Unified description of the double β decay to the first quadrupole phonon state in spherical and deformed nuclei

C. M. Raduta[1] and A. A. Raduta[1,2]
[1]*Institute of Physics and Nuclear Engineering, Bucharest, P. O. Box MG6, Romania*
[2]*Department of Theoretical Physics and Mathematics, Bucharest University, P. O. Box MG11, Romania*
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The Gamow-Teller transition operator is written as a polynomial in the dipole proton-neutron and quadrupole charge-conserving quasiparticle random-phase approximation boson operators, using the prescription of the boson expansion technique. Then, the $2\nu\beta\beta$ process ending on the first 2^+ state in the daughter nucleus is allowed through one-, two-, and three-boson states describing the odd-odd intermediate nucleus. The approach uses a single particle basis that is obtained by projecting out the good angular momentum from an orthogonal set of deformed functions. The basis for mother and daughter nuclei may have different deformations. The GT transition amplitude as well as the half-lives were calculated for 18 transitions. Results are compared with the available data as well as with the predictions obtained with other methods.

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I. INTRODUCTION

One of the most exciting subjects of nuclear physics is that of $\beta\beta$ decay. The interest is generated by the fact that to describe the decay rate quantitatively one has to treat consistently the neutrino properties as well as the nuclear structure features. The process may take place in two distinct ways: (a) by a $2\nu\beta\beta$ decay where the initial nuclear system, the mother nucleus, is transformed in the final stable nuclear system, usually called the daughter nucleus (two electrons and two antineutrinos) and (b) by the $0\nu\beta\beta$ process where the final state involves no neutrino. The latter decay mode is especially interesting because one hopes that its discovery might provide a definite answer to the question whether the neutrino is a Majorana or a Dirac particle. The contributions over several decades have been reviewed by many authors [1–6]. Although none of the $\beta\beta$ emitters is a spherical nucleus, most formalisms use a single-particle spherical basis.

In the mid-1990s we treated the $2\nu\beta\beta$ process in a proton-neutron quasiparticle random-phase approximation (pnQRPA) formalism using a projected spherical single-particle basis that resulted in having a unified description of the process for spherical and deformed nuclei [7,8]. Recently the single-particle basis [9,10] has been improved by accounting for the volume conservation while the mean field is deformed [11,12]. The improved basis has been used for describing quantitatively the ground state to ground state $\beta\beta$ decay rates as well as the corresponding half-lives [13,14]. The results were compared with the available data as well as with the predictions of other formalisms. The manners in which the physical observable is influenced by the nuclear deformations of mother and daughter nuclei are in detail commented. Two features of the deformed basis are essential: (a) the single-particle energy levels exhibit no gap and (b) the pairing properties of the deformed system are different from those of spherical system. These two aspects of the deformed nuclei affect the overlap matrix of the pnQRPA states of mother and daughter nuclei. Moreover, considering the Gamow-Teller (GT) transition operator in the single-particle space generated by the deformed mean field, one obtains an inherent renormalization with respect to the one acting in a spherical basis.

In Ref. [15] we studied the higher pnQRPA effects on the GT transition amplitude by means of the boson expansion technique for a spherical single-particle basis. Considering higher-order boson expansion terms in the transition operator, significant corrections to the GT transition amplitude are obtained, especially when the strength of the two-body particle-particle (*pp*) interaction approaches its critical value where the lowest dipole energy is vanishing. As we showed in the quoted reference, there are transitions that are forbidden at the pnQRPA level but allowed once the higher pnQRPA corrections are included. An example of this type is the $2\nu\beta\beta$ decay leaving the daughter nucleus in a collective excited state 2^+ [16]. The electrons resulting in this process can be distinguished from the ones associated to the ground-to-ground transition by measuring, in coincidence, the γ rays due to the transition $2^+ \rightarrow 0^+$ in the daughter nucleus [17].

The aim of this work is to study the $2\nu\beta\beta$ decay $0^+ \rightarrow 2^+$, where 0^+ is the ground state of the emitter, whereas 2^+ is a single quadrupole phonon state describing the daughter nucleus. The adopted procedure is the boson expansion method as formulated in our previous article [15] but using a projected spherical single-particle basis. It is worth mentioning that despite the fact the boson expansion approach has been widely used for two alike fermion operators, the procedure has been extended for proton-neutron operators only since the beginning of 1990s ([15]) for a spherical single-particle basis and recently for a deformed mean field [16].

In conclusion, our formalism involves two basic ingredients defined in some of our previous publications. (a) The first is the boson expansion approach for the Gamow-Teller transition operator; in this way the $2\nu\beta\beta$ decay $0^+ \rightarrow 2^+$, forbidden in the framework of the pnQRPA formalism, becomes an allowed process. (b) The second is the projected spherical single-particle basis. Using such a basis one may unitarily treat the transitions of spherical and deformed nuclei. Moreover, situations when the mother and daughter nuclei have different nuclear deformations could be accounted for. Because here we present for the first time results for $\beta\beta$ decay to excited states in an extended article, we devote a separate section to each of the mentioned ingredients. In this way we hope to get a self-sustained description of the results.

The formalism and the results will be described along several sections as follows. In Sec. II the projected single-particle basis to be used is presented. The model Hamiltonian, written in second quantization for the projected spherical basis, is defined in Sec. II. The states of mother, daughter, and intermediate odd-odd nucleus, which participate at the considered process, are eigenstates of the chosen many-body Hamiltonian. Section IV deals with the boson expansion of the Gamow-Teller $\beta\beta$ transition operator. Results for the $\beta\beta$ transition amplitude are presented in Sec. V. Numerical applications to 18 nuclei are commented on in Sec. VI. The main results and conclusions are summarized in Sec. VII.

II. A PROJECTED SPHERICAL SINGLE-PARTICLE BASIS

The single-particle mean field is determined by a particle-core Hamiltonian:

$$\tilde{H} = H_{\rm sm} + H_{\rm core} - M\omega_0^2 r^2 \sum_{\lambda=0,2} \sum_{-\lambda \leqslant \mu \leqslant \lambda} \alpha^*_{\lambda\mu} Y_{\lambda\mu}, \quad (2.1)$$

where $H_{\rm sm}$ denotes the spherical shell-model Hamiltonian and $H_{\rm core}$ is a harmonic quadrupole boson (b^+_μ) Hamiltonian associated to a phenomenological core. The interaction of the two subsystems is accounted for by the third term of the above equation, written in terms of the shape coordinates $\alpha_{00}, \alpha_{2\mu}$. The quadrupole shape coordinates and the corresponding momenta are related to the quadrupole boson operators by the canonical transformation:

$$\begin{aligned} \alpha_{2\mu} &= \frac{1}{k\sqrt{2}}[b^\dagger_{2\mu} + (-)^\mu b_{2,-\mu}], \\ \pi_{2\mu} &= \frac{ik}{\sqrt{2}}[(-)^\mu b^\dagger_{2,-\mu} - b_{2\mu}], \end{aligned} \quad (2.2)$$

where k is an arbitrary C number. The monopole shape coordinate is determined from the volume conservation condition. In the quantized form, the result is:

$$\alpha_{00} = \frac{1}{2k^2\sqrt{\pi}} \left\{ 5 + \sum_\mu [2b^\dagger_\mu b_\mu + (b^\dagger_\mu b^\dagger_{-\mu} + b_{-\mu} b_\mu)(-)^\mu] \right\}. \quad (2.3)$$

Averaging $\tilde{H}$ on the eigenstates of $H_{\rm sm}$, hereafter denoted by $|nljm\rangle$, one obtains a deformed boson Hamiltonian whose ground state is, in the harmonic limit, described by a coherent state

$$\Psi_g = \exp[d(b^+_{20} - b_{20})]|0\rangle_b, \quad (2.4)$$

with $|0\rangle_b$ standing for the vacuum state of the boson operators and d a real parameter that simulates the nuclear deformation. However, the average of $\tilde{H}$ on Ψ_g is similar to the Nilsson Hamiltonian [18]. Due to these properties, it is expected that the best trial functions to generate a spherical basis are

$$\Psi^{pc}_{nlj} = |nljm\rangle \Psi_g. \quad (2.5)$$

The projected states are obtained by acting on these deformed states with the projection operator

$$P^I_{MK} = \frac{2I+1}{8\pi^2} \int D^{I*}_{MK}(\Omega) \hat{R}(\Omega) d\Omega, \quad (2.6)$$

where $D^I_{MK}(\Omega)$ denotes the rotation matrix corresponding to the Euler angles Ω. The subset of projected states

$$\Phi^{IM}_{nlj}(d) = \mathcal{N}^I_{nlj} P^I_{MI}[|nljI\rangle \Psi_g] \equiv \mathcal{N}^I_{nlj} \Psi^{IM}_{nlj}(d), \quad (2.7)$$

are orthogonal with the normalization factor denoted by $\mathcal{N}^I_{nlj}$.

Although the projected states are associated to the particle-core system, they can be used as a single-particle basis. Indeed, when a matrix element of a particle-like operator is calculated, the integration on the core collective coordinates is performed first, which results in obtaining a final factorized expression: one factor carries the dependence on deformation and one is a spherical shell-model matrix element.

The single-particle energies are approximated by the average of the particle-core Hamiltonian $H' = \tilde{H} - H_{\rm core}$ on the projected spherical states defined by Eq. (2.7):

$$\epsilon^I_{nlj} = \langle \Phi^{IM}_{nlj}(d) | H' | \Phi^{IM}_{nlj}(d) \rangle. \quad (2.8)$$

The off-diagonal matrix elements of H' is ignored at this level. Their contribution is, however, considered when the residual interaction is studied. It is an open interesting question how to determine the mean-field operator that admits the energies given by Eq. (2.8) as eigenvalues.

As shown in Ref. [9], the dependence of the new single-particle energies on deformation is similar to that shown by the Nilsson model [18]. The quantum numbers in the two schemes are, however, different. Indeed, here we generate from each j a multiplet of $(2j+1)$ states distinguished by the quantum number I, which plays the role of the Nilsson quantum number Ω and runs from 1/2 to j and, moreover, the energies corresponding to the quantum numbers K and $-K$ are equal to each other. However, for a given I there are $2I+1$ degenerate substates, whereas the Nilsson states are only double degenerate. As explained in Ref. [9], the redundancy problem can be solved by changing the normalization of the model functions:

$$\langle \Phi^{IM}_\alpha | \Phi^{IM}_\alpha \rangle = 1 \Longrightarrow \sum_M \langle \Phi^{IM}_\alpha | \Phi^{IM}_\alpha \rangle = 2. \quad (2.9)$$

Due to this weighting factor the particle-density function provides the consistency result that the number of particles that can be distributed on the (2I + 1) substates is at most 2, which agrees with the Nilsson model. Here α stands for the set of shell-model quantum numbers *nlj*. Due to this normalization, the states Φ^{IM}_α used to calculate the matrix elements of a given operator should be multiplied with the weighting factor $\sqrt{2/(2I+1)}$.

Finally, we recall a fundamental result, obtained in Ref. [12], concerning the product of two projected states, which comprises a product of two core components. Therein we have proved that the matrix elements of a two-body interaction corresponding to the present scheme are very close to the matrix elements corresponding to spherical states projected from a deformed state consisting of two spherical single-particle states times a single collective core wave function. The small discrepancies of the two types of matrix elements could be washed out by using slightly different strengths for the two-body interaction in the two methods. This feature is caused by the coherent state properties.

III. THE MODEL HAMILTONIAN FOR THE STATES INVOLVED IN THE PROCESS

In the present work we are interested in describing the Gamow-Teller two-neutrino $\beta\beta$ decay of an even-even deformed nucleus. In our treatment the Fermi transitions, contributing about 20% to the total rate, and the "forbidden" transitions are ignored, which is a reasonable approximation for the two-neutrino $\beta\beta$ decay in medium and heavy nuclei. The $2\nu\beta\beta$ process is conceived as two successive single β^- virtual transitions. The first transition connects the ground state of the mother nucleus to a magnetic dipole state 1^+ of the intermediate odd-odd nucleus, which subsequently decays to the first state 2^+ of the daughter nucleus. The second leg of the transition is forbidden within the pnQRPA approach but nonvanishing within a higher pnQRPA approach [15]. The states, involved in the $2\nu\beta\beta$ process are described by the following many-body Hamiltonian:

$$\begin{aligned}
H = &\sum_{\tau\alpha IM} \frac{2}{2I+1}(\epsilon_{\tau\alpha I} - \lambda_{\tau\alpha})c^\dagger_{\tau\alpha IM}c_{\tau\alpha IM} \\
&- \sum_{\tau\alpha\alpha' I} \frac{G_\tau}{4} P^\dagger_{\tau\alpha I}P_{\tau\alpha' I} \\
&+ 2\chi \sum_{pn;p'n';\mu} \beta^-_\mu(pn)\beta^+_{-\mu}(p'n')(-)^\mu \\
&- 2\chi_1 \sum_{pn;p'n';\mu} P^-_{1\mu}(pn)P^+_{1,-\mu}(p'n')(-)^\mu \\
&- \sum_{\tau,\tau'=p,n} X_{\tau,\tau'} Q_\tau Q^\dagger_{\tau'}.
\end{aligned} \tag{3.1}$$

The operator $c^\dagger_{\tau\alpha IM}(c_{\tau\alpha IM})$ creates (annihilates) a particle of type $\tau(=p,n)$ in the state Φ^{IM}_α, when acting on the vacuum state $|0\rangle$. To simplify the notations, hereafter the set of quantum numbers $\alpha(=nlj)$ will be omitted. The two-body interaction consists of three terms, the pairing, the dipole-dipole particle-hole (ph), and the particle-particle (pp) interactions. The corresponding strengths are denoted by G_τ, χ, χ_1, respectively. All of them are separable interactions, with the factors defined by the following expressions:

$$\begin{aligned}
P^\dagger_{\tau I} &= \sum_M \frac{2}{2I+1} c^\dagger_{\tau IM} c^\dagger_{\widetilde{\tau IM}}, \\
\beta^-_\mu(pn) &= \sum_{M,M'} \frac{\sqrt{2}}{\hat{I}} \langle pIM|\sigma_\mu|nI'M'\rangle \frac{\sqrt{2}}{\hat{I'}} c^\dagger_{pIM} c_{nI'M'}, \\
P^-_{1\mu}(pn) &= \sum_{M,M'} \frac{\sqrt{2}}{\hat{I}} \langle pIM|\sigma_\mu|nI'M'\rangle \frac{\sqrt{2}}{\hat{I'}} c^\dagger_{pIM} c^\dagger_{\widetilde{nI'M'}}, \\
\hat{I} &= \sqrt{2I+1} \\
Q^{(\tau)}_{2\mu} &= \sum_{i,k} q^{(\tau)}_{ik}\left(c^\dagger_i c_k\right)_{2\mu}, \quad q^{(\tau)}_{ik} = \frac{\sqrt{2}}{\hat{I}_k}\langle I_i||r^2Y_2||I_k\rangle.
\end{aligned} \tag{3.2}$$

The remaining operators from Eq. (3.1) can be obtained from the above operators by Hermitian conjugation.

The one-body term and the pairing interaction terms are treated first through the standard BCS formalism and consequently replaced by the quasiparticle one-body term $\sum_{\tau IM} E_\tau a^\dagger_{\tau IM} a_{\tau IM}$. In terms of quasiparticle creation ($a^\dagger_{\tau IM}$) and annihilation ($a_{\tau IM}$) operators, related to the particle operators by means of the Bogoliubov-Valatin transformation, the two-body interaction terms, involved in the model Hamiltonian, can be expressed just by replacing the operators (3.2) by their quasiparticle images. Thus, the Hamiltonian terms describing the quasiparticle correlations become a quadratic expression in the dipole and quadrupole two-quasiparticle and quasiparticle density operators:

$$\begin{aligned}
A^\dagger_{1\mu}(pn) &= \sum_{m_p,m_n} C^{I_p\ I_n\ 1}_{m_p\ m_n\ \mu} a^\dagger_{pI_pm_p} a^\dagger_{nI_nm_n}, \\
B^\dagger_{1\mu}(pn) &= \sum_{m_p,m_n} C^{I_p\ I_n\ 1}_{m_p\ -m_n\ \mu} a^\dagger_{pI_pm_p} a_{nI_nm_n}(-)^{I_n-m_n}, \\
A^\dagger_{2\mu}(\tau\tau') &= \sum_{m_\tau,m_{\tau'}} C^{I_\tau\ I_{\tau'}\ 2}_{m_\tau\ m_{\tau'}\ \mu} a^\dagger_{\tau I_\tau m_\tau} a^\dagger_{\tau' I_{\tau'} m_{\tau'}}, \\
B^\dagger_{2\mu}(\tau\tau') &= \sum_{m_\tau,m_{\tau'}} C^{I_\tau\ I_{\tau'}\ 2}_{m_\tau\ -m_{\tau'}\ \mu} a^\dagger_{\tau I_\tau m_\tau} a_{\tau' I_{\tau'} m_{\tau'}}(-)^{I_{\tau'}-m_{\tau'}}, \\
&\tau, \tau' = p, n.
\end{aligned} \tag{3.3}$$

Because the pnQRPA treatment of the dipole-dipole interaction in the particle-hole (ph) and pp channels run in an identical way as in our previous publications [13,14], here we provide no details about building the dipole proton-neutron phonon operator:

$$\Gamma^\dagger_{1\mu} = \sum_k [X_1(k)A^\dagger_{1\mu}(k) - Y_1(k)A_{1,-\mu}(k)(-)^{1-\mu}]. \tag{3.4}$$

We just mention that the amplitudes X and Y are determined by the pnQRPA equations and the normalization condition.

The charge-conserving QRPA bosons

$$\begin{aligned}
\Gamma^\dagger_{2\mu} &= \sum_k [X_2(k)A^\dagger_{2\mu}(k) - Y_1(k)A_{2,-\mu}(k)(-)^\mu], \\
k &= (p, p'), \quad (n, n')
\end{aligned} \tag{3.5}$$

are determined by the QRPA equations associated with the matrices:

$$\begin{aligned}\mathcal{A}_{\tau\tau'}(ik;i'k') &= \delta_{\tau\tau'}\delta_{ii'}\delta_{kk'}\left(E_i^\tau + E_k^\tau\right)\\ &\quad - X_{\tau\tau'}\left(q_{ik}^{(\tau)}\xi_{ik}^{(\tau)}\right)\left(q_{i'k'}^{(\tau)}\xi_{i'k'}^{(\tau)}\right),\\ \mathcal{B}_{\tau\tau'}(ik;i'k') &= -X_{\tau\tau'}\left(q_{ik}^{(\tau)}\xi_{ik}^{(\tau)}\right)\left(q_{i'k'}^{(\tau)}\xi_{i'k'}^{(\tau)}\right),\\ & i \leqslant k, \quad i' \leqslant k',\end{aligned} \tag{3.6}$$

where

$$\xi_{ik}^{(\tau)} = \frac{1}{\sqrt{1+\delta_{i,k}}}\left(U_i^\tau V_k^\tau + U_k^\tau V_i^\tau\right). \tag{3.7}$$

Here V_i^τ and U_i^τ denote the square roots of occupation and nonoccupation probabilities of the state i of $\tau\,(=p,n)$ type, respectively, given by the BCS equations. To distinguish between the phonon operators acting in the RPA space associated to the mother and daughter nuclei, respectively, one needs an additional index. Also, an index labeling the solutions of the RPA equations is necessary. Thus, the two kinds of bosons will be denoted by

$$\begin{aligned}&{}_j\Gamma_{1\mu}^\dagger(k), \quad j = i, f; \quad k = 1, 2, \ldots N_s^{(1)}; \quad {}_j\Gamma_{2\mu}^\dagger(k),\\ &\qquad j = i, f; \quad k = 1, 2, \ldots N_s^{(2)}.\end{aligned} \tag{3.8}$$

Acting with ${}_i\Gamma_{1\mu}^\dagger(k)$ and ${}_f\Gamma_{1\mu}^\dagger(k)$ on the vacuum states $|0\rangle_i$ and $|0\rangle_f$, respectively, one obtains two sets of nonorthogonal states describing the intermediate odd-odd nucleus. By contrast, the states ${}_i\Gamma_2^\dagger(k)|0\rangle_i$ and ${}_f\Gamma_2^\dagger(k)|0\rangle_f$ describe different nuclei, namely the initial and final ones, participating in the process of $2\nu\beta\beta$ decay. The mentioned indices are, however, omitted whenever their presence is not necessary.

IV. THE BOSON EXPANSION (BE) PROCEDURE

Within the boson expansion formalism, the basic operators $A_{1\mu}^\dagger(p,n)$, $A_{1\mu}$, $B_{1\mu}^\dagger(p,n)$, $B_{1\mu}$ are written as polynomial expansions in terms of the QRPA boson operators with the expansion coefficients determined such that their mutual commutation relations are preserved in each order of approximation [19]. Based on this criterion the boson expansions of the quadrupole two-quasiparticle and quadrupole quasiparticle density charge conserving operators were obtained by Belyaev and Zelevinsky in Ref. [19]. For charge nonconserving two-quasiparticle and quasiparticle density dipole operators the expansion has been derived by one of us (A.A.R., in Collaboration) in Ref. [15]. The latter expansion has the peculiarity that the commutator algebra cannot be satisfied restricting the expansion to the proton-neutron dipole bosons. However, this goal can be touched if the boson operators space is enlarged by adding the charge-conserving quadrupole two-quasiparticle bosons. The last step consists in expressing the quasiboson operators $\overset{0}{A}{}^\dagger_{1\mu}(pn)$, $\overset{0}{A}_{1\mu}(pn)$, $\overset{0}{A}{}^\dagger_{2\mu}(pp)$, $\overset{0}{A}{}^\dagger_{2\mu}(pp)$, $\overset{0}{A}{}^\dagger_{2\mu}(nn)$, $\overset{0}{A}_{2\mu}(nn)$ (these are, in fact the operators denoted by the same symbol but without the index of the superscript zero, with the commutators approximated to be of boson type) as linear combinations of the QRPA bosons. In this way the basic operators mentioned above are written as polynomials of pn and $pp+nn$ QRPA bosons. The expansions involve not only the collective but also noncollective QRPA bosons. The final expressions obtained in this way are

$$\begin{aligned}A_{1\mu}^\dagger(j_pj_n) &= \sum_{k_1}\left\{\mathcal{A}_{k_1}^{(1,0)}(j_pj_n)\Gamma_{1\mu}^\dagger(k_1)\right.\\ &\quad\left. + \mathcal{A}_{k_1}^{(0,1)}(j_pj_n)\Gamma_{1-\mu}(k_1)(-)^{1-\mu}\right\} + \sum_{k_1,k_2,k_3;l=0,2}\\ &\quad\times\left(\mathcal{A}_{k_3k_2k_1}^{(3,0);l}(j_pj_n)\{[\Gamma_2^\dagger(k_3)\Gamma_2^\dagger(k_2)]_l\Gamma_1^\dagger(k_1)\}_{1\mu}\right.\\ &\quad\left. + \mathcal{A}_{k_3k_2k_1}^{(0,3);l}(j_pj_n)\{[\Gamma_2(k_3)\Gamma_2(k_2)]_l\Gamma_1(k_1)\}_{1\mu}\right)\\ &\quad + \sum_{k_1,k_2,k_3;l=0,2}\left(\mathcal{A}_{k_1k_2k_3}^{1;(2\bar{2})l}(j_pj_n)\{\Gamma_1^\dagger(k_1)\right.\\ &\quad\times[\Gamma_2^\dagger(k_2)\Gamma_2(k_3)]_l\}_{1\mu} + \mathcal{A}_{k_3k_2k_1}^{(2\bar{2})l;1}(j_pj_n)\\ &\quad\left.\times\{[\Gamma_2^\dagger(k_3)\Gamma_2(k_2)]_l\Gamma_1(k_1)\}_{1\mu}\right)\\ B_{1\mu}^\dagger(j_pj_n) &= \sum_{k_1k_2}\left\{\mathcal{B}_{k_1k_2}^{(2,0)}(j_pj_n)[\Gamma_1^\dagger(k_1)\Gamma_2^\dagger(k_2)]_{l\mu}\right.\\ &\quad + \mathcal{B}_{k_1k_2}^{(0,2)}(j_pj_n)[\Gamma_1(k_1)\Gamma_2(k_2)]_{l\mu}\\ &\quad + \mathcal{B}_{k_1k_2}^{11;12}(j_pj_n)[\Gamma_1^\dagger(k_1)\Gamma_2(k_2)]_{l\mu}\\ &\quad\left. + \mathcal{B}_{k_1k_2}^{11;2l}(j_pj_n)[\Gamma_1^\dagger(k_2)\Gamma_1(k_1)]_{l\mu}\right\},\end{aligned} \tag{4.1}$$

where the expansion coefficients are those given in Ref. [15], whereas the notations for the dipole and quadrupole bosons introduced in the previous section have been used. The boson expansions associated to the two-quasiparticle and quasiparticle density proton-neutron operators have the property that the two sides of Eqs. (4.1) have the same matrix elements in a boson basis. Actually, this can be used as a criterion to determine the expansion coefficients. For example the first expansion coefficients in the above expression can be determined as:

$$\begin{aligned}\mathcal{A}_{k_1}^{(1,0)}(j_pj_n) &= \langle 0|[\Gamma_{1\mu}(k_1), A_{1\mu}^\dagger(j_p,j_n)]|0\rangle,\\ \mathcal{B}_{k_1k_2}^{(2,0)}(j_pj_n) &= \sum_{\mu_1,\mu_2} C_{\mu_1\mu_2\mu}^{121}\langle 0|\left\{\Gamma_{1\mu_1}(k_1),\right.\\ &\qquad\left.\times\left[\Gamma_{1\mu_2}(k_2), B_{1\mu}^\dagger(j_p,j_n)\right]\right\}|0\rangle.\end{aligned} \tag{4.2}$$

The properties of the nested commutators determine vanishing values for the coefficients accompanying the operators involving an even number of bosons in the $A^\dagger$ expansion and an odd number of bosons in the $B^\dagger$ expansion. Thus, the $A_{1\mu}^\dagger$ has an odd-order boson expansion, whereas $B_{1\mu}^\dagger$ exhibits an even-order expansion in bosons. It is worth mentioning that the matrix element of the double commutator involved in Eq. (4.2) does not depend on the order in which the commutators are performed. Indeed, the same result is obtained when (a) first the k_2 boson is commuted with $B^\dagger$ and the result is commuted with the k_1 boson and (b) first the dipole boson is commuted with $B^\dagger$ and the result is commuted with the quadrupole boson. However, the commutation order is important when one determines the remaining expansion coefficients. The ordering in the mentioned commutators is chosen such that the mutual commutator equations of the basic operators $A^{\dagger}_{1\mu}$, $B^{\dagger}_{1\mu}$ are satisfied in each order of approximation. The comparison of the boson expansion formulated in Ref. [15] and other approaches [20–23] may be found in Ref. [24].

V. THE GAMOW-TELLER TRANSITION AMPLITUDE

If the energy carried by leptons in the intermediate state is approximated by the sum of the rest energy of the emitted electron and half the Q value of the $\beta\beta$ decay process from the ground state of the mother nucleus to the first excited state 2^+ of the daughter nucleus,

$$\Delta E = m_e c^2 + \tfrac{1}{2} Q^{(0\to 2)}_{\beta\beta}, \tag{5.1}$$

the reciprocal value of the $2\nu\beta\beta$ half-life can be factorized as

$$T^{2\nu}_{1/2}(0^+_i \to 2^+_f)^{-1} = G_{02}\left|M^{(02)}_{\rm GT}\right|^2, \tag{5.2}$$

where G_{02} is the Fermi integral which characterizes the phase space of the process, whereas the second factor is the GT transition amplitude which, in the second order of perturbation theory, has the expression:

$$M^{(02)}_{\rm GT} = \sqrt{3}\sum_{k,m} \times \frac{{}_i\langle 0\|\beta^+\|k,m\rangle_{ii}\langle k,m|k',m'\rangle_{ff}\langle k',m'\|\beta^+\|2^+_1\rangle_f}{(E_{k,m}+\Delta E_2)^3}. \tag{5.3}$$

Here $\Delta E_2 = \Delta E + E_{1^+}$, with E_{1^+} standing for the experimental energy for the first state 1^+. The intermediate states $|k,m\rangle$ are k-boson states with $k = 1, 2, 3$ labeled by the index m, specifying the spin and the ordering label of the RPA roots. Note that by contrast to the case of ground-to-ground transition here the denominator has a cubic power which results in obtaining a suppression of the corresponding GT amplitude. Inserting the boson expansions from Eq. (4.1) into the expression of the β^+ transition operator one can check that the following nonvanishing factors, at numerator, show up:

$$\begin{aligned}
&{}_i\langle 0||{}_i\Gamma_1(k_1)||1,1_{k_1}\rangle_{if}\langle 1,1_{k_2}||{}_f\Gamma^{\dagger}_1(k_2)_f\Gamma_2(1)||1,2_1\rangle_f,\\
&{}_i\langle 0||{}_i\Gamma_1(k_1)_i\Gamma_2(ik_2)||2,1_{k_1}2_{k_2}\rangle_{if}\langle 2,1_{j_1}2_1||{}_f\Gamma^{\dagger}_1(j_1)||1,2_1\rangle_f,\\
&{}_i\langle 0||{}_i\Gamma_1(k_1)_i\Gamma_2(k_2)||2,1_{k_1}2_{k_2}\rangle_{if}\langle 2,1_{j_1}2_{j_2}\\
&||{}_f\Gamma^{\dagger}_1(j_1)_f\Gamma^{\dagger}_2(j_2)_f\Gamma_2(1)||1,2_1\rangle_f,\\
&{}_i\langle 0||{}_i\Gamma_1(k_1)_i\Gamma_2(k_2)_i\Gamma_2(k_3)||3,1_{k_1}2_{k_2}2_{k_3}\rangle_{if}\\
&\langle 3,1_{j_1}2_{j_2}2_1||{}_f\Gamma^{\dagger}_1(j_1)_f\Gamma^{\dagger}_2(j_2)||1,2_1\rangle_f.
\end{aligned} \tag{5.4}$$

The term $E_{k,m}$ from the denominator of Eq. (5.3) is the average of the energies of the mother and daughter states $|k,m\rangle$ normalized to the average energy of the first pnQRPA states 1^+ in the initial and final nuclei. The left low indices i and f suggest that the phonon operators are built up with quasiparticle operators characterizing the initial and final nuclei, respectively. Acting with the i and f dipole single or dipole multiphonon operators on the states $|0\rangle_i$ and $|2^+_f\rangle$ (or $|0\rangle_f$) one populates two sets of states $|1^+\rangle_i$ and $|1^+\rangle_f$, respectively, characterizing the odd-odd intermediate nuclei. The two sets are not orthogonal onto each other.

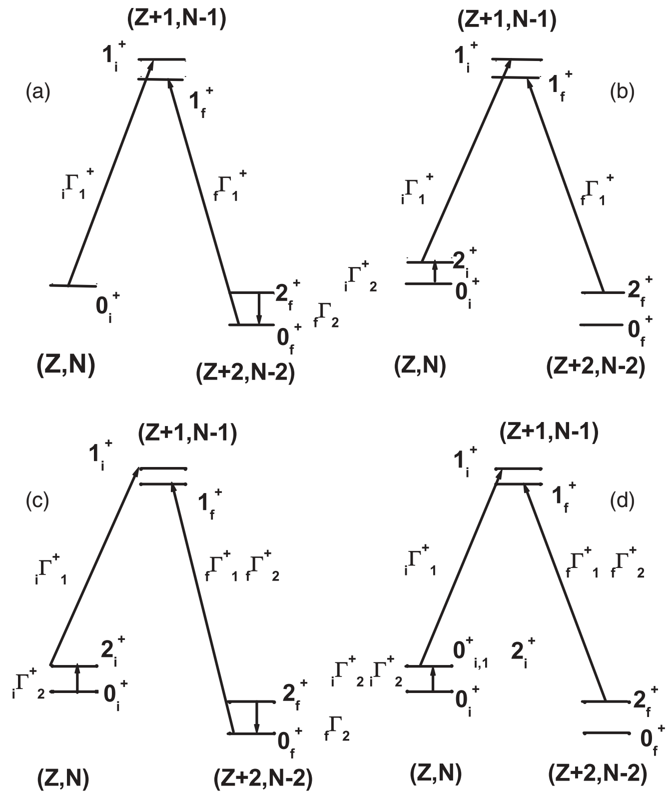

FIG. 1. One illustrates various GT transitions $0^+ \to 2^+$ via one (a) two (b) and (c) and three (d) phonon states.

The matrix elements, listed above, are associated to partial transition amplitudes represented pictorially in Fig. 1.

VI. NUMERICAL APPLICATION

Calculations were performed for 18 nuclei which have been previously considered in Refs. [13,14] for studying the $\beta\beta$ ground to ground decay. Among these, 11 are proved to be, indeed, $\beta\beta$ ground to ground emitters, whereas the remaining ones are suspected to have this property due to the corresponding positive Q value. Because the excitation energies for the states 2^+ in the daughter nuclei are not large, the Q values characterizing the $\beta\beta$ transition $0^+ \to 2^+$ are also positive. For some of the selected nuclei, experimental data either for the half-life of the process or for the low bounds of the half-lives are available.

The single-particle space, the pairing interaction treatment and the pnQRPA description of the dipole states describing the intermediate odd-odd nuclei used in the present article are identical with those from Refs. [13,14] for ground-to-ground transition. Therefore, to save the space, we do not present them again. The parameters determining the single-particle energies and the strengths of the pairing and dipole interactions are taken from our previous publications. However, the microscopic Hamiltonian used here involves in addition to the terms considered in the quoted references the quadrupole-quadrupole interaction between alike nucleons.

TABLE I. The experimental and calculated energies for the first 2^+ states in mother and daughter nuclei are given. The strength parameter of the quadrupole-quadrupole interaction was fixed such that the experimental energies are reproduced. In our calculations we considered $X_{pp} = X_{nn} = X_{pn}$. The oscillator length is denoted by $b = (\hbar/M\omega)^{1/2}$.

Nucleus	$E_{2^+}^{\text{exp.}}$ (keV)	$E_{2^+}^{\text{th.}}$ (keV)	b^4X_{pp}(keV)	Nucleus	$E_{2^+}^{\text{exp.}}$ (keV)	$E_{2^+}^{\text{th.}}$ (keV)	b^4X_{pp}(keV)
^{48}Ca	983.00	983.00	71.30	^{130}Te	839.49	831.03	12.12
^{48}Ti	983.52	979.02	42.80	^{130}Xe	536.07	534.2	17.28
^{76}Ge	562.93	558.88	50.80	^{134}Xe	847.04	841.75	20.00
^{76}Se	559.10	558.87	65.20	^{134}Ba	604.72	607.98	17.56
^{82}Se	654.75	654.73	19.10	^{136}Xe	1313.027	1314.90	16.37
^{82}Kr	776.52	776.84	25.84	^{136}Ba	818.49	810.30	14.82
^{96}Zr	1750.49	1465.62	2.00	^{148}Nd	301.702	298.00	24.29
^{96}Mo	778.24	776.81	38.10	^{148}Sm	550.250	553.00	24.64
^{100}Mo	535.57	534.43	31.50	^{150}Nd	130.21	135.24	27.32
^{100}Ru	539.5	536.11	19.70	^{150}Sm	330.86	333.12	22.45
^{104}Ru	358.03	358.45	29.80	^{154}Sm	81.976	83.05	22.62
^{104}Pd	555.81	561.83	20.90	^{154}Gd	123.070	123.35	20.37
^{110}Pd	373.8	370.45	44.65	^{160}Gd	75.26	73.13	18.70
^{110}Cd	657.76	662.85	25.10	^{160}Dy	86.788	87.25	19.03
^{116}Cd	513.49	514.50	30.50	^{232}Th	49.369	48.32	15.25
^{116}Sn	1293.56	1179.16	7.00	^{232}U	47.572	45.22	14.94
^{128}Te	743.22	746.12	12.12	^{238}U	44.916	47.34	12.91
^{128}Xe	442.91	449.58	19.43	^{238}Pu	44.076	46.15	14.83

As we already mentioned this interaction is needed to define the charge conserving quadrupole phonon operators used by the boson expansion procedure. Moreover, this interaction is used to describe the final state, i.e., 2^+, in the daughter nucleus. The strength of the QQ interaction was fixed by requiring that the first root of the QRPA equation for the quadrupole charge conserving boson is close to the experimental energy of the first 2^+ state. The results of the fitting procedure are given in Table I.

Having the RPA states defined, the GT amplitude has been calculated by means of Eq. (5.3), whereas the half-life with Eq. (5.2). The Fermi integral for the transition $0^+ \to 2^+$, denoted by G_{02}, was computed by using the analytical result given in Ref. [4].

The final results are collected in Table II. Therein one may find also the available experimental data as well as some theoretical results obtained with other approaches. One notices that the half-life is influenced by both the phase-space integral (through the Q value) and the single-particle properties that determine the transition amplitude. Indeed, for ^{128}Te and ^{134}Xe the small Q value causes a very large half-life, whereas in ^{48}Ca the opposite situation is met. By contrary, the Q value of ^{110}Pd is about the same as for ^{76}Ge but, due to the specific single-particle and pairing properties of the orbits participating coherently to the process, the half-life for the former case is more than three orders of magnitude less than in the later situation.

To isolate the deformation effect on the process half-life we repeated the calculations in the spherical limit, i.e., $d \to 0$. In this limit the single-particle energies coincide with the spherical shell-model energies. The process of going to the spherical limit alters the pairing properties as well as the energies of the quadrupole collective states 2^+ in mother and daughter nuclei. We modified the the strengths for pairing and QQ interactions such that the pairing gaps and the 2^+ energies are the same as in the deformed picture. Also we preserved the dimension for single-particle space for both protons and neutrons. For illustration we give here the result for the case of ^{100}Mo where the half-life for the deformed situation, 1.21×10^{25}, becomes in the spherical limit equal to 0.46×10^{24}. Thus, one may conclude that the nuclear deformation enhances the decay half-life. The same effect of deformation on the GT matrix elements was pointed out by Zamick and Auerbach in Ref. [26]. Indeed, they calculated the GT transition matrix elements for the neutrino capture $\nu_\mu + {}^{12}\text{C} \to {}^{12}\text{N} + \mu^-$ using different structures for the ground states of ^{12}C and ^{12}N: (a) spherical ground states, (b) asymptotic limits of the wave functions, and (c) deformed states with an intermediate deformation of $\delta = -0.3$. The results for the transition rate were $\frac{16}{3}$, 0, and 0.987, respectively. Similar results are obtained also for the spin $M1$ transitions in ^{12}C. The ratio between the transition rates obtained with spherical and deformed basis explains the factor of 5 overestimate in the calculations of Ref. [27], where a spherical basis is used. It is worth mentioning the good agreement between our prediction for ^{100}Mo and that of Ref. [23] obtained with a deformed SU(3) single-particle basis.

As we mentioned in Ref. [13], the experimental data for the energy of first 1^+ in the odd-odd isotopes of ^{150}Pm and ^{238}Np are not yet available. For these cases we took the values 137 and 1000 keV, respectively. These values are suggested by the energies in the neighboring odd-odd isotopes. Although the GT amplitude $M_{\text{GT}}^{0\to 0}$ is not very sensitive to the experimental energy to which the first pnQRPA energy is normalized, the situation for $M_{\text{GT}}^{0\to 2}$ is different because in this case the energy denominator has a cubic power. Indeed we repeated the calculations for ^{150}Nd by taking for the first 1^+, an energy equal to 1 MeV. The half-life that corresponds to this choice

TABLE II. The GT transition amplitudes and the half lives of the $\beta\beta$ decay $0^+ \to 2^+$ are given. Also, the Q values are given in units of m_ec^2. ΔE_2 is the energy shift defined in the text. For comparison, we give also the available experimental results as well as some theoretical predictions obtained with other formalisms: [a,b]Ref. [23], [c]Ref. [21],[d,e] [23] for different nuclear deformations, $\beta = 0.28$ and $\beta = 0.19$, respectively. The M_{GT} values for the ground to ground transitions are also listed. For ^{100}Mo we mention the result of Ref. ([23]) obtained with an SU(3) deformed single particle basis[a] and with a spherical basis[b].

Nucleus	$Q_{\beta\beta}^{2^+}$ (m_ec^2)	ΔE_2 (MeV)	$\lvert M_{\text{GT}}^{(0\to 0)}\rvert$ (MeV^{-1})	$\lvert M_{\text{GT}}^{(0\to 2)}\rvert$ (MeV^{-3})	$T_{1/2}^{(0\to 2)}$(yr) Present	Exp.	Ref. [20]
^{48}Ca	6.432	2.473	0.043	0.901×10^{-3}	1.72×10^{24}		
^{76}Ge	2.894	1.295	0.222	0.558×10^{-3}	5.75×10^{28}	$>1.1 \times 10^{21}$	1.0×10^{26}
^{82}Se	4.37	1.708	0.096	0.259×10^{-3}	1.7×10^{27}	5.8×10^{23}	3.3×10^{26}[c]
^{96}Zr	5.033	2.913	0.113	0.834×10^{-3}	2.27×10^{25}	$>7.9 \times 10^{19}$	4.8×10^{21}
^{100}Mo	4.874	1.756	0.305	0.136×10^{-2}	1.21×10^{25}	$>1.6 \times 10^{21}$	3.9×10^{24}
							2.5×10^{25}[a]
							1.2×10^{26}[b]
^{104}Ru	1.456	0.883	0.781	0.028	6.2×10^{28}		
^{110}Pd	2.646	1.182	0.263	0.050	1.48×10^{25}		
^{116}Cd	2.967	1.269	0.116	0.507×10^{-2}	3.4×10^{26}	$>2.3 \times 10^{21}$	1.1×10^{24}
^{128}Te	0.836	1.305	0.090	0.229×10^{-2}	4.7×10^{33}	$>4.7 \times 10^{21}$	1.6×10^{30}
^{130}Te	3.902	2.358	0.055	0.620×10^{-3}	6.94×10^{26}	$>4.5 \times 10^{21}$	2.7×10^{23}
^{134}Xe	0.460	0.806	0.039	0.621×10^{-2}	5.29×10^{35}		
^{136}Xe	3.251	1.518	0.039	0.249×10^{-2}	3.88×10^{26}		2.0×10^{24}
^{148}Nd	2.7	1.99	0.559	$1.408 \cdot 10^{-3}$	9.97×10^{27}		
^{150}Nd	5.9	3.087	0.546	2.668×10^{-3}	1.5×10^{23}	$>8 \times 10^{18}$	7.2×10^{24}[d]
							1.2×10^{25}[e]
^{154}Sm	2.2	1.172	0.311	1.238×10^{-3}	1.41×10^{29}		
^{160}Gd	3.2	1.739	0.624	6.799×10^{-3}	4.56×10^{25}		
^{232}Th	1.6	1.084	0.262	2.142×10^{-3}	1.399×10^{30}		
^{238}U	2.2	1.344	0.152	0.692×10^{-3}	2.84×10^{29}		

is 1.08×10^{24}. Thus, the discrepancy between our result and that from Ref. [23] is likely to be caused by different values for the first 1^+ energy and nuclear deformations.

The transition matrix elements reported in Refs. [20,21] are larger than those given here. The discrepancies are caused by the differences between the two approaches: (a) in the quoted references one uses a spherical single-particle basis, whereas here a deformed one is considered; and (b) the single-particle energies used there are Woods-Saxon energies adjusted so that the quasiparticle spectrum in the odd-odd system be realistically described. We recall that the spherical limit of our model provides spherical shell-model single-particle energies. Also, the single-particle spaces are different in the two formalisms; (c) the higher RPA approach from Ref. [20] is the multiple commutator method (MCM) applied to the pnQRPA bosons or, alternatively [21], the renormalized pnQRPA bosons. A detailed comparison of the boson expansion formalism and MCM were performed in Ref. [24]. It is a difficult task to make explicit the quantitative effect brought by the factors (a), (b), (c), which, as a matter of fact, is beyond the scope of the present article. However, concerning the sources (a) and (c) for the deviations one could draw some qualitative conclusions. Indeed, as we have already seen before, the nuclear deformation decreases the transition matrix element and consequently enhances the process half-life. The MCM and boson expansion approaches provide different expressions for the terms that are cubic in bosons, involved in the transition operator. Indeed, the coefficients of these terms given by MCM are cubic in the forward amplitudes (X), whereas in the boson expansion formalism the expansion coefficients of the mentioned terms are at most quadratic in the amplitudes X. One expects, therefore, that MCM provides larger matrix elements for these terms, which results in having a shorter half-life. Thus, the effects caused by the factors (a) and (c) are consistent with the sign of the discrepancies of results corresponding to the two approaches.

To have a reference value for the matrix elements associated to the transition $0^+ \to 2^+$, Table II also lists the M_{GT} values for the ground-to-ground transitions [14]. The ratio of the transition $0^+ \to 0^+$ and $0^+ \to 2^+$ matrix elements is quite large for ^{76}Ge (398), ^{100}Mo (224), and ^{96}Zr (136) but small for ^{110}Pd (5.26), ^{134}Xe (6.3), and ^{150}Nd (20.46). However, these ratios are not directly reflected in the half-lives, because the phase-space factors for the two transitions are very different from each other and, moreover, the differences depend on the atomic mass of the emitter.

The composing terms of the transition amplitude are represented in Fig. 1 as suggestions. The term corresponding to Fig. 1(d) has a negligible contribution and, therefore, has been ignored. The terms corresponding to the panels (a), (b), and (c) of Fig. 1 are denoted by $M_{\text{GT}}^{(1)}$, $M_{\text{GT}}^{(2)}$, $M_{\text{GT}}^{(3)}$ respectively. To see the relative contribution of the three terms to the

TABLE III. The values of the partial Gamow-Teller transition amplitudes $M_{\rm GT}^{(1)}$, $M_{\rm GT}^{(2)}$, $M_{\rm GT}^{(3)}$, are given for some of the nuclei studied in the present article. Their sum is denoted by $M_{\rm GT}^{(0\to 2)}$.

Nucleus	$M_{\rm GT}^{(0\to 2)}\times 10^6$ (MeV^{-3})	$M_{\rm GT}^{(1)}\times 10^6$ (MeV^{-3})	$M_{\rm GT}^{(2)}\times 10^6$ (MeV^{-3})	$M_{\rm GT}^{(3)}\times 10^6$ (MeV^{-3})
^{48}Ca	901.100	900.000	0.07	−0.004
^{76}Se	−258.722	−269.412	9.380	1.31
^{105}Ru	28281.530	26110.000	1583.810	587.72
^{148}Nd	−1408.148	3734.201	−5360.949	218.60
^{150}Nd	2668.170	3909.451	−1417.082	175.80
^{154}Sm	1237.572	358.452	916.240	−37.12
^{160}Gd	−6799.463	−5813.393	−820.580	−165.49
^{232}Th	2142.316	1780.647	−165.981	527.65
^{238}U	692.181	−88.013	815.784	−35.59

total amplitude, in Table III we give the partial contribution for some of the nuclei. We see that for some nuclei the term $M_{\rm GT}^{(1)}$ prevails, whereas for others $M_{\rm GT}^{2}$ is the dominant partial amplitude. Similar situations are met for the nuclei not listed here. None for the cases has $M_{\rm GT}^{(3)}$ as a dominant term. The leading contributions coming from $M_{\rm GT}^{(1)}$ and $M_{\rm GT}^{(2)}$ have opposite sign. There are, however, two exceptions, the cases of ^{104}Ru and ^{154}Sm, where the two contributions add coherently.

VII. CONCLUSIONS

In the previous sections we presented the formalism as well as the numerical results for the two neutrino $\beta\beta$ decay to the collective excited state 2^+. The Gamow-Teller transition rate has been calculated within a boson expansion formalism that is essentially a higher random-phase approximation approach. The single-particle basis is generated through an angular momentum projection procedure from a deformed set of states. The projected basis depends on a real parameter d that simulates the nuclear deformation. In the limit of $d \to 0$ the spherical shell-model basis is obtained, whereas for d different of zero, the single-particle energies depend on the deformation parameter in a similar manner as the energies predicted by the Nilsson model. Due to these features the present formalism is able to describe in an unified fashion the spherical and deformed nuclei. In our previous publications we treated various situations when the mother and daughter had different deformations, in the context of the ground-to-ground $\beta\beta$ transition. We have seen that deformation causes a fragmentation of the single β decays strength among the pnQRPA states. One expects that for the transition $0^+ \to 2^+$ the nuclear deformation is even more important. This can be understood even at the first glance because the larger the deformation of the daughter nucleus the lower the energy of the first 2^+ state. Consequently, the Q value is expected to be larger.

It is worth noticing that during the transition $0^+ \to 2^+$ several symmetries might be broken. Indeed, the second leg of the transition connects a magnetic state 1^+ from the intermediate odd-odd nucleus to an electric state 2^+ in the daughter nucleus. Among the nuclei considered in the present work there are situations when the mother nucleus is spherical, whereas the daughter is a quadrupole deformed system. Moreover, in the case of ^{160}Gd decay there are suspicions that the mother has not a good space reflection symmetry [25], whereas the daughter satisfies this symmetry. Because the GT transition operator involves quadrupole phonon operators it may excite states whose isospin is different from that characterizing the mother ground state by $\Delta T = 1, 2$. The isospin mixing is also favored by the inclusion of the *pp* interaction. However, each symmetry breaking causes a new nuclear phase with specific properties. To our knowledge it is still an open question how these symmetry breaking are reflected in the decay rate. On this line, the results of the present work suggest to what direction the decay rate is modified by the nuclear deformation.

Concerning the quantitative description, the results presented in Table II reveal the following features. There are five nuclei whose half-lives fall in the range accessible to experiment. These are ^{48}Ca, ^{96}Zr, ^{100}Mo, ^{110}Pd, ^{150}Nd, ^{160}Gd. Comparing with the results obtained by Toivanen and Suhonen [20] or Civitarese and Suhonen [22], the half-lives obtained in the present work are larger. The reason is that we use a deformed single-particle basis, whereas the quoted authors use a spherical one. The agreement we obtain for ^{100}Mo with the calculations from Ref. [23], where a deformed SU(3) basis is used, support the above statement.

It is worth mentioning that the $\beta\beta$ transitions to excited states have been considered by several authors in the past, but the calculations emphasized the role of the transition operator and some specific selection rules. Many calculations regarded the neutrinoless process. Thus, in Ref. [28] it was shown that the neutrinoless transition to the excited 0^+ for medium heavy nuclei might be characterized by matrix elements that are larger than that of ground-to-ground transition and that happens because in the first transition, the change of the K quantum number is less. In Ref. [29] it has been stated that the $0^+ \to 2^+$ matrix element depends on the left-right current coupling and not on the neutrino mass. This could provide a way of fixing the strength of the left-right coupling if the transition matrix element is experimentally known. However, according to the calculations of Haxton *et al.* [2], the matrix element is strongly suppressed and, therefore, the mentioned method of fixing the coupling parameter would not be reliable. Although the transition operator might have a complex structure, many calculations have been performed with the approximate interaction $[\sigma(1)\times\sigma(2)]^{\lambda=2}t_+(1)t_+(2)$ to test some selection rules. Thus, this interaction was used in Ref. [30] for the transition $0^+ \to 2^+$ of ^{48}Ca, using a single j calculation. It has been proved that the matrix element for this transition is suppressed due to the signature selection rules. Actually, this result confirms the feature of suppression for the $0^+ \to 2^+$ $\beta\beta$ transition matrix element pointed out by Vergados [31] and Haxton *et al.* [2].

The transition to 0_1^+ was examined for $A = 76, 82, 100, 136$ nuclei by assuming light and heavy Majorana neutrino exchange mechanism and trilinear R-parity contribution. Higher

RPA as well as renormalization effects for the nuclear matrix elements were included [32].

Here we show that the transition $0^+ \to 2^+$ in a $2\nu\beta\beta$ process is allowed by renormalizing the GT transition operator with some higher RPA corrections, which results in making the matrix elements from Eq. (5.4) nonvanishing.

The calculated $M_{\rm GT}$ values of the present work are smaller than those from Ref. [20] obtained with a spherical single-particle basis, which agrees with the earlier calculations of Zamick and Auerbach for ^{12}C, showing that the nuclear deformation suppresses the GT matrix elements.

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