Transition probabilities in the U(6) limit of the symplectic interacting vector boson model

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The tensor properties of the algebra generators and the basis are determined in respect to the reduction chain Sp(12, R) ⊃ U(6) ⊃ U(3) ⊗ U(2) ⊃ O(3) ⊗ U(1), which defines one of the dynamical symmetries of the interacting vector boson model. The action of the Sp(12, R) generators as transition operators between the basis states is presented. Analytical expressions for their matrix elements in the symmetry-adapted basis are obtained. As an example the matrix elements of the $E2$ transition operator between collective states of the ground band are determined and compared with the experimental data for the corresponding intraband transition probabilities of nuclei in the actinide and rare-earth region. On the basis of this application the important role of the symplectic extension of the model is analyzed.

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I. INTRODUCTION

In the algebraic models the use of the dynamical symmetries defined by a certain reduction chain of the group of dynamical symmetry yields exact solutions for the eigenvalues and eigenfunctions of the model Hamiltonian, which is constructed from the invariant operators of the subgroups in the chain.

Moreover, it is very simple and straightforward to calculate matrix elements of transition operators between the eigenstates, as both the basis states and the operators can be defined as tensor operators in respect to the considered dynamical symmetry. Then the calculation of matrix elements is simplified by the use of a respective generalization of the Wigner-Eckart theorem, which requires the calculation of the isoscalar factors and reduced matrix elements. By definition such matrix elements give the transition probabilities between the collective states attributed to the basis states of the Hamiltonian.

The comparison of the experimental data with the calculated transition probabilities is one of the best tests of the validity of the considered algebraic model. With the aim of such applications of the symplectic extension of one of the dynamical symmetries in IVBM, we develop in this article a practical mathematical approach for explicit evaluation of the matrix elements of transitional operators in the model.

The algebraic interacting vector bosons model (IVBM) was developed [1] initially for the description of the low-lying bands of the well deformed even-even nuclei [2]. Recently this approach was adapted to incorporate the newly observed higher collective states, both in the first positive- and negative-parity bands [3] by considering the basis states a "yrast" states for the different values of the number of bosons, N, that built them. This was achieved by extending the dynamical symmetry group U(6) to the noncompact Sp(12, R). The excellent results obtained for the energy spectrum motivated the present investigation of the transition probabilities in the framework of the generalized IVBM with Sp(12, R) as a group of dynamical symmetry. Thus we consider the tensorial properties of the algebra generators in respect to the reduction chain:

$$\mathrm{Sp}(12, R) \supset \mathrm{U}(6) \supset \mathrm{U}(3) \otimes \mathrm{U}(2) \supset \mathrm{O}(3) \otimes \mathrm{U}(1). \quad (1)$$

and also classify the basis states by the quantum numbers corresponding to the irreducible representations (irreps) of its subgroups (Sec. II). In this way we are able to define the transition operators between the basis states and then to evaluate analytically their matrix elements (Sec. III).

Transition probabilities are by definition SO(3) reduced matrix elements of transition operators T^{E2} between the $|i\rangle$−initial and $|f\rangle$−final collective states

$$B(E2; L_i \rightarrow L_f) = \frac{1}{2L_i + 1} |\langle f \,\|\, T^{E2} \,\|\, i\rangle|^2. \quad (2)$$

As a first step we will test the theory on the transitions between the states belonging to the ground bands in the even-even nuclei from the rare earths and the actinides, where the energies and the staggering between the states are rather well reproduced in our model approach [3]. This proves the correct mapping of the basis states to the experimentally observed ones and their band systematic, which is very important for the theoretical reproduction of the behavior of the physical observables in the framework of the considered model.

II. TENSORIAL PROPERTIES OF THE GENERATORS OF THE SP(12, R) GROUP AND CONSTRUCTION OF THE SYMPLECTIC BASIS STATES OF IVBM

The basic building blocks of the IVBM [1] are the creation and annihilation operators of the vector bosons $u_m^{\dagger}(\alpha)$ and $u_m(\alpha)$ ($m = 0, \pm 1; \alpha = \pm\frac{1}{2}$), which can be considered components of a six-dimensional vector that transforms according to the fundamental U(6) irreducible representations $[1, 0, 0, 0, 0, 0]_6 \equiv [1]_6$ and $[0, 0, 0, 0, 0, -1]_6 \equiv [1]_6^*$, respectively. These irreducible representations become reducible along the chain of subgroups (1) defining the dynamical symmetry [2]. This means that along with the quantum number characterizing the representations of U(6), the operators are also characterized by the quantum numbers of the subgroups of chain (1).

The only possible representation of the direct product of $U(3) \otimes U(2)$ belonging to the representation $[1]_6$ of U(6) is $[1]_3.[1]_2$, i.e., $[1]_6 = [1]_3.[1]_2$. According to the reduction rules for the decomposition $U(3) \supset O(3)$ the representation $[1]_3$ of U(3) contains the representation $(1)_3$ of the group O(3) giving the angular momentum of the bosons $l = 1$ with a projection $m = 0, \pm 1$. The representation $[1]_2$ of U(2) defines the "pseudospin" of the bosons $T = \frac{1}{2}$, whose projection is given by the corresponding representation of U(1), i.e., $\alpha = \pm\frac{1}{2}$. In this way the creation and annihilation operators $u_m^{\dagger}(\alpha)$ and $u_m(\alpha)$ are defined as irreducible tensors along the chain (1) and the used phase convention and commutation relations are as follows [4]:

$$\left(u^{[1]_6}_{[1]_3[1]_2 m\alpha}\right)^{\dagger} = u^{[1]^*_6 \quad m\alpha}_{[1]^*_3[1]^*_2} = (-1)^{m+\frac{1}{2}-\alpha} u^{[1]^*_6}_{[1]^*_3[1]^*_2 -m-\alpha} \tag{3}$$

$$\left[u^{[1]^*_6 \quad m\alpha}_{[1]^*_3[1]^*_2}, u^{[1]_6}_{[1]_3[1]_2 n\beta}\right] = \delta_{m,n}\delta_{\alpha,\beta}.$$

We do not consider here the microscopic structure of the so-introduced vector bosons. In the IVBM they serve as a convenient mathematical tool and in the present work only their tensor properties are important, as they generate the transition operators and the basis states.

Initially the generators of the symplectic group Sp(12, R) were written as double tensors [5] with respect to the $O(3) \supset O(2)$ and $U(2) \supset U(1)$ reductions

$$A^{LM}_{TT_0} = \sum_{m,n}\sum_{\alpha,\beta} C^{LM}_{1m1n} C^{TT_0}_{\frac{1}{2}\alpha\frac{1}{2}\beta} \quad u^{[1]_6}_{[1]_3[1]_2 m\alpha} u^{[1]^*_6 \quad \beta n}_{[1]^*_3[1]^*_2}, \tag{4}$$

$$F^{LM}_{TT_0} = \sum_{m,n}\sum_{\alpha,\beta} C^{LM}_{1m1n} C^{TT_0}_{\frac{1}{2}\alpha\frac{1}{2}\beta} \quad u^{[1]_6}_{[1]_3[1]_2 m\alpha} u^{[1]_6}_{[1]_3[1]_2 n\beta}, \tag{5}$$

$$G^{LM}_{TT_0} = \sum_{m,n}\sum_{\alpha,\beta} C^{LM}_{1m1n} C^{TT_0}_{\frac{1}{2}\alpha\frac{1}{2}\beta} \quad u^{[1]^*_6 \quad \alpha m}_{[1]^*_3[1]^*_2} u^{[1]^*_6 \quad \beta n}_{[1]^*_3[1]^*_2}. \tag{6}$$

Further, they can be defined as irreducible tensor operators according to the whole chain (1) of subgroups and expressed in terms of Eqs. (4), (5), and (6)

$$A^{[\chi]_6 \quad LM}_{[\lambda]_3[2T]_2 \, TT_0} = C^{[1]_6 \qquad [1]^*_6 \qquad [\chi]_6}_{[1]_3[1]_2[1]^*_3[1]^*_2 \quad [\lambda]_3[2T]_2} C^{[1]_3[1]^*_3[\lambda]_3}_{(1)_3(1)_3(L)_3} A^{LM}_{TT_0}, \tag{7}$$

$$F^{[\chi]_6 \quad LM}_{[\lambda]_3[2T]_2 \, TT_0} = C^{[1]_6 \qquad [1]_6 \qquad [\chi]_6}_{[1]_3[1]_2[1]_3[1]_2 \quad [\lambda]_3[2T]_2} C^{[1]_3[1]_3[\lambda]_3}_{(1)_3(1)_3(L)_3} F^{LM}_{TT_0}, \tag{8}$$

$$G^{[\chi]_6 \quad LM}_{[\lambda]_3[2T]_2 \, TT_0} = C^{[1]^*_6 \qquad [1]^*_6 \qquad [\chi]_6}_{[1]^*_3[1]^*_2[1]^*_3[1]^*_2 \quad [\lambda]_3[2T]_2} C^{[1]^*_3[1]^*_3[\lambda]_3}_{(1)_3(1)_3(L)_3} G^{LM}_{TT_0}, \tag{9}$$

where, according to the lemma of Racah [6], the Clebsch-Gordan coefficients along the chain are factorized by means of the isoscalar factors (IF), defined for each step of decomposition (1). It should be pointed out [4] that the U(6) and U(3) IFs, entering in Eqs. (7), (8), and (9), are equal to ± 1.

The tensors [Eq. (7)] transform according to the direct product $[\chi]_6$ of the corresponding U(6) representations $[1]_6$ and $[1]^*_6$ [4], namely

$$[1]_6 \times [1]^*_6 = [1, -1]_6 + [0]_6, \tag{10}$$

where $[1, -1]_6 = [2, 1, 1, 1, 1, 0]_6$ and $[0]_6 = [1, 1, 1, 1, 1, 1]_6$ is the scalar U(6) representation. Further we multiply the two conjugated fundamental representations of $U(3) \otimes U(2)$

$$\begin{aligned}
&[1]_3[1]_2 \times [1]^*_3[1]^*_2 \\
&\quad = ([1]_3 \times [1]^*_3)([1]_2 \times [1]^*_2) \\
&\quad = ([210]_3 \oplus [1, 1, 1]_3) \times ([2, 0]_2 \oplus [1, 1]_2) \\
&\quad = [210]_3[2]_2 \oplus [210]_3[0]_2 \oplus [0]_3[2]_2 \oplus [0]_3[0]_2.
\end{aligned} \tag{11}$$

Obviously the first three $U(3) \otimes U(2)$ irreducible representations in the resulting decomposition (11) belong to the $[1, -1]_6$ of U(6) and the last one to $[0]_6$.

Introducing the notations $u_i^{\dagger}(\frac{1}{2}) = p_i^{\dagger}$ and $u_i^{\dagger}(-\frac{1}{2}) = n_i^{\dagger}$, the scalar operator

$$A^{[0]_6 00}_{[0]_3[0]_2 00} = \widehat{N} = \frac{1}{\sqrt{2}} \sum_m C^{00}_{1m1-m}(p_m^{\dagger} p_{-m} + n_m^{\dagger} n_{-m}) \tag{12}$$

has the physical meaning of the total number of bosons operator $\widehat{N} = \widehat{N_p} + \widehat{N_n}$, where $\widehat{N_p} = \sum p_m^{\dagger} p_m$, $\widehat{N_n} = \sum n_m^{\dagger} n_m$ and is obviously the first-order invariant of all the unitary groups, U(6), U(3), and U(2). Hence it reduces them to their respective unimodular subgroups, SU(6), SU(3), and SU(2). Moreover, the invariant operator $(-1)^N$ decomposes the action space $\mathcal{H}$ of the Sp(12, R) generators to the even $\mathcal{H}_+$ with $N = 0, 2, 4, \ldots,$ and odd $\mathcal{H}_-$ with $N = 1, 3, 5, \ldots,$ subspaces of the boson representations of Sp(12, R) [7].

In terms of Elliott's notations [8](λ, μ), where $\lambda = n_1 - n_2$, $\mu = n_2 - n_3$, we have $[210]_3 = (1, 1)$ and $[0]_3 = (0, 0)$. The corresponding values of L from the $SU(3) \supset O(3)$ reduction rules are $L = 1, 2$ in the (1, 1) irrep and $L = 0$ in the (0, 0). The values of T are 1 and 0 for the U(2) irreps $[2]_2$ and $[0]_2$, respectively. Hence, the U(2) irreps in the direct product distinguish the equivalent U(3) irreps that appear in this reduction and there is not degeneracy. The tensors with $T = 0$ correspond to the SU(3) generators

$$A^{[1-1]_6 \quad 1M}_{[210]_3[0]_2 \quad 00} = \frac{1}{\sqrt{2}} \sum_{m,k} C^{1M}_{1m1k}(p_m^{\dagger} p_k + n_m^{\dagger} n_k) \tag{13}$$

$$A^{[1-1]_6 \quad 2M}_{[210]_3[0]_2 \quad 00} = \frac{1}{\sqrt{2}} \sum_{m,k} C^{2M}_{1m1k}(p_m^{\dagger} p_k + n_m^{\dagger} n_k), \tag{14}$$

representing the components of the angular L_M and Elliott's quadrupole Q_M momenta operators [8]. The tensors

$$\begin{aligned}
A^{[1-1]_6 \quad 00}_{[0]_3[2]_2 \quad 11} &= \sqrt{\frac{3}{2}} \sum_m p_m^{\dagger} n_{-m} \sim T_1, \\
A^{[1-1]_6 \quad 00}_{[0]_3[2]_2 \quad 1-1} &= -\sqrt{\frac{3}{2}} \sum_m n_m^{\dagger} p_{-m} \sim T_{-1} \\
A^{[1-1]_6 \quad 00}_{[0]_3[2]_2 \quad 10} &= -\frac{\sqrt{3}}{2} \sum_m (p_m^{\dagger} p_{-m} - n_m^{\dagger} n_{-m}) \sim T_0,
\end{aligned} \tag{15}$$

correspond to the SU(2) generators, which are the components of the pseudospin operator $\widehat{T}$. And finally the tensors

$$A^{[1-1]_6}_{[210]_3[2]_2}{}^{LM}_{11} = \sum_{m,k} C^{LM}_{1m1k} p^+_m n_k, \tag{16}$$

$$A^{[1-1]_6}_{[210]_3[2]_2}{}^{LM}_{1-1} = \sum_{m,k} C^{LM}_{1m1k} n^+_m p_k \tag{17}$$

and

$$A^{[1-1]_6}_{[210]_3[2]_2}{}^{LM}_{10} = \frac{1}{\sqrt{2}} \sum_{m,k} C^{LM}_{1m1k}(p^+_m p_k - n^+_m n_k), \tag{18}$$

with $L = 1, 2$ and $M = -L, -L+1, \ldots, L$ extend the U(3) $\otimes$ U(2) algebra to the U(6) one.

By analogy, the tensors described in Eqs. (8) and (9) transform according to [4]

$$[1]_6 \times [1]_6 = [2]_6 + [1,1]_6 \tag{19}$$

and

$$[1]^*_6 \times [1]^*_6 = [-2]_6 + [-1,-1]_6,$$

respectively. But because the basis states of the IVBM are fully symmetric, we consider only the fully symmetric U(6) representations $[2]_6$ and its conjugated $[-2]_6$, because for the operators (8) and (9) we have $(F^{[\chi]_6}_{[\lambda]_3[2T]_2}{}^{LM}_{TT_0})^\dagger = (-1)^{\lambda+\mu+L-M+T-T_0} G^{[\chi]^*_6}_{[\lambda]^*_3[2T]^*_2}{}^{L-M}_{T-T_0}$, where $[\lambda]_3 = (\lambda, \mu)$. Hence we present the next decompositions only for the F tensors (19). According to the decomposition rules for the fully symmetric U(6) irreps [4] we have

$$[2]_6 = [2]_3[2]_2 + [1,1]_3[0]_2 = (2,0)[2]_2 + (0,1)[0]_2, \tag{20}$$

which further contain in (2, 0) $L = 0, 2$ with $T = 1$ and in (0, 1) $- L = 1$ with $T = 0$. Their explicit expressions in terms of the creation $p^\dagger_i, n^\dagger_i$ and annihilation operators p_i, n_i at $i = 0, \pm 1$ are

$$F^{[2]_6}_{[2]_3[2]_2}{}^{LM}_{11} = \sum_{m,k} C^{LM}_{1m1k} p^\dagger_m p^\dagger_k,$$

$$F^{[2]_6}_{[2]_3[2]_2}{}^{LM}_{1-1} = \sum_{m,k} C^{LM}_{1m1k} n^\dagger_m n^\dagger_k \tag{21}$$

$$F^{[2]_6}_{[2]_3[2]_2}{}^{LM}_{10} = \frac{1}{\sqrt{2}} \sum_{m,k} C^{LM}_{1m1k}(p^\dagger_m n^\dagger_k - n^\dagger_m p^\dagger_k),$$

$$F^{[2]_6}_{[1,1]_3[0]_2}{}^{LM}_{00} = \frac{1}{\sqrt{2}} \sum_{m,k} C^{LM}_{1m1k}(p^\dagger_m n^\dagger_k + n^\dagger_m p^\dagger_k). \tag{22}$$

In addition to the SU(3) raising generators (21) $F^{[2]_6}_{(2,0)}$ we have the operator $F^{[2]_6}_{(0,1)}$ (22), which is a new one compared to the generators of the Sp(6, R) model of Rosensteel and Rowe [9].

The above operators and their conjugated ones, $G^{[\chi]^*_6}_{[\lambda]^*_3[2T]^*_2}{}^{LM}_{TT_0}$, change the number of bosons by two and realize the symplectic extension of the U(6) algebra. In this way we have listed all the irreducible tensor operators in respect to the reduction chain (1) that correspond to the infinitesimal operators of the Sp(12, R) algebra.

TABLE I. Tensor products of two raising operators.

$[2]_6$ $[\lambda_1]_3[2T_1]_2$	$[2]_6$ $[\lambda_2]_3[2T_1]_2$	$[4]_6$ $[\lambda]_3[2T]_2$	O(3) $K; L$	U(2) T	U(1) T_0
$(2,0)[2]_2$	$(2,0)[2]_2$	$(4,0)[4]_2$	0; 0, 2, 4	2	$0, \pm1, \pm2$
$(2,0)[2]_2$	$(2,0)[2]_2$	$(2,1)[2]_2$	1; 1, 2, 3	1	$0, \pm1$
$(2,0)[2]_2$	$(2,0)[2]_2$	$(0,2)[0]_2$	0; 0, 2	0	0
$(2,0)[2]_2$	$(0,1)[0]_2$	$(2,1)[2]_2$	1; 1, 2, 3	1	$0, \pm1$
$(0,1)[0]_2$	$(0,1)[0]_2$	$(0,2)[0]_2$	0; 0, 2	0	0

Next we can introduce the tensor products

$$T^{([\chi_1]_6[\chi_2]_6)}{}^{\omega[\chi]_6}_{[\lambda]_3[2T]_2}{}^{LM}_{TT_0} = \sum T^{[\chi_1]_6}_{[\lambda_1]_3[2T_1]_2}{}^{L_1M_1}_{T_1(T_0)_1} T^{[\chi_2]_6}_{[\lambda_2]_3[2T_2]_2}{}^{L_2M_2}_{T_2(T_0)_2} \times C^{[\chi_1]_6}_{[\lambda_1]_3[T_1]_2}{}^{[\chi_2]_6}_{[\lambda_2]_3\,[T_2]_2}{}^{\omega[\chi]_6}_{[\lambda]_3[2T]_2} C^{[\lambda_1]_3[\lambda_2]_3[\lambda]_3}_{(L_1)_3(L_2)_3\,(L)_3} \times C^{L_1}_{M_1}{}^{L_2}_{M_2}{}^{L}_{M} C^{T_1}_{(T_0)_1}{}^{T_2}_{(T_0)_2}{}^{T}_{T_0} \tag{23}$$

of two tensor operators, $T^{[\chi]_6}_{[\lambda]_3[2T]_2}{}^{LM}_{TT_0}$, which are as well tensors in respect to the considered reduction chain. We use Eq. (23) to obtain the tensorial properties of the operators in the enveloping algebra of Sp(12,R), containing the products of the algebra generators. In this particular case we are interested in the transition operators between states differing by four bosons, $T^{[4]_6}_{[\lambda]_3[2T]_2}{}^{LM}_{TT_0}$, expressed in terms of the products of two operators, $F^{[2]_6}_{[\lambda]_3[2T]_2}{}^{LM}_{TT_0}$. Making use of the decomposition (20) and the reduction rules in the chain (1), we list in Table I all the representations of the chain subgroups that define the transformation properties of the resulting tensors.

To clarify the role of the tensor operators introduced in this section as transition operators and to simplify the calculation of their matrix elements, the basis for the Hilbert space must be symmetry adapted to the algebraic structure along the considered subgroup chain (1). It is evident from Eqs. (21) and (22) that the basis states of the IVBM in the $\mathcal{H}_+$ ($N-$even) subspace of the boson representations of Sp(12, R) can be obtained by a consecutive application of the raising operators $F^{[2]_6}_{[\lambda]_3[2T]_2}{}^{LM}_{TT_0}$ on the boson vacuum $\mid 0\rangle$ (ground state), annihilated by the tensor operators $G^{[\chi]_6}_{[\lambda]_3[2T]_2}{}^{LM}_{TT_0} \mid 0\rangle = 0$ and $A^{[\chi]_6}_{[\lambda]_3[2T]_2}{}^{LM}_{TT_0} \mid 0\rangle = 0$.

Thus, in general a basis for the considered dynamical symmetry of the IVBM can be constructed by applying the multiple symmetric coupling [Eq. (23)] of the raising tensors $F^{[2]_6}_{[\lambda_i]_3[2T_i]_2}{}^{L_iM_i}_{T_iT_{0i}}$ with itself: $[F \times \cdots \times F]^{[\chi]_6}_{[\lambda]_3[2T]_2}{}^{LM}_{TT_0}$. Note that only fully symmetric tensor products $[\chi]_6 \equiv [N]_6$ are nonzero, because the raising operator commutes with itself. The possible U(3) couplings are enumerated by the set $[\lambda]_3 = \{[n_1, n_2, 0] \equiv (\lambda = n_1 - n_2, \mu = n_2); n_1 \geqslant n_2 \geqslant 0\}$. The number of copies of the operator F in the symmetric product tensor $[N]_6$ is $N/2$, where $N = \lambda + 2\mu$ [3]. Each raising operator will increase the number of bosons N by two. Then, the resulting infinite basis is denoted by:

$$|[N](\lambda, \mu); KLM; TT_0\rangle, \tag{24}$$

where KLM are the quantum numbers for the nonorthonormal basis of the irrep (λ, μ).

The Sp(12, R) classification scheme for the SU(3) boson representations obtained by applying the reduction rules [3] for the irreps in the chain (1) for even value of the number of bosons N is shown on Table II. Each row (fixed N) of the table corresponds to a given irreducible representation of the U(6) algebra. Then the possible values for the pseudospin, given in the column next to the respective value of N, are $T=\frac{N}{2}, \frac{N}{2}-1, \ldots 0$. Thus when N and T are fixed, $2T+1$ equivalent representations of the group SU(3) arise. Each of them is distinguished by the eigenvalues of the operator T_0 : $-T, -T+1, \ldots, T$, defining the columns of Table II. The same SU(3) representations (λ, μ) arise for the positive and negative eigenvalues of T_0.

Now it is clear which of the tensor operators act as transition operators between the basis states ordered in the classification scheme presented on Table II. The operators $F^{[2]_6}_{[\lambda]_3[2T]_2}{}^{LM}_{TT_0}$ with $T_0=0$ (22) give the transitions between two neighboring cells ($\downarrow$) from one column, whereas the ones with $T_0=\pm 1$(21) change the column as well ($\swarrow$). The tensors $A^{[1-1]_6}_{[2,1]_3[0]_2}$ (13) and (14), which correspond to the SU(3) generators do not change the SU(3) representations (λ, μ) but can change the angular momentum L inside it ($\Longrightarrow$). The SU(2) generating tensors $A^{[1-1]_6}_{[0]_3[2]_2}$ [Eq. (15)] change the projection $T_0(\rightarrow)$ of the pseudospin T and in this way distinguish the equivalent SU(3) irreps belonging to the different columns of the same row of Table II. Inside a given cell the transition between the different SU(3) irreps ($\Downarrow$) is realized by the operators $A^{[1-1]_6}_{[2,1]_3[2]_2}$ [Eqs. (16), (17), and (18)] that represent the U(6) generators. The action of the tensor operators on the SU(3) vectors inside a given cell or between the cells of Table II. is also schematically presented on it with corresponding arrows, given above in parentheses.

TABLE II. Classification of the basis states.

N	T	$T_0 \backslash \ldots \pm 4$	± 3	± 2	± 1	0
0	0				$\swarrow F^{[2]_6}_{[2]_3[2]_2}$	(0,0)
2	1			$F^{[2]_6}_{[1,1]_3[0]_2}\ \downarrow$	$\Longrightarrow$ (2,0)	(2,0)
	0				$A^{[1-1]_6}_{[2,1]_3[0]_2}$ –	(0,1)
4	2			(4,0)	(4,0)	(4,0)
	1			–	$A^{[1-1]_6}_{[2,1]_3[2]_2}\ \Downarrow$ (2,1)	(2,1)
	0			–	–	(0,2)
6	3	$A^{[1-1]_6}_{[0]_3[2]_2}$	(6,0)	(6,0)	(6,0)	(6,0)
	2		–	(4,1)	(4,1)	(4,1)
	1	$\rightarrow$	–	–	(2,2)	(2,2)
	0		–	–	–	(0,3)
8	4	(8,0)	(8,0)	(8,0)	(8,0)	(8,0)
	3	–	(6,1)	(6,1)	(6,1)	(6,1)
	2	–	–	(4,2)	(4,2)	(4,2)
	1	–	–	–	(2,3)	(2,3)
	0	–	–	–	–	(0,4)
...		...	...	...	...	...

III. MATRIX ELEMENTS OF THE TRANSITION OPERATORS IN SYMMETRY-ADAPTED BASIS

Physical applications are based on the correspondence of sequences of SU(3) vectors to sequences of collective states belonging to different bands in the nuclear spectra. The above analysis permits the definition of the appropriate transition operators as corresponding combinations of the tensor operators given in Sec. II.

Matrix elements of the Sp(12, R) algebra can be calculated in several ways. A direct method is to use the Sp(12, R) commutation relations [1] to derive recursion relations. Another is to start from approximate matrix element and proceed by successive approximations, adjusting the matrix elements until the commutation relations are precisely satisfied [10]. The third method is to make use of a vector-valued coherent-state representation theory [5,11] to relate the matrix elements to the known matrix elements of a much simpler Weyl algebra.

However, in the present article we use another technique for calculation of the matrix elements of the Sp(12, R) algebra, based on the fact that the representations of the SU(3) subgroup in IVBM are built with the help of the two kinds of vector bosons, which is in some sense simpler than the construction of the SU(3) representations in the IBM and the Sp(6, R) symplectic model.

In the preceding section we expressed the Sp(12, R) generators $F^{LM}_{TT_0}$, $G^{LM}_{TT_0}$, $A^{LM}_{TT_0}$ and the basis states as components of irreducible tensors in respect to the reduction chain (1). Thus, for calculating their matrix elements, we have the advantage of using the Wigner-Eckart theorem in two steps. For the SU(3) $\rightarrow$ SO(3) and SU(2) $\rightarrow$ U(1) reduction we need the standard SU(2) Clebsch-Gordan coefficient (CGC)

$$\begin{aligned}&\langle [N'](\lambda',\mu'); K'L'M'; T'T'_0 \big| T^{[\chi]_6}_{[\sigma]_3[2t]_2}{}^{lm}_{tt_0} \big| \\ &\quad \times [N]\,(\lambda,\mu); KLM; TT_0\,\rangle \\ &\quad = \langle [N'](\lambda',\mu'); K'L' \big\| T^{[\chi]_6}_{[\sigma]_3[2t]_2}{}^{lm}_{tt_0} \big\| \\ &\quad \times [N](\lambda,\mu); KL\rangle C^{L'M'}_{LMlm} C^{T'T'_0}_{TT_0tt_0}. \end{aligned} \tag{25}$$

For the calculation of the double-barred reduced matrix elements in Eq. (25) we use the next step:

$$\begin{aligned}&\langle [N'](\lambda',\mu'); K'L' \| T^{[\chi]_6}_{[\sigma]_3[2t]_2}{}^{lm}_{tt_0} \| [N]\,(\lambda,\mu); KL\rangle \\ &\quad = \langle [N'] ||| T^{[\chi]_6}_{[\sigma]_3[2t]_2} ||| [N]\rangle C^{[N]_6}_{(\lambda,\mu)[2T]_2}{}^{[\chi]_6}_{[\sigma]_3[2t]_2}{}^{[N']_6}_{(\lambda',\mu')[2T']_2} \\ &\quad \times C^{(\lambda,\mu)}_{KL}{}^{[\lambda]_3}_{k(l)_3}{}^{(\lambda',\mu')}_{K'L'}, \end{aligned} \tag{26}$$

where $C^{[N]_6}_{(\lambda,\mu)[2T]_2}{}^{[\chi]_6}_{[\sigma]_3[2t]_2}{}^{[N']_6}_{(\lambda',\mu')[2T']_2}$ and $C^{(\lambda,\mu)}_{KL}{}^{[\lambda]_3}_{k(l)_3}{}^{(\lambda',\mu')}_{K'L'}$ are U(6) and SU(3) IFs. Obviously the practical value of the application of the generalized Wigner-Eckart theorem for the calculation of the matrix elements of the Sp(12, R) generators and the construction of the symplectic basis depends on the knowledge of the isoscalar factors for the reductions U(6) $\supset$ U(3) $\otimes$ U(2) and U(3) $\supset$ O(3), respectively. For the evaluation of the matrix elements [Eq. (25)] of the Sp(12, R) operators in respect to the chain show in Eq. (1) the reduced triple-barred U(6) matrix elements are also required [Eq. (26)].

IV. $B(E2)$ TRANSITION PROBABILITIES FOR THE GROUND BAND

In the symplectic extension of the IVBM the complete spectrum of the system is obtained in all the even subspaces with fixed N, even of the UIR $[N]_6$ of U(6), belonging to a given even UIR of Sp(12, R). The classification scheme of the SU(3) boson representations for even values of the number of bosons N is presented in Table II. The equivalent use of the (λ,μ) labels, resulting from the connections $T=\lambda/2$ and $N=\lambda+2\mu$ facilitates the final reduction to the SO(3) representations, which define the angular momentum L and its projection M. The multiplicity index K appearing in this reduction is related to the projection of L on the body fixed frame and is used with the parity (π) to label the different bands (K^{π}) in the energy spectra of the nuclei. We have defined the parity of the states as $\pi=(-1)^T$ [3]. This allowed us to describe both positive and negative bands.

In this article we give as an example the evaluation of the $E2$ transition probabilities of the ground-state band (GSB) [3], whose states were identified with the SU(3) multiplets $(0,\mu)$. In terms of (N,T) this choice corresponds to $(N=2\mu, T=0)$. We define the energies of each state with given L as yrast energy with respect to N in the considered bands. Hence for the ground band their minimum values are obtained at $N=2L$. Using the tensorial properties of the Sp(12, R) generators it is easy to define the $E2$ transition operator between the states of the considered band:

$$T^{E2}=e\big[A^{[1-1]_6\ \ 20}_{[210]_3[0]_2\ 00}+\theta\big([F\times F]^{[4]_6\ \ 20}_{(0,2)[0]_2\ 00}+[G\times G]^{[-4]_6\ \ 20}_{(2,0)[0]_2 00}\big)\big], \tag{27}$$

where the first tensor operator is expressed in terms of the boson creation $p^{\dagger}_m, n^{\dagger}_m$ and annihilation $p_m, n_m, m=\pm1$ operators in Eq. (14) and, as part of the SU(3) generators, actually changes only the angular momentum with $\Delta L=2$.

The tensor product

$$\begin{aligned}&[F\times F]^{[4]_6\ \ 20}_{(0,2)[0]_2\ 00}\\&\quad=\sum C^{[2]_6\qquad [2]_6\qquad [4]_6}_{(2,0)[2]_2(2,0)[2]_2\ (0,2)[0]_2}C^{(2,0)\ (2,0)\ \ (0,2)}_{(2)_3\ \ \ (2)_3\ \ \ (2)_3}\\&\qquad\times C^{20}_{20\ 20}C^{10}_{11\ 1-1}F^{[2]_6\ \ 20}_{(2,0)[2]_2\ 11}F^{[2]_6\ \ 20}_{(2,0)[2]_2 1-1}\end{aligned} \tag{28}$$

of the operators (21) that are the pair raising Sp(12, R) generators changes the number of bosons by $\Delta N=4$ and $\Delta L=2$. Thus, for calculating the matrix elements of Eq. (27) between the basis states shown in Eq. (24), we have the advantage of using the Wigner-Eckart theorem in two steps [Eqs. (25) and (26)], where only their reduced triple-barred U(6) matrix elements are required.

However, the SU(3) generators [Eqs. (13) and (14)] are scalars with respect to the isospin group U(2), so they act only on the SU(3) part of the wave function and the Wigner-Eckart theorem is applied in respect to the SU(3) subgroup [12]

$$\begin{aligned}&\langle[N],(\lambda',\mu');K'L'M';T'T'_0\big|\ A^{[1,-1]_6\ \ lm}_{(1,1)[0]_2\ 00}\big|[N],\\&\quad\times(\lambda,\mu);KLM;TT_0\rangle\\&\quad=\delta_{TT'}\delta_{T_0T'_0}\delta_{\lambda\lambda'}\delta_{\mu\mu'}\sum_{\rho=1,2}C^{(\lambda,\mu)\quad (1,1)\quad \rho(\lambda',\mu')}_{K(L)\quad k(l)\quad K'(L')}\\&\qquad\times C^{L'M'}_{LM\ lm}\langle[N],(\lambda',\mu')\big|\big|\big|\ A^{[1,-1]_6}_{(1,1)[0]_2}\ \big|\big|\big|[N],(\lambda,\mu)\rangle.\end{aligned}$$

The sum over ρ runs over terms containing products of IFs of SU(3) and U(6), respectively. The reduced triple-barred matrix elements are well known and are given for $\rho=1$ by [9]

$$\langle[N],(\lambda,\mu)|||A^{[1-1]_6}_{(1,1)_3[0]_2}\quad|||[N],(\lambda,\mu)\rangle_1=\begin{cases}g_{\lambda\mu},\ \mu=0\\-g_{\lambda\mu},\ \mu\neq0\end{cases}, \tag{29}$$

where

$$g_{\lambda\mu}=2\left(\frac{\lambda^2+\mu^2+\lambda\mu+3\lambda+3\mu}{3}\right)^{1/2} \tag{30}$$

and the phase convention is chosen to agree with that of Draayer and Akiyama [13]. For $\rho=2$ we have $\langle[N],(\lambda,\mu)|||A^{[1-1]_6}_{[210]_3[0]_2}\quad|||[N],(\lambda,\mu)\rangle_2=0$. Thus, for the matrix elements of $A^{[1-1]_6\ \ 20}_{[210]_3[0]_2\ 00}$ between the states attributed to the GSB we obtain

$$\begin{aligned}&\langle[N],(0,\mu);0L-20;00|A^{[1-1]_6\ \ 20}_{(1,1)[0]_2\ 00}|[N],(0,\mu);0L0;00\rangle\\&\quad=C^{(0,\mu)\ (1,1)\ (0,\mu)}_{L-2\quad 2\quad\ L}C^{L,0}_{L-2,0\ 2,0}\\&\qquad\times\langle[N],(0,\mu)\big|\big|\big|A^{[1-1]_6}_{(1,1)\ [0]_2}\big|\big|\big|[N],(0,\mu)\rangle\\&\quad=2\left[\frac{(\mu-L+2)(\mu+L+1)(L-1)L}{2(2L-1)(2L+1)}\right]^{1/2}C^{L,0}_{L-2,0\ 2,0}.\end{aligned} \tag{31}$$

The value of the reduced SU(3) Clebsch-Gordan coefficient (IF) is taken from Ref. [14]. Actually, we are interested in the SO(3) reduced matrix elements which enter in Eq. (2). Thus taking into account the yrast conditions $\mu=L$ we obtain

$$\begin{aligned}&\langle[N],(0,\mu);0L-2;00\big\|A^{[1-1]_6}_{(1,1)[0]_2}\big\|[N],(0,\mu);0L;00\rangle\\&\quad=2\left[\frac{(L-1)L}{(2L-1)}\right]^{1/2}.\end{aligned} \tag{32}$$

For the calculation of the matrix element

$$\begin{aligned}&\langle[N+4],(0,\mu+2);0L+20;00|\\&\times[F\times F]^{[4]_6\ \ \ 20}_{(0,2)[0]_2\ \ 00}|[N],(0,\mu);0L0;00\rangle\\&\quad=C^{[N]_6\qquad [4]_6\qquad [N+4]_6}_{(0,\mu)[0]_2\ (0,2)[0]_2\ (0,\mu+2)[0]_2}C^{(0,\mu)\ \ (0,2)\ \ (0,\mu+2)}_{L\qquad 2\qquad L+2}C^{L+2,\ 0}_{L,0\ \ 2,0}\\&\qquad\times\langle[N+4],(0,\mu+2)|||[F\times F]^{[4]_6}_{(0,2)[0]_2}\ \ |||[N],(0,\mu)\rangle\end{aligned} \tag{33}$$

we use the standard recoupling technique for two coupled U(6) tensors [15]:

$$\begin{aligned}&\langle[N']|||[T^{[\alpha]_6}\times T^{[\beta]_6}]^{\sigma[\gamma]_6}|||[N]\rangle\\&\quad=\sum_{c,\rho_1,\rho_2}U([N]_6;[\beta]_6;[N']_6;[\alpha]_6|[N_c]_6\rho_2\rho_1;[\gamma]_6\sigma)\\&\qquad\times\langle[N']|||T^{[\alpha]_6}\ |||[N_c]\rangle\langle[N_c]|||T^{[\beta]_6}|||[N]\rangle,\end{aligned} \tag{34}$$

where U(...) are the U(6) Racah coefficients in unitary form [16]. For the reduced triple-bared matrix element in our case, which is multiplicity free and hence there is no sum, we have

$$\langle [N+4] |||[F \times F]^{[4]_6}_{(0,2)[0]_2} |||[N]\rangle = U([N]_6;[2]_6;[N+4]_6;[2]_6|[N+2]_6;[4]_6) \times \langle [N+4]|||F^{(2,0)}|||[N+2]\rangle\langle[N+2]|||F^{(2,0)}|||[N]\rangle,$$

where the corresponding Racah coefficient for maximal coupling representations is equal to unity ([15]; see also formula A9 of Ref. [16]). Applying again the formula (34) with respect to coupled tensor $F^{[2]_6}$ and using the fact that in the case of vector bosons which span the fundamental irrep [1] of $u(n)$ algebra the $u(n)$-reduced matrix element of raising generators has the well known form [17]

$$\langle [N+1]|||u^\dagger_m(\alpha)|||[N]\rangle = \sqrt{N+1}. \tag{35}$$

we obtain

$$\langle [N+2]|||F^{[2]_6}|||[N]\rangle = U([N]_6;[1]_6;[N+2]_6;[1]_6|[N+1]_6;[2]_6) \times \langle [N+2]|||p^{\dagger[1]_6}|||[N+1]\rangle\langle[N+1]|||p^{\dagger[1]_6}|||[N]\rangle = \sqrt{(N+1)(N+2)}$$

and in analogy

$$\langle [N+4]|||F^{[2]_6}|||[N+2]\rangle = \sqrt{(N+3)(N+4)}.$$

Introducing in Eq. (33) the above results and the value of the coefficient $C^{(0,\mu)(2,0)(0,\mu+2)}_{L\ \ 2\ \ L+2}$ from Ref. [14] (the corresponding/fully stretched [15]/ U(6) IF for maximal coupling representations is equal to 1), we finally derive for the SO(3) reduced matrix element

$$\langle [N+4],(0,\mu+2);0L+2;00\|[F\times F]^{[4]_6\ \ 20}_{(0,2)[0]_2\ 00}\| \times [N],(0,\mu);0L;00\rangle = \left[\frac{(\mu+L+3)(\mu+L+5)(L+1)(L+2)}{(\mu+1)(\mu+2)(2L+3)(2L+5)}\right]^{1/2} \times\sqrt{(N+1)(N+2)(N+3)(N+4)}, = \sqrt{(2L+1)(2L+2)(2L+3)(2L+4)}, \tag{36}$$

where $N = 2\mu + \lambda$ and for the last row the yrast condition $\mu = L$ is taken into account. For the calculation of the matrix element of $[G\times G]^{[-4]_6\ \ 20}_{(2,0)[0]_2\ 00}$ we use the conjugation property

$$\langle [N-4],(0,\mu-2);0L-20;00|[G\times G]^{[-4]_6\ \ 20}_{(2,0)[0]_2\ 00}| \times [N],(0,\mu);0L0;00\rangle = (\langle [N],(0,\mu);0L0;00|[F\times F]^{[4]_6\ \ 20}_{(0,2)[0]_2\ 00}| \times [N-4],(0,\mu-2);0L-20;00\rangle)^* = C^{[N-4]_6\ \ [4]_6\ \ [N]_6}_{(0,\mu-2)[0]_2\ (0,2)[0]_2\ (0,\mu)[0]_2} C^{(0,\mu-2)\ (0,2)(0,\mu)}_{L-2\ \ \ 2\ \ \ L} C^{L,0}_{L-2,020} \times\sqrt{(N-3)(N-2)(N-1)N} = C^{L,0}_{L-2,0\ 20}\sqrt{(2L-3)(2L-2)(2L-1)2L}. \tag{37}$$

With the help of the above analytic expressions [Eqs. (31), (35), and (36)] for the matrix elements of the tensor operators forming the E2 transition operator we can calculate the transition probabilities [Eq. (2)] between the states in the ground band as attributed to the SU(3) symmetry-adapted basis states of the model [Eq. (24)]. It is obvious that the second term in T^{E2} (27) comes from the symplectic extension of the model. The behavior of each term of the transition operator is plotted as a function of the angular momentum L in Fig. 1 where for comparison typical experimental data for the GSB of ^{236}U are also shown. It can also be seen that because of the yrast conditions ($\mu = L$), the well-known parabolic behavior corresponding to the Elliott' s quadrupole operator is modified and looks like a rigid rotor curve (see also the curve corresponding to $\theta = 0$ in Fig. 2). In this case, the rigid rotor predictions are asymptotically determined by the ordinary SO(3) Clebsch-Gordan coefficient. Such type of curve is obtained in the limit of large-dimensional irreducible representations $2\lambda + \mu \to \infty$ when su(3) algebra contracts to the rigid rotor algebra rot(3) = $[R^5]$so(3) [18]. It is obvious that the experimental points are well reproduced by the modified SU(3) term up to $L \approx 20$, whereas for the description of the states with $L > 20$ the symplectic term is appropriate.

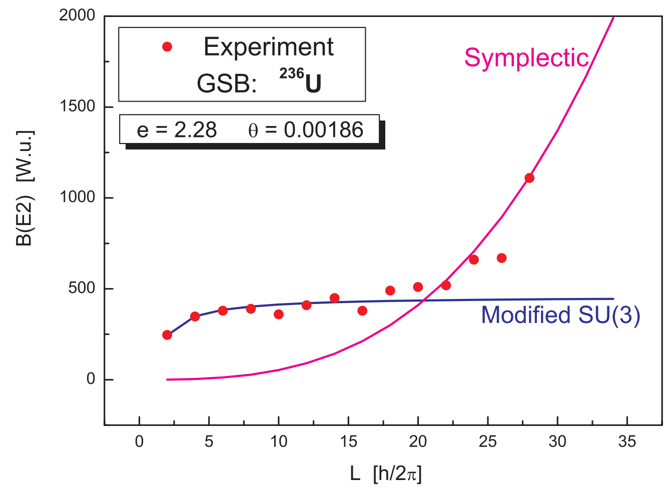

FIG. 1. (Color online) The behavior of the number conserving and symplectic terms of the matrix elements of the transition operator T^{E2} [Eq. (27)].

To see what type of $B(E2)$ behavior can be obtained in our theoretical predictions we give in Fig. 2 the results for various values of the parameters θ and e.It is clearly seen that the two main types of $B(E2)$ behavior—the enhancement or the reduction of the $B(E2)$ values can be reproduced. The strongly enhanced values that are an indication for increased collectivity in the high angular momentum domain are easily obtained for positive values of the parameter θ. For negative values of the parameter θ we obtain behavior similar to that of the standard SU(3) one and it can be used to reproduce the well-known cutoff effect. Such saturation effect is also characteristic feature of the IBM-based calculations in its SU(3) limit. Although the coefficient in front of symplectic term is about four orders of magnitude smaller than the SU(3) contribution to the transition operator its role in reproducing the correct behavior (with or without cutoff) of the transition probabilities between the states of the GSB band is very important.

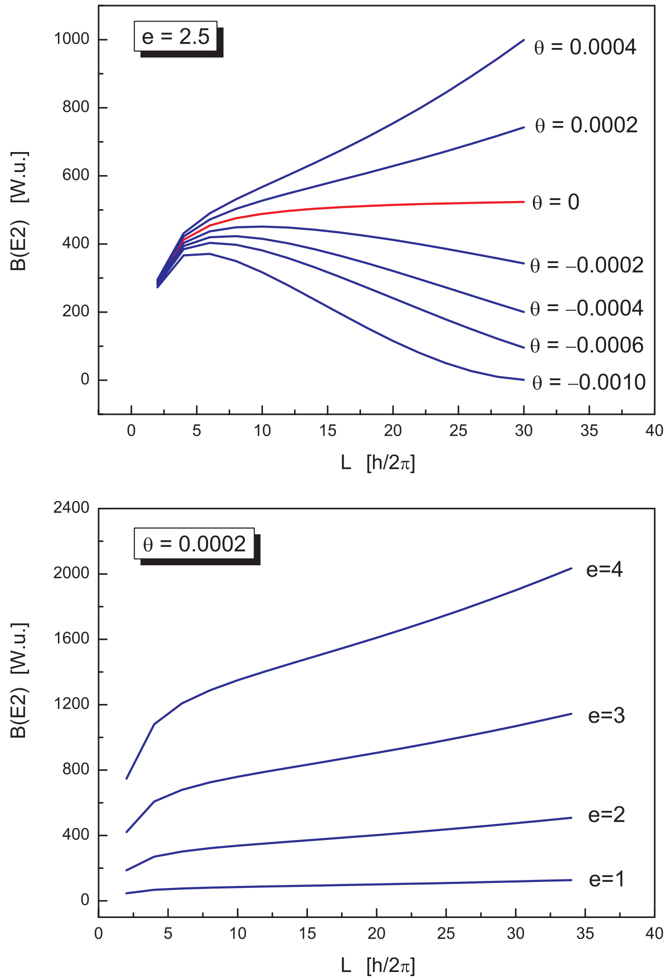

FIG. 2. (Color online) Study of particular dependence of the yrast $B(E2)$ values, using the $E2$ operator [Eq. (27)] as a function of the parameters θ and e.

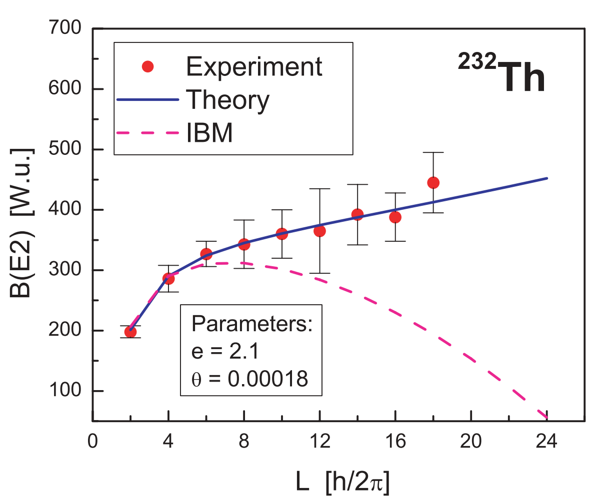

FIG. 3. (Color online) Comparison of theoretical and experimental values for the $B(E2)$ transition probabilities for the ^{232}Th.

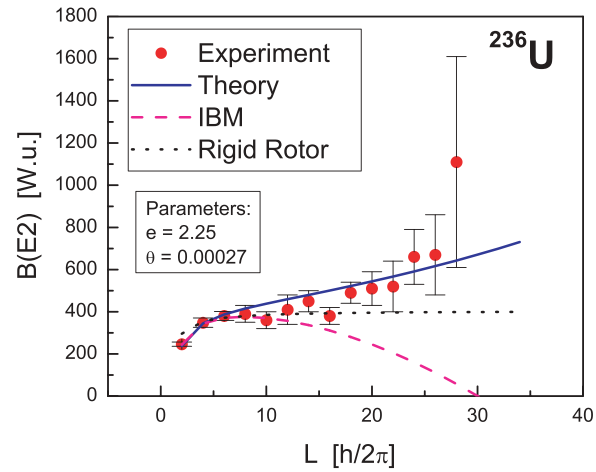

FIG. 4. (Color online) The same as in Fig. 3 but for ^{236}U.

V. APPLICATION TO REAL NUCLEI

To prove the correct predictions following from our theoretical results we apply the theory to real nuclei for which there is available experimental data for the transition probabilities [19–21] between the states of the ground bands up to very high angular momenta. The application actually consists of fitting the two parameters of the transition operator T^{E2} (27) to the experiment for each of the considered bands.

As a first example we consider the intraband $B(E2)$ transitions in the GSB for the nucleus ^{232}Th. The experimental data for it are compared with the corresponding theoretical results of the symplectic IVBM and the SU(3) limit of the IBM in Fig. 3. We see the standard SU(3) behavior for the latter and because the IBM involves a small number of quadrupole bosons the cutoff effect is observed at low spins and hence only the transitions between the first few excited states are well reproduced by it. From Fig. 3 one can see the enhanced $B(E2)$ values in the high-spin region and the good reproduction of the experimental data [19] by our theoretical predictions.

Next the ^{236}U case is presented. For it there are a lot of experimental data, reaching high angular momenta up to $L = 28$ [20]. The $B(E2)$ values for transitions between members of the GSB compared with the theoretical results of the IVBM, the IBM and the rigid rotor are shown in Fig. 4. One can see that the IBM works well for the transitions between the first excited states ($L = 2 - 10$). The rigid rotor describes well the experimental states in the middle spin region ($L = 4 - 16$), whereas for the high spins the $B(E2)$ values must be enhanced due to the observed collectivity excess. Thus, at high spins in the yrast band the calculations of the IBM and the rigid rotor model cannot reproduce the fine structure of the $B(E2)$ data. As mentioned in the preceding section, such an enhancement can be obtained for slightly positive values of the parameter θ in the transition operator T^{E2} [Eq. (27)] (see Fig. 2). From Fig. 4 one can see that the experimental points lay very close to the theoretical curves.

As a final example we consider the ^{156}Dy nucleus. The results (values of e and θ) obtained for the yrast band compared with that of the IBM and the experimental data [21] are presented in Fig. 5. From it the saturation effect is clearly

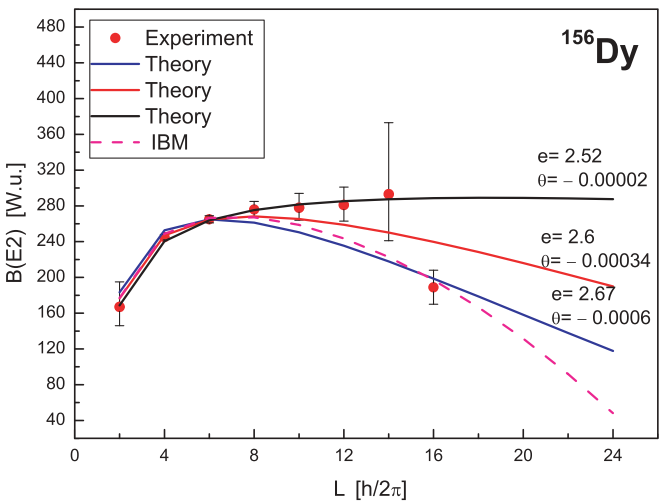

FIG. 5. (Color online) Comparison of theoretical and experimental values for the $B(E2)$ transition probabilities for several closed values of the parameters θ and e for the nucleus ^{156}Dy.

observed at $L = 16$. The calculated results (for the IVBM with negative value of the parameter θ) illustrate characteristics of the generic $B(E2)$ curve discussed in connection with Fig. 2. We see that the two models (for IVBM the blue curve) give identical results with about the same level of accuracy. As one can see, better overall reproduction of the experimental data can be obtained if the parameters θ and e are slightly modified, which is also illustrated in Fig. 5.

From the presented examples we see how sensitive the theory is to the term coming from the symplectic extension and in particular from the sign of the parameter θ entering in the transition operator [Eq. (27)].

VI. CONCLUSIONS

In the present article we investigated the tensor properties of the algebra generators of Sp(12, R) with respect to the reduction chain (1). Sp(12, R) is the group of dynamical symmetry of the IVBM and the considered chain of subgroups was applied in Ref. [3] for the description of positive- and negative-parity bands in well-deformed nuclei. The basis states of the model Hamiltonian are also classified by the quantum numbers corresponding to the irreducible representations of the subgroups from the chain and in this way the symmetry adapted basis is constructed in this limit of the model. The action of the symplectic generators as transition operators between the basis states is analyzed. Analytical expressions for the matrix elements of Sp(12, R) generators in the U(6) symmetry-adapted basis are obtained as well.

In the present new application of the rotational limit of the symplectic extension of the IVBM, the model was tested on the more complicated and complex problem of reproducing the $B(E2)$ transition probabilities between the states of the ground band up to very high spins. In developing the theory the advantages of the algebraic approach were used first for the proper assignment of the basis states to the experimentally observed states of the collective bands. Here the construction of the $E2$ transition operator as linear combination of tensor operators representing the generators of the subgroups of the respective chain is a basic result that allows the application of a specific version of the Wigner-Eckart theorem and consecutively leads to analytic results for their matrix elements in the U(6) symmetry-adapted basis that gives the transition probabilities.

Analyzing the terms taking part in the construction of the $E2$ transition operator the important role of the symplectic extension of the model is revealed. In the application to real nuclei the parameters of the transition operator are evaluated in a fitting procedure for GSB of the considered nuclei. The experimental data for the presented examples is reproduced rather well, although the results are very sensitive to the values of the parameters.

A further investigation of these bands in other nuclei from other nuclear regions will clarify better the development of collectivity in the symplectic extension of the IVBM, for which more experiment on transition probabilities is needed. The presented approach is rather general and universal and can be used for the calculation of transitions in other collective bands, in particular in the similarly constructed negative-parity bands and the excited β bands, which are of great interest lately in the nuclear structure.

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