\mathrm{Tr}\,(R_0 R_0) - \frac{1}{8}\,\mathrm{Tr}\,([R_i, R_0][R_i, R_0]) \right\} d^3x, \quad (5)$$

and this is quadratic in the time derivative of the Skyrme field. The rest of the Lagrangian (2) is (minus) the potential energy:

$$E = \int \left\{ -\frac{1}{2}\,\mathrm{Tr}\,(R_i R_i) - \frac{1}{16}\,\mathrm{Tr}\,([R_i, R_j][R_i, R_j]) - m^2 \mathrm{Tr}(U - 1_2) \right\} d^3x. \quad (6)$$

Static Skyrmion solutions can be obtained by solving the variational equations derived from E, or in practice by numerically minimizing E in the sector with given B.

The parameters e and F_π can be fixed in a number of ways. It has been common practice to use the set of parameters given in Ref. [3] with the physical pion mass taken into account, specifically

$$e = 4.84, \quad F_\pi = 108\,\mathrm{MeV}, \quad \text{and} \quad m_\pi = 138\,\mathrm{MeV} \quad (\text{which implies } m = 0.528). \quad (7)$$

In Ref. [3], the values of e and F_π were tuned to reproduce the masses of the proton and the delta resonance, but this can be criticized on the grounds mentioned before. This parameter set was adjusted to optimize the predictions of the model in the $B = 1$ sector at the expense of the $B = 0$ sector, which requires $F_\pi = 186$ MeV. It is not, therefore, the optimal parameter set globally.

In Ref. [13], we showed that for the Skyrme model to more closely model nuclear properties, a reparametrization is necessary. We performed such a reparametrization by matching the model in the $B = 6$ sector with properties of the lithium-6 nucleus, obtaining

$$e = 3.26, \quad F_\pi = 75.2\,\mathrm{MeV}, \quad \text{and} \quad m_\pi = 138\,\mathrm{MeV}\,(\text{which implies } m = 1.125). \quad (8)$$

Figure 1 shows graphs of nuclear masses, M_B, and static Skyrmion masses, $\mathcal{M}_B$, per unit baryon number, using this new parameter set. The Skyrmion quantum energies are not included. We observe that the graphs intersect at $B = 6$, as expected. It is clear that it is not possible by a single parameter choice to correctly match nuclear and Skyrmion masses for all baryon numbers and this remains the case when the quantum spin and isospin energies are included. We believe that calibrating the model in the $B = 6$ sector is a promising way to describe the properties of nuclei with $B \geqslant 4$. For $B = 1, 2, 3$ the Skyrmion energies are now too high, and neither the nucleon mass nor delta resonance will be accurately fitted. Ideally, one would like a variant of the Skyrme model in which the nuclear mass per unit baryon number declines more slowly with baryon number. Equivalently, one would like a variant in which the Faddeev-Bogomolny bound ($\mathcal{M}_B \geqslant cB$, where c is a constant) is almost saturated. Perhaps a low-energy model of QCD based on the AdS/CFT correspondence would achieve this.

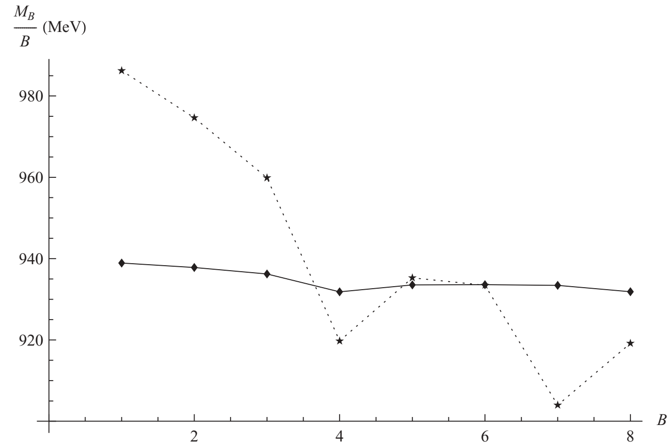

FIG. 1. Nuclear masses per unit baryon number (M_B/B) (solid), compared with static Skyrmion masses per unit baryon number (dotted).

For consistency, in the following sections we use the new parameter set [Eqs. (8)] throughout, making a few remarks about the old parameters in Appendix B.

III. THE RATIONAL MAP ANSATZ

We describe here the ansatz for Skyrme fields that uses rational maps between Riemann spheres to describe their angular behavior [17]. This has been shown to give good approximations to several known Skyrmions, including all the minimal-energy solutions up to $B = 7$ (and much higher B when $m = 0$). The rational maps have exactly the same symmetries as the numerically known Skyrmions in almost all cases (with $B = 14$ as an exception [24]).

Via stereographic projection, the complex coordinate z encodes the conventional polar coordinates as $z = \tan(\theta/2)e^{i\phi}$. Equivalently, the point z on a sphere corresponds to the radial unit vector

$$\mathbf{n}_z = \frac{1}{1+|z|^2}[z+\bar{z}, i(\bar{z}-z), 1-|z|^2], \quad (9)$$

and inversely

$$z = \frac{(\mathbf{n}_z)_1 + i(\mathbf{n}_z)_2}{1+(\mathbf{n}_z)_3}. \quad (10)$$

The ansatz for the Skyrme field depends on a rational map $R(z) = p(z)/q(z)$, where p and q are polynomials in z, and a radial profile function $f(r)$. The target value R is associated with a point in the unit 2-sphere of the Lie algebra of SU(2), given by the unit vector

$$\mathbf{n}_R = \frac{1}{1+|R|^2}[R+\bar{R}, i(\bar{R}-R), 1-|R|^2]. \quad (11)$$

The ansatz is then

$$U(r, z) = \exp\,[if(r)\mathbf{n}_{R(z)}\cdot\boldsymbol{\tau}], \quad (12)$$

where τ_1, τ_2, and τ_3 are Pauli matrices and $f(r)$ satisfies $f(0) = \pi$ and $f(\infty) = 0$.

Using this ansatz gives a baryon number of

$$B = \int \frac{-f'}{2\pi^2}\left(\frac{\sin f}{r}\right)^2 \left(\frac{1+|z|^2}{1+|R|^2}\left|\frac{dR}{dz}\right|\right)^2 \frac{2i\,dz\,d\bar{z}}{(1+|z|^2)^2} r^2\,dr, \tag{13}$$

and it can be shown that this is an integer equal to the degree of the rational map R.

The energy E for a field of form (12) is

$$E = 4\pi \int_0^\infty \left(r^2 f'^2 + 2B\sin^2 f(f'^2+1) + \mathcal{I}\frac{\sin^4 f}{r^2} + 2m^2r^2(1-\cos f)\right) dr, \tag{14}$$

in which $\mathcal{I}$ denotes the angular integral

$$\mathcal{I} = \frac{1}{4\pi}\int \left(\frac{1+|z|^2}{1+|R|^2}\left|\frac{dR}{dz}\right|\right)^4 \frac{2i\,dz\,d\bar{z}}{(1+|z|^2)^2}. \tag{15}$$

To minimize E one first minimizes $\mathcal{I}$ over all maps of degree B. The profile function $f(r)$ is then found by solving the second-order ordinary differential equation that is the Euler-Lagrange equation for the expression (14) with B and $\mathcal{I}$ as fixed parameters. Given the profile function, the energy is determined by numerical integration. This gives the optimized rational map ansatz, and we denote the minimized energy by $\mathcal{M}_B$. This is our estimate for the true Skyrmion mass, for baryon number B. To obtain a physical value to compare to a nuclear mass, one multiplies by the energy unit $F_\pi/4e = 5.76$ MeV, obtaining a classical Skyrmion mass in MeV.

IV. QUANTIZING THE SKYRME MODEL

Given a static Skyrmion $U_0(\mathbf{x})$, there is generically a nine-parameter set of solutions, each with the same energy, obtained by acting with the Euclidean group and isorotations:

$$U(\mathbf{x}) = A_1 U_0[D(A_2)(\mathbf{x}-\mathbf{X})]A_1^{-1}, \tag{16}$$

where A_1, A_2 are SU(2) matrices and A_2 is recast in the SO(3) form $D(A_2)_{ij} = \frac{1}{2}\mathrm{Tr}(\tau_i A_2 \tau_j A_2^{-1})$. Semiclassical quantization is performed by promoting the collective coordinates $A_1, A_2, \mathbf{X}$ to dynamical degrees of freedom [5]. As we shall only be concerned with the computation of spin and isospin, we shall ignore the translational degrees of freedom $\mathbf{X}$ and quantize the solitons in their zero-momentum frame.

Making the replacement $U(\mathbf{x}) \to \hat{U}(\mathbf{x},t) = A_1(t)U_0[D(A_2(t))\mathbf{x}]A_1(t)^{-1}$, and inserting this into the Skyrme Lagrangian, one obtains the kinetic contribution to the total energy:

$$T = \tfrac{1}{2}a_i U_{ij} a_j - a_i W_{ij} b_j + \tfrac{1}{2} b_i V_{ij} b_j, \tag{17}$$

where

$$a_j = -i\,\mathrm{Tr}\,\tau_j A_1^{-1}\dot{A}_1, \quad b_j = i\,\mathrm{Tr}\,\tau_j \dot{A}_2 A_2^{-1}. \tag{18}$$

Here $\mathbf{b}$ is the angular velocity in physical space, and $\mathbf{a}$ is the angular velocity in isospace. The inertia tensors U_{ij}, V_{ij}, and W_{ij} are given by

$$U_{ij} = -\int \mathrm{Tr}\left(T_iT_j + \frac{1}{4}[R_k, T_i][R_k, T_j]\right) d^3x, \tag{19}$$

$$V_{ij} = -\int \epsilon_{ilm}\epsilon_{jnp} x_l x_n \mathrm{Tr}\left(R_m R_p + \frac{1}{4}[R_k, R_m][R_k, R_p]\right) \times d^3x, \tag{20}$$

$$W_{ij} = \int \epsilon_{jlm} x_l \,\mathrm{Tr}\left(T_i R_m + \frac{1}{4}[R_k, T_i][R_k, R_m]\right) d^3x, \tag{21}$$

where $R_k = (\partial_k U_0)U_0^{-1}$ is the right invariant $\mathfrak{su}(2)$ current defined previously and $T_i = \frac{i}{2}[\tau_i, U_0]U_0^{-1}$ is also an $\mathfrak{su}(2)$ current. The total energy, in terms of collective coordinates, is just T plus the constant $\mathcal{M}_B$, the static mass of the Skyrmion.

We may write

$$T = \tfrac{1}{2}c^{\mathrm{T}}\mathcal{W}c, \tag{22}$$

where $c^{\mathrm{T}} = (a_1, a_2, a_3, b_1, b_2, b_3)$ and the 6×6 symmetric matrix $\mathcal{W}$ is given by

$$\mathcal{W} = \begin{pmatrix} U & -W \\ -W^{\mathrm{T}} & V \end{pmatrix}. \tag{23}$$

The momenta corresponding to b_i and a_i are the body-fixed spin and isospin angular momenta L_i and K_i [5]:

$$L_i = -W_{ij}^{\mathrm{T}} a_j + V_{ij} b_j, \tag{24}$$

$$K_i = U_{ij} a_j - W_{ij} b_j. \tag{25}$$

The usual space-fixed spin and isospin angular momenta J_i and I_i are related to the body-fixed momenta by

$$J_i = -D(A_2)_{ij}^{\mathrm{T}} L_j, \quad I_i = -D(A_1)_{ij} K_j. \tag{26}$$

Defining $H^{\mathrm{T}} = (K_1, K_2, K_3, L_1, L_2, L_3)$, and using the relation $H^{\mathrm{T}} = c^{\mathrm{T}}\mathcal{W}$, we find, provided $\det\mathcal{W} \neq 0$,

$$T = \tfrac{1}{2}H^{\mathrm{T}}\mathcal{W}^{-1}H. \tag{27}$$

We now promote the four sets of classical momenta just introduced to quantum operators, each individually satisfying the $\mathfrak{su}(2)$ commutation relations. The Casimir invariants satisfy $\mathbf{J}^2 = \mathbf{L}^2$ and $\mathbf{I}^2 = \mathbf{K}^2$.

The basic FR constraints, which apply to any Skyrmion, are that physical quantum states $|\Psi\rangle$ should satisfy

$$e^{2\pi i \mathbf{n}\cdot\mathbf{L}}|\Psi\rangle = e^{2\pi i\mathbf{n}\cdot\mathbf{K}}|\Psi\rangle = (-1)^B|\Psi\rangle, \tag{28}$$

for any unit vector $\mathbf{n}$, which implies that for even B the spin and isospin are integral, and for odd B they are half-integral. There are further FR constraints on states if the Skyrmion has symmetries, and these are simple to determine if the Skyrmion is described by the rational map ansatz. A rational map, and hence the corresponding Skyrmion, has a rotational symmetry if it satisfies an equation of the form

$$R[M_2(z)] = M_1[R(z)], \tag{29}$$

for some combination of SU(2) Möbius transformations M_2 and M_1. M_2 corresponds to a rotation in physical space, and M_1 to an isorotation. In general there will be a group $\mathcal{S}$ of such symmetries. We say that the map R is $\mathcal{S}$-symmetric if, for each $M_2 \in \mathcal{S}$, there exists an M_1 such that Eq. (29) holds.

For consistency, pairs (M_2, M_1) must have the same composition rule as in $\mathcal{S}$, so $R[M_2 M_2'(z)] = M_1 M_1'[R(z)]$. The map $M_2 \to M_1$ is therefore a homomorphism. Note that it is not possible to construct such a map from M_1 to M_2. This is related to the fact that a Skyrmion may be invariant under a rotation alone, but it cannot be invariant under an isorotation alone.

Consider a rotation in physical space by an angle θ_2 about an axis $\mathbf{n}_2$, and an isorotation by an angle θ_1 about an axis $\mathbf{n}_1$. We recall that under such a rotation, z transforms to $M_2(z)$, given by [11]

$$M_2(z) = \frac{\left[\cos\frac{\theta_2}{2} + i(\mathbf{n}_2)_3 \sin\frac{\theta_2}{2}\right]z + \left[(\mathbf{n}_2)_2 - i(\mathbf{n}_2)_1\right]\sin\frac{\theta_2}{2}}{\left[-(\mathbf{n}_2)_2 - i(\mathbf{n}_2)_1\right]\sin\frac{\theta_2}{2} z + \left[\cos\frac{\theta_2}{2} - i(\mathbf{n}_2)_3 \sin\frac{\theta_2}{2}\right]}. \tag{30}$$

Similarly, under such an isorotation, R transforms to $M_1(R)$, given by

$$M_1(R) = \frac{\left[\cos\frac{\theta_1}{2} + i(\mathbf{n}_1)_3 \sin\frac{\theta_1}{2}\right]R + [(\mathbf{n}_1)_2 - i(\mathbf{n}_1)_1]\sin\frac{\theta_1}{2}}{[-(\mathbf{n}_1)_2 - i(\mathbf{n}_1)_1]\sin\frac{\theta_1}{2} R + \left[\cos\frac{\theta_1}{2} - i(\mathbf{n}_1)_3 \sin\frac{\theta_1}{2}\right]}. \tag{31}$$

So given a specific symmetry [Eq. (29)] of a rational map, we use these formulas to determine the corresponding angles and axes of rotation and isorotation. These are the data that are used in the conventional formulas describing the effect of rotations on quantized rigid bodies [25].

For θ_2 not an integer multiple of 2π, M_2 only leaves the points

$$z_{\mathbf{n}_2} = \frac{(\mathbf{n}_2)_1 + i(\mathbf{n}_2)_2}{1 + (\mathbf{n}_2)_3} \quad \text{and} \quad z_{-\mathbf{n}_2} = \frac{-(\mathbf{n}_2)_1 - i(\mathbf{n}_2)_2}{1 - (\mathbf{n}_2)_3} \tag{32}$$

fixed. Similarly, M_1 only leaves $R_{\pm\mathbf{n}_1}$ fixed, where $R_{\pm\mathbf{n}_1}$ are defined similarly. Therefore, for the symmetry (29) to hold, we have

$$R(z_{-\mathbf{n}_2}) = R_{\mathbf{n}_1} \quad \text{or} \quad R_{-\mathbf{n}_1}. \tag{33}$$

Krusch showed that to correctly determine the FR constraint it is important to choose the direction of the axis $\mathbf{n}_1$ so as to satisfy the base point condition[1]

$$R(z_{-\mathbf{n}_2}) = R_{-\mathbf{n}_1}. \tag{34}$$

The symmetry (29) then leads to the following FR constraint on the wave function:

$$e^{i\theta_2 \mathbf{n}_2\cdot\mathbf{L}} e^{i\theta_1 \mathbf{n}_1\cdot\mathbf{K}}|\Psi\rangle = \chi_{\mathrm{FR}}|\Psi\rangle, \tag{35}$$

a representation-independent statement, in which $\mathbf{L}$ and $\mathbf{K}$ are the body-fixed spin and isospin operators, respectively, and the FR sign $\chi_{\mathrm{FR}} = \pm 1$. The combined rotation and isorotation corresponding to M_2 and M_1, respectively, at most change the state by a sign factor, χ_{FR}. It was proved in Ref. [11] that the value of χ_{FR} for a given symmetry of a rational map only depends on θ_2 and θ_1, where the angles have unambiguous signs because of the base point condition [Eq. (34)], and is given by

$$\chi_{\mathrm{FR}} = (-1)^{\mathcal{N}}, \quad \text{where } \mathcal{N} = \frac{B}{2\pi}(B\theta_2 - \theta_1). \tag{36}$$

The FR signs χ_{FR} form a one-dimensional representation of the symmetry group $\mathcal{S}$ of the Skyrmion.

A basis for the wave functions is given by $|J, L_3\rangle \otimes |I, K_3\rangle$, the tensor product of states of a rigid body in space and a rigid body in isospace. Here we suppress the additional labels J_3 and I_3, which can take any values in the usual ranges allowed by J and I. J_3 is the physically meaningful projection of spin on the third space axis, and I_3 is the conventional third component of isospin. When we come to consider specific rational maps and the associated FR constraints, we seek low-energy states that are allowed by the FR constraints, and may represent the spin and isospin operators appearing on the left-hand side of Eq. (35) by Wigner D matrices acting on the $(2J+1)\times(2I+1)$-dimensional space of wave functions.

Another advantage of the rational map ansatz is that it clearly illustrates any reflection symmetries of the Skyrmion, which enables one to determine the effect of the inversion operator $\mathcal{P}$, which acts as

$$\mathcal{P}: U(\mathbf{x}) \to U^\dagger(-\mathbf{x}) \tag{37}$$

and is baryon number preserving. For rational maps, the inversion $\mathbf{x} \to -\mathbf{x}$ corresponds to $z \to -1/\bar{z}$, and the inversion $U \to U^\dagger$ corresponds to $R \to -1/\bar{R}$. A rational map, and hence the corresponding Skyrmion, has a reflection symmetry if it satisfies an equation of the form

$$-1/\overline{R[M_2(z)]} = M_1[R(-1/\bar{z})]. \tag{38}$$

In this case, $\mathcal{P}$ is equivalently given by the combination of rotation and isorotation corresponding to M_2 and M_1 occurring here. The parity of a quantum state is then the eigenvalue of the state when acted upon by the operator $e^{i\theta_2\mathbf{n}_2\cdot\mathbf{L}}e^{i\theta_1\mathbf{n}_1\cdot\mathbf{K}}$ derived from M_2 and M_1. There is, however, an ambiguity in the definition of $\mathcal{P}$, which was first explained in Ref. [10]. Given a candidate parity operator $\mathcal{P}_0$ for a given Skyrmion, we can also represent the operator by $\mathcal{P}_0$ times any element of the symmetry group of the classical solution. If the FR sign of a particular symmetry element is -1, then these two choices for $\mathcal{P}$ give different results. In particular, there is this problem for odd B: Given a parity operator $\mathcal{P}_0$, we can also represent the operator by $\mathcal{P}_0 e^{2\pi i\mathbf{n}\cdot\mathbf{L}}$, where $\mathbf{n}$ is any unit vector. Because 2π rotations have associated FR signs of -1 for odd B, we see that these two choices differ when acting on states. For each of the cases $B = 1$ to 8, we make particular choices for the parity operators that we believe to be the most natural. We note that despite the ambiguity in the definition of $\mathcal{P}$ for a given Skyrmion, the relative parities of the Skyrmion's quantum states are fixed.

[1]We note that this differs slightly from the condition given in Ref. [11]. This is because we are working with the inverse of the isospatial Möbius transformation that was considered there.

V. TENSORS OF INERTIA FOR RATIONAL MAP SKYRMIONS

Kopeliovich [18] first presented general formulas for the inertia tensors of rational map Skyrmions. By writing $U_0 = \exp[if(r)\mathbf{n}\cdot\boldsymbol{\tau}]$, these can be expressed as follows [18]:

$$U_{ij} = 2\int \sin^2 f[(\delta_{ij} - n_i n_j)(1+f'^2) + \sin^2 f\, \partial_k n_i \partial_k n_j] \times d^3x, \quad (39)$$

$$V_{ij} = 2\int \sin^2 f\{(1+f'^2 + \sin^2 f\, \partial_k n_s \partial_k n_s) \times [\partial_m n_r \partial_m n_r (r^2\delta_{ij} - x_i x_j) - \partial_i n_r \partial_j n_r\, r^2] - \sin^2 f[\partial_m n_s \partial_k n_s \partial_m n_r \partial_k n_r (r^2\delta_{ij} - x_i x_j) - r^2 \partial_i n_r \partial_k n_r \partial_j n_s \partial_k n_s]\} d^3x, \quad (40)$$

$$W_{ij} = 2\int \epsilon_{jlm}\epsilon_{isp} x_l n_s \sin^2 f[(1+f'^2)\,\partial_m n_p + \sin^2 f\, \partial_k n_r(\partial_k n_r \partial_m n_p - \partial_m n_r \partial_k n_p)]\, d^3x. \quad (41)$$

These formulas for the inertia tensors assume that f depends only on r, and $\mathbf{n}$ depends only on the angular coordinates θ, ϕ; further simplifications can be made if we assume that $\mathbf{n}$ depends just on a rational function $R(z)$ as in Ref. (11). To obtain these simplified formulas, we find it helpful to write the $\mathbb{R}^3$ metric and volume element in terms of r, z, and $\bar{z}$:

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 = dr^2 + \frac{4r^2\, dz\, d\bar{z}}{(1+|z|^2)^2} = g_{\alpha\beta} dx^\alpha dx^\beta, \quad (42)$$

$$d^3x = \frac{4r^2\, dr\, dz\, d\bar{z}}{(1+|z|^2)^2}, \quad (43)$$

and to replace Cartesian derivatives with derivatives with respect to r, z, and $\bar{z}$. The products of commutators in Eqs. (19)–(21) may then be rewritten in these coordinates:

$$[R_k,\cdots][R_k,\cdots] = g^{rr}[R_r,\cdots][R_r,\cdots] + g^{z\bar{z}}[R_z,\cdots] \times [R_{\bar{z}},\cdots] + g^{\bar{z}z}[R_{\bar{z}},\cdots][R_z,\cdots], \quad (44)$$

where $R_z = (\partial_z U_0)U_0^{-1}$ etc. We also have

$$-i\epsilon_{jlm} x_l R_m = (l_j U_0)U_0^{-1} = \mu_j R_z - \bar{\mu}_j R_{\bar{z}}, \quad (45)$$

where

$$l_1 = -\frac{1}{2}\left((1-z^2)\frac{\partial}{\partial z} - (1-\bar{z}^2)\frac{\partial}{\partial \bar{z}}\right), \quad (46)$$

$$l_2 = -\frac{i}{2}\left((1+z^2)\frac{\partial}{\partial z} + (1+\bar{z}^2)\frac{\partial}{\partial \bar{z}}\right), \quad (47)$$

$$l_3 = z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}, \quad (48)$$

so

$$\mu_j = \left(-\frac{1}{2}(1-z^2),\ -\frac{i}{2}(1+z^2),\ z\right). \quad (49)$$

We ultimately find that, for the rational map ansatz, the tensors of inertia U_{ij}, V_{ij}, and W_{ij} can be expressed in the following form:

$$\Sigma_{ij} = 2\int \sin^2 f\, \frac{C_{\Sigma_{ij}}}{(1+|R|^2)^2} \times \left(1 + f'^2 + \frac{\sin^2 f}{r^2}\left(\frac{1+|z|^2}{1+|R|^2}\left|\frac{dR}{dz}\right|\right)^2\right) d^3x, \quad (50)$$

where $\Sigma = (U, V, W)$ and the quantities $C_{\Sigma_{ij}}$ (which are given explicitly in Appendix A) are functions of the variables z and $\bar{z}$ only. In what follows, we use this formula to numerically determine the elements of the inertia tensors, for a given rational map and profile function. The numerical values we obtain are, of course, in Skyrme units. To convert to physical values we must multiply these by the mass scale and by the square of the length scale: $(F_\pi/4e) \times (2/eF_\pi)^2 = 1/e^3 F_\pi$, obtaining quantities in inverse MeV. With the new parameter set [Eqs. (8)], $e^3 F_\pi = 2613$ MeV. Although our optimal rational maps are familiar [17], the profile functions have all been calculated anew using a shooting method.[2] In Fig. 2 we plot the profile functions for $B = 1$ to 8, using the new dimensionless pion mass parameter $m = 1.125$.

VI. $B = 1$

The rational map describing the single Skyrmion is given by $R(z) = z$, which is $O(3)$ symmetric. We find that the inertia tensors are each proportional to the unit matrix, satisfying $U_{ij} = V_{ij} = W_{ij} = \lambda\delta_{ij}$, where λ is given by

$$\lambda = \frac{16\pi}{3}\int r^2 \sin^2 f\left(1 + f'^2 + \frac{\sin^2 f}{r^2}\right) dr. \quad (51)$$

Numerically we compute $\lambda = 45.1$.

The FR constraints associated with spherical symmetry are

$$e^{i\theta\mathbf{n}\cdot\mathbf{L}} e^{i\theta\mathbf{n}\cdot\mathbf{K}}|\Psi\rangle = |\Psi\rangle, \quad (52)$$

where θ and $\mathbf{n}$ are arbitrary, leading to the following constraint on the space of physical states:

$$(\mathbf{L} + \mathbf{K})|\Psi\rangle = 0. \quad (53)$$

[2] This shooting method is from Bernard M. A. G. Piette, University of Durham, UK.

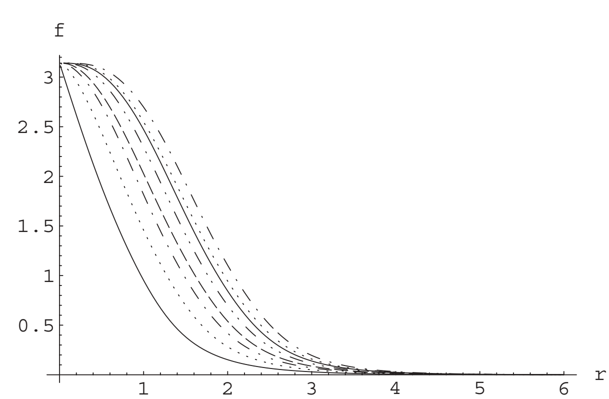

FIG. 2. The profile functions $f(r)$ for $B = 1$ to 8. B increases from left to right.

The "grand spin" $\mathbf{M} = \mathbf{L} + \mathbf{K}$ (or its components) will appear quite frequently in what follows. The states satisfying Eq. (53) are linear combinations of the states $|J, L_3\rangle \otimes |I, K_3\rangle$, whose grand spin is zero. These are of the form $|J, I; M, M_3\rangle = |J, J; 0, 0\rangle$, in the standard notation for adding angular momenta, where $J = L$ and $I = K$. So the spin J and isospin I have to have the same magnitude. In addition, J must be half-integral.

The kinetic energy operator is given by

$$T = \frac{1}{2\lambda}\mathbf{J}^2 = \frac{1}{2\lambda}\mathbf{I}^2. \tag{54}$$

The eigenvalue of $\mathbf{J}^2$ in states of spin J is $J(J+1)$, a standard result we will use frequently. Similarly, $\mathbf{I}^2$ has eigenvalues $I(I+1)$. For the lowest energy states, the nucleons with spin $\frac{1}{2}$ and isospin $\frac{1}{2}$, the spin energy in physical units is

$$\frac{1}{2(45.1)}\frac{3}{4}e^3 F_\pi = 21.7\,\text{MeV}, \tag{55}$$

and the total energy is

$$\begin{aligned} E_{J=1/2,\, I=1/2} &= \mathcal{M}_1 + 21.7\,\text{MeV} = 986.2\,\text{MeV} + 21.7\,\text{MeV} \\ &= 1008\,\text{MeV}. \end{aligned} \tag{56}$$

This is 69 MeV above the nucleon mass of 939 MeV. The spin energy is approximately one-quarter of its value with the old parameters, but the higher classical Skyrmion mass makes the total energy too high. In the following sections, we obtain ground-state energies that, when compared to the masses of constituent single Skyrmions, suggest that the corresponding nuclei are much too tightly bound. This is due to our overestimate of the nucleon mass.

The spin $\frac{3}{2}$, isospin $\frac{3}{2}$ delta resonances come out with an energy of 1095 MeV, which is too low and so the nucleon-delta splitting is too small by roughly a factor of 3. However, as mentioned previously, we are not too concerned that the delta resonance is not accurately fitted, as we believe it to be poorly described as a rotating rigid body.

Because the rational map $R(z) = z$ satisfies $-1/\overline{R(z)} = R(-1/\bar{z})$, the parity operator $\mathcal{P}$ is naturally represented by the identity operator. We then find that each of the states just described has positive parity, in agreement with experiment.

VII. $B = 2$

The symmetry of the $B = 2$ Skyrmion is $D_{\infty h}$, and the rational map that approximates this Skyrmion is $R(z) = z^2$. The tensors of inertia U_{ij}, V_{ij}, and W_{ij} are all diagonal, with $U_{11} = U_{22}, V_{11} = V_{22}$, and $W_{11} = W_{22} = 0$. We also have that $U_{33} = \frac{1}{2}W_{33} = \frac{1}{4}V_{33}$, relations that make the inertia tensor degenerate, a consequence of the axial symmetry. The degeneracy is resolved by imposing the following FR constraint on physical states:

$$(L_3 + 2K_3)|\Psi\rangle = 0. \tag{57}$$

The discrete symmetry $R(1/z) = 1/R(z)$ leads to the FR constraint

$$e^{i\pi L_1}e^{i\pi K_1}|\Psi\rangle = -|\Psi\rangle. \tag{58}$$

The ground state is then the $J = 1, I = 0$ state $|1, 0\rangle \otimes |0, 0\rangle$, which has the quantum numbers of the deuteron. The first excited state $|0, 0\rangle \otimes |1, 0\rangle$ may be identified with the isovector ${}^1S_0$ state of the two-nucleon system.

Using the expressions for the inertia tensors given in Appendix A, we find that, numerically,

$$U_{11} = 96.58, U_{33} = 62.94, \quad \text{and} \quad V_{11} = 160.61. \tag{59}$$

The kinetic energy operator is given by [5]

$$T = \frac{1}{2V_{11}}\mathbf{J}^2 + \frac{1}{2U_{11}}\mathbf{I}^2 - \left(\frac{1}{2U_{11}} + \frac{2}{V_{11}} - \frac{1}{W_{33}}\right)K_3^2. \tag{60}$$

For the ground state, we find (with the conversion factor e^3F_π implied from now on)

$$\begin{aligned} E_{J=1,\, I=0} &= \mathcal{M}_2 + 16.3\,\text{MeV} = 1949.3\,\text{MeV} + 16.3\,\text{MeV} \\ &= 1966\,\text{MeV}. \end{aligned} \tag{61}$$

For the first excited state, we get

$$E_{J=0,\, I=1} = \mathcal{M}_2 + 27.1\,\text{MeV} = 1976\,\text{MeV}. \tag{62}$$

The experimentally determined mass of the deuteron is 1876 MeV, with the proton and neutron constituents only very weakly bound by 2 MeV. The ${}^1S_0$ state is marginally unbound, with a mass of 1880 MeV [5,26]. As our energies [Eqs. (61) and (62)] exceed the sum of the masses of a proton and a neutron, it would appear that we have predicted states that are unbound. However, when we compare $E_{J=1,\, I=0}$ and $E_{J=0,\, I=1}$ to the sum of the masses of two quantized single Skyrmions with spin $\frac{1}{2}$ (calculated in the previous section), these states appear bound (with binding energies of 50 and 39 MeV, respectively).

Thus the new parameters are clearly not ideal in the $B = 2$ sector, but the old parameters more strongly overestimate the binding energies of these two states [5]. Also, we calculate the excitation energy of the ${}^1S_0$ state to be 11 MeV relative to the deuteron, which is of the correct order of magnitude and better than that obtained in Ref. [5] (35 MeV).

To determine the parities of these two states we observe that the rational map $R(z) = z^2$ has the reflection symmetry $-1/\overline{R(z)} = -R(-1/\bar{z})$, and so $\mathcal{P} = e^{i\pi K_3}$. Applying $\mathcal{P}$ to the allowed states, we find that both have positive parity, in agreement with experiment.

VIII. $B = 3$

The tetrahedrally symmetric $B = 3$ Skyrmion was first quantized in Ref. [7]. Here we use the rational map ansatz to simplify the analysis. The Skyrmion is approximated by using the map

$$R(z) = \frac{\sqrt{3}iz^2 - 1}{z^3 - \sqrt{3}iz}. \tag{63}$$

The symmetry group is generated by two elements. These correspond to the following symmetries of the rational map:

$$R(-z) = -R(z), \tag{64}$$

$$R\left(\frac{iz+1}{-iz+1}\right) = \frac{iR(z)+1}{-iR(z)+1}. \tag{65}$$

A π rotation about the x_3-axis in space is equivalent to a π isorotation about the 3-axis in isospace; and a $2\pi/3$ rotation about the $(x_1+x_2+x_3)$-axis in space is equivalent to a $2\pi/3$ isorotation about the $(1+2+3)$-axis in isospace. This leads to the FR constraints

$$e^{i\pi L_3}e^{i\pi K_3}|\Psi\rangle = |\Psi\rangle, \tag{66}$$

$$e^{i\frac{2\pi}{3\sqrt{3}}(L_1+L_2+L_3)}e^{i\frac{2\pi}{3\sqrt{3}}(K_1+K_2+K_3)}|\Psi\rangle = |\Psi\rangle. \tag{67}$$

There is a spin $\frac{1}{2}$, isospin $\frac{1}{2}$ (unnormalized) solution of these constraints,

$$|\Psi\rangle = \left|\frac{1}{2},\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2},-\frac{1}{2}\right\rangle - \left|\frac{1}{2},-\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2},\frac{1}{2}\right\rangle. \tag{68}$$

This is the unique state with the same quantum numbers as the hydrogen-3/helium-3 isodoublet of nuclei in their ground states. The FR constraints also allow for two distinct states with spin $\frac{3}{2}$ and isospin $\frac{3}{2}$, given by

$$|\Psi\rangle = \left|\frac{3}{2},\frac{3}{2}\right\rangle \otimes \left|\frac{3}{2},-\frac{3}{2}\right\rangle - \left|\frac{3}{2},\frac{1}{2}\right\rangle \otimes \left|\frac{3}{2},-\frac{1}{2}\right\rangle + \left|\frac{3}{2},-\frac{1}{2}\right\rangle \otimes \left|\frac{3}{2},\frac{1}{2}\right\rangle - \left|\frac{3}{2},-\frac{3}{2}\right\rangle \otimes \left|\frac{3}{2},\frac{3}{2}\right\rangle \tag{69}$$

and

$$|\Psi\rangle = \left|\frac{3}{2},-\frac{1}{2}\right\rangle \otimes \left|\frac{3}{2},-\frac{3}{2}\right\rangle + \left|\frac{3}{2},-\frac{3}{2}\right\rangle \otimes \left|\frac{3}{2},-\frac{1}{2}\right\rangle - \left|\frac{3}{2},\frac{3}{2}\right\rangle \otimes \left|\frac{3}{2},\frac{1}{2}\right\rangle - \left|\frac{3}{2},\frac{1}{2}\right\rangle \otimes \left|\frac{3}{2},\frac{3}{2}\right\rangle. \tag{70}$$

The first of these has the correct quantum numbers to allow for its interpretation as a nucleus in which one of the nucleons is excited to a delta isobar [27].

The inertia tensors have been numerically determined for the rational map given in Eq. (63). They are all diagonal and proportional to the unit matrix: $U_{ij} = u\delta_{ij}$, $V_{ij} = v\delta_{ij}$, and $W_{ij} = w\delta_{ij}$. This was to be expected because of the irreducibility of the action of the tetrahedral group on $\mathbb{R}^3$. Numerically,

$$u = 121.80, \quad v = 418.83, \quad \text{and} \quad w = -80.34. \tag{71}$$

The kinetic energy operator then takes the following form:

$$T = \frac{1}{2}\frac{1}{uv-w^2}[(u-w)\mathbf{J}^2 + (v-w)\mathbf{I}^2 + w\mathbf{M}^2], \tag{72}$$

where $\mathbf{M} = \mathbf{L} + \mathbf{K}$. Each of the three states [Eqs. (68), (69), and (70)] can be rewritten in terms of the basis states $|J, I; M, M_3\rangle$: The first is proportional to $|\frac{1}{2},\frac{1}{2};0,0\rangle$, the second to $|\frac{3}{2},\frac{3}{2};0,0\rangle$, and the third to $|\frac{3}{2},\frac{3}{2};3,2\rangle - |\frac{3}{2},\frac{3}{2};3,-2\rangle$. They are thus eigenstates of $\mathbf{M}^2$ with eigenvalues 0, 0, and 12, respectively. The energies of the three states are then

$$\begin{aligned} E_{J=1/2,\, I=1/2,\, M=0} &= \mathcal{M}_3 + \frac{3}{8}\frac{u+v-2w}{uv-w^2} = \mathcal{M}_3 + 15.4\,\text{MeV} \\ &= 2895\,\text{MeV}, \end{aligned} \tag{73}$$

$$\begin{aligned} E_{J=3/2,\, I=3/2,\, M=0} &= \mathcal{M}_3 + \frac{15}{8}\frac{u+v-2w}{uv-w^2} = \mathcal{M}_3 + 77.1\,\text{MeV} \\ &= 2957\,\text{MeV}, \end{aligned} \tag{74}$$

$$\begin{aligned} E_{J=3/2,\, I=3/2,\, M=3} &= \mathcal{M}_3 + \frac{3}{8}\frac{5u+5v+6w}{uv-w^2} = \mathcal{M}_3 + 48.8\,\text{MeV} \\ &= 2929\,\text{MeV}. \end{aligned} \tag{75}$$

These formulas are identical to those obtained in Ref. [7], although the numerical values of u, v, and w are different because of the rational map approximation. The average mass of a helium-3 nucleus and a hydrogen-3 nucleus is 2809 MeV. Our ground state comes to within 4% of this value. However, our second state, with an excitation energy of 62 MeV, is rather too low in energy to have an $NN\Delta$ interpretation.

To determine the parities of these three states we observe that there is the reflection symmetry $-1/\overline{R(iz)} = iR(-1/\bar{z})$, and so $\mathcal{P} = e^{i\frac{\pi}{2}(L_3+K_3)}$. Applying $\mathcal{P}$ to the allowed states [Eqs. (68)–(70)], we find that they have parities +, +, and −, respectively. We note that the helium-3 and hydrogen-3 ground states have positive parity.

IX. $B = 4$

The minimal-energy $B = 4$ Skyrmion has O_h symmetry and a cubic shape, and it is described by the rational map

$$R(z) = \frac{z^4 + 2\sqrt{3}iz^2 + 1}{z^4 - 2\sqrt{3}iz^2 + 1}. \tag{76}$$

This map has the generating symmetries

$$R(iz) = 1/R(z), \tag{77}$$

$$R\left(\frac{iz+1}{-iz+1}\right) = e^{i\frac{2\pi}{3}}R(z), \tag{78}$$

which lead to the FR constraints

$$e^{i\frac{\pi}{2}L_3}e^{i\pi K_1}|\Psi\rangle = |\Psi\rangle, \tag{79}$$

$$e^{i\frac{2\pi}{3\sqrt{3}}(L_1+L_2+L_3)}e^{i\frac{2\pi}{3}K_3}|\Psi\rangle = |\Psi\rangle. \tag{80}$$

Seeking simultaneous solutions of these, we obtain the ground state $|0,0\rangle \otimes |0,0\rangle$. There exists a spin 2, isospin 1 state given by

$$\begin{aligned} &\left(|2,2\rangle + \sqrt{2}i|2,0\rangle + |2,-2\rangle\right) \otimes |1,1\rangle \\ &\quad - \left(|2,2\rangle - \sqrt{2}i|2,0\rangle + |2,-2\rangle\right) \otimes |1,-1\rangle \end{aligned} \tag{81}$$

and a spin 4, isospin 0 state given by [10]

$$\left(|4,4\rangle + \sqrt{\frac{14}{5}}|4,0\rangle + |4,-4\rangle\right) \otimes |0,0\rangle. \tag{82}$$

The cubic symmetry excludes a spin 2, isospin 0 state.

The tensors of inertia are found to be diagonal, satisfying $U_{11} = U_{22}$, $V_{ij} = v\delta_{ij}$, and $W_{ij} = 0$. Although the cubic group acts irreducibly on spatial $\mathbb{R}^3$, the associated isospin rotations are reducible, with the $\mathbb{R}^3$ of isospace decomposing into a two-dimensional and a one-dimensional subspace. This is why the inertia tensor U has two independent diagonal entries, whereas V only has one, and why the cross-term W vanishes. Numerically,

$$U_{11} = 142.84, U_{33} = 169.41, \quad \text{and} \quad v = 663.16. \tag{83}$$

The kinetic energy operator is given by

$$T = \frac{1}{2v}\mathbf{J}^2 + \frac{1}{2U_{11}}\mathbf{I}^2 + \frac{1}{2}\left(\frac{1}{U_{33}} - \frac{1}{U_{11}}\right)K_3^2. \tag{84}$$

For the spin 0, isospin 0 ground state, the energy is simply the static mass of the Skyrmion, $\mathcal{M}_4 = 3679$ MeV. Comparing this to the mass of the helium-4 nucleus, 3727 MeV, we see that our prediction comes to within 2% of the experimental value. The classical binding energy of the $B = 4$ Skyrmion is significantly larger than that of the $B = 3$ or $B = 5$ Skyrmion (see the next section). The mean charge radius of the quantized $B = 4$ Skyrmion was calculated by using the new parameter set in Ref. [13] to be 2.13 fm, which agrees reasonably well with the experimental value of 1.71 fm. Walhout [9] calculated this quantity using the old parameter set and taking into account a number of the vibrational modes, obtaining 1.58 fm.

For the state given by Eq. (81) with spin 2 and isospin 1, the energy is

$$E_{J=2,\,I=1} = \mathcal{M}_4 + 28.7\,\text{MeV} = 3679.0\,\text{MeV} + 28.7\,\text{MeV} = 3708\,\text{MeV}. \tag{85}$$

We note here that hydrogen-4, helium-4, and lithium-4 form an isospin triplet, whose lowest energy state has spin 2 and whose average excitation energy is 23.7 MeV relative to the ground state of helium-4 [28], so here the Skyrmion picture works well.

Finally, for the predicted spin 4, isospin 0 state, we find

$$E_{J=4,\,I=0} = \mathcal{M}_4 + 39.4\,\text{MeV} = 3679.0\,\text{MeV} + 39.4\,\text{MeV} = 3718\,\text{MeV}. \tag{86}$$

Such a state of helium-4 has not yet been experimentally observed. However, predictions for such a state with an excitation energy of 24.6 MeV have been made [29,30]. Our calculation suggests a slightly larger energy, in the range 30–40 MeV (allowing for the discrepancy between our calculation and the data for the isospin 1 state). The energy levels are summarized in Fig. 3.

To determine the parities of these three states we observe that the rational map [Eq. (76)] has the reflection symmetry $-1/\overline{R(z)} = -R(-1/\bar{z})$, and so $\mathcal{P} = e^{i\pi K_3}$. By acting with this operator on the physical states, we find that the spin 0, isospin 0 state and the spin 4, isospin 0 state both have positive parity. However, the spin 2, isospin 1 state has negative parity, and so we find no contradiction with experiment.

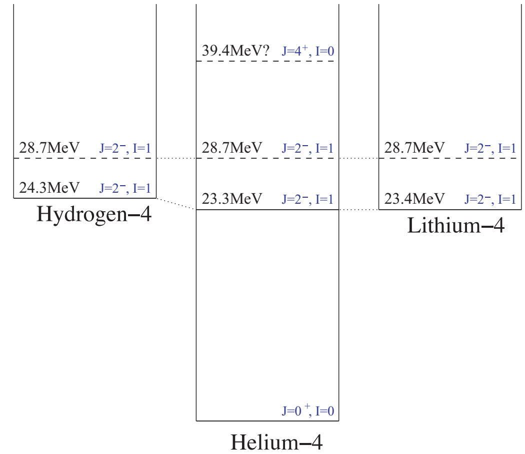

FIG. 3. (Color online) Energy level diagram for the quantized $B = 4$ Skyrmion. Solid lines indicate experimentally observed states; dashed lines indicate our predictions.

X. $B = 5$

Finding a quantized Skyrmion description of the ground and first excited states of the helium-5/lithium-5 isodoublet, with spins $\frac{3}{2}$ and $\frac{1}{2}$, has proved difficult. It still remains to determine the symmetries and FR constraints that might give a lowest energy state of spin $\frac{3}{2}$. Here we explore in detail the idea floated in Ref. [19], that one should consider variants of the rational map, and not just the one that optimizes the classical Skyrmion energy. The minimal-energy $B = 5$ Skyrmion has D_{2d} symmetry, and it can be approximated by the rational map

$$R(z) = \frac{z(z^4 + ibz^2 + a)}{az^4 + ibz^2 + 1}, \quad a = -3.07, \quad b = 3.94. \tag{87}$$

The ground state obtained from this map [10] has spin $\frac{1}{2}$ and isospin $\frac{1}{2}$, which is inconsistent with the observed spin $\frac{3}{2}$ ground states of helium-5 and lithium-5. The Skyrmion has this shape up to a pion mass $m \simeq 1$. However, if higher m is considered, the symmetry of the Skyrmion might change, and this ground state become unstable. Therefore more symmetric solutions, which apparently have higher energy, might be the ones describing the true Skyrmion and are worth investigating.

When $b = 0$ the rational map [Eq. (87)] has D_{4h} symmetry, and it acquires octahedral symmetry when in addition $a = -5$. Octahedral symmetry has been previously considered [11] and it leads to a ground state with spin $\frac{5}{2}$ and isospin $\frac{1}{2}$. Let us therefore consider the D_{4h}-symmetric map (which could in fact be restricted to C_4 symmetry)

$$R(z) = \frac{z(z^4 + a)}{az^4 + 1}, \quad a \neq -5. \tag{88}$$

This map has the generating symmetries

$$R(iz) = iR(z), \tag{89}$$

$$R(1/z) = 1/R(z), \tag{90}$$

which lead to the FR constraints

$$e^{i\frac{\pi}{2}L_3}e^{i\frac{\pi}{2}K_3}|\Psi\rangle = -|\Psi\rangle, \tag{91}$$

$$e^{i\pi L_1}e^{i\pi K_1}|\Psi\rangle = |\Psi\rangle. \tag{92}$$

Seeking simultaneous solutions, we obtain a ground state with $J = \frac{3}{2}$ and isospin $I = \frac{1}{2}$, given by

$$|\Psi\rangle = \left|\frac{3}{2}, \frac{3}{2}\right\rangle \otimes \left|\frac{1}{2}, \frac{1}{2}\right\rangle + \left|\frac{3}{2}, -\frac{3}{2}\right\rangle \otimes \left|\frac{1}{2}, -\frac{1}{2}\right\rangle. \quad (93)$$

This is the spin we are looking for. There are two excited states with $J = \frac{5}{2}$ and $I = \frac{1}{2}$, most easily written in terms of $|J, I; M, M_3\rangle$, where $\mathbf{M} = \mathbf{L} + \mathbf{K}$:

$$|\Psi\rangle = \left(\left|\frac{5}{2}, \frac{1}{2}; 3, 2\right\rangle - \left|\frac{5}{2}, \frac{1}{2}; 3, -2\right\rangle\right) + c_{\pm}\left(\left|\frac{5}{2}, \frac{1}{2}; 2, 2\right\rangle + \left|\frac{5}{2}, \frac{1}{2}; 2, -2\right\rangle\right), \quad (94)$$

with $c_{\pm}$ evaluated in Appendix C. The FR constraints allow for a further excited state with $J = \frac{1}{2}$ and $I = \frac{3}{2}$.

The D_4 symmetry implies that the tensors of inertia are diagonal, with $U_{11} = U_{22}$, $V_{11} = V_{22}$, and $W_{11} = W_{22}$, which leads to the expression for the kinetic energy operator

$$T = \frac{1}{2}\Bigg\{\frac{1}{\left(U_{11}V_{11} - W_{11}^2\right)}\left[U_{11}\left(\mathbf{J}^2 - L_3^2\right) + V_{11}\left(\mathbf{I}^2 - K_3^2\right) + W_{11}(\mathbf{M}^2 - \mathbf{J}^2 - \mathbf{I}^2 - 2L_3K_3)\right] + \frac{1}{\left(U_{33}V_{33} - W_{33}^2\right)}\left[U_{33}L_3^2 + V_{33}K_3^2 + 2W_{33}L_3K_3\right]\Bigg\}. \quad (95)$$

The energy of the ground state is therefore

$$E_{J=3/2,\, I=1/2} = \mathcal{M}_5 + \frac{3U_{11} + V_{11}}{4\left(U_{11}V_{11} - W_{11}^2\right)} + \frac{9U_{33} + V_{33} + 6W_{33}}{8\left(U_{33}V_{33} - W_{33}^2\right)}. \quad (96)$$

The numerical value of the energy depends on the parameter a in the rational map, which has yet to be determined.

We now argue that the D_{4h} symmetry we are considering is justified even if octahedral symmetry ($a = -5$) provides us with a slightly lower classical energy. The dependence of the classical energy on a is shown in Fig. 4, whereas the quantum energy is a strictly increasing function of a near $a = -5$ (see Fig. 5). Therefore the total energy achieves its minimum just below $a = -5$ (see Fig. 6), the quantum energy being much smaller than the classical one. Taking $a = -5.0025$ we find

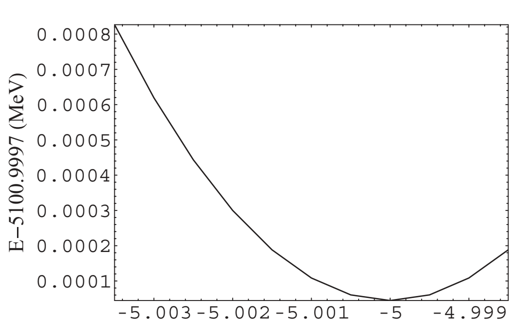

FIG. 4. Classical energy of the $B = 5$ Skyrmion as a function of a.

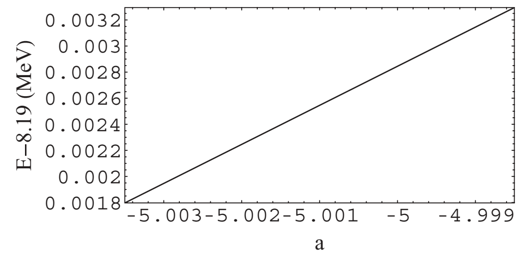

FIG. 5. Quantum energy of the $B = 5$ Skyrmion as a function of a.

that

$$U_{11} = 203.41, \quad U_{33} = 203.36, \quad V_{11} = 1333.49, \quad V_{33} = 1332.96, \quad W_{11} = -186.54, \quad W_{33} = -186.61. \quad (97)$$

The static Skyrmion mass $\mathcal{M}_5$, for this value of a, is calculated to be 5101 MeV. The energies of the above four states are then

$$E_{J=3/2,\, I=1/2} = \mathcal{M}_5 + 8.2\,\text{MeV} = 5109\,\text{MeV}, \quad (98)$$

$$E_{J=5/2,\, I=1/2, c_-} = \mathcal{M}_5 + 12.5\,\text{MeV} = 5114\,\text{MeV}, \quad (99)$$

$$E_{J=5/2,\, I=1/2, c_+} = \mathcal{M}_5 + 12.9\,\text{MeV} = 5114\,\text{MeV}, \quad (100)$$

$$E_{J=1/2,\, I=3/2} = \mathcal{M}_5 + 26.9\,\text{MeV} = 5128\,\text{MeV}. \quad (101)$$

The D_{4h}-symmetric map [Eq. (88)] satisfies $-1/\overline{R(z)} = R(-1/\bar{z})$. The parity operator could therefore be represented by the identity operator. However, we may also choose $\mathcal{P} = e^{2\pi i \mathbf{n}\cdot\mathbf{L}}$, where $\mathbf{n}$ is any unit vector. If we make this choice, then each of the states just described has negative parity. We note that the ground states of helium-5 and lithium-5 have negative parities.

In conclusion, the achievement of the correct spin $\frac{3}{2}$ for the ground state comes at a price. First, the slightly excited $J = \frac{1}{2}$ state is not allowed by the FR constraints. Second, by comparing $E_{J=3/2,\, I=1/2}$ to the average mass of the helium-5 and lithium-5 nuclei (4668 MeV), we see that our prediction is some 10% from the experimental value, whereas for the D_{2d}-symmetric Skyrmion there was a better match to the helium-5/lithium-5 ground-state energy. Physically, the ground and first excited states are unbound, and may better be described as a cubic $B = 4$ Skyrmion loosely attracted to a single Skyrmion.

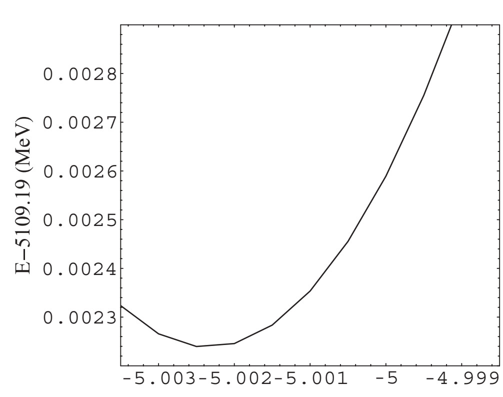

FIG. 6. Total energy of the $B = 5$ Skyrmion as a function of a.

XI. $B = 6$

The minimal-energy $B = 6$ Skyrmion has D_{4d} symmetry and is well-approximated by using the rational map

$$R(z) = \frac{z^4 + ia}{z^2(iaz^4 + 1)}. \tag{102}$$

This map has the generating symmetries

$$R(iz) = -R(z), \tag{103}$$
$$R(1/z) = 1/R(z), \tag{104}$$

which lead to the FR constraints

$$e^{i\frac{\pi}{2}L_3} e^{i\pi K_3}|\Psi\rangle = |\Psi\rangle, \tag{105}$$
$$e^{i\pi L_1} e^{i\pi K_1}|\Psi\rangle = -|\Psi\rangle. \tag{106}$$

These constraints allow for the existence of states $|1, 0\rangle \otimes |0, 0\rangle$, $|3, 0\rangle \otimes |0, 0\rangle$, $|0, 0\rangle \otimes |1, 0\rangle$, $|2, 0\rangle \otimes |1, 0\rangle$, and $|5, 0\rangle \otimes |0, 0\rangle$.

A numerical search over the parameter a in the rational map shows that the integral $\mathcal{I}$ is minimized at $a = 0.16$. However, it was suggested in Ref. [13] that allowing a slight deformation of the rational map would lead to more accurate predictions. In particular, it was found that by setting $a = 0.1933$, and using the new parameter set, one obtains a quantum quadrupole moment in agreement with experiment. In what follows, we set $a = 0.1933$.

The inertia tensors have been computed for this rational map, and are found to be diagonal, satisfying $U_{11} = U_{22}$, $V_{11} = V_{22}$, and $W_{11} = W_{22} = 0$. Numerically, we get

$$U_{11} = 215.84, \quad U_{33} = 230.77, \quad V_{11} = 1525.99, \quad V_{33} = 1493.66, \quad \text{and} \quad W_{33} = -105.45. \tag{107}$$

The kinetic energy operator is given by

$$T = \frac{1}{2V_{11}}\left(\mathbf{J}^2 - L_3^2\right) + \frac{1}{2U_{11}}\left(\mathbf{I}^2 - K_3^2\right) + \frac{1}{2\left(U_{33}V_{33} - W_{33}^2\right)} \times \left(U_{33}L_3^2 + V_{33}K_3^2 + 2W_{33}L_3K_3\right). \tag{108}$$

The static Skyrmion mass, $\mathcal{M}_6$, is calculated to be 5601 MeV, which is precisely equal to the mass of the lithium-6 nucleus. (The new parameter set was determined such that this would be the case–in Ref. [13] we estimated the spin energy for spin 1, isospin 0 to be approximately 1 MeV, and then neglected this small quantity.) The energy eigenvalues corresponding to the above five states are then

$$E_{J=1,\,I=0} = \mathcal{M}_6 + \frac{1}{V_{11}} = \mathcal{M}_6 + 1.7\,\text{MeV} = 5602\,\text{MeV}, \tag{109}$$

$$E_{J=3,\,I=0} = \mathcal{M}_6 + \frac{6}{V_{11}} = \mathcal{M}_6 + 10.3\,\text{MeV} = 5611\,\text{MeV}, \tag{110}$$

$$E_{J=0,\,I=1} = \mathcal{M}_6 + \frac{1}{U_{11}} = \mathcal{M}_6 + 12.1\,\text{MeV} = 5613\,\text{MeV}, \tag{111}$$

$$E_{J=2,\,I=1} = \mathcal{M}_6 + \frac{1}{U_{11}} + \frac{3}{V_{11}} = \mathcal{M}_6 + 17.2\,\text{MeV} = 5618\,\text{MeV}, \tag{112}$$

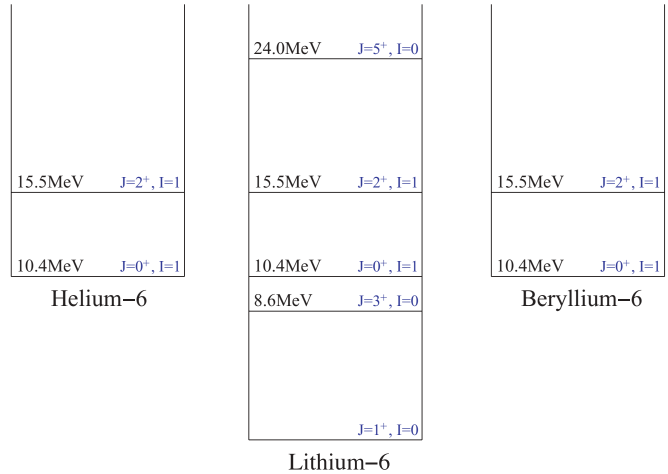

FIG. 7. (Color online) Energy level diagram for the quantized $B = 6$ Skyrmion. Energies are given relative to the spin 1, isospin 0 ground state.

$$E_{J=5,\,I=0} = \mathcal{M}_6 + \frac{15}{V_{11}} = \mathcal{M}_6 + 25.7\,\text{MeV} = 5626\,\text{MeV}. \tag{113}$$

We may identify these with isospin 0 states of lithium-6, and with states of the helium-6, lithium-6, and beryllium-6 nuclei, which together form an isospin triplet (see Fig. 7). The assumption in Ref. [13] that the spin kinetic energy of the state $|1, 0\rangle \otimes |0, 0\rangle$ is of order 1 MeV is clearly justified.

Spin and isospin excitation energies relative to the lithium-6 ground state are experimentally known for these nuclei [31] (see Fig. 8). The ground state of the lithium-6 nucleus is identified with the state $|1, 0\rangle \otimes |0, 0\rangle$, and there is an excited state $|3, 0\rangle \otimes |0, 0\rangle$, with excitation energy 2.2 MeV. Lithium-6 has a further spin 0 excited state with excitation energy 3.6 MeV; this is joined by the lowest energy states of the helium-6 and beryllium-6 nuclei, which relative to the ground state of lithium-6 have energies of 4.1 and 3.1 MeV,

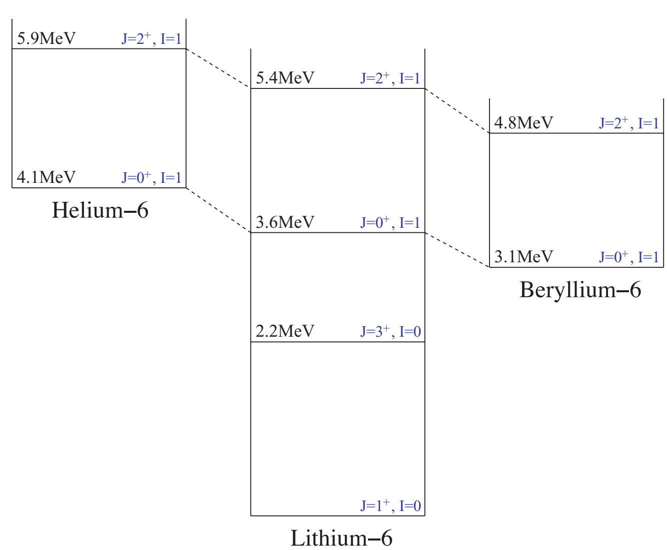

FIG. 8. (Color online) Energy level diagram for nuclei with $B = 6$.

respectively, to form the spin 0 isotriplet $|0, 0\rangle \otimes |1, 0\rangle$. Similarly, $|2, 0\rangle \otimes |1, 0\rangle$ is identified with the spin 2 excited states of the isotriplet. A spin 5 excited state of lithium-6 has not been seen experimentally. We, however, predict the existence of such a state with excitation energy higher than those of the other states so far discussed.

The splitting between the various spin and isospin states of the Skyrmion is clearly too large; the predicted quantum energies are roughly four times the experimental values. This may be connected to the fact that lithium-6 is an odd-odd nucleus. Alternatively, it may be because we have not considered the possibility of the nucleus splitting into an α particle and a deuteron. However, we have performed the same calculation using the old parameter set and have found that this gives even wider gaps between the energy levels. The new parameter set is therefore not perfect, but it is certainly an improvement. Furthermore, the ratios of the relative excitation energies given by

$$\frac{E_{J=0,\,I=1} - E_{J=1,\,I=0}}{E_{J=3,\,I=0} - E_{J=1,\,I=0}} = 1.2 \tag{114}$$

and

$$\frac{E_{J=2,\,I=1} - E_{J=1,\,I=0}}{E_{J=0,\,I=1} - E_{J=1,\,I=0}} = 1.5 \tag{115}$$

correspond well to experimental data for these nuclei, for which the first ratio is $3.6/2.2 = 1.6$ and the second is $5.4/3.6 = 1.5$.

To determine the parities of these states we first observe the reflection symmetry

$$-1/\overline{R(e^{i\frac{\pi}{4}}z)} = -iR(-1/\bar{z}). \tag{116}$$

The parity operator can therefore be represented as $\mathcal{P} = e^{i\frac{\pi}{4}L_3}e^{-i\frac{\pi}{2}K_3}$. If we make this choice, then each of the states given here has positive parity, in agreement with experiment.

XII. $B = 7$

Here, as for $B = 5$, quantizing the Skyrmion of lowest energy gives states with the wrong spins to match the nuclear data. The minimal-energy $B = 7$ Skyrmion has icosahedral symmetry, and it is described by the rational map

$$R(z) = \frac{7z^5 + 1}{z^2(z^5 - 7)}. \tag{117}$$

This map leads to a ground state with $J = \frac{7}{2}$, $I = \frac{1}{2}$, a spin that appears experimentally as the second excited state of the lithium-7/beryllium-7 isospin doublet. Experimentally, the ground state has spin $\frac{3}{2}$.

There are many ways in which the icosahedral symmetry might be broken, allowing for the appearance of a $J = \frac{3}{2}$, $I = \frac{1}{2}$ state in the spectrum. The most interesting possibility, in our opinion, is the breaking of the C_3 symmetry, while preserving D_5 symmetry. This leads to a ground state with $J = \frac{3}{2}$ and $I = \frac{1}{2}$. So let us consider the D_5-symmetric map

$$R(z) = \frac{az^5 + 1}{z^2(z^5 - a)}, \quad a \neq 7, \tag{118}$$

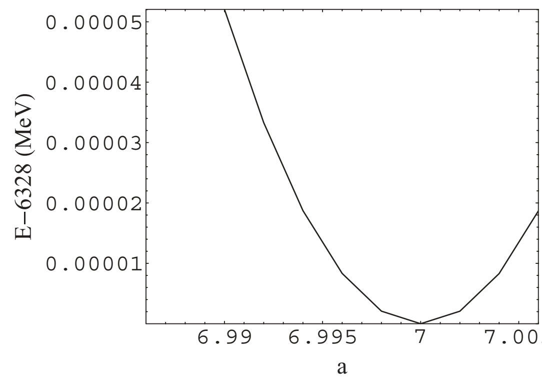

FIG. 9. Classical energy of the $B = 7$ Skyrmion as a function of a.

where $a = 7$ restores the icosahedral symmetry. The generating symmetries of this map are

$$R(e^{i\frac{2\pi}{5}}z) = e^{-i\frac{4\pi}{5}}R(z), \tag{119}$$

$$R(-1/z) = -1/R(z), \tag{120}$$

which lead to the FR constraints

$$e^{i\frac{2\pi}{5}L_3}e^{-i\frac{4\pi}{5}K_3}\,|\Psi\rangle = -\,|\Psi\rangle\,, \tag{121}$$

$$e^{i\pi L_2}e^{i\pi K_2}\,|\Psi\rangle = -\,|\Psi\rangle\,. \tag{122}$$

The ground state with $J = \frac{3}{2}$ and $I = \frac{1}{2}$ is

$$|\Psi\rangle = \left|\frac{3}{2},\frac{3}{2}\right\rangle \otimes \left|\frac{1}{2},-\frac{1}{2}\right\rangle + \left|\frac{3}{2},-\frac{3}{2}\right\rangle \otimes \left|\frac{1}{2},\frac{1}{2}\right\rangle, \tag{123}$$

the first excited state with $J = \frac{5}{2}$ and $I = \frac{1}{2}$ is

$$|\Psi\rangle = \left|\frac{5}{2},\frac{3}{2}\right\rangle \otimes \left|\frac{1}{2},-\frac{1}{2}\right\rangle - \left|\frac{5}{2},-\frac{3}{2}\right\rangle \otimes \left|\frac{1}{2},\frac{1}{2}\right\rangle, \tag{124}$$

and there exist two further excited states with $J = \frac{7}{2}$, $I = \frac{1}{2}$, given by

$$|\Psi^1\rangle = \left|\frac{7}{2},\frac{3}{2}\right\rangle \otimes \left|\frac{1}{2},-\frac{1}{2}\right\rangle + \left|\frac{7}{2},-\frac{3}{2}\right\rangle \otimes \left|\frac{1}{2},\frac{1}{2}\right\rangle, \tag{125}$$

$$|\Psi^2\rangle = \left|\frac{7}{2},\frac{7}{2}\right\rangle \otimes \left|\frac{1}{2},\frac{1}{2}\right\rangle - \left|\frac{7}{2},-\frac{7}{2}\right\rangle \otimes \left|\frac{1}{2},-\frac{1}{2}\right\rangle. \tag{126}$$

States with $I = \frac{3}{2}$ are also allowed. In particular, there is one spin $\frac{1}{2}$ state:

$$|\Psi\rangle = \left|\frac{1}{2},\frac{1}{2}\right\rangle \otimes \left|\frac{3}{2},\frac{3}{2}\right\rangle - \left|\frac{1}{2},-\frac{1}{2}\right\rangle \otimes \left|\frac{3}{2},-\frac{3}{2}\right\rangle, \tag{127}$$

and two spin $\frac{3}{2}$ states:

$$|\Psi^1\rangle = \left|\frac{3}{2},\frac{3}{2}\right\rangle \otimes \left|\frac{3}{2},-\frac{1}{2}\right\rangle - \left|\frac{3}{2},-\frac{3}{2}\right\rangle \otimes \left|\frac{3}{2},\frac{1}{2}\right\rangle, \tag{128}$$

$$|\Psi^2\rangle = \left|\frac{3}{2},\frac{1}{2}\right\rangle \otimes \left|\frac{3}{2},\frac{3}{2}\right\rangle + \left|\frac{3}{2},-\frac{1}{2}\right\rangle \otimes \left|\frac{3}{2},-\frac{3}{2}\right\rangle. \tag{129}$$

The inertia tensors are found to be diagonal, with $U_{11} = U_{22}$, $V_{11} = V_{22}$, and $W_{11} = W_{22} = 0$, leading to the kinetic

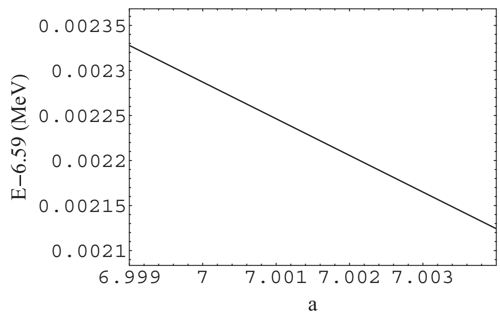

FIG. 10. Quantum energy of the $B = 7$ Skyrmion as a function of a.

energy operator

$$T = \frac{1}{2V_{11}}\left(\mathbf{J}^2 - L_3^2\right) + \frac{1}{2U_{11}}\left(\mathbf{I}^2 - K_3^2\right) + \frac{1}{2\left(U_{33}V_{33} - W_{33}^2\right)} \times \left(U_{33}L_3^2 + V_{33}K_3^2 + 2W_{33}L_3K_3\right). \tag{130}$$

The energy of the ground state is given by

$$E_{J=3/2,\, I=1/2} = \mathcal{M}_7 + \frac{3}{4V_{11}} + \frac{1}{4U_{11}} + \frac{9U_{33} + V_{33} - 6W_{33}}{8\left(U_{33}V_{33} - W_{33}^2\right)}. \tag{131}$$

The static Skyrmion mass, $\mathcal{M}_7$, is found to be close to 6328 MeV. The dependencies of the classical and quantum energies on a are shown in Figs. 9 and 10, respectively. Looking for the value of a giving the minimum of the total energy, we obtain $a = 7.002$ (see Fig. 11). For this value of a, we find numerically

$$U_{11} = 246.27, \quad U_{33} = 246.26, \quad V_{11} = 1873.03, \quad V_{33} = 1872.76, \quad W_{33} = 0.04, \tag{132}$$

and

$$E_{J=3/2,\, I=1/2} = \mathcal{M}_7 + 6.6\,\text{MeV} = 6335\,\text{MeV}, \tag{133}$$

to be compared to the average mass of the lithium-7 and beryllium-7 nuclei, which is 6534 MeV.

For the excited states we find that

$$E_{J=5/2,I=1/2} = \mathcal{M}_7 + 10.1\,\text{MeV} = 6338\,\text{MeV}, \tag{134}$$

$$E^1_{J=7/2,I=1/2} = \mathcal{M}_7 + 15.0\,\text{MeV} = 6343\,\text{MeV}, \tag{135}$$

$$E^2_{J=7/2,I=1/2} = \mathcal{M}_7 + 15.0\,\text{MeV} = 6343\,\text{MeV}, \tag{136}$$

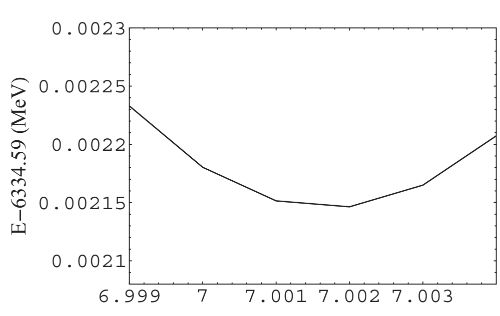

FIG. 11. Total energy of the $B = 7$ Skyrmion as a function of a.

$$E_{J=1/2,I=3/2} = \mathcal{M}_7 + 20.4\,\text{MeV} = 6348\,\text{MeV}, \tag{137}$$

$$E^1_{J=3/2,I=3/2} = \mathcal{M}_7 + 22.5\,\text{MeV} = 6351\,\text{MeV}, \tag{138}$$

$$E^2_{J=3/2,I=3/2} = \mathcal{M}_7 + 22.5\,\text{MeV} = 6351\,\text{MeV}. \tag{139}$$

There are two main problems with this spectrum. One is the absence of the $J = \frac{1}{2}, I = \frac{1}{2}$ state, and the other is the appearance of the $J = \frac{5}{2}, I = \frac{1}{2}$ state as the first excitation. We could try to overcome this problem by noticing that the first two excited states in the experimental lithium-7 and beryllium-7 spectra are in a sense *anomalous*: They have very low excitation energy, and the spin-energy correspondence is reversed. As was discussed in the introduction it is possible that such excitations cannot be described by our usual approach, and we need to allow for some vibrational modes or consider a Skyrmion of a different shape. Possibly, the states we find here could correspond to the ones lying above the lowest energy $J = \frac{7}{2}, I = \frac{1}{2}$ excited state. This interpretation fits rather well to the experimental data. The second problem is more difficult to tackle within this framework. The value of a being very close to 7 leads to a configuration that is nearly C_3-symmetric. This fact is reflected in the spectrum: We have two spin $\frac{7}{2}$ and two isospin $\frac{3}{2}$ states whose energies are almost indistinguishably close. This is not reflected in the experimental data. Let us therefore consider a smaller a and see whether there is a better fit to the spectrum. Another advantage of this approach is that it helps to partially overcome the first problem as well. Indeed, by looking through a large range of a we find that at $a = 2$ the energies of the states given here are, in increasing order,

$$E_{J=3/2,I=1/2} = \mathcal{M}_7 + 6.3\,\text{MeV}, \tag{140}$$

$$E^2_{J=7/2,I=1/2} = \mathcal{M}_7 + 9.3\,\text{MeV}, \tag{141}$$

$$E_{J=5/2,\,I=1/2} = \mathcal{M}_7 + 9.4\,\text{MeV}, \tag{142}$$

$$E^1_{J=7/2,I=1/2} = \mathcal{M}_7 + 13.7\,\text{MeV}, \tag{143}$$

$$E_{J=1/2,I=3/2} = \mathcal{M}_7 + 19.7\,\text{MeV}, \tag{144}$$

$$E^1_{J=3/2,I=3/2} = \mathcal{M}_7 + 19.9\,\text{MeV}, \tag{145}$$

$$E^2_{J=3/2,I=3/2} = \mathcal{M}_7 + 21.6\,\text{MeV}. \tag{146}$$

The energy of the $J = \frac{5}{2}, I = \frac{1}{2}$ state is now higher than the energy of one of the $J = \frac{7}{2}, I = \frac{1}{2}$ states, and lower than that of the other, in agreement with experiment. This achievement, however, comes at a price as the classical energy is now some 10% higher than the experimental value. Figures 12 and 13 are energy level diagrams for the quantized D_5-symmetric $B = 7$ Skyrmion, with $a = 2$, and for the $B = 7$ nuclei, respectively.

The symmetry breaking we have just considered is only one of the ways in which icosahedral symmetry might be broken. It is possible that a different breaking has to be considered to better understand the spectrum of the excited states, in particular the low-lying $J = \frac{1}{2}, I = \frac{1}{2}$ state. It is also possible that with the increase of the pion mass the configuration will eventually break up into a $B = 4$ and a $B = 3$ part, which is suggested by the very low energy for breakup of lithium-7 into helium-4 plus a triton.

The D_5-symmetric map [Eq. (118)] satisfies $-1/\overline{R(z)} = R(-1/\bar{z})$. As for $B = 5$, we find it favorable to choose $\mathcal{P} = e^{2\pi i \mathbf{n}\cdot\mathbf{L}}$, where $\mathbf{n}$ is any unit vector. If we make this choice,

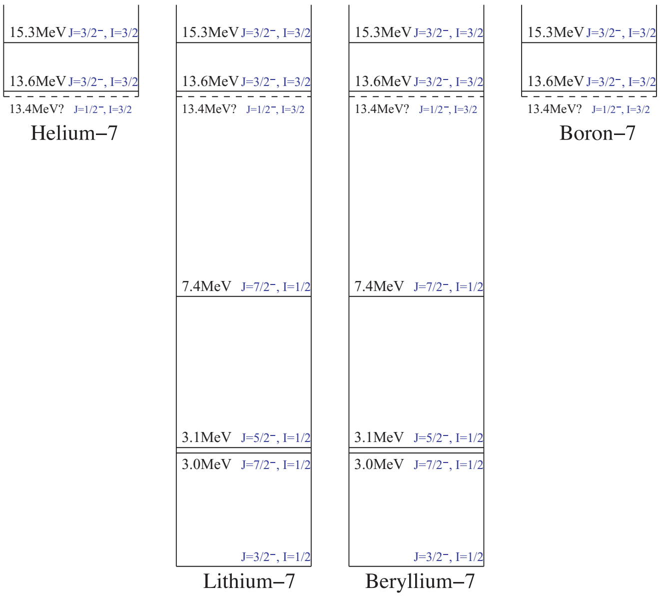

FIG. 12. (Color online) Energy level diagram for the quantized $B = 7$ Skyrmion. A putative $J = \frac{1}{2}^-$ isoquartet is represented by dashed lines.

then each of the states just described has negative parity, in agreement with experiment.

XIII. $B = 8$

In this section we introduce some new ideas for estimating the moments of inertia of the $B = 8$ Skyrmion and hence the excitation energies of the quantum states. It is believed that for our new parameter set, the minimal-energy classical solution resembles two touching $B = 4$ cubes (see Fig. 14) [15]. Here the rational map ansatz is not a good approximation, so our previous methods of calculation are no longer valid. Despite this, it is convenient to note that a field that is qualitatively of

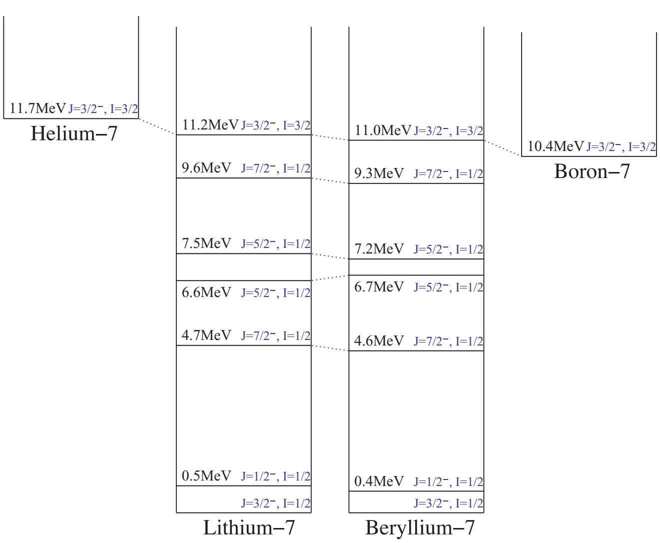

FIG. 13. (Color online) Energy level diagram for nuclei with $B = 7$.

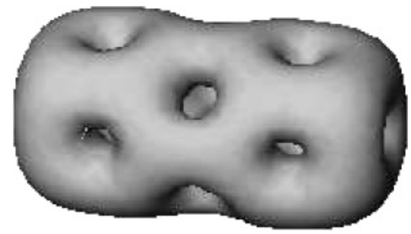

FIG. 14. Baryon density isosurface for the numerically relaxed $B = 8$ Skyrmion with $m \approx 1$, resembling two touching $B = 4$ Skyrmions.

the right form, with the correct symmetries, can be obtained from a rational map, and this enables one to determine the allowed spin/isospin/parity states. There are also classical Skyrmion solutions that are well approximated by the rational map ansatz and have only very slightly greater energy than the double cube. We consider these first.

For pion mass parameter between 0 and approximately 1, the minimal-energy $B = 8$ Skyrmion has D_{6d} symmetry, and it is well-approximated by the rational map

$$R(z) = \frac{z^6 - ia}{z^2(iaz^6 - 1)}, \quad a = 0.14, \tag{147}$$

which has the symmetries

$$R\left(e^{i\frac{\pi}{3}}z\right) = e^{-i\frac{2\pi}{3}}R(z), \tag{148}$$

$$R(1/z) = 1/R(z), \tag{149}$$

leading to the FR constraints

$$e^{i\frac{\pi}{3}L_3}e^{-i\frac{2\pi}{3}K_3}|\Psi\rangle = |\Psi\rangle, \tag{150}$$

$$e^{i\pi L_1}e^{i\pi K_1}|\Psi\rangle = |\Psi\rangle. \tag{151}$$

The ground state is then determined to be $|0,0\rangle \otimes |0,0\rangle$, and the first excited state is $|2,0\rangle \otimes |0,0\rangle$, in agreement with states of the beryllium-8 nucleus. However, this Skyrmion becomes unstable once the pion mass parameter exceeds 1. The true minimum is then described by two $B = 4$ cubes placed together, and as a first approximation to this in terms of a rational map we consider the O_h-symmetric map (whose Wronskian vanishes on the 14 faces of a truncated octahedron)

$$R(z) = \frac{z^8 + 4\sqrt{3}z^6 - 10z^4 + 4\sqrt{3}z^2 + 1}{z^8 - 4\sqrt{3}z^6 - 10z^4 - 4\sqrt{3}z^2 + 1}, \tag{152}$$

whose symmetries

$$R(iz) = 1/R(z), \tag{153}$$

$$R\left(\frac{iz+1}{-iz+1}\right) = \frac{-\sqrt{3} + R(z)}{1 + \sqrt{3}\,R(z)}, \tag{154}$$

lead to the FR constraints

$$e^{i\frac{\pi}{2}L_3}e^{i\pi K_1}|\Psi\rangle = |\Psi\rangle, \tag{155}$$

$$e^{i\frac{2\pi}{3\sqrt{3}}(L_1+L_2+L_3)}e^{-i\frac{2\pi}{3}K_2}|\Psi\rangle = |\Psi\rangle. \tag{156}$$

Here the ground state is again $|0,0\rangle \otimes |0,0\rangle$, but the $|2,0\rangle \otimes |0,0\rangle$ state is not allowed. The inertia tensors for this rational map are found to be diagonal, satisfying $U_{11} = U_{33}$, $V_{11} = V_{22} = V_{33}$, and $W_{ij} = 0$.

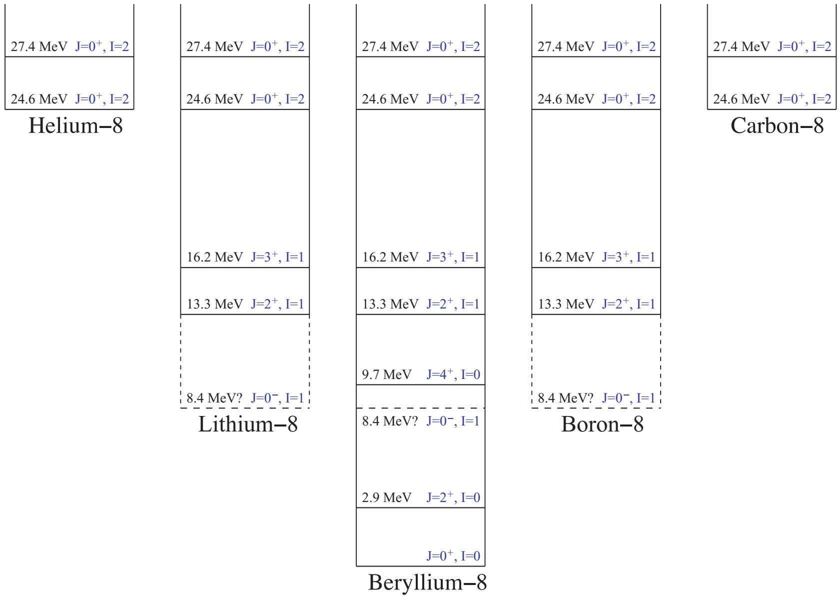

FIG. 15. (Color online) Energy level diagram for the quantized $B = 8$ Skyrmion, using the double cube approach. A putative $J = 0^-$ isotriplet is represented by dashed lines.

However, the O_h symmetry is too strong for the description of two cubes and has to be relaxed to D_{4h} symmetry. Therefore we consider next

$$R(z) = \frac{z^8 + bz^6 - az^4 + bz^2 + 1}{z^8 - bz^6 - az^4 - bz^2 + 1}, \tag{157}$$

where $a = 10$ and $b = 4\sqrt{3}$ restore the O_h symmetry. The rational map ansatz then gives a better approximation to the double cube Skyrmion, but only slightly because, for example, $U = -1$ at the origin with the rational map ansatz, whereas for the true solution, $U = -1$ at points near the cube centres. However, it has the right symmetry and is good enough to determine the allowed spin/isospin states. The FR constraints are now

$$e^{i\frac{\pi}{2}L_3} e^{i\pi K_1}|\Psi\rangle = |\Psi\rangle, \tag{158}$$

$$e^{i\pi L_1}|\Psi\rangle = |\Psi\rangle, \tag{159}$$

which again allows a $|2, 0\rangle \otimes |0, 0\rangle$ state. The inertia tensors have the same symmetry properties as for the octahedral map, with the exceptions that $U_{11} \neq U_{33}$ and $V_{33} \neq V_{11} = V_{22}$. This leads to the kinetic energy operator

$$T = \frac{1}{2V_{11}}\left(\mathbf{J}^2 - L_3^2\right) + \frac{L_3^2}{2V_{33}} + \frac{K_1^2}{2U_{11}} + \frac{K_2^2}{2U_{22}} + \frac{K_3^2}{2U_{33}}. \tag{160}$$

To determine the parities of states, we observe that the rational map [Eq. (157)] has the reflection symmetry $-1/\overline{R(z)} = -1/R(-1/\bar{z})$. The parity operator can therefore be represented by $\mathcal{P} = e^{i\pi K_2}$.

To progress, we now work directly with two cubic $B = 4$ Skyrmions separated along the x_3-axis and find the moments of inertia of the resulting structure using the parallel axis theorem (ignoring the interaction of the cubes). The top cube is rotated by $\frac{\pi}{4}$ about the x_3-axis relative to the standard orientation

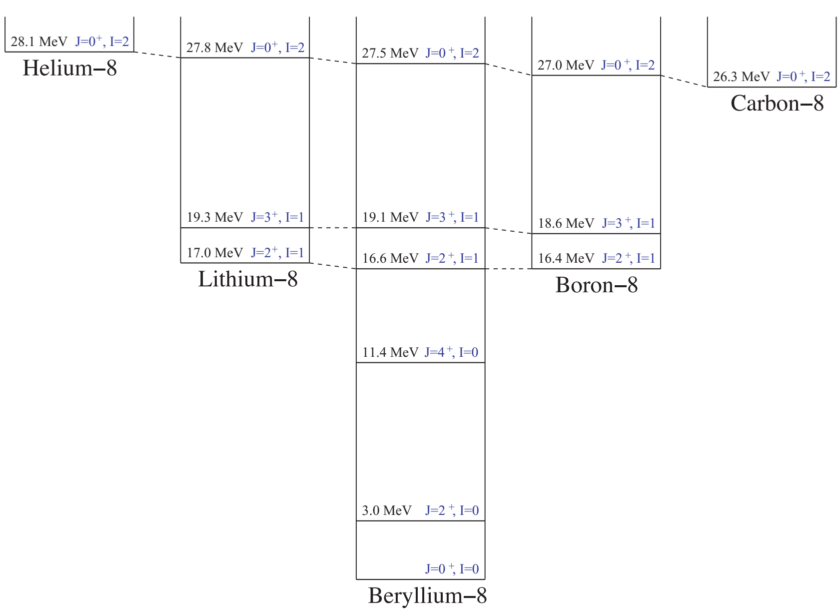

FIG. 16. (Color online) Energy level diagram for nuclei with $B = 8$.

TABLE I. Energies and parities of the $B = 8$ allowed states. $E_{\rm dc}$ and $E_{\rm rm}$ are the quantum energies obtained using the double cube approach and rational map ansatz, respectively.

J, I	Wave function	Parity	$E_{\rm dc}$ (MeV)	$E_{\rm rm}$ (MeV)
0, 0	$\|0,0\rangle \otimes \|0,0\rangle$	+	0	0
2, 0	$\|2,0\rangle \otimes \|0,0\rangle$	+	2.9	2.7
4, 0	$\|4,0\rangle \otimes \|0,0\rangle$	+	9.7	9.0
	$(\|4,4\rangle + \|4,-4\rangle) \otimes \|0,0\rangle$	+	17.7	11.2
0, 1	$\|0,0\rangle \otimes (\|1,1\rangle - \|1,-1\rangle)$	−	8.4	9.5
2, 1	$\|2,0\rangle \otimes (\|1,1\rangle - \|1,-1\rangle$	−	11.3	12.2
	$(\|2,2\rangle + \|2,-2\rangle) \otimes (\|1,1\rangle + \|1,-1\rangle)$	+	13.3	12.1
	$(\|2,2\rangle + \|2,-2\rangle) \otimes \|1,0\rangle$	−	14.1	12.4
3, 1	$(\|3,2\rangle - \|3,-2\rangle) \otimes (\|1,1\rangle + \|1,-1\rangle)$	+	16.2	14.8
	$(\|3,2\rangle - \|3,-2\rangle) \otimes \|1,0\rangle$	−	16.9	15.1
4, 1	$\|4,0\rangle \otimes (\|1,1\rangle - \|1,-1\rangle)$	−	18.1	18.5
	$(\|4,2\rangle + \|4,-2\rangle) \otimes (\|1,1\rangle + \|1,-1\rangle)$	+	20.1	18.4
	$(\|4,2\rangle + \|4,-2\rangle) \otimes \|1,0\rangle$	−	20.8	18.7
	$(\|4,4\rangle + \|4,-4\rangle) \otimes (\|1,1\rangle - \|1,-1\rangle)$	−	26.1	20.7
0, 2	$\|0,0\rangle \otimes (\|2,2\rangle + \|2,-2\rangle)$	+	24.6	26.3
	$\|0,0\rangle \otimes (\|2,1\rangle + \|2,-1\rangle)$	−	26.7	26.4
	$\|0,0\rangle \otimes \|2,0\rangle$	+	27.4	28.6
2, 2	$\|2,0\rangle \otimes (\|2,2\rangle + \|2,-2\rangle)$	+	27.5	29.0
	$(\|2,2\rangle + \|2,-2\rangle) \otimes (\|2,2\rangle - \|2,-2\rangle)$	−	29.5	30.8
	$\|2,0\rangle \otimes (\|2,1\rangle + \|2,-1\rangle)$	−	29.6	29.1
	$\|2,0\rangle \otimes \|2,0\rangle$	+	30.3	31.3
	$(\|2,2\rangle + \|2,-2\rangle) \otimes (\|2,1\rangle - \|2,-1\rangle)$	+	31.6	31.6

corresponding to Eq. (76). The bottom cube is rotated by $-\frac{\pi}{4}$ about the x_3-axis relative to the standard orientation. One difficulty here is in determining the separation of the cubes. The picture in Fig. 14 suggests that the separation is the value of r where the profile function becomes close to zero. From Fig. 2 we see that it is reasonable to take $r = 1.8$, leading to the separation in question being $d = r/\sqrt{3} = 1.04$ in dimensionless units. Then

$$V_{11}^{(B=8)} = V_{22}^{(B=8)} = 2V_{11}^{(B=4)} + \mathcal{M}d^2 = 2706, \quad (161)$$

$$V_{33}^{(B=8)} = 2V_{33}^{(B=4)} = 1326, \quad (162)$$

where $\mathcal{M} = 1277$ (in dimensionless units) is the classical mass of two $B = 4$ Skyrmions. The isospin moments of inertia are simply given by

$$U_{11}^{(B=8)} = U_{22}^{(B=8)} = 2U_{11}^{(B=4)} = 286, \quad (163)$$

$$U_{33}^{(B=8)} = 2U_{33}^{(B=4)} = 339. \quad (164)$$

The equality of U_{11} and U_{22}, which we do not expect to be exactly satisfied by the true $B = 8$ solution, simplifies Eq. (160) to

$$T = \frac{1}{2V_{11}}(\mathbf{J}^2 - L_3^2) + \frac{1}{2U_{11}}(\mathbf{I}^2 - K_3^2) + \frac{L_3^2}{2V_{33}} + \frac{K_3^2}{2U_{33}}. \quad (165)$$

The ground state has quantum energy zero, so its total energy is simply the classical Skyrmion mass. The additional quantum energy of the spin 2, isospin 0 state is 2.9 MeV, which is a very good match to the experimental value of 3 MeV [32]. There are a lot of further excited states, consistent with the FR constraints [Eqs. (158) and (159)], whose wave functions, energies, and parities are presented in Table I. Figures 15 and 16 are energy level diagrams for the quantized $B = 8$ Skyrmion and for the $B = 8$ nuclei, respectively. We see a good agreement with experiment for positive-parity states and the appearance of some negative-parity states that have not yet been observed experimentally. Of particular interest is the appearance of the $J = 0$, $I = 1$ negative-parity state. If found, it could be a new ground state of the lithium-8 nucleus. The detection of the latter might be very difficult experimentally. We have also found quintets of $I = 2$ states. The lowest of these, with spin 0, have been detected experimentally with excitation energies very close to our predictions and include the helium-8 and carbon-8 ground states.

It remains worthwhile to find the optimal values of a and b in the rational map [Eq. (157)]. Ideally, the classical mass should not be very far away from the experimental mass of the beryllium-8 ground state which is 7455 MeV, and the moments of inertia should be comparable with the ones we get from the double cube approach. The second condition is more difficult to achieve since the rational map is defined on a sphere, and it cannot exactly reproduce a double cube configuration. Looking through a range of possible a and b values we find that the optimal map is given approximately by

$$R(z) = \frac{z^8 + \frac{13\sqrt{3}}{2}z^6 - 20z^4 + \frac{13\sqrt{3}}{2}z^2 + 1}{z^8 - \frac{13\sqrt{3}}{2}z^6 - 20z^4 - \frac{13\sqrt{3}}{2}z^2 + 1}, \quad (166)$$

leading to the following moments of inertia:

$$V_{11} = V_{22} = 2901, \quad (167)$$

$$V_{33} = 2214, \quad (168)$$

$$U_{11} = 308, \tag{169}$$
$$U_{22} = 268, \tag{170}$$
$$U_{33} = 283. \tag{171}$$

We have recalculated the energies of the states in Table I, using the kinetic energy operator [Eq. (160)] and formulas for the energy levels of an asymmetrical top [25]. The quantum energy of the first excited state is 2.7 MeV, which is only slightly worse than the double cube approach. However, for further excited states the discrepancy in results increases, making the advantages of the double cube approach more evident.

It is interesting to consider the $B = 8$ binding energy relative to two cubic $B = 4$ Skyrmions. Using results from Ref. [15], in which the pion mass m was set equal to 1, we have that the energy per baryon of the $B = 8$ solution is 883 MeV, and the energy per baryon of the $B = 4$ cube is 892 MeV. So the quantized $B = 8$ Skyrmion appears bound from breaking up into two $B = 4$ cubes by roughly 70 MeV, which is rather small by our standards. Experimentally, beryllium-8 is slightly unbound (by 0.1 MeV). Perhaps the inclusion of vibrational effects may cause the $B = 8$ double cube to become unbound.

XIV. CONCLUSION

The rational map ansatz simplifies the classification of the allowed spin and isospin states of quantized Skyrmions and has enabled us to estimate their moments of inertia and energy spectra. The results are promising and provide support for the interpretation of Skyrmions as nuclei. We have obtained the correct spin, parity, and isospin quantum numbers for the ground states and various excited states in most cases, and the quantum energies of excited states are reasonably close to the experimental values. We have also been able to predict some excited states that have not yet been observed. The new parameter set for the Skyrme model, with which we have been working throughout, has provided better results than the traditional parameter set for the larger values of B. We have also put into effect a new approach for some Skyrmions of odd baryon number, in particular for $B = 7$. By deforming the highly symmetric minimal-energy Skyrmion, we have been able to reproduce the spins of the experimental ground state and several excited states. We have given the first estimates of the energies of quantum states based on the double cube $B = 8$ Skyrmion, and similar methods should be applicable to the multicube solutions for $B = 12$, 16, and beyond, presented in Ref. [15].

The calculations presented here are subject to a number of limitations. First, we consider the semiclassical quantization, in which only the collective coordinates for rotations and isospin rotations are considered. A more accurate procedure would have to take into account further degrees of freedom, which we refer to as vibrational modes. Allowing the individual Skyrmions, or subclusters of Skyrmions, to move relative to each other, and performing a quantization of these degrees of freedom, would be a significant refinement to our approach. In so doing, some missing low-lying experimentally observed states of nuclei may appear. These include the low-lying excited states with $J = \frac{1}{2}$ and $I = \frac{1}{2}$ that are present for $B = 5$ and $B = 7$. Second, our current understanding is that the Skyrme model provides a description of nuclear physics in which nucleons are partially merged, and their orientations in space and isospace are highly correlated. In a sense, this is the opposite of a naive shell model, in which nucleons move in a potential and are to first approximation uncorrelated. A more realistic model would possibly lie somewhere between these two extremes. An example of a model with correlated nucleons is described in Ref. [33]. It leads in some cases to pictures of baryon density isosurfaces that are similar to those obtained using the Skyrme model and also illustrates α-clustering. In particular, a study of two-nucleon density distributions reveals a toroidal density isosurface for the deuteron, as predicted by the Skyrme model.

The work here should be taken further by working with the exact Skyrmion solutions, and not just the rational map approximation to these solutions. Classical energies and moments of inertia will change, though we hope not drastically. Further investigation of the effect of varying the dimensionless pion mass parameter is also warranted. The length scale of the Skyrmions is quite sensitive to this. Possibly, an increased parameter will create an instability in the Skyrmions with $B = 5$ or $B = 7$, thereby justifying our arguments for changing the symmetries. Finally, it will be interesting to calculate electromagnetic form factors within our version of the Skyrme model and compare to experiment.

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APPENDIX A: INERTIA TENSORS

The tensors of inertia for rational map Skyrmions may be expressed in the form

$$\Sigma_{ij} = 2\int \sin^2 f \,\frac{C_{\Sigma_{ij}}}{(1+|R|^2)^2} \times \left(1 + f'^2 + \frac{\sin^2 f}{r^2}\left(\frac{1+|z|^2}{1+|R|^2}\left|\frac{dR}{dz}\right|\right)^2\right) d^3x, \tag{A1}$$

where $\Sigma = (U, V, W)$ and the quantities $C_{U_{ij}}$ are given by

$$C_{U_{11}} = |1 - R^2|^2, \tag{A2}$$
$$C_{U_{22}} = |1 + R^2|^2, \tag{A3}$$
$$C_{U_{33}} = 4|R|^2, \tag{A4}$$
$$C_{U_{12}} = C_{U_{21}} = -2\,\Im R^2, \tag{A5}$$

$$C_{U_{13}} = C_{U_{31}} = 2\,(|R|^2 - 1)\Re R, \tag{A6}$$
$$C_{U_{23}} = C_{U_{32}} = 2\,(|R|^2 - 1)\Im R, \tag{A7}$$

the quantities $C_{V_{ij}}$ are given by

$$C_{V_{11}} = |1 - z^2|^2 \left|\frac{dR}{dz}\right|^2, \tag{A8}$$

$$C_{V_{22}} = |1 + z^2|^2 \left|\frac{dR}{dz}\right|^2, \tag{A9}$$

$$C_{V_{33}} = 4|z|^2 \left|\frac{dR}{dz}\right|^2, \tag{A10}$$

$$C_{V_{12}} = C_{V_{21}} = -2\Im z^2 \left|\frac{dR}{dz}\right|^2, \tag{A11}$$

$$C_{V_{13}} = C_{V_{31}} = 2\Re\left(|z|^2 z - \bar{z}\right) \left|\frac{dR}{dz}\right|^2, \tag{A12}$$

$$C_{V_{23}} = C_{V_{32}} = 2\Im\left(|z|^2 z + \bar{z}\right) \left|\frac{dR}{dz}\right|^2, \tag{A13}$$

and, finally, the quantities $C_{W_{ij}}$ are given by

$$C_{W_{11}} = \Re\left((1 - z^2)(1 - \bar{R}^2)\frac{dR}{dz}\right), \tag{A14}$$

$$C_{W_{22}} = \Re\left((1 + z^2)(1 + \bar{R}^2)\frac{dR}{dz}\right), \tag{A15}$$

$$C_{W_{33}} = 4\Re\left(\bar{R}z\frac{dR}{dz}\right), \tag{A16}$$

$$C_{W_{12}} = -\Im\left((1 + z^2)(1 - \bar{R}^2)\frac{dR}{dz}\right), \tag{A17}$$

$$C_{W_{13}} = -2\Re\left(z(1 - \bar{R}^2)\frac{dR}{dz}\right), \tag{A18}$$

$$C_{W_{23}} = -2\Im\left(z(1 + \bar{R}^2)\frac{dR}{dz}\right), \tag{A19}$$

$$C_{W_{21}} = \Im\left((1 - z^2)(1 + \bar{R}^2)\frac{dR}{dz}\right), \tag{A20}$$

$$C_{W_{31}} = -2\Re\left(\bar{R}(1 - z^2)\frac{dR}{dz}\right), \tag{A21}$$

$$C_{W_{32}} = 2\Im\left(\bar{R}(1 + z^2)\frac{dR}{dz}\right). \tag{A22}$$

APPENDIX B: OLD PARAMETERS

Here we collect some data on moments of inertia, in Skyrme units, calculated with the dimensionless pion mass parameter $m = 0.528$ that emerges from the calibration of Ref. [3]. The following results are novel, as they were obtained by using the rational map ansatz and the formulas in Appendix A, and extend from $B = 1$ up to $B = 4$. For $B = 1$ the rational map ansatz is exact, so our result should agree with that of Ref. [3], and indeed it does. For $B = 2, 3$ our results can be compared with the moments of inertia calculated from the exact Skyrmion solutions (with the same m) as given by Refs. [5,7]. This allows us to investigate the accuracy of the rational map ansatz for these Skyrmions.

The notation is as in the Secs. VI–IX. For $B = 1$

$$\lambda = 62.85. \tag{B1}$$

For $B = 2$

$$U_{11} = 135.43, \quad U_{33} = 86.59, \quad \text{and} \quad V_{11} = 221.88. \tag{B2}$$

Comparing these numbers to those obtained in Ref. [5] using the exact numerical solution ($U_{11} = 127.8, U_{33} = 86.9,$ and $V_{11} = 200.2$), we see that the rational map ansatz has enabled us to obtain quite accurate moments of inertia. We recall that the old parameter set led to a model of the deuteron that was much too tightly bound.

For $B = 3$

$$u = 170.01, \ v = 576.09, \ \text{ and } \ w = -109.47. \tag{B3}$$

These were evaluated in Ref. [7], by using the exact numerical solution ($u = 136, v = 435,$ and $w = -91$).

For $B = 4$

$$U_{11} = 197.60, \quad U_{33} = 236.49, \quad \text{and} \quad v = 911.45. \tag{B4}$$

These numbers were calculated by using the same procedure that we have used throughout, but with the old parameters. Walhout [9] performed a different style of analysis for the $B = 4$ Skyrmion, and unfortunately we are unable to directly compare our results for the individual components of the inertia tensors.

APPENDIX C: COEFFICIENTS OF WAVE FUNCTIONS

The FR constraints do not determine all coefficients in the wave functions. Usually finding these constants is trivial, but in some cases (as in the $B = 5$ first and second excited states) one has to be more careful. As an illustration let us consider the constants $c_\pm$ in Eq. (94). The solutions of Eqs. (91) and (92) form a subspace of Hilbert space, which is transformed into itself when acted upon by the operator of the kinetic energy [Eq. (95)]. Therefore, the eigenvectors of the operator will define the wave functions we are looking for. In terms of the moments of inertia, $c_\pm$ is given by

$$c_\pm = \frac{b_2 + 5b_1 - a_1 - 5a_2 \pm \sqrt{(b_2 + 5b_1 - a_1 - 5a_2)^2 + 20(a_1 - a_2)(b_1 - b_2)}}{2\sqrt{5}(b_1 - b_2)}, \tag{C1}$$

where

$$a_1 = \frac{1}{8}\left(\frac{10U_{11} + 2V_{11} + 20W_{11}}{U_{11}V_{11} - W_{11}^2} + \frac{25U_{33} + V_{33} - 10W_{33}}{U_{33}V_{33} - W_{33}^2}\right), \tag{C2}$$

$$a_2 = \frac{1}{8}\left(\frac{26U_{11} + 2V_{11} + 4W_{11}}{U_{11}V_{11} - W_{11}^2} + \frac{9U_{33} + V_{33} + 6W_{33}}{U_{33}V_{33} - W_{33}^2}\right), \tag{C3}$$

$$b_1 = \frac{1}{8}\left(\frac{10U_{11} + 2V_{11} - 4W_{11}}{U_{11}V_{11} - W_{11}^2} + \frac{25U_{33} + V_{33} - 10W_{33}}{U_{33}V_{33} - W_{33}^2}\right), \tag{C4}$$

$$b_2 = -\frac{1}{8}\left(\frac{26U_{11} + 2V_{11} - 20W_{11}}{U_{11}V_{11} - W_{11}^2} + \frac{9U_{33} + V_{33} + 6W_{33}}{U_{33}V_{33} - W_{33}^2}\right). \tag{C5}$$

The quantum energy of these states is given by

$$E = \frac{a_1 + 5a_2 + c_\pm\sqrt{5}(b_1 - b_2)}{6}. \tag{C6}$$

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